CONTENTS

Preface x

SECTION 1: ALGEBRA 1 1-29

1. SETS
   - Algebra of sets
   - De' Morgan's laws
   - Two-set problems
   - Three-set problems
   - Review exercise 1 30-45

2. SURDS
   - Definition, Reduction to basic form
   - Addition and subtraction of surds
   - Rules of surds, Multiplication of surds
   - Conjugate surds
   - Surds of the form $a+b\sqrt{n}$ where n is not a perfect square
   - Rationalizing surds with binomial denominators
   - Review exercise 2 46-63

3. BINARY OPERATIONS
   - Definition
   - Closure
   - Properties of binary operations
   - Identity and inverse elements
   - Review exercise 3 64-91

4. RELATIONS AND FUNCTIONS
   - Relations, mappings and functions
   - The functional notation
   - Types of functions
   - Domain of a function, range of a function
   - Graphs of functions
   - Inverse of a function
   - Composite functions
   - Review exercise 4

5. **POLYNOMIAL FUNCTIONS** 92-129
Linear and quadratic functions
Sketching curves of quadratic functions
Solving quadratic equations
Completing the square
Quadratic formula
Roots of quadratic equations
Definition of polynomial functions
Cubic functions
Algebra of polynomials
Remainder and factor theorems
The factors and roots of polynomial functions/equations
Review exercise 5

6. **RATIONAL FUNCTIONS** 130-151
Rational functions of the form $Q(x) = \dfrac{f(x)}{g(x)}$, $g(x) \neq 0$
Operations on rational functions
Partial fractions
Review exercise 6

7. **INDICES AND LOGARITHMIC FUNCTIONS** 152-182
Laws of indices
Equations involving indices
Laws of logarithms
Equations involving logarithm and change of base
Graphs of exponential relations and their applications
Review exercise 7

8. **BINOMIAL THEOREM** 183-210
Binomial Theorem
Combination method
Expansion and use of $(1+x)^n$ to approximate exponential values
Review exercise 8

9. **INEQUALITIES AND LINEAR PROGRAMMING** 211-231
Linear inequalities in two variables
Linear programming
Quadratic inequalities
Review exercise 9

SECTION 2 : COORDINATE GEOMETRY I

10. **STRAIGHT LINE** 232-256
Distance between two points
Gradient of a line joining two points
Midpoint of a line segment
Division of a line segment in a given ratio
Equations of a straight line
Parallel and perpendicular lines
Perpendicular distance from a line
Acute angle between two intersecting lines
Review exercise 10

SECTION 3 : STATISTICS AND PROBABILITY I

11. **STATISTICS** 257-303
Definition of terms, frequency distribution tables
Grouped frequency tables, class intervals, class boundaries
Histogram,
Cumulative frequency diagrams
Measures of central tendency (mean, mode, median, quartiles, percentiles)
Measures of dispersion (mean and standard deviations, variance, range, inter-quartile range)
Review exercise 11

12. **PROBABILITY 1** 304-337
Equally likely events
Mutually exclusive and independent events
Conditional probability
Review exercise 12

SECTION 1 : CO-ORDINATE GEOMETRY II

13. **THE CIRCLE** 338-357
Equation of a circle
Finding the centre and radius given the equation of a circle
Tangent and normal to a circle
Loci
Review exercise 13

14. **PARABOLA** 358-364
Equation of a parabola
Sketching a parabola
Tangent and normal to a parabola
Review exercise 14

SECTION 2 : ALGEBRA II

15. SEQUENCES AND SERIES 365-395
Application of linear sequence
Exponential Sequence (Geometric Progression GP)
Recurrence sequence
Review exercise 15

SECTION 3 : TRIGONOMETRY

16. TRIGONOMETRIC RATIOS AND RULES 396-414
Basic trigonometric ratios and their reciprocals
Angles in radians
Sine and Cosine rules
Application of sine and cosine rules to bearings
Review exercise 16

17. COMPOUND AND MULTIPLE ANGLES 415-424
Compound angles
Multiple angles
Review exercise 17

18. TRIGONOMETRIC FUNCTIONS 425-436
Graphs of trigonometric functions – tan *x*
Solving trigonometric equations (up to quadratic)
Maximum and minimum points of trigonometric functions
Review exercise 18

SECTION 4 : CALCULUS

19. DIFFERENTIATION 437-465
Limit of a function
Limit of simple trigonometric functions
Differentiation from first principles
Rules of differentiation
Product rule, quotient rule, chain rule
Higher derivatives, differentiation of implicit functions
Differentiation of trigonometric functions
Review exercise 19

20. APPLICATIONS OF DIFFERENTIATION 466-504
  - Gradient function
  - Equations of tangent and normal to a curve
  - Rates of changes, small changes and percentage changes
  - Maxima and minima
  - Increasing and decreasing functions,
  - Curve sketching
  - Linear kinematics
  - Review exercise 20

21. INTEGRATION 505-514
  - Indefinite integrals
  - Definite Integrals
  - Integration by substitution
  - Integration of simple trigonometric functions
  - Review exercise 21

22. APPLICATIONS OF INTEGRATION 515-545
  - Finding the equation of a curve when the gradient function is given
  - Kinematics
  - Areas under curves
  - Volumes of revolution
  - The Trapezium Rule
  - Review exercise 22

SECTION 5 : VECTORS AND MECHANICS

23. VECTORS IN A PLANE 546-572
  - Representation of vectors in standard basis form
  - Zero vector, negative vectors, magnitude of a vector
  - Equality of vectors, Addition and subtraction of vectors,
  - Scalar multiplication
  - Resultant of vectors
  - Triangle, parallelogram and polygon laws of addition
  - Scale drawing and resolution of vectors
  - Properties of addition of vectors
  - Position vectors and unit vectors
  - Review exercise 23

24. SCALAR (DOT) PRODUCT 573-586
  - Concept of Scalar (Dot) product
  - Angle between two vectors
  - The sine and cosine rules
  - Compound angles for cosine and sine
  - Direction vectors
  - Review exercise 24

SECTION 1 : VECTORS AND MECHANICS II

25. **STATICS** 587-624
Scalar and vector quantities
Coplanar forces acting at a point
Resultant of forces
Equilibrium of particles
Triangle of forces, Lami's theorem
Moments of forces
Friction, ladder problems
Equilibrium on an inclined plane
Review exercise 25

26. **DYNAMICS** 625-656
Concepts of motion
Equations of motion
Motion under Gravity
Relative velocity, resultant velocity
Newton's laws of motion
Motion along inclined planes
Momentum, impulse
Principle of conservation of linear momentum
Direct impact
Review exercise 26

SECTION 2 : MATRICES AND LINEAR TRANSFORMATIONS

27. **MATRICES** 657-671
Idea of a matrix
Equal matrices
Addition and subtraction of matrices
Multiplication of matrices
Multiplication of a matrix by a scalar
Multiplication of matrices
Determinant of a 2 x 2 matrix
Determinant of a 3 x 3 matrix
Inverse of a 2 x 2 matrix
Review exercise 27

28. **LINEAR TRANSFORMATIONS** 672-694
Images of points and objects under linear transformation
Matrix of a linear transformation
Inverse of a linear transformation
Composition of linear transformations
Identity transformations
Linear transformation of a line
Review exercise 28

SECTION 3: LOGIC

29. LOGIC 695-702
Compound statements and converses of statements
Implications
The truth table
Validity of arguments
Review exercise 29

SECTION 4 : STATISTICS AND PROBABILITY II

30. CORRELATION 703-726
The scatter diagram
Forms of correlation
Line of best fit
Equation of line of best fit
Spearman's rank correlation coefficient
Review exercise 30

31. PERMUTATION AND COMBINATION 727-744
Permutation- ordered choices
Circular arrangement
Combination
Drawing with replacement
Drawing without replacement
Review exercise 31

32. THE BINOMIAL PROBABILITY DISTRIBUTION 745-754
Application of combination
Binomial probability
Review exercise 32

33. SELF ASSESSMENT QUESTIONS 755-774

34. HOW PREPARED ARE YOU? 775- 828

ANSWERS TO REVIEW EXERCISES 829-848

Preface

This book has been specifically written to cater for the needs of students who are preparing for the West Africa Senior School Certificate Examination (WASSCE) in **Elective** or **Further Mathematics**. As a revision guide, special care has been taken in the presentation of steps in problem solving for easy understanding. *The book covers in detail, all the topics in the new **WASSCE** syllabuses.*

The treatment is straightforward and does not attempt to cover more than what the syllabus requires but rather to represent the material as clearly as possible. Only the most essential theory is dealt with and a basic knowledge of arithmetic algebra is all that is needed – a welcome change from most other books on the subject.

The book has also been so structured that brief and concise notes are given under each topic before solutions to standard questions are presented. The author's many years of teaching Mathematics has convinced him that, a combination of careful explanation, worked examples and adequate supply of exercises will help the student to understand the concepts and principles underlying each topic.

The mathematical steps are also so simple that even the average student can follow them independently.

It is, therefore, earnestly hoped that this book, when used alongside classroom tuition, will train the student to systematically attempt any question, however complicated, with the utmost confidence. Although the primary objective of the book is to help Senior High School Students, it may prove helpful to those studying in other equivalent schools or colleges.

Misprints are a plaque to authors of Mathematical books. Although every care has been taken to check mistakes and misprints, it is not too much to expect imperfections, especially when I am reminded of the phrase "to err is human". All such imperfections, when brought to my notice, will be duly corrected in the subsequent editions.

It has been God's grace that this book has come out successfully, I therefore give God all the praise and gratitude. Amen!

I wish to acknowledge my indebtedness to my dear wife, Mrs. Wendy Asiedu, for her unfailing assistance and encouragement.

I also thank my students, readers and numerous reviewers for their help with the previous and current editions. For this edition, I am grateful to Henric Attah Baah–Yeboah for his immense contribution.

Peter Asiedu

CHAPTER 1

SETS AND OPERATIONS ON SETS

Definition of a set

A **set** is a well-defined collection of objects of the same kind. For example, the set of colours of the rainbow i.e. {red, orange, yellow, green, blue, indigo, violet} or the set of the first six letters of the alphabet. i.e. $\{a, b, c, d, e, f\}$.

Set notation

A set is denoted or represented by a capital letter. For example, A, B, P, Q, etc. and the members of a set are enclosed in curly brackets { }.

The components or objects of a set are called **members** or **elements** of the set.
For example, if $A = \{2, 4, 6, 8\}$, then 2 is a member or an element of A i.e. $2 \in A$.
Also $4 \in A$. The symbol $\in$ means "is a member of".
We can also say that 5 is not a member of A i.e. $5 \notin A$.
The symbol $\notin$ means "is not a member of".

The ***number of members or elements*** in set $A = \{2, 4, 6, 8\}$ is 4 and this is denoted as $n(A) = 4$.

Ways of describing sets

A set may be described by;

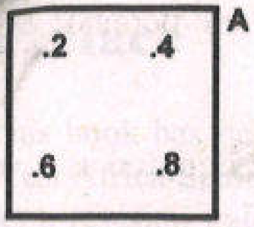
1. *listing its members.*
For example,
$P = \{2, 3, 5, 7, 11\}$ or $Q = \{1, 2, 3, 4, 5 \ldots\}$.
The three dots after the 5 represent all the missing members of Q. It means the set Q has other members which we have not written down. There is always a clear pattern which will help us obtain the missing members. For example, the missing members of the set $N = \{1, 3, 5, 7, \ldots, 17\}$ are 9, 11, 13 and 15.

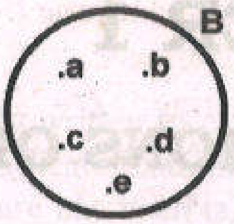
2. *giving a word description of its members* or *defining a property of the set,*
For example, the sets P and Q above can be written as
P = {prime numbers less than 12} and
Q = {positive integers}.

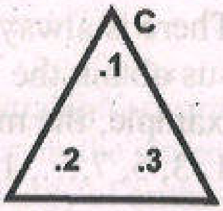
3. *using a set-builder notation.*
For example, the set $P = \{2, 3, 5, 7, 11\}$ can be written as
$P = \{x : 1 < x < 12$, where x is a prime number$\}$.
Also, the set $N = \{1, 2, 3, \ldots, 10\}$ can be written in the set-builder notation as
$N = \{x : 1 \leq x \leq 10$, where x is an integer$\}$.
Similarly, $Q = \{a, e, i, o, u\}$ can be written in the set-builder notation as:
$Q = \{x : x$ is a vowel in English alphabet$\}$
The symbol x is simply used to represent any member of the set under consideration.
Other symbols such as n, or y can also be used.

Venn diagrams

A set can be represented by any closed figure, such as a rectangle, circle, triangle etc. The diagrams representing sets are called Venn diagrams. Venn diagrams are therefore pictorial representations of sets. For example, $A = \{2, 4, 6, 8\}$, $B = \{a, b, c, d, e\}$ and $C = \{1, 2, 3\}$ can be represented by the diagrams in the next page.







The Venn diagram also helps you to give a clearer picture of the relationship between one set and another. In a Venn diagram, the universal set is represented by a set of points in a rectangle and every subset is represented by a circle. The points inside the diagram represent members of the set. Sometimes, the diagram can be used to represent the *number of members* of various sets, from which deductions can be made (see examples under two-set and three-set problems).

Types of sets

Finite set

A set whose last member can be found is a **finite set**. For example, the set $A = \{1, 2, 3, 4, 5\}$ is a finite set with five members i.e. a definite number of elements.
Also $B = \{1, 3, 5, 7\}$ is a finite set with four members. Note that the elements of a finite set can be counted.

Infinite set

A set whose last member cannot be found is an **infinite set**. For example, the set $Q = \{1, 2, 3, 4, 5, 6, ...\}$, which is the set of positive integers is an infinite set. The three dots means the set continues indefinitely or has no end. Note that the objects of an infinite set cannot be counted as it continues indefinitely.

Null or empty set

A set which has *no* member or element is called a **null set** or an **empty set**.
A null set is denoted by $\varnothing$ or $\{\}$.
Note that the set $A = \{0\}$ is not an empty set since 0 is a member of set A.

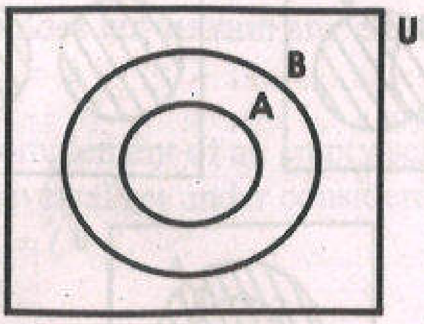
Unit set

A **unit** set is a set which has only *one* member. For example, $P = \{5\}$ or $Q = \{2\}$ is a unit set.
Also $A = \{0\}$ is a unit set since zero is the only member.

Subsets

If all the members of a set A belong to a set B, then A is said to be a **subset** of B written as $A \subset B$ (also read as A is contained in B).
Equivalently, we can write $B \supset A$ which means A is a **superset** of B or B contains A.
The symbols $\subset$ and $\supset$ mean "subset of" and "superset of", respectively.
In order not to be confused about how to use the symbols it is advisable to stick to the use of one. It is easier to use $\subset$.
For example, if $A = \{2, 4, 6\}$ and $B = \{2, 4, 6, 8, 10, 12\}$ then $A \subset B$ since all the members of set A belong to set B. Also if $P = \{1, 3\}$ and $Q = \{1, 2, 3, 4\}$ then $P \subset Q$.
If $C = \{0, 1\}$, then C has 4 subsets $\{0\}, \{1\}, \{0, 1\}$ and $\{\ \}$.
Note that the null set $\varnothing$ is a subset of any set and we allow any set to be a subset of itself. So $\{0, 1\} \subseteq \{0, 1\}$ even though they are equal. This idea is only of use in higher mathematics. In general, if a set has n members then it has 2^n subsets.
For example, $A = \{a, b, c\}$ has 3 members and therefore has $2^3 = 8$ subsets i.e. $\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \varnothing$ and $\{a, b, c\}$.

The following Venn diagram shows $A \subset B$.



Universal set

It is the set of all the members or elements under discussion. It is denoted by the symbol U or ξ. When a particular set is defined, we consider it as a set of a larger set which is referred to as the **universal set**. For example, if the set of triangles is under discussion then the universal set is taken to be the set of all polygons. The universal set is represented by a rectangle in a Venn diagram.

Equal sets

Two sets are said to be **equal** if they have the *same members*. For example, if $A = \{2, 3, 5\}$ and $B = \{5, 2, 3\}$ then $A = B$ since each member of set A is also a member of set B. Also if $A = \{1, 3, 5, 7\}$, $B = \{3, n, 5, 1\}$ and $A = B$ then $n = 7$.

Note that the order of listing the members may or may not be the same.

Equivalent sets

Two sets are said to be **equivalent** if they have the same *number of members*.

For example, if $A = \{2, 3, 5\}$ and $B = \{11, 12, 13\}$ then A is said to be equivalent to B since they both have *three* members.

Listing the elements of a set

Given that $Z = \{x : 2 \leq x \leq 6$, where x is an integer$\}$, we can **list** the elements of the set and determine the number of elements of as follows:

Listing: $Z = \{2, 3, 4, 5, 6\}$;

$\therefore$ Number of elements of Z denoted as $n(Z) = 5$

The number of elements in a set is called the **cardinality** of the set. Therefore the cardinality of the set Z is 5. The cardinality of the null set is 0 and the cardinality of the unit set is 1.

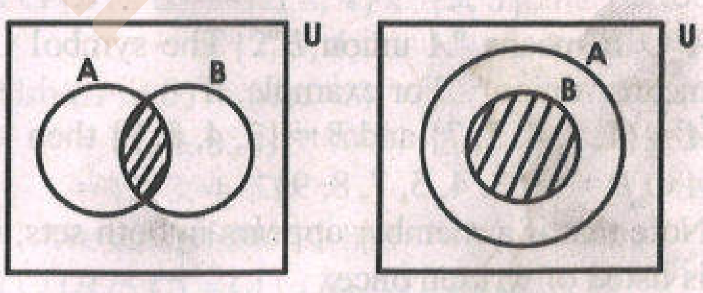
Algebra of sets

The three operations on sets are intersection ($\cap$), union ($\cup$) and complement ($'$).

Intersection of sets

The set of objects which belong to set A and at the same time set B is called the **intersection** of A and B, i.e. $A \cap B$. The symbol $\cap$ means "intersection". Therefore $P \cap Q$ means the intersection of the sets P and Q. For example, if $A = \{2, \mathbf{4}, 6, \mathbf{8}\}$ and $B = \{1, \mathbf{4}, \mathbf{8}\}$ then $A \cap B = \{4, 8\}$ since 4 and 8 belong to both sets A and B.

The shaded regions in the figure below represent $A \cap B$.



Illustrative examples

Example 1

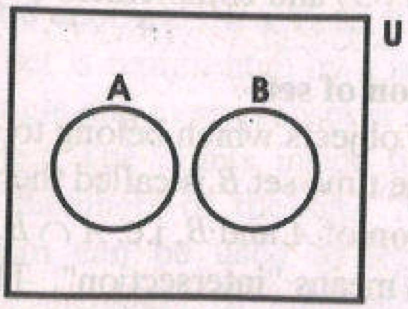
If $P = \{\mathbf{3}, 6, \mathbf{9}\}$ and $Q = \{1, \mathbf{3}, 5, 7, \mathbf{9}\}$ then $P \cap Q = \{3, 9\}$

Example 2

If $Q = \{4, 8, \mathbf{12}, 16\}$ and $R = \{\mathbf{12}, 13, 14\}$ then $Q \cap R = \{12\}$

Disjoint sets

If *no* member of a set A belongs to a set B then the sets A and B are **disjoint sets**. In particular, if A and B do not intersect then they are disjoint sets. For example, if $P = \{2, 4, 6, 8\}$ and $Q = \{1, 3, 5, 7\}$then P and Q are disjoint sets since no member of P belongs to Q, i.e. $P \cap Q = \varnothing$. Similarly, $X = \{1, 3, 5\}$ and $Y = \{2, 6\}$ are disjoint sets since there are no common elements and so $X \cap Y = \varnothing$. Disjoint sets are said to be mutually exclusive. The Venn diagram below shows two disjoint sets A and B.



Union of two sets

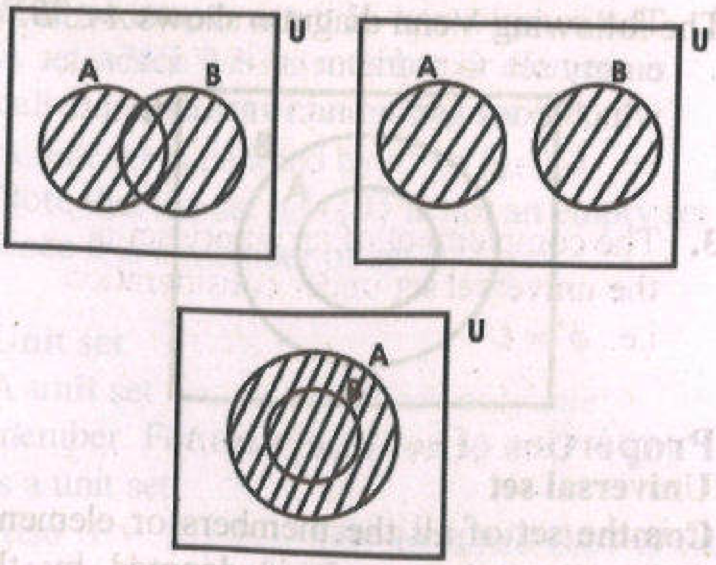
The **union** of two sets A and B is the set formed by putting the two sets together. In particular, the union of sets A and B is the set of elements that belong to either A or B or both.

$A \cup B$ means "A union B". The symbol $\cup$ means "union". For example, if $A = \{1, 3, 4, 5, 7\}$ and $B = \{3, 4, 8, 9\}$ then $A \cup B = \{1, 3, 4, 5, 7, 8, 9\}$.

Note that if a member appears in both sets, it is listed or written once.

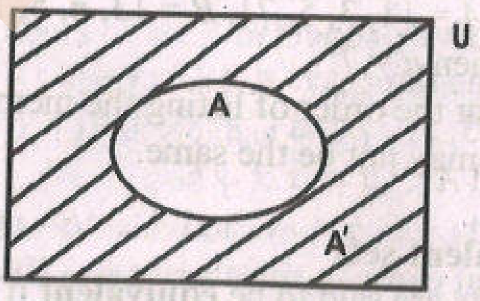
For the above example, 3 and 4 appeared in both A and B but they were written once in $A \cup B$.

Note also that members of a set are usually listed or written in ascending order (smallest members first). The shaded regions in the following diagram represent $A \cup B$.



Complement of a set

If A is a subset of the universal set U or ξ then the **complement** of A is the set of members which belong to the universal set U but *do not* belong to A. The complement of A is written as A'. In particular, the complement of a set is equal to elements in the universal set *minus* elements in the set. For example, if $\xi = \{1, 2, 3, 4, 5, 6\}$ and $A = \{3, 4, 5\}$, then $A' = \{1, 2, 6\}$. Clearly if we did not specify the universal set, ξ, there would be rather a lot of elements not in A. The shaded region in the diagram below is A'.



Properties of the complement of a set

1. The complement of the complement of set A is itself.
   For example, $U = \{1, 2\ 3, 4, 5, 6\}$ and $A = \{1, 3, 5\}$ then $A' = \{2, 4, 6\}$
   Again taking the complement of A', we get, $(A')' = A$

2. The complement of the universal set is an empty set. Complement of U is the set which does not contain any element.
i.e. $U' = \phi$

3. The complement of an empty set is the universal set under consideration.
i.e. $\phi' = U$

Properties of set operations

Commutative property

Given the two sets $A = \{2, 3, 4\}$ and $B = \{1, 3, 5, 7\}$

i) Find $A \cap B$
ii) Find $B \cap A$
iii) Find $A \cup B$
iv) Find $B \cup A$
v) What can you say about $A \cap B$ and $B \cap A$
vi) What can you say about $A \cup B$ and $B \cup A$

From the above activity we can conclude that
$A \cap B = B \cap A = \{3\}$ and
$A \cup B = B \cup A = \{1, 2, 3, 4, 5, 7\}$

Therefore intersection ($\cap$) and union ($\cup$) of sets are commutative.

Associative property

Given the three sets $A = \{2, 3, 4\}$, $B = \{1, 3, 5, 7\}$ and $C = \{3, 4, 5\}$

i) Find $A \cup B$ and $B \cup C$
ii) Find $(A \cup B) \cup C$ and $A \cup (B \cup C)$
iii) What can you say about $(A \cup B) \cup C$ and $A \cup (B \cup C)$?
iv) Find $A \cap B$ and $B \cap C$
v) Find $(A \cap B) \cap C$ and $A \cap (B \cap C)$
vi) What can you say about $(A \cap B) \cap C$ and $A \cap (B \cap C)$?

From the above activity,
$(A \cap B) \cap C = \{3\} \cap \{3, 4, 5\} = \{3\}$
[Note: Work out the operation in the bracket first]

Also $A \cap (B \cap C) = \{2, 3, 4\} \cap \{3, 5\} =$

Similarly, $(A \cup B) \cup C = A \cup (B \cup C)$
$= \{1, 2, 3, 4, 5, 7\}$

We can conclude that
$(A \cap B) \cap C = A \cap (B \cap C)$ and
$(A \cup B) \cup C = A \cup (B \cup C)$.

Therefore intersection ($\cap$) and union ($\cup$) of sets are both associative.

Distributive property

Let $A = \{2, 3, 4\}$, $B = \{1, 3, 5, 7\}$ and $C = \{3, 4, 5\}$

i) Find $B \cup C$ and $A \cap (B \cup C)$
ii) Find $A \cap B$ and $A \cap C$
iii) Find $(A \cap B) \cup (A \cap C)$
iv) What can you say about $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$?
v) Find $B \cap C$ and $A \cup (B \cap C)$
vi) Find $A \cup B$ and $A \cup C$
vii) Find $(A \cup B) \cap (A \cup C)$
viii) What can you say about $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$?

From the above activity,
$A \cup (B \cap C) = \{2, 3, 4\} \cup \{3, 5\}$
$= \{2, 3, 4, 5\}$

Also $(A \cup B) \cap (A \cup C)$
$= \{1, 2, 3, 4, 5, 7\} \cap \{2, 3, 4, 5\}$
$= \{2, 3, 4, 5\}$

$\therefore$ We can conclude that:
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Therefore union ($\cup$) of sets is distributive over intersection ($\cap$) of sets.

Similarly, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$= \{3, 4\}$

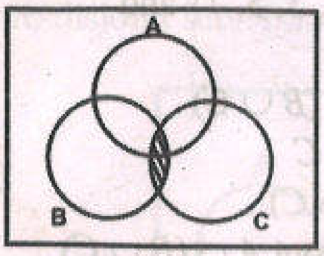
Therefore intersection ($\cap$) of sets is distributive over union ($\cup$) of sets.

Illustrating properties of set operations using Venn diagrams

The relationship:

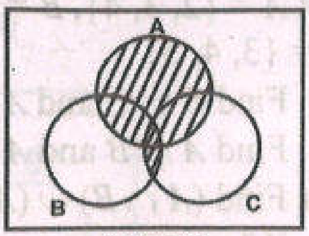
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ can be verified using the diagrams below.

The operations in these diagrams are shaded in stages, so as to get a clear idea of what is happening. The operation represented by the shaded portion is indicated below each diagram.



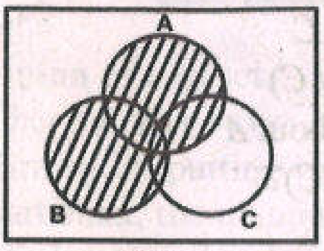
$(B \cap C)$

(i)



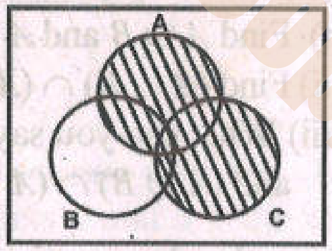
$A \cup (B \cap C)$

(ii)



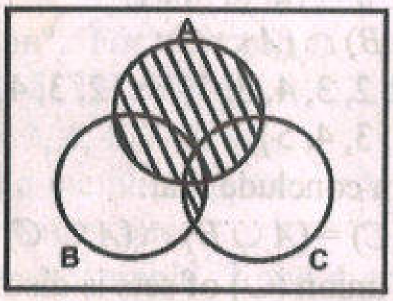
$A \cup B$

(iii)



$A \cup C$

(iv)



$(A \cup B) \cap (A \cup C)$

(v)

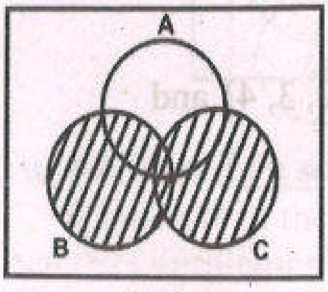
Shaded region in (ii) = Shaded region in (v)

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

The relationship:

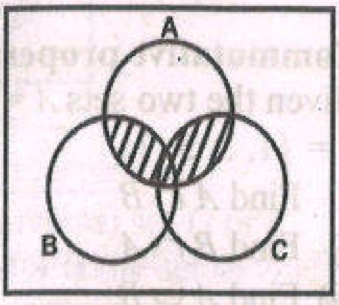
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ can also be verified using the diagrams below.

The operations in these diagrams are shaded in stages, so as to get a clear idea of what is happening. The operation represented by the shaded portion is indicated below each diagram.



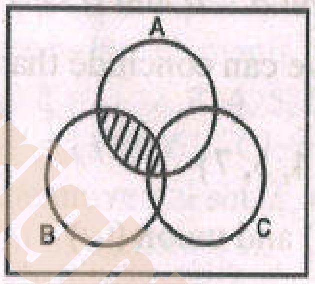
$B \cup C$

(i)



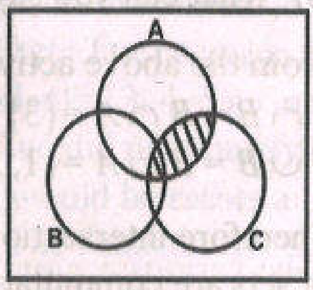
$A \cap (B \cup C)$

(ii)



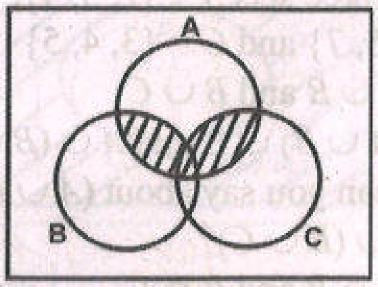
$A \cap B$

(iii)



$A \cap C$

(iv)



$(A \cap B) \cup (A \cap C)$

(v)

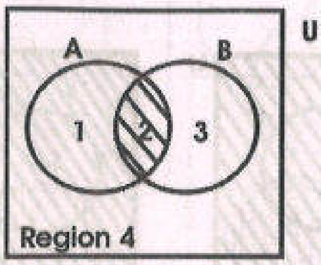
Shaded region in (ii) = Shaded region in (v)

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Description and identification of regions of Venn diagrams

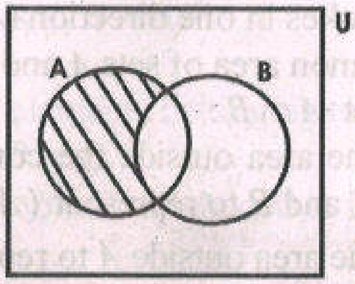
Two intersecting sets

If U is the universal set and A and B are two intersecting subsets of the universal set then the figure below shows how to illustrate or represent the sets A and B in a Venn diagram.

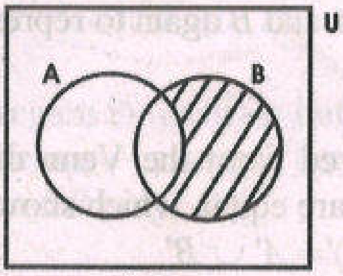


There are **four** regions.
Region 1 is $A \cap B'$,
i.e. elements belonging to ***only*** A.
Region 2 is $A \cap B$,
i.e. elements belonging to ***both*** A and B.
Region 3 is $A' \cap B$
i.e. elements belonging to ***only*** B
Region 4 is $(A \cup B)'$
i.e. elements belonging to neither A nor B.

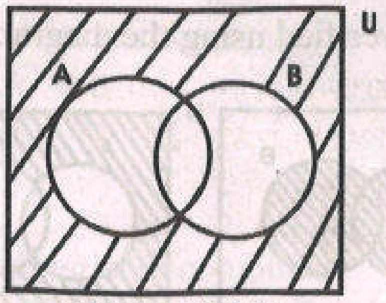
The various regions for two intersecting sets are shown in diagram (i) to (vi).



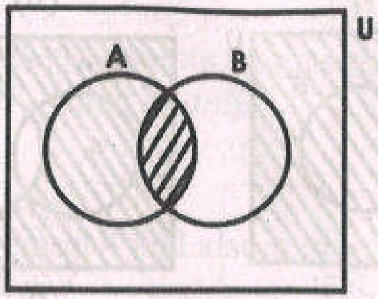
$A \cap B'$
(i)



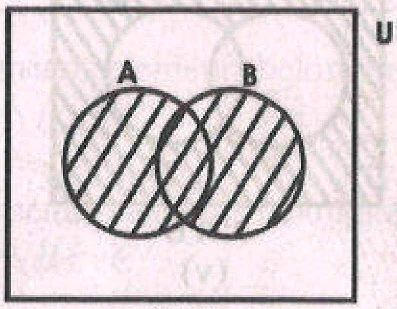
$A' \cap B$
(ii)



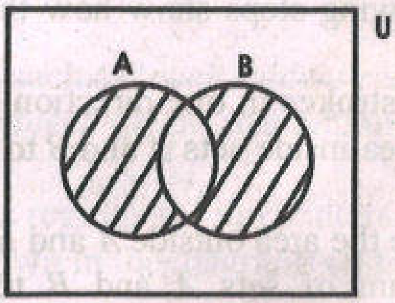
$A' \cap B'$ or $(A \cup B)'$
(iii)



$A \cap B$
(iv)



$A \cup B$
(v)



$(A \cap B') \cup (B \cap A')$
(vi)

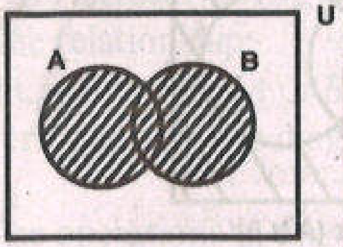
De' Morgan's laws

De' Morgans laws are two results involving union, intersection and complement of two intersecting sets. They are stated below.

1. $(A \cup B)' = A' \cap B'$

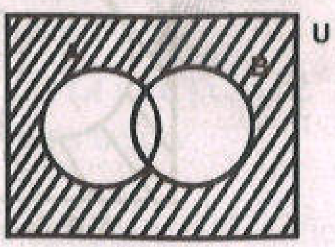
2. $(A \cap B)' = A' \cup B'$

Law **1** is verified using the diagrams below.



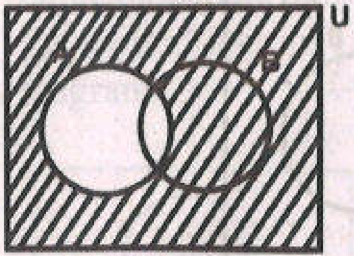
$A \cup B$

(i)



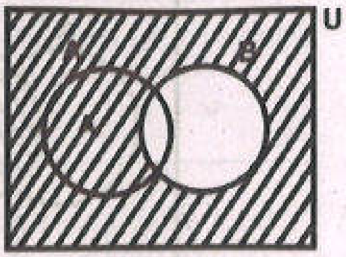
$(A \cup B)'$

(ii)



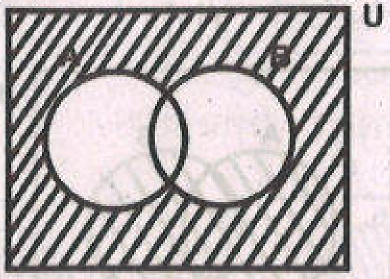
A'

(iii)



B'

(iv)



$A' \cap B'$

(v)

The operations are shaded in stages.

The following steps show how the shadings are done.

i) With strokes in one direction (///), shade the area inside sets A and B to represent $A \cup B$.

ii) Shade the area outside A and B in a Venn diagram of sets A and B to represent $(A \cup B)'$.

iii) Shade the area outside A to represent A'.

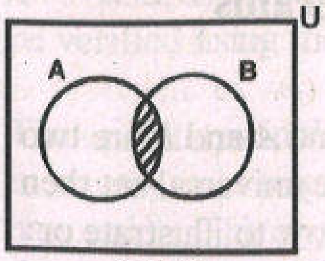
iv) Shade the area outside B to represent B'.

v) Shade the area outside A and B again to represent $A' \cap B'$.

It is observed from the Venn diagrams that (ii) and (v) are equal which, shows that

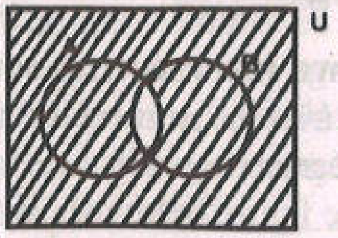
$$(A \cup B)' = A' \cap B'$$

Law **2** is verified using the diagrams below.



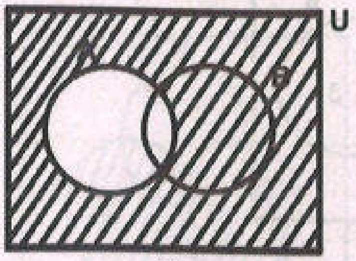
$A \cap B$

(i)



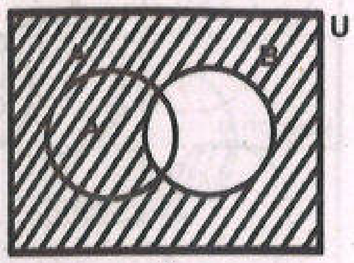
$(A \cap B)'$

(ii)



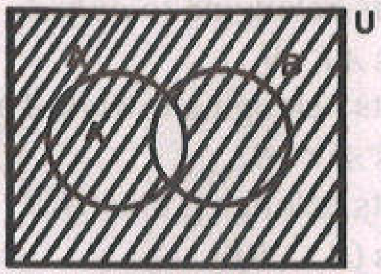
A'

(iii)



B'

(iv)



$A' \cup B'$

(v)

The following steps show how the shadings are done.

i) With strokes in one direction (///), shade the common area of sets A and B to represent $A \cap B$.

ii) Shade the area outside the common area of sets A and B to represent $(A \cap B)'$.

iii) Shade the area outside A to represent A'.

iv) Shade the area outside B to represent B'.

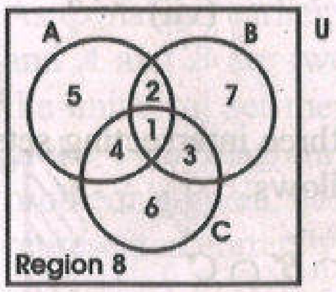
v) Shade the area outside the common area of sets A and B again to represent $A' \cup B'$.

It is observed from the Venn diagrams that (ii) and (v) are equal, which shows that

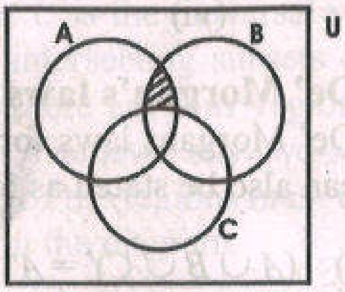
$$(A \cap B)' = A' \cup B'$$

Three intersecting sets

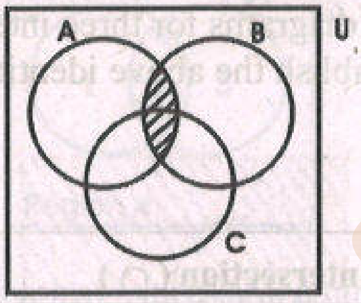
If U is the universal set and A, B and C are three intersecting subsets of U, then the diagrams below show how to illustrate or represent the three sets A, B, and C in a Venn diagram.



(i)



(ii) $A\cap B\cap C'$



(iii)

For three intersecting sets there are **eight** (8) regions.

Region 1 = $A \cap B \cap C$

Note: A member in region 1 belongs to all the three sets A, B and C.

Region 2 = elements belonging to both A and B ***only***
= $A\cap B\cap C'$ as shaded in fig(ii) above

Region 1 and region 2 combined
= elements belonging to both A and B
= $A \cap B$ as shaded in fig(ii) above

The shaded region in fig(iii) is both A and B. Note the difference between 'both A and B' and 'both A and B **only**' etc.

Region 3 = elements belonging to both B
C ***only***
= $A' \cap B \cap C$

Similarly,
Region 1 and region 3 combined
= elements belonging to both B and C
= $B \cap C$.

But region 3 = elements belonging to both B and C ***only***
= $A' \cap B \cap C$.

What are region 4 and also regions 1 and 4 combined?

Region 5 means elements belonging to A *only* = $A \cap B' \cap C'$

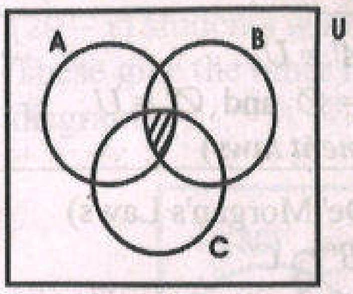
Region 6 means elements belonging to C *only* = $A' \cap B' \cap C$

Region 7 means elements belonging to B *only* = $A' \cap B \cap C'$

Region 8 means elements belonging to neither *A or B* nor $C = (A \cup B \cup C)'$

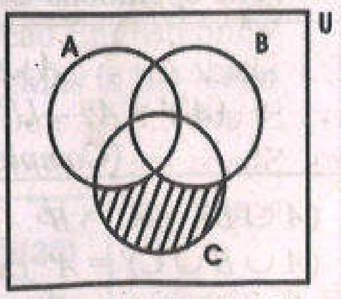
Statements such as '*each student reads at least one subject*' means that region 8 is zero.

The various regions for three intersecting sets are shown in the diagrams (i) to (vii).



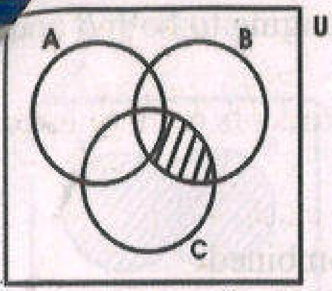
$A \cap B \cap C$ or $(A\cap B)\cap(B\cap C)$

(i)



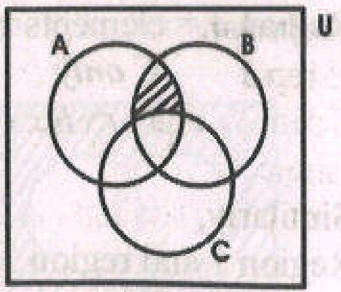
$A\cap(B \cap C') \cup (A \cup B') \cap C)$

(ii)



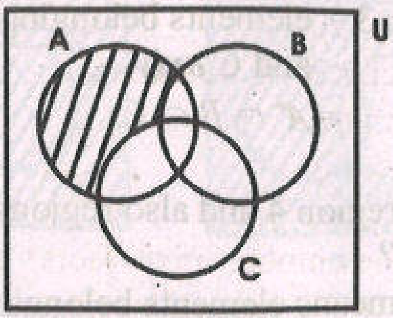
$B \cap C \cap A'$

(iii)



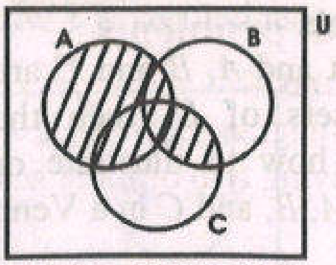
$A \cap B \cap C'$

(iv)



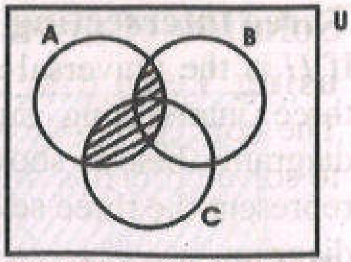
$A \cap B' \cap C'$

(v)



$A \cup (B \cap C)$ or $(A \cup B) \cap (A \cup C)$

(vi)



$A \cap (B \cup C)$ or $(A \cap B) \cup (A \cap C)$

(vii)

De' Morgan's laws

De' Morgans laws for three intersecting sets can also be stated as follows:

i) $(A \cup B \cup C)' = A' \cap B' \cap C'$

ii) $(A \cap B \cap C)' = A' \cup B' \cup C'$

Use Venn diagrams for three intersecting sets to establish the above identities.

Summary of some important set identities

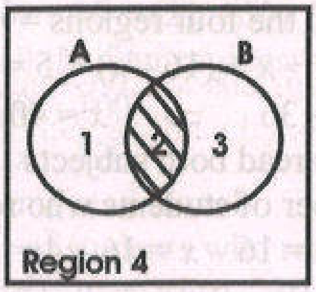
Union ($\cup$)	Intersection ($\cap$)
$A \cup A = A$	$A \cap A = A$
$A \cup \varnothing = A$	$A \cap \varnothing = \varnothing$
$A \cup U = U$	$A \cap U = A$
$A \cup B = B \cup A$ (*Commutative property*)	$A \cap B = B \cap A$ (*Commutative property*)
$A \cup (B \cup C) = (A \cup B) \cup C$ (*Associative property*)	$A \cap (B \cap C) = (A \cap B) \cap C$ (*Associative property*)
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (The operations $\cup$ is *distributive* over $\cap$)	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (The operations $\cap$ is *distributive* over $\cup$)
$A \cup A' = U$ $(A')' = A$; $U' = \varnothing$ and $\varnothing' = U$ (*Complement laws*)	$A \cap A' = \varnothing$ (*Complement law*)
$(A \cup B)' = A' \cap B'$ (De' Morgan's Laws) $(A \cup B \cup C)' = A' \cap B' \cap C'$ *Note:* $(A \cup B')' = A' \cap B$ since $(B')' = B$	$(A \cap B)' = A' \cup B'$ (De' Morgan's Laws) $(A \cap B \cap C)' = A' \cup B' \cup C'$
$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$

Solution of practical problems using Venn diagrams

The concept of Venn diagrams can be used to solve problems involving two or three sets.

Two - set problems

As explained earlier, if U is the universal set and A and B are two intersecting subsets of the universal set then there are **four** regions in the Venn diagram. Whenever any two-set problem is given, each of the regions must be represented correctly on the diagram.



The mathematical expression connecting these regions is

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$ ……..(1)

where $n(A \cup B) = n(U) - n(A \cup B)'$

If there is no complement, then $n(A \cup B)' = 0$ and $n(U) = n(A \cup B)$

Also

$n(U)$ = region 1 + region 2 + region 3 + region 4 ………..(2)

Statements such as '*each student reads at least one subject*' means that region 4 is zero. Note that $n(A)$ means the number of elements of set A.

Illustrative examples

Example 1

In a class of 40 students, 30 read Mathematics and 20 read English. Each student reads at least one subject. How many students read both Mathematics and English?

Solution

Step 1

Identify the number of students reading each subject.

Let U = {students in the class}

$n(U) = 40$

M = {students who read Mathematics}

$n(M) = 30$

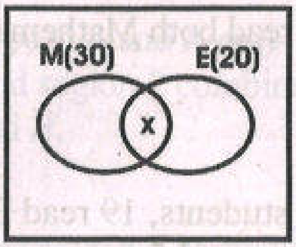
E = {students who read English}

$n(E) = 20$

x = number of students who read both subjects

Step 2

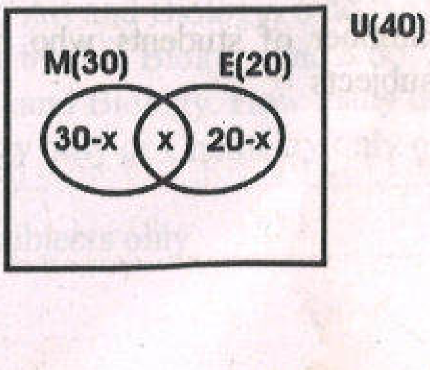
Identify the number of subjects. There are two subjects. Draw a Venn diagram of two intersecting sets and label them as shown below.



Note: Number of students who read both subjects is represented with a variable say x i.e. $n(M \cap E) = x$

Step 3

There are 30 students in the Mathematics subset, so there are $(30 - x)$ students who read Mathematics only. Similarly, there are 20 students in the English subset, so there are $(20 - x)$ students who read English only. These give the other regions in the Venn diagram.



Region 4 is zero since each student reads at least one subject.

Sum of all the four regions = $n(U) = 40$

$\therefore$ From the Venn diagram i.e. Fig 1.9(ii)

$$(30 - x) + x + (20 - x) = 40$$
$$30 - x + x + 20 - x = 40$$
$$50 - x = 40$$
$$x = 50 - 40 = 10$$

$\therefore$ 10 students read both Mathematics and English.

Alternative method

$n(M \cup E) = n(M) + n(E) - n(M \cap E)$

Since region 4 is zero ;

$n(U) = n(M \cup E) = 40;\ n(M) = 30;\ n(E) = 20$

$$40 = 30 + 20 - x$$
$$40 = 50 - x$$
$$x = 50 - 40 = 10$$

$\therefore$ 10 students read both Mathematics and English.

Example 2

In a class of 36 students, 19 read Biology, 16 read Chemistry and 5 were not allowed to read both subjects. Illustrate the information on a Venn diagram. Find how many students read

i) Both Biology and Chemistry

ii) Only Chemistry

Solution

Let $U = \{$students in the class$\}$

$n(U) = 36$

$B = \{$students who read Biology$\}$

$n(B) = 19$

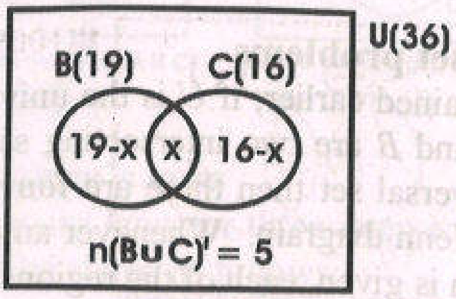
$C = \{$students who read Chemistry$\}$

$n(C) = 16$

x = number of students who read both subjects

Since 5 students were not allowed to read both subjects it implies region 4 is **5** as shown in the Venn diagram below

i.e. $n(B \cup C)' = 5$.



From the Venn diagram,

The sum of all the four regions = $n(U) = 36$

i.e. $(19 - x) + x + (16 - x) + 5 = 36$

$40 - x = 36 \quad \Rightarrow \quad x = 40 - 36 = 4$

i) 4 students read both subjects

ii) The number of students who read only chemistry = $16 - x = 16 - 4 = 12$

Example 3

A survey of the reading habits of 100 students showed that 30 read both Comics and Novels, 10 read neither Comics nor Novels and twice as many students read Comics as read Novels.

i) Illustrate the above information in a Venn diagram

ii) How many students read novels?

iii) How many students read comics only?

Solution

Let $U = \{$students$\} \quad \Rightarrow n(U) = 100$

$N = \{$students who read Novels$\}$

$n(N) = x$

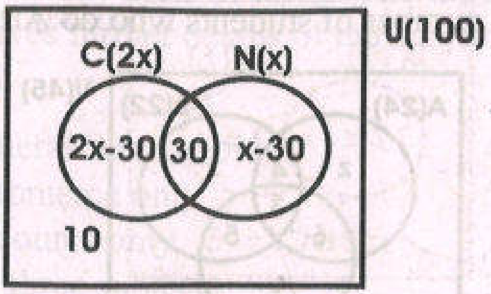
$C = \{$students who read Comics$\}$

$n(C) = 2x$

Also from the question, $n(C \cap N) = 30$

Note: x = number of students who read novels

Since 10 students read neither Comics nor Novels it means region 4 is **10** as shown in the diagram below. i.e. $n(C \cup N)' = 10$. Note this very well



From the Venn diagram,

$$(x - 30) + 30 + (2x - 30) + \mathbf{10} = 100$$

i.e. sum of all the four regions $= n(U) = 100$

$$3x - 20 = 100$$

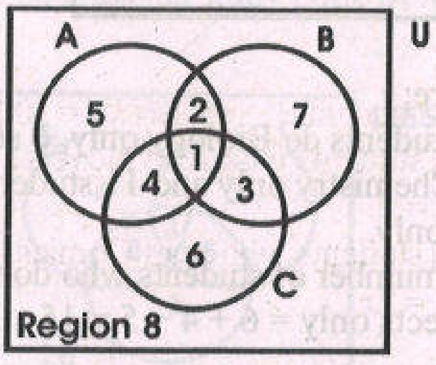
$$3x = 120 \qquad \therefore\ x = 40$$

ii) 40 students read Novels

iii) Number of students who read Comics only $= 2x - 30 = 2(40) - 30 = 50$

Three - set problems

As explained earlier, if U is the universal set and A, B and C are three intersecting subsets of the universal set then there are **eight** regions in the Venn diagram below. Whenever any three-set problem is given, each of the regions must be represented correctly on the diagram.



Note that we can also use the following *mathematical relationships* for three-set problems.

The sum of all the regions $= n(U)$
i.e. Region $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = n(U)$ ……………(1)

Also

$n(A \cup B \cup C)$
$= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$……………….(2)

$n(A \cup B \cup C) + n(A \cup B \cup C)' = n(U)$…..(3)

When there is no complement (i.e. when region 8 is zero), then $n(A \cup B \cup C)' = 0$ and
$n(A \cup B \cup C) = n(U)$…………………(4)

Again

Region $1 + 2 + 4 + 5 = n(A)$
Region $1 + 2 + 3 + 7 = n(B)$
Region $1 + 3 + 4 + 6 = n(C)$

Note the meanings of the following carefully:

Region 2 = both *A* and *B* ***only***
Region 1 and region 2 combined = both *A* and *B*.

Similarly,
Region 1 and region 3 combined = Both *B* and *C*.
But region 3 is both *B* and *C* ***only***.
Region 8 = none of *A*, *B* or *C*

Statements such as '*each student reads at least one subject*' means that region 8 is zero.

Illustrative examples

Example 1

In a class of 45 students, it is known that 24 of them do Art, 20 do Chemistry and 22 do Biology. All the students do at least one of the three subjects. 3 do all the three subjects while 7 do Art and Biology. 6 do Art and Chemistry but not Biology and 8 do Chemistry and Biology. How many do

i) Biology only or Chemistry only or Art only

ii) Two subjects only

Solution

Step 1

Identify the number of students reading each subject.

Let $U = \{\text{Students in the class}\}$

$n(U) = 45$

$A = \{\text{Students who do Art}\}$

$n(A) = 24$

$B = \{\text{Students who do Biology}\}$

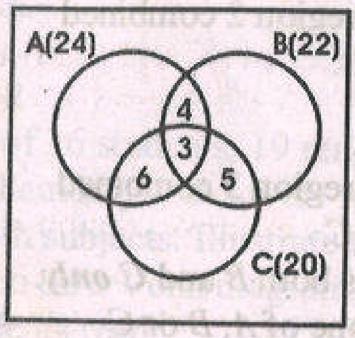
$n(B) = 22$

$C = \{\text{Students who do Chemistry}\}$

$n(C) = 20$

Step 2

Identify the number of subjects. There are three subjects. Draw a Venn diagram of three intersecting sets and label them as shown below using the information given in the problem.



Note that it is advisable to represent the intersection of all the three sets first. (In this example, it is 3)

7 students do both Arts and Biology. Hence the region of intersection of Arts and Biology subsets is 7. But 3 students do all three subjects. Therefore 4 students (i.e. 7 – 3) do Arts and Biology *only*.

Similar analysis is done to obtain the other regions.

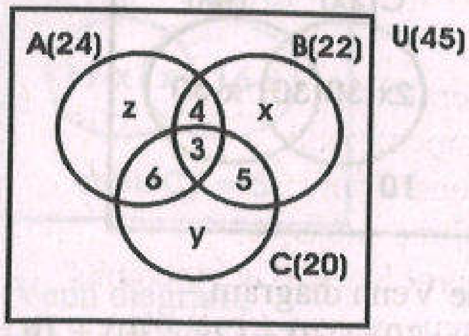
Since all the students do at least one of the three subjects, it means region 8 in fig.1.18 is zero.

Step 3

Let x = number of students who do Biology only

y = number of students who do Chemistry only

z = number of students who do Art only



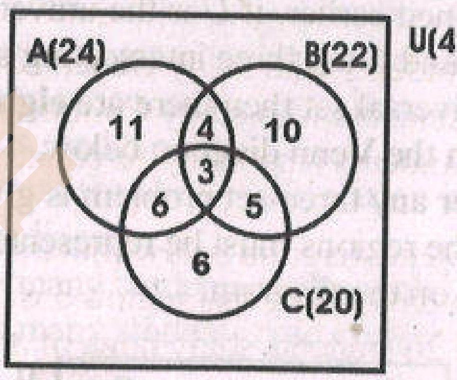
From the Venn diagram, we can write:

$4 + 3 + 5 + x = 22 \Rightarrow x = 22 - 12 = 10$

$6 + 3 + 5 + y = 20 \Rightarrow y = 20 - 14 = 6$

$4 + 3 + 6 + z = 24 \Rightarrow z = 24 - 13 = 11$

The complete Venn diagram showing the information is shown below.



Therefore;

i) 10 students do Biology only, 6 students do Chemistry only and 11 students do Art only.

ii) The number of students who do two subjects only = 6 + 4 + 5 = 15

Note that when solving problems, information given on one particular region may involve other regions as in the above example.

Example 2

In a class of 55 students, some study at least one of the following subjects: General Science, Commerce and Accounts. 20 students study none of them. The following table gives further details of the subject studied.

General Science only	4
Commerce only	5
Accounts only	7
All three subjects	3
General Science & Accounts	7
Commerce and Accounts	8

a) Illustrate the above data in a Venn diagram

b) Find the number of students who study
  i) General Science or Accounts or both but not Commerce
  ii) Commerce

Solution

Let U = {student in the class}

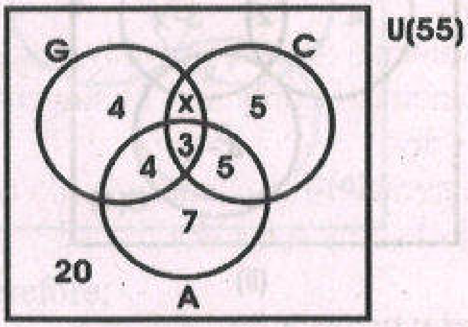
$n(U) = 55$

G = {students who study General Science}

C = {students who study Commerce}

A = {students who study Accounts}

a) The Venn diagram below illustrates the given data. Note that region 8 is 20, since 20 students study none of A, G and C.



b) i) The number of students who study General Science or Accounts or both but not Commerce is
$7 + 4 + 4 = 15$

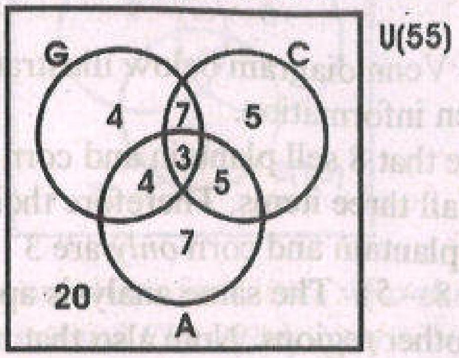
ii) Let the number of students who st
both Commerce and General Science =
(as shown in the Venn diagram)

$\therefore$ From the Venn diagram, the sum of all the regions = U

$7 + 4 + 4 + 5 + 3 + 5 + 20 + x = 55$

$48 + x = 55 \Rightarrow x = 55 - 48 = 7$

The complete Venn diagram showing the information is shown below.



Therefore;

Hence the number of students who study Commerce $= 5 + 3 + 5 + x = 5 + 3 + 5 + 7$
$= 20$

Note that for this example, region is *not zero* but 20 unlike the example above.

If each student studies at least one of the subjects then region 8 is zero.

Example 3

A group of market women sell at least one of the following items of foodstuffs; yam, corn and plantain. 18 of them sell corn, 17 sell yam and 19 sell plantain. 8 sell plantain and corn, 9 sell yam and corn, 12 sell yam and plantain and 5 sell all three items of foodstuff.

a) Draw a Venn diagram to illustrate the given information.

b) From your diagram, find:
  i) The total number of women in the group
  ii) The number of women who sell one item of foodstuff.

Solution

All women sell at least one foodstuff therefore there is no complement.

Let U = {Group of market women}

Y = {Those women who sell yam}

$n(Y) = 17$

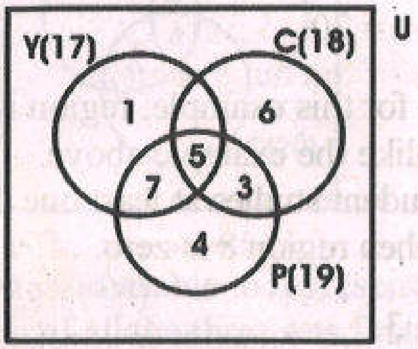
C = {Those women who sell corn}

$n(C) = 18$

P = {Those women who sell plantain}

$n(P) = 19$

a) The Venn diagram below illustrates the given information.
Note that 8 sell plantain and corn and 5 sell all three items. Therefore those who sell plantain and corn *only* are 3 (i.e. 8 – 5). The same analysis applies to the other regions. Note also that it is advisable to represent the intersection of all the three sets first. (In this example it is 5).



b) i) The total number of women in the group = sum of all the regions)

$\therefore \; n(U) = 1 + 4 + 6 + 7 + 5 + 3 + 4 = 30$

ii) The number of women who sell only one item of foodstuff = 4 + 6 + 1 = 11
Thus 4 sell only plantain, 6 sell only corn and 1 sells only yam.

Example 4

A science sixth form in a school has 12 teachers. Of these, 6 teach Mathematics, 5 teach Physics and 4 teach Chemistry; 2 teach Mathematics and Physics but no one teaches both Mathematics and Chemistry.

a) Draw a Venn diagram to illustrate the given information.

b) Find the number of teachers who teach;
   i) Chemistry and Physics
   ii) Only Physics

Solution

Let U ={Science sixth form teachers in the school}

$n(U) = 12$

M = {Mathematics teachers]

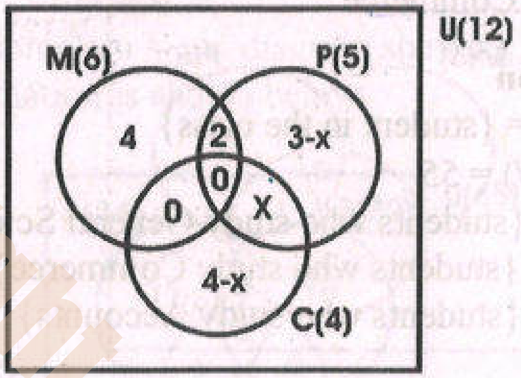
$n(M) = 6$

P = {Physics teachers}

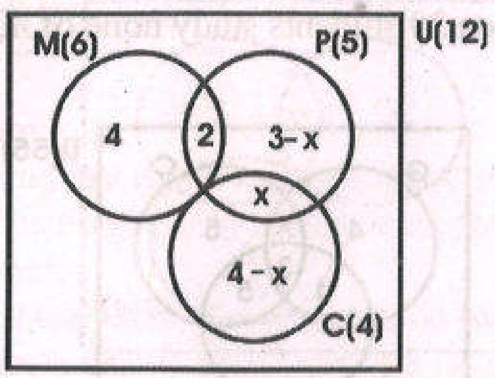
$n(P) = 5$

C = {Chemistry teachers}

$n(C) = 4$



(i)



(ii)

a) Fig (i) or (ii) above is the Venn diagram illustrating the given information.

No one teaches both Maths and Chemistry therefore the region of intersection of M and C is zero as indicated.

Note also that $n(M \cup P \cup C) = 12$

Let x = the number of teachers who teach Chemistry and Physics *only*

$(3 - x)$ teach Physics only,

$(4 - x)$ teach Chemistry only.

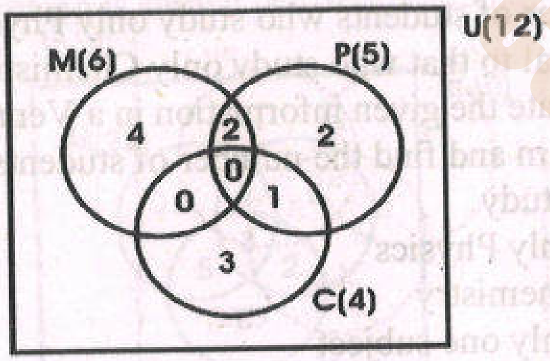
From the Venn diagram,

$4 + 2 + 0 + 0 + (4 - x) + x + (3 - x) = 12$

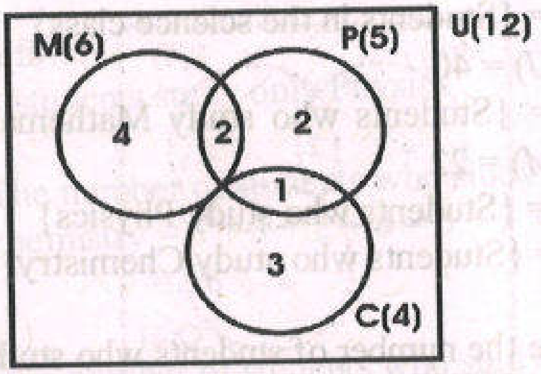
$13 - x = 12$

$x = 13 - 12 = 1$

The complete Venn diagram showing the information is shown below.



OR



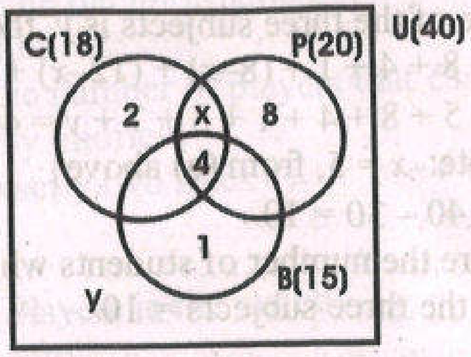
Therefore;

i) number of teachers who teach both Chemistry and Physics = 1

ii) number of teachers who teach only Physics = 2

Example 5

The Venn diagram below shows choices of subjects by 40 students. 18 chose Chemistry (C), 20 chose Physics (P) and 15 chose Biology (B). 2 chose Chemistry only, 8 chose Physics only and 1 chose Biology only. 4 chose all three subjects.



Find the number of students who chose

a) Physics and Chemistry only
b) Physics and Chemistry
c) Physics and Biology only
d) Physics and Biology
e) Chemistry and Biology only
f) Chemistry and Biology
g) None of the three subjects

Solution

From the Venn diagram;

The number of students who offer Physics and Biology only $= 20 - (8 + x + 4) = 8 - x$

The number of students who offer Chemistry and Biology only

$= 18 - (2 + x + 4) = 12 - x$

Therefore for Biology,

$(12 - x) + 4 + 1 + (8 - x) = 15$

$25 - 2x = 15 \Rightarrow x = 5$

a) The number of students who chose Physics and Chemistry only $= x = 5$

b) The number of students who chose Physics and Chemistry
$= 4 + x = 4 + 5 = 9$

c) The number of students who chose Physics and Biology only
$= 8 - x = 8 - 5 = 3$

d) The number of students who chose Physics and Biology $= 4 + 3 = 7$

e) The number of students who chose Chemistry and Biology only
$= 12 - x = 12 - 5 = 7$

f) The number of students who chose Chemistry and Biology $= 7 + 4 = 11$

g) If the number of students who chose none of the three subjects is y, then

$2 + x + 8 + 4 + 1 + (8 - x) + (12 - x) + y = 40$

$2 + 5 + 8 + 4 + 1 + 3 + 7 + y = 40$

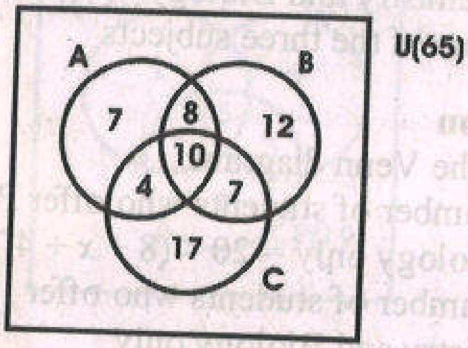
{Note: $x = 5$, from (a) above}

$y = 40 - 30 = 10$

Therefore the number of students who chose none of the three subjects = 10

Example 6

65 teachers applied for employment to teach Mathematics in three schools A, B and C. The number of teachers who gained one or more schools is shown in the Venn diagram below.



i) How many teachers were offered employment in schools A, B and C.

ii) Find the number of teachers who were offered employment by schools B and C but not A.

iii) How many teachers were offered employment in two schools only?

iv) If a teacher is selected at random, what is the probability that he was offered employment in all three schools?

Solution

i) 29, 38 and 37 teachers were offered employment in schools A, B and C respectively.

ii) The number of teachers who were offered employment by schools B and C but not A is $12 + 7 + 17 = 36$

iii) The number of teachers who were offered employment in two schools only is $8 + 7 + 4 = 19$

Example 7

In a science class of 40 students, 22 study Mathematics, 5 study Mathematics and Physics, 8 study Mathematics and Chemistry, 5 study Physics and Chemistry and 3 study all the three subjects. If the number of students who study only Physics is equal to that who study only Chemistry, illustrate the given information in a Venn diagram and find the number of students who study

i) Only Physics
ii) Chemistry
iii) Only one subject

Solution

Let U = {Students in the science class}

$n(U) = 40$

M = {Students who study Mathematics}

$n(M) = 22$

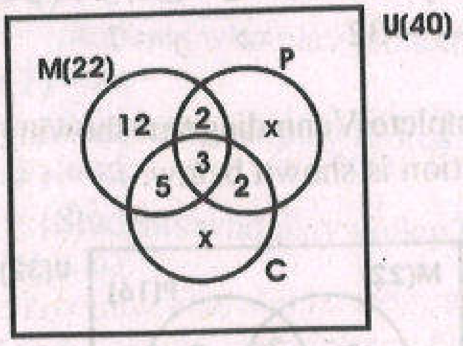
P = {Students who study Physics}

C = {Students who study Chemistry}

Let x be the number of students who study only Physics

$\therefore$ the number of students who study only Chemistry $= x$.

The Venn diagram below illustrates the given information



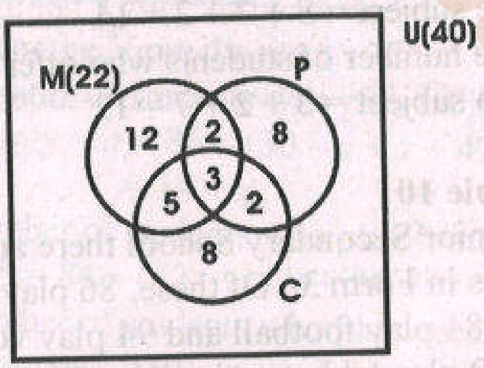
From the Venn diagram
$2 + x + 2 + 3 + 12 + 5 + x = 40$
[i.e. $n(M \cup P \cup C) = 40$]

$2x + 24 = 40$

$2x + 24 = 40 - 24 = 16$

$x = 8$

The complete Venn diagram showing the information is shown below.



Therefore;

i) 8 students study only Physics

ii) The number of students who study chemistry = 5 + 3 + 2 + 8 = 5 + 3 + 2 + 8 = 18

iii) The number of students who study only one subject = 8 + 8 + 12 = 8 + 8 + 12 = 28

Example 8

A national soccer coach invited 20 players for a match. 8 of them were attackers, 12 of them were defenders and 12 were midfielders. The coach realized that 5 could play both in the attack and in the midfield, 6 defence and midfield, 2 only in attack and 3 play all three roles.

a) Illustrate the information in a Venn diagram

b) Find the number of players that can play
   i) only midfield
   ii) exactly two roles

Solution

Let $U =$ {Players in the match}

$n(U) = 20$

$A =$ {Players who are Attackers}

$n(A) = 8$

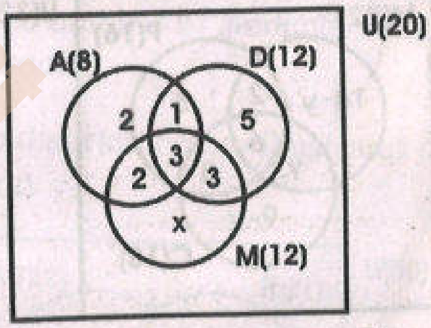
$D =$ {Players who are Defenders}

$n(D) = 12$

$M =$ {Players who are Midfielders}

$n(M) = 12$

a) The Venn diagram below illustrates the given information



i) Let x play in only midfield then from the Venn diagram

$2 + 3 + 3 + x = 12$

$x = 12 - 8 = 4$

Therefore 4 can play only in midfield.

ii) Those who can play in attack and defence only = $8 - (2 + 2 + 3) = 8 - 7 = 1$

Therefore those who can play exactly two roles = $2 + 3 + 1 = 6$

Example 9

In a class of 32 students, 22 offer Mathematics, 18 offer Chemistry and 16 offer Physics. 6 offer all the three subjects, 3 offer Physics and Chemistry only and 5 offer Physics only. Each student offers at least one subject. Illustrate the information in a Venn diagram

Find the number of students who offer

i) Chemistry only

ii) Only one subject

iii) Only two subjects.

Solution

Let $U = \{\text{Students in the class}\}$

$n(U) = 32$

$M = \{\text{Students who offer Mathematics}\}$

$n(M) = 22$

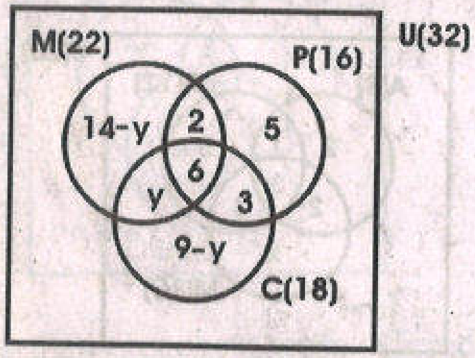
$P = \{\text{Students who offer Physics}\}$

$n(P) = 16$

$C = \{\text{Students who offer Chemistry}\}$

$n(C) = 18$

The Venn diagram below illustrates the given information



From the Venn diagram, the number of students who offer Mathematics and Physics only $= 16 - (6 + 5 + 3) = 2$

Let number of students who offer Mathematics and Chemistry only be y then the number of students who offer Mathematics only $= 22 - (y + 6 + 2) = 14 - y$

Similarly, the number of students who offer Chemistry only $= 18 - (y + 6 + 3) = 9 - y$

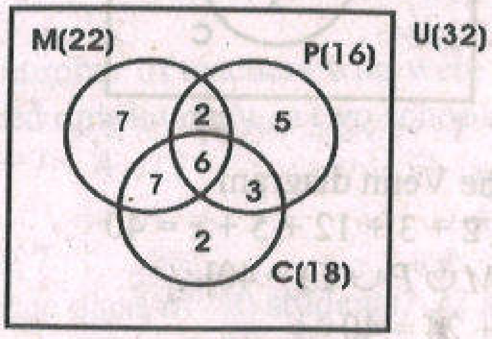
Since each student offers at least one subject

$n(M \cup P \cup C) = 32$

$(14 - y) + y + 6 + 2 + 5 + 3 + (9 - y) = 32$

$39 - y = 32 \quad \Rightarrow \quad y = 7$

The complete Venn diagram showing the information is shown below.



Therefore;

i) The number of students who offer Chemistry = 2

ii) The number of students who offer only one subject = 5 + 7 + 2 = 14

iii) The number of students who offer only two subject = 3 + 2 + 7 = 12

Example 10

In a Senior Secondary School there are 174 students in Form 3. Of these, 86 play table tennis, 84 play football and 94 play volley-ball; 30 play table tennis and volley-ball, 34 play volley-ball and football and 42 play table tennis and football. Each student plays at least one of the three games and x students play all three games.

i) Illustrate the information in a Venn diagram

ii) Write down an equation in x and hence find x.

iii) If a student is chosen at random from Form 3, what is the probability that he plays two games.

Solution

Let $U = \{\text{Students in the Form 3}\}$

$n(U) = 174$

$T = \{\text{Students who play table tennis}\}$

$n(T) = 86$

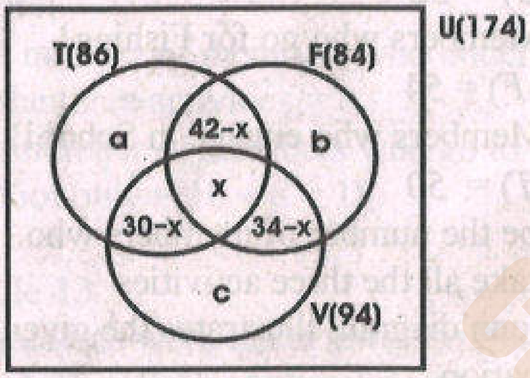
$F = \{\text{Students who play football}\}$

$n(F) = 84$

$V = \{\text{Students who play volley ball}\}$

$n(V) = 94$

i) The Venn diagram below illustrates the given information



ii) From the Venn diagram,

The number of students who play table tennis only $= a = 86 - (30 - x + x + 42 - x)$

$= 14 + x$

The number of students who play football only $= b = 84 - (42 - x + x + 34 - x) = 8 + x$

The number of students who play volley ball only $= c = 94 - (30 - x + x + 34 - x)$

$= 30 + x$

Since each student plays at least one of the three games; $n(T \cup F \cup V) = 174$

$(14 + x) + (42 - x) + (8 + x) + (30 - x) + x + (34 - x) + (30 + x) = 174$

$158 + x = 174$

$x = 174 - 158 = 16$

iii) The number of students who play two games $= (42 - x) + (30 - x) + (34 - x)$

$= 26 + 14 + 18 = 58$

The number of students in Form 3 = 174

$\therefore$ If a student is chosen at random the probability that he plays two games $= \frac{58}{174} = \frac{1}{3}$

Example 11

Mathematics, English and Life Skills books were distributed to 50 students in a class. 22 had Mathematics books, 21 English books and 25 Life Skills books. 7 had Mathematics and English books, 6 Mathematics and Life skills books and 9 had English and Life Skills books.

Illustrate the information in a Venn diagram. Find the number of students who had

i) all three books

ii) exactly two of the books

iii) only Life Skills books

Solution

Let $U = \{\text{Students in the class}\}$

$n(U) = 50$

$M = \{\text{Students who had Mathematics books}\}$

$n(M) = 22$

$E = \{\text{Students who had English books}\}$

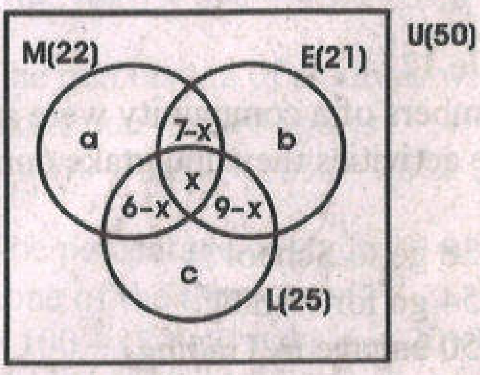
$n(E) = 21$

$L = \{\text{Students who had Life skills books}\}$

$n(L) = 25$

Let x be the number of students who had all the three books.

The Venn diagram below illustrates the given information



From the Venn diagram,

The number of students who had Mathematics books only

$= a = 22 - (6 - x + x + 7 - x) = 9 + x$

The number of students who had English books only

$= b = 21 - (7 - x + x + 9 - x) = 5 + x$

The number of students who had Life skills books only

$= c = 25 - (6 - x + x + 9 - x) = 10 + x$

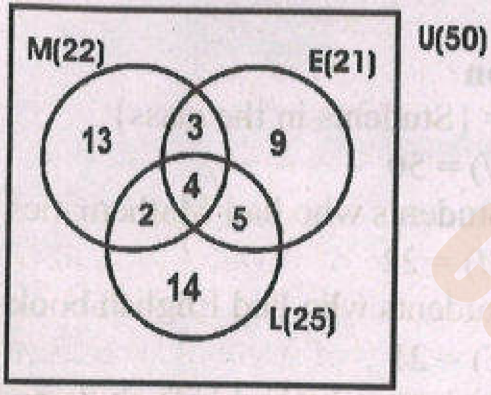
$n(M \cup E \cup L) = n(U) = 50$

$(9 + x) + (7 - x) + (5 + x) + x + (6 - x) +$
$(9 - x) + (10 + x) = 50$

$46 + x = 50$

$x = 50 - 46 = 4$

The complete Venn diagram showing the information is shown below.



Therefore;

i) The number of students who had all the three books = 4

ii) The number of students who had exactly two books = 2 + 3 + 5 = 10

iii) The number of students who had only Life skills books = 14

Example 12

100 members of a community were asked to state the activities they undertake during the day.

38 go to School
54 go for Fishing
50 engage in Trading
10 go to School and also Fish
18 go to School and also Trade
22 go for Fishing and also Trade

Each of these members undertakes at least one of the activities. The number of people who go to School only is the same as the number who engage in Trading only.

Illustrate the information in a Venn diagram and use it to find the number of people who

a) undertake all the three activities;
b) go to School only

Solution

Let $U =$ {Members of the community}

$n(U) = 100$

$S =$ {Members who go to School}

$n(S) = 38$

$F =$ {Members who go for Fishing}

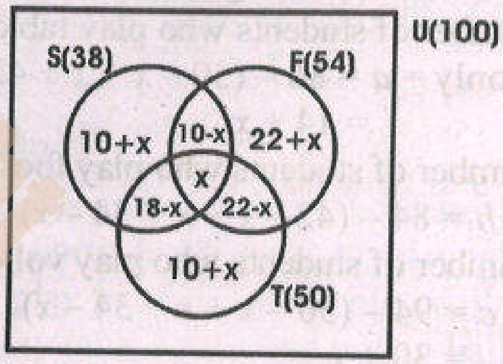
$n(F) = 54$

$T =$ {Members who engage in School}

$n(T) = 50$

Let x be the number of members who undertake all the three activities

The Venn diagram illustrates the given information.



From the Venn diagram,

The number of members who go to School only $= 38 - (10 - x + x + 18 - x) = 10 + x$

Therefore from the question, the number of members who engage in Trading only $= 10 + x$

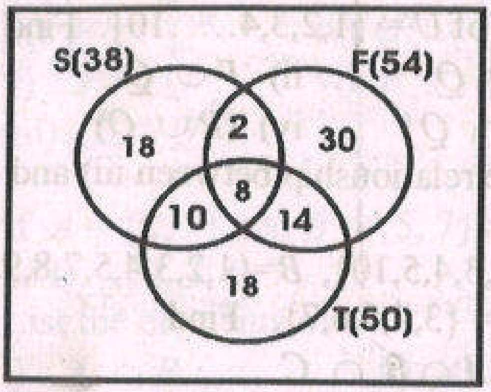
The number of members who go to Fishing only $= 54 - (10 - x + x + 22 - x) = 22 + x$

Since each member undertakes at least one of the activities

$(10 + x) + (10 - x) + (22 + x) + x +$
$(22 - x) + (18 - x) + (10 + x) = 100$

$92 + x = 100 \quad \Rightarrow \quad x = 8$

The complete Venn diagram showing the information is shown below.



Therefore;

a) The number of members who undertake all the three activities = 8

b) The number of members who go to School only = 10 + 8 = 18

Example 13

Some students were interviewed to find out which of the following three sports they liked: football, boxing and volleyball. 70% of the students liked football, 60% boxing and 45% volleyball. 45% liked football and boxing, 15% boxing and volleyball, 25% football and volleyball and 5% liked all three sports.

a) Draw a Venn diagram to illustrate this information

b) Use your diagram to find the percentage of students who liked

  i) football and boxing but not volleyball

  ii) exactly two sports

  iii) none of the three sports

Solution

Let U = {Students interviewed}

$n(U) = 100\%$

(Since we are working in percentages)

F = {Students who liked Football}

$n(F) = 70\%$

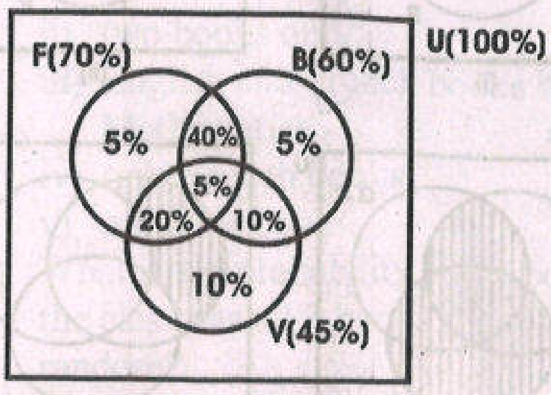
B = {Students who liked Boxing}

$n(B) = 60\%$

V = {Students who liked Volleyball}

$n(V) = 45\%$

a) The Venn diagram below illustrates the given information.



b) From the Venn diagram,

The number of students who liked Football only = 70 – (40 + 5 + 20) = 5%

The number of students who liked Boxing only = 60 – (40 + 5 + 10) = 5%

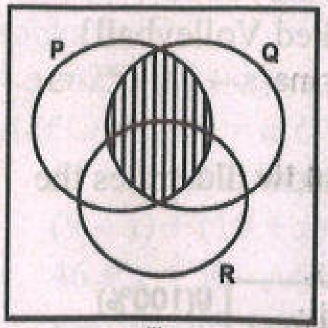
The number of students who liked Volley-ball only = 45 – (20 + 5 + 10) = 10%

i) The percentage of students who liked football and boxing but not volleyball = 5% + 40% + 5% = 50%

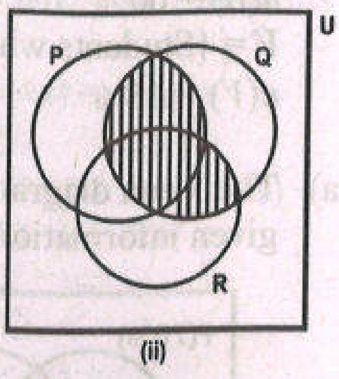
ii) The percentage of students who liked exactly two sports = 40% + 20% + 10% = 70%

iii) The percentage of students who liked none of the three sports = 100 – (5 + 40 + 5 + 5 + 20+10+10) = 100 – 95 = 5%

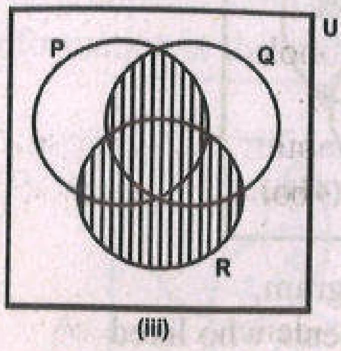
Example 14
Describe the shaded region in the Venn diagrams below.



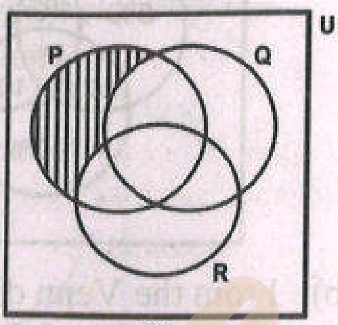
(i)



(ii)



(iii)



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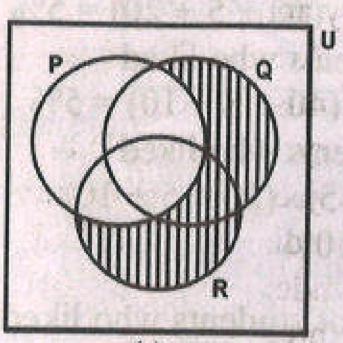
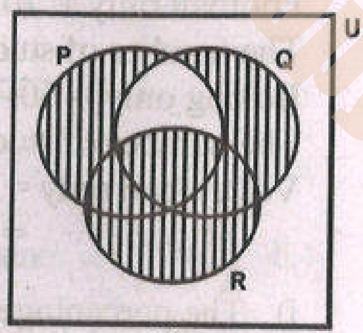


(v)



(vi)

Solution

i) $P \cap Q$

ii) $Q \cap (P \cup R)$ or $(Q \cap P) \cup (Q \cap R)$

iii) $R \cup (P \cap Q)$ or $(R \cup P) \cap (R \cup Q)$

iv) $P \cap (Q \cup R)'$ or $P \cap Q' \cap R'$

v) $(R \cup Q) \cap P'$

vi) $R \cup (P \cap Q)'$ or $R \cup P' \cup Q'$

Review exercise 1

1. $P = \{1,2,4,5,7\}$ and $Q = \{3,4,7,8\}$ are the subsets of $U = \{1,2,3,4 \ldots, 10\}$. Find
i) $P \cap Q$ ii) $P \cup Q$
iii) $P' \cap Q'$ iv) $(P \cup Q)'$
State the relationship between iii) and iv)

2. If $A = \{2,3,4,5,10\}$, $B = \{1,2,3,4,5,7,8,9,10\}$ and $C = \{3,4,5,6,7\}$. Find
i) $A \cap B \cap C$
ii) $A \cap B$
iii) $A \cap C$
iv) $B \cap C$
v) $A \cap (B \cup C)$
vi) $(A \cap B) \cup (A \cap C)$
vii) $A \cup (B \cap C)$
viii) $(A \cup B) \cap (A \cup C)$
State the relationship between
a) (v) and (vi) b) (vii) and (viii)

3. Given that $A = \{5, 10, 15, 20, 25\}$, $B = \{5, 25, 125, 625\}$ and $U = A \cup B$
List the elements of the following
i) U ii) A'
iii) $A' \cup B'$ iv) $A' \cap B'$

4. If P, Q and R are subsets of the universal set $U = \{$prime numbers less than 20$\}$ and $P = \{x : 1 \le x \le 10\}$, $Q = \{x : x > 5\}$ and $R = \{x : x < 7\}$ Find
a) i) $P \cup Q$ ii) $P' \cap R'$
iii) $Q \cap R$ iv) $(P \cap Q')'$
v) $P' \cup Q$
b) Show that
i) $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$
What property is this?
ii) $P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$
What property is this?

5. If $P = \{a, b, d, e\}$ and $Q = \{c, d, e\}$ are the subsets of $U = \{a, b, c, d, e\}$. Find

i) $P \cap Q$ ii) $P \cup Q$
iii) $P' \cap Q'$ iv) $(P \cup Q)'$
v) $P \cap Q'$ vi) $P' \cap Q$
vii) $(P \cap Q)'$

6. If $A = \{2, 5\}$ and $B = \{5, 7\}$ are subsets of $U = \{2, 3, 5, 7\}$.
List the elements of

i) $A \cap B$ ii) $A \cup B$
iii) $A \cap B'$ iv) $A' \cap B$
v) $A' \cap B'$ vi) $(A \cap B)'$
vii) $A' \cup B'$ viii) $(A \cup B)'$

State the relationship between (a) v) and viii) (b) vi) and vii)

7. The universal set $U = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$. $A = \{1, 2, 3, 4, 6, 12\}$ and $B = \{1, 2, 3, 6, 9, 18\}$ are subsets of U.

a) List the elements of
   i) $A' \cap B$ ii) $A \cup B'$
   iii) $A' \cap B'$ iv) $(A' \cap B)'$
b) Which of the sets in a) are equal

8. The universal set U = {whole numbers from 2 to 25 inclusive} and P, Q and R are subsets of U such that P = {multiples of 3}, Q = {square numbers} and R = {numbers whose sum of digits is 8}. Find

a) i) $P \cap Q$ ii) $P \cap Q'$
   iii) $Q \cap R$ iv) $(P \cap Q')'$
   v) $P' \cup Q$ vi) $(R \cup Q) \cap P'$
b) Show that
   $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$.
   What property is this?

9. A class of 26 boys were each required to have certain textbooks in English, French and Mathematics. 19 boys had the English book, 23 had the French book and 15 the Mathematics book. 16 boys had both the English and French books, 14 had the French and Mathematics books and 13 had the Mathematics and English books.

a) Illustrate the information in a Venn diagram
b) How many boys in the class possessed:
   i) all the 3 books.
   ii) two books only
   iii) English and French books but not Mathematics
   iv) only French books
   v) only one book
c) What is the probability that a boy in the class has all 3 books if chosen at random?

10. In a school of 200 students, 100 study Art, 70 study Biology and 46 study Chemistry. 30 study Art and Biology, 28 study Art and Chemistry, 23 study Biology and Chemistry. If 18 students study all the three subjects, illustrate the information in a Venn diagram and find

i) The number of students who do not study any of the three subjects
ii) The number of students who study only Art, only Biology and only Chemistry
iii) How many students study only Art and Biology, only Chemistry and Art, only Biology and Chemistry.

11. A student interviewed 70 people in Accra to ask which channels they watched on TV the night before.
He discovered the following:
52 watched GTV
24 watched KTV
22 watched MTV
14 watched GTV and KTV only

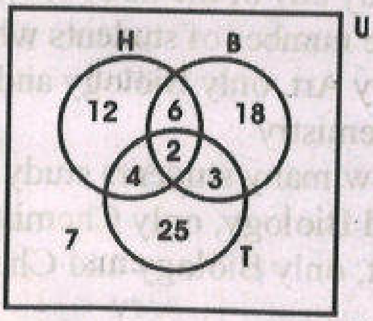
8 watched KTV and MTV
13 watched GTV and MTV
2 watched all three channels

a) Illustrate the information in a Venn diagram

b) Find from the people interviewed the number who
   i) did not watch TV at all
   ii) watched MTV only

12. Two newspapers, Graphic and Times are delivered to 40 houses in Adenta. Each house receives at least one copy of a newspaper. In all 26 copies of Graphic and 24 copies of Times are delivered. Find the number of houses which receive Times only.

13. The figure below shows the results of interviewing some women students at a Teacher Training school.
B = {women who play badminton},
H = {women who play hockey} and
T = {women who play tennis}.
i) how many women were interviewed?
ii) how many women play tennis?
iii) how many women play only two of the sports?



14. In a class of 80 students, 53 study Art, 60 study Biology, 36 study Art and Biology, 34 study Art and Chemistry, 6 study Biology only and 18 study Biology but not Chemistry. Illustrate the information in a Venn diagram. Determine the number of students who study:
i) Art only ii) Chemistry

15. In a class of 100 students, 37 study Mathematics, 32 study Physics and 31 study Chemistry. 9 students study Mathematics and Physics, 12 students study Mathematics and Chemistry, 11 students study Physics and Chemistry and 28 students study none of these subjects. Illustrate the information in a Venn diagram and find;
a) the number of students who study all the three subjects
b) the number of students who study only Mathematics, only Physics and only Chemistry
c) how many students study exactly two subjects?

16. A group of 22 travellers were each required to obtain a passport, a health certificate and convertible currency equivalent to 50 dollars. 12 have the passports, 14 the health certificates and 11 the currency. 6 had passports and currency, 6 passports and certificates and 7 health certificates and currency. Each traveller had at least one of the requirements.

a) Draw a Venn diagram to illustrate the given information

b) Find the number who had
   i) all the three requirements
   ii) exactly two requirements

17. In a class, 30 offer Biology, 21 offer Chemistry and 22 offer Physics. 15 offer Physics and Biology, 10 offer Physics and Chemistry and 13 offer Biology and Chemistry. 2 offer Physics only, 3 offer Chemistry only and 7 offer Biology only.

i) Illustrate this information in a Venn diagram

ii) Find the number of students
  a) who offer all the three subjects
  b) in the class
  c) If a student is selected at random, what is the probability that he studies either Physics or Chemistry.

18. In a survey of the 100 out-patients who reported at a hospital one day, it was found out that 70 complained of fever, 50 complained of stomach ache and 30 were injured. All 100 patients had at least one of the complaints and 44 had exactly two of the complaints. How many patients had all three complaints?

19. Illustrate the following on a Venn diagram.
  i) $(A' \cup B)'$
  ii) $A \cap B'$
  iii) $(A \cup B) \cup C$
  iv) $A \cup (B \cup C)$
  v) $A \cup (B \cap C)$
  vi) $(A \cup B) \cap (A \cup C)$
  vii) $A \cap B' \cap C$
  viii) $A' \cap B \cap C'$

20. In a class of 70 students, 6 offer Economics only, 18 offer Economics but not Mathematics, 36 offer Economics and Geography, 53 offer Economics, 50 offer Geography and 34 offer Mathematics and Geography. All students offer at least one subject.
  a) Illustrate this information in a Venn diagram
  b) Determine the number of students who offer:
    i) Mathematics.
    ii) Mathematics and Geography but not Economics
    iii) The probability that a student selected at random from the class offers Geography only.

21. 50 Children had a choice of beans, plantain and rice. 21 of them took beans, 24 took plantain and 18 took rice. 3 took beans only, 9 took plantain only, 2 took rice only and 5 took all three items of food.
  a) Illustrate this information in a Venn diagram
  b) Use your diagram to find the number of children who took
    i) plantain and beans only
    ii) rice and beans only
    iii) None of the three items

22. In a class of 40 students, the number of students who study French is 10 more than the number of students who study History. If 8 students study both French and History, how many students study
  i) History ii) French

23. In a class of 38 students, each student studies at least one of the following subjects: French, Mathematics and History. The following table gives further details of the subject studied.

French only	9
Mathematics only	8
History only	1
French and Mathematics	7
French and History	9
History and Mathematics	8

  a) If x is the number of students who study all the three subjects, illustrate the above information in a Venn diagram.
  b) Write an equation in x involving the number of students in the region of the diagram.

c) Find the number of students who study
   i) all the three subjects
   ii) French

24. The universal set ξ is the set of all integers. P, Q and R are subsets of ξ defined as follows:
$P=\{x : x < 0\}$, $Q = \{\ldots,-5,-3,-1,1,3,5\ldots\}$ and $R = \{x : -2 \leq x < 7\}$
a) Find $Q \cap R$
b) Find R', where R' is the complement of R with respect to ξ.
c) Find $P' \cap R'$
d) List the members of $(P \cap Q)'$

25. In a certain class, 22 pupils take one or more of Chemistry, Economics and Government. 12 take Economics(E), 8 take Government(G) and 7 take Chemistry(C). Nobody takes Economics and Chemistry and 4 pupils take Economics and Government.
a) i) Using set notation and the letters indicated above, write down the two statements in the last sentence.
   ii) Draw a Venn diagram to illustrate the information
b) How many pupils take?
   i) Both Chemistry and Government
   ii) Government only

26. In a market survey, 100 traders sell fruits. 40 sell apples, 46 sell oranges, 50 sell mangoes, 14 sell apples and oranges, 15 sell apples and mangoes and 10 sell all the three fruits. Each of the traders at least sells one of the three fruits.
i) Represent the information in a Venn diagram
ii) Find the number of traders that sell oranges and mangoes only.

27. If ξ is the set $\{1, 2, 3,\ldots, 19, 20\}$ and A, B and C are subsets of ξ such that
$A = \{\text{multiples of } 5\}$,
$B = \{\text{multiples of } 4\}$ and
$C = \{\text{multiples of } 3\}$.
a) List the elements of A, B and C.
b) Find i) $A \cap B$
   ii) $A \cap C$
   iii) $B \cup C$
c) Using your results in b), show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

28. In a class of 40 students, 25 speak Twi, 16 speak Hausa and 21 speak Ewe and each of the students speak at least one of these languages. If 8 speak Twi and Hausa, 11 speak Twi and Ewe and 6 speak Hausa and Ewe.
a) Draw a Venn diagram to illustrate this information, using x to represent the number of students who speak all three languages
b) Calculate the value of x

29. There are 30 students in a class. 20 of them play football, 16 play hockey and 16 play volley. 9 play all the three games, 15 play football and volley, 11 play football and hockey, while 10 play hockey and volley.
a) Illustrate the information on a Venn diagram
b) Using your Venn diagram, find the number of students who play **at least** two games
c) What is probability that a student chosen at random from the class **does not** play any of the three games?

30. In a youth club with 94 members, 60 like modern music, 50 like traditional music. The number of members who like both traditional and modern music is three times those who do not like any type of music. How many members like only one type of music?

31. In a survey of 100 students in an institution in Nigeria, 80 students speak Hausa and 20 students speak Igbo. 9 students speak both Hausa and Igbo. How many students speak either Hausa or Igbo?

32. Four members of a school's first eleven cricket team are also in the first fourteen rugby team. How many members play for at least one of the two teams?

33. In a school of 80 students, every student studies, Economics or Geography or both. If 65 students study Economics and 50 study Geography, how many study both subjects?

34. If ξ is the set $\{1, 2, 3, \ldots, 29, 30\}$ and A, B and C are subsets of ξ such that
$A = \{\text{multiples of } 7\}$,
$B = \{\text{multiples of } 5\}$ and
$C = \{\text{multiples of } 3\}$.

a) List the elements of A, B and C.

b) Find
   i) $A \cup B$
   ii) $B \cap C$
   iii) $A \cup C'$

35. In a class of 47 students, 31 study Physics, 26 study Chemistry and 16 study Biology. 3 study all the three subjects, 2 study Chemistry and Biology only and 6 study Biology only.

a) Illustrate the information in a Venn diagram.

b) Find the number of students who study
   i) Chemistry only
   ii) only one subject
   iii) only two subjects

36. Out of the 400 students in the final year in a Senior Secondary School, 300 are offering Biology and 190 are offering Chemistry.

i) How many students are offering both Biology and Chemistry, if only 70 students are offering neither Biology nor Chemistry?

ii) How many students are offering at least one of Biology or Chemistry?

37. There are 70 women in a club. Each plays at least one of the following games: volleyball, basketball and table-tennis. 20 play volleyball only, 10 play basketball only and 6 play table-tennis only. 4 play all three games and an equal number play two games only.

a) Illustrate this information on a Venn diagram.

b) Find the number of women who lay volleyball.

38. Use Venn diagrams to show that;

i) $(A \cup B \cup C)' = A' \cap B' \cap C'$

ii) $(A \cap B \cap C)' = A' \cup B' \cup C'$

iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

CHAPTER 2

SURDS

Definition

Taking only the positive square root, expressions such as $\sqrt{9}$, $\sqrt{25}$ etc. give exact numerical values 3 and 5 respectively called rational numbers. But expressions such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$ ... cannot be written as numerically exact quantities. For example, $\sqrt{2} = 1.44142136...$ and $\sqrt{20} = 4.472....$

Numbers such as $\sqrt{2}$ and $\sqrt{3}$ are called irrational numbers, generally called **surds**. So if a calculation requires an exact answer, **irrational numbers** must be left in the surd form. Other examples of surds are $\sqrt{11}$, $\sqrt{13}$, $\sqrt{17}$ etc. In general, the square roots of prime numbers are all surds. A rational multiple of a surd is also a surd. For example, $2\sqrt{3}$, $2\sqrt{2}$ $3\sqrt{5}$ etc. The sign '$\sqrt{\ }$' means the positive square root.

Reduction to basic form

If $\sqrt{x}$ is such that it cannot be reduced further then it is said to be in the basic form. i.e. x does not contain a factor, which is a perfect square. For example, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{7}$ are in the basic form and cannot be simplified any further.

However, $\sqrt{24}$ can be simplified as $\sqrt{24} = \sqrt{4 \times 6} = \sqrt{4} \times \sqrt{6} = 2\sqrt{6}$.

Similarly,

$$\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}$$

$$\sqrt{98} = \sqrt{49 \times 2} = \sqrt{49} \times \sqrt{2} = 7\sqrt{2}$$

$$\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2} \quad \text{and}$$

$$\sqrt{200} = \sqrt{100 \times 2} = \sqrt{100} \times \sqrt{2} = 10\sqrt{2}$$

Note the following perfect squares very carefully:

$\sqrt{9} = 3$ $\quad \sqrt{16} = 4$ $\quad \sqrt{25} = 5$

$\sqrt{36} = 6$ $\quad \sqrt{49} = 7$ $\quad \sqrt{64} = 8$

$\sqrt{81} = 9$ $\quad \sqrt{100} = 10$ $\quad \sqrt{121} = 11$

$\sqrt{144} = 12$ $\quad \sqrt{169} = 13$ $\quad \sqrt{196} = 14$

$\sqrt{225} = 15$ $\quad \sqrt{256} = 16$ $\quad \sqrt{289} = 17$

$\sqrt{324} = 18$ $\quad \sqrt{361} = 19$ $\quad \sqrt{400} = 20$

Illustrative example

Simplify

i) $\sqrt{27}$ ii) $\sqrt{32}$ iii) $\sqrt{8}$ iv) $\sqrt{128}$

Solution

i) $\sqrt{27} = \sqrt{(9 \times 3)} = \sqrt{9} \times \sqrt{3} = 3 \times \sqrt{3} = 3\sqrt{3}$

ii) $\sqrt{32} = \sqrt{(16 \times 2)}$
$= \sqrt{16} \times \sqrt{2} = 4 \times \sqrt{2} = 4\sqrt{2}$

iii) $\sqrt{8} = \sqrt{(4 \times 2)}$
$= \sqrt{4} \times \sqrt{2} = 2 \times \sqrt{2} = 2\sqrt{2}$

iv) $\sqrt{128} = \sqrt{(64 \times 2)}$
$= \sqrt{64} \times \sqrt{2} = 8 \times \sqrt{2} = 8\sqrt{2}$

Addition and subtraction of surds

We can only add or subtract surds which are alike or have the same form.

1. $m\sqrt{k} + n\sqrt{k} = (m+n)\sqrt{k}$
2. $m\sqrt{k} - n\sqrt{k} = (m-n)\sqrt{k}$

Note 1: First reduce the surds to their basic forms if they are not.

Note 2: $\sqrt{5} + \sqrt{2} \neq \sqrt{7}$ or

$\sqrt{5} - \sqrt{2} \neq \sqrt{3}$.

Thus we cannot add or subtract $\sqrt{5}$ and $\sqrt{2}$ since they are not alike.

Illustrative examples

Example 1

i) $5\sqrt{2} + 3\sqrt{2} = 8\sqrt{2}$

ii) $4\sqrt{3} + \sqrt{3} = 5\sqrt{3}$

iii) $\sqrt{5} + \sqrt{5} = 2\sqrt{5}$

iv) $5\sqrt{3} + 4\sqrt{3} = 9\sqrt{3}$

v) $6\sqrt{5} + 7\sqrt{5} = 13\sqrt{5}$

vi) $5\sqrt{2} - 3\sqrt{2} = 2\sqrt{2}$

vii) $4\sqrt{3} - \sqrt{3} = 3\sqrt{3}$

viii) $2\sqrt{5} - \sqrt{5} = \sqrt{5}$

ix) $\sqrt{3} - 2\sqrt{3} = -\sqrt{3}$

x) $2\sqrt{5} - 5\sqrt{5} = -3\sqrt{5}$

Example 2

Simplify the following:

i) $3\sqrt{8} + 3\sqrt{200} + \frac{2}{3}\sqrt{162}$

ii) $\sqrt{50} + \sqrt{32} - \sqrt{162}$

Solution

i) $3\sqrt{8} + 3\sqrt{200} + \frac{2}{3}\sqrt{162}$

$= 3\sqrt{(4 \times 2)} + 3\sqrt{(100 \times 2)} + \frac{2}{3}\sqrt{(81 \times 2)}$

$= 6\sqrt{2} + 30\sqrt{2} + 6\sqrt{2}$

$= 42\sqrt{2}$

ii) $\sqrt{50} + \sqrt{32} - \sqrt{162}$

$= 5\sqrt{2} + 4\sqrt{2} - 9\sqrt{2} = 0$

Note: Simplify each surd before performing the addition or subtraction.

Example 3

Evaluate the following:

i) $\sqrt{147} - \sqrt{75} + \sqrt{27}$

ii) $\sqrt{24} - 3\sqrt{6} - \sqrt{216} + \sqrt{294}$

Solution

i) Simplifying the surds, we have

$\sqrt{147} = \sqrt{(49 \times 3)} = \sqrt{49} \times \sqrt{3} = 7\sqrt{3}$

$\sqrt{75} = \sqrt{(25 \times 3)} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}$

$\sqrt{27} = \sqrt{(9 \times 3)} = \sqrt{9} \times \sqrt{3} = 3\sqrt{3}$

$\therefore \sqrt{147} - \sqrt{75} + \sqrt{27}$

$= 7\sqrt{3} - 5\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}$

ii) $\sqrt{24} - 3\sqrt{6} - \sqrt{216} + \sqrt{294}$

$= 2\sqrt{6} - 3\sqrt{6} - 6\sqrt{6} + 7\sqrt{6} = 0$

Example 4

Simplify the following:

i) $3\sqrt{500} - 2\sqrt{125}$

ii) $\sqrt{50{,}000} - 25\sqrt{125} + 5\sqrt{5}(\sqrt{5} - 5)$

Solution

i) $3\sqrt{500} - 2\sqrt{125}$

$= 3\sqrt{(100 \times 5)} - 2\sqrt{(25 \times 5)}$

$= 30\sqrt{5} - 10\sqrt{5} = 20\sqrt{5}$

ii) Simplifying each term, we have

$\sqrt{50{,}000} = \sqrt{(10{,}000 \times 5)}$

$= \sqrt{10{,}000} \times \sqrt{5} = 100\sqrt{5}$

$25\sqrt{125} = 25\sqrt{(25 \times 5)}$

$= 25 \times \sqrt{25} \times \sqrt{5} = 125\sqrt{5}$

$5\sqrt{5}(\sqrt{5} - 5) = 5(\sqrt{5})^2 - 25\sqrt{5}$

$= 25 - 25\sqrt{5}$

$\therefore \sqrt{50{,}000} - 25\sqrt{125} + 5\sqrt{5}(\sqrt{5} - 5)$

$= 100\sqrt{5} - 125\sqrt{5} + (25 - 25\sqrt{5})$

$= 25 - 50\sqrt{5}$

Example 5

If $\sqrt{32}+\sqrt{128}-\sqrt{18}+\sqrt{50}=k\sqrt{2}$,
find the value of *k*.

Solution

$\sqrt{32}+\sqrt{128}-\sqrt{18}+\sqrt{50}$

$=\sqrt{(16\times 2)}+\sqrt{(64\times 2)}-\sqrt{(9\times 2)}+\sqrt{(25\times 2)}$

$=4\sqrt{2}+8\sqrt{2}-3\sqrt{2}+5\sqrt{2}\;=14\sqrt{2}=k\sqrt{2}$

By comparison, $k=14$

Example 6

Find the value of *y* if
$\sqrt{24}+\sqrt{96}-\sqrt{600}=y\sqrt{6}$

Solution

$\sqrt{24}+\sqrt{96}-\sqrt{600}$

$=\sqrt{(4\times 6)}+\sqrt{(16\times 6)}-\sqrt{(100\times 6)}$

$=2\sqrt{6}+4\sqrt{6}-10\sqrt{6}$

$=-4\sqrt{6}=y\sqrt{6}$

$\therefore$ By comparison, $y=-4$

Example 7

Given that $\sqrt{3k}+\sqrt{18}-\sqrt{98}=\sqrt{50}$,
find the value of the constant *k*.

Solution

$k=54$ [Work out the solution]

Rules of surds

The following properties of surds should be noted carefully.

Rule	Example
1. $\sqrt{a}\times\sqrt{a}=(\sqrt{a})^2$ $=a$	$\sqrt{2}\times\sqrt{2}=(\sqrt{2})^2=2$
2. $\sqrt{a^2}=a$	$\sqrt{3^2}=3$
3. $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$	$\sqrt{2}\times\sqrt{3}=\sqrt{(2\times 3)}=\sqrt{6}$
4. $\dfrac{\sqrt{a}}{\sqrt{b}}=\sqrt{\left(\dfrac{a}{b}\right)}$	$\dfrac{\sqrt{6}}{\sqrt{2}}=\sqrt{\left(\dfrac{6}{2}\right)}=\sqrt{3}$
5. $a\sqrt{b}\times\sqrt{c}=a\sqrt{bc}$	$2\sqrt{3}\times\sqrt{5}=2\sqrt{(3\times 5)}$ $=2\sqrt{15}$
6. $a\times\sqrt{b}=a\sqrt{b}$	$2\times\sqrt{3}=2\sqrt{3}$
7. $a\sqrt{b}\times c\sqrt{d}=ac\sqrt{bd}$	$2\sqrt{3}\times 4\sqrt{5}$ $=(2\times 4)\sqrt{(3\times 5)}=8\sqrt{15}$
8. $a\times b\sqrt{c}=ab\sqrt{c}$	$2\times 3\sqrt{5}=6\sqrt{5}$

Multiplication of surds

When multiplying two surds in brackets, the rules of surds must be used.

Illustrative examples

Example 1

Expand and simplify the following surds

i) $(\sqrt{3}-1)(2-\sqrt{3})$
ii) $(2+\sqrt{3})(\sqrt{3}+1)$
iii) $(2+2\sqrt{7})(5-\sqrt{7})$
iv) $\sqrt{18}\times\sqrt{75}$

Solution

i) $(\sqrt{3}-1)(2-\sqrt{3})=2\sqrt{3}-(\sqrt{3})^2-2+\sqrt{3}$

(Expanding)

$(\sqrt{3}-1)(2-\sqrt{3})=2\sqrt{3}-3-2+\sqrt{3}$

(Adding like surds)

$(\sqrt{3}-1)(2-\sqrt{3})=-5+3\sqrt{3}$

ii) $(2+\sqrt{3})(\sqrt{3}+1)=2\sqrt{3}+2+(\sqrt{3})^2+\sqrt{3}$

$=2\sqrt{3}+2+3+\sqrt{3}$

$=5+3\sqrt{3}$

iii) $(2+2\sqrt{7})(5-\sqrt{7})$

$=10-2\sqrt{7}+10\sqrt{7}-14$

$=8\sqrt{7}-4$

iv) $\sqrt{18}\times\sqrt{75}=\sqrt{(9\times 2)}\times\sqrt{(25\times 3)}$

$=3\sqrt{2}\times 5\sqrt{3}=15\sqrt{6}$

Example 2

Simplify $(1-\sqrt{3})(\frac{1}{3}+\sqrt{3})$ giving your answer in the form $p+q\sqrt{3}$.

Solution

$(1-\sqrt{3})(\frac{1}{3}+\sqrt{3})$

$= \frac{1}{3}+\sqrt{3}-\frac{1}{3}\sqrt{3}-(\sqrt{3})^2$

$= \frac{1}{3}+\sqrt{3}-\frac{1}{3}\sqrt{3}-3$

$= -\frac{8}{3}+\frac{2}{3}\sqrt{3} = p+q\sqrt{3}$,

where $p=-\frac{8}{3}$ and $q=\frac{2}{3}$

Example 3

Expand and simplify the following surds

i) $(2-3\sqrt{3})(3+2\sqrt{3})$

ii) $(5-2\sqrt{7})(5+2\sqrt{7})$

iii) $(4-\sqrt{5})(3+\sqrt{5})$

iv) $(4-\sqrt{3})(4+\sqrt{3})$

v) $\sqrt{2}(3-\sqrt{2})$

vi) $\sqrt{2}(3-2\sqrt{2})$

vii) $\sqrt{3}(\sqrt{27}-1)$

viii) $(\sqrt{2}-1)(\sqrt{2}+1)$

ix) $(\sqrt{3}-2)(\sqrt{3}-1)$

x) $(2\sqrt{2}+1)(\sqrt{2}-2)$

xi) $(3\sqrt{3}-2)(3\sqrt{3}+2)$

xii) $(2\sqrt{5}+3)(3\sqrt{5}-2)$

xiii) $(\sqrt{3}-1)(\sqrt{2}+1)$

xiv) $(2\sqrt{6}-3)^2$

Solution

Note: In each case the final answer has been provided. Work out the missing steps.

i) $(2-3\sqrt{3})(3+2\sqrt{3})=-12-5\sqrt{3}$

ii) $(5-2\sqrt{7})(5+2\sqrt{7})=-3$

iii) $(4-\sqrt{5})(3+\sqrt{5})=7+\sqrt{5}$

iv) $(4-\sqrt{3})(4+\sqrt{3})=13$

v) $\sqrt{2}(3-\sqrt{2})=3\sqrt{2}-2$

vi) $\sqrt{2}(3-2\sqrt{2})=3\sqrt{2}-4$

vii) $\sqrt{3}(\sqrt{27}-1)=9-\sqrt{3}$

viii) $(\sqrt{2}-1)(\sqrt{2}+1)=1$

ix) $(\sqrt{3}-2)(\sqrt{3}-1)=5-3\sqrt{3}$

x) $(2\sqrt{2}+1)(\sqrt{2}-2)=2-3\sqrt{2}$

xi) $(3\sqrt{3}-2)(3\sqrt{3}+2)=23$

xii) $(2\sqrt{5}+3)(3\sqrt{5}-2)=24+5\sqrt{5}$

xiii) $(\sqrt{3}-1)(\sqrt{2}+1)=\sqrt{6}+\sqrt{3}-\sqrt{2}-1$

xiv) $(2\sqrt{6}-3)^2=33-12\sqrt{6}$

Conjugate surds

Given the surd $(\sqrt{a}+\sqrt{b})$, its **conjugate surd** is $(\sqrt{a}-\sqrt{b})$.

Similarly, given $(4+\sqrt{5})$, its conjugate surd is $(4-\sqrt{5})$. The product of two conjugate surds **is always a rational number** but not a surd. For example,

$$(4+\sqrt{5})(4-\sqrt{5}) = 4^2-(\sqrt{5})^2 = 16-5 = 11.$$

Note that we treat the product as difference of two squares i.e. $(a+b)(a-b) = a^2-b^2$ This property has an important application in the next section.

Illustrative examples

Multiply the following by a bracket which will make the product rational.

i) $(2-\sqrt{3})$ ii) $(\sqrt{7}+2)$

iii) $(2\sqrt{6}+\sqrt{5})$ iv) $(5-2\sqrt{7})$

Solution

i) $(2-\sqrt{3})(2+\sqrt{3})=1$

ii) $(\sqrt{7}+2)(\sqrt{7}-2)=3$

iii) $(2\sqrt{6}+\sqrt{5})(2\sqrt{6}-\sqrt{5})=19$

iv) $(5-2\sqrt{7})(5+2\sqrt{7})=-3$

Example 5

If $\sqrt{32}+\sqrt{128}-\sqrt{18}+\sqrt{50}=k\sqrt{2}$ find the value of k.

Solution

$\sqrt{32}+\sqrt{128}-\sqrt{18}+$

$=\sqrt{(16\times 2)}+$

$=4\sqrt{2}+8$

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($\sqrt{2}$ or $\sqrt{3}$) from the ator.

Where denominators such as $(\sqrt{a}+\sqrt{b})$ and $(\sqrt{a}-\sqrt{b})$ are involved, it is always important to multiply both the *numerator* and the *denominator* by the **conjugate surd**.

For example, to simplify $\frac{3\sqrt{2}}{5-\sqrt{2}}$, we multiply the numerator and denominator by $5+\sqrt{2}$. Thus

$$\frac{3\sqrt{2}}{5-\sqrt{2}}=\frac{3\sqrt{2}}{5-\sqrt{2}}\times\frac{5+\sqrt{2}}{5+\sqrt{2}}=\frac{15\sqrt{2}+6}{5^2-(\sqrt{2})^2}$$

$$=\frac{15\sqrt{2}+6}{25-2}=\frac{15\sqrt{2}+6}{23}$$

Illustrative examples

Example 1

Simplify the following:

i) $\frac{1}{4-\sqrt{5}}$ ii) $\frac{2+\sqrt{3}}{\sqrt{3}-1}$

iii) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ iv) $\frac{1}{3-2\sqrt{2}}$

v) $\frac{5-\sqrt{3}}{2+\sqrt{3}}$ vi) $\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$

vii) $\frac{2}{3-\sqrt{7}}$ viii) $\frac{1}{(1-\sqrt{3})^2}$

ix) $\frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}$

Solution

i) The conjugate surd of $(4-\sqrt{5})$ is $(4+\sqrt{5})$, therefore multiply numerator and denominator by $(4+\sqrt{5})$.

$$\frac{1}{4-\sqrt{5}}=\frac{1}{4-\sqrt{5}}\times\frac{4+\sqrt{5}}{4+\sqrt{5}}=\frac{4+\sqrt{5}}{4^2-(\sqrt{5})^2}$$

$$=\frac{4+\sqrt{5}}{16-5}=\frac{4+\sqrt{5}}{11}=\frac{4}{11}+\frac{\sqrt{5}}{11}$$

ii) The conjugate surd of $(\sqrt{3}-1)$ is $(\sqrt{3}+1)$, therefore multiply numerator and denominator by $(\sqrt{3}+1)$.

$$\frac{2+\sqrt{3}}{\sqrt{3}-1}=\frac{2+\sqrt{3}}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$=\frac{2\sqrt{3}+2+3+\sqrt{3}}{3-1}$$

$$=\frac{5+3\sqrt{3}}{2}=\tfrac{5}{2}+\tfrac{3}{2}\sqrt{3}$$

iii) The conjugate surd of $(\sqrt{3}+\sqrt{2})$ is $(\sqrt{3}-\sqrt{2})$, therefore multiply numerator and denominator by $(\sqrt{3}-\sqrt{2})$.

$$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}\times\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$$

$$=\frac{(\sqrt{3}-\sqrt{2})(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}$$

$$=\frac{3-\sqrt{6}-\sqrt{6}+2}{(\sqrt{3})^2-(\sqrt{2})^2}$$

$$=\frac{5-2\sqrt{6}}{3-2}=5-2\sqrt{6}$$

iv) The conjugate surd of $(3-2\sqrt{2})$ is $(3+2\sqrt{2})$, therefore multiply numerator and denominator by $(3+2\sqrt{2})$.

$$\frac{1}{3-2\sqrt{2}}=\frac{1}{3-2\sqrt{2}}\times\frac{3+2\sqrt{2}}{3+2\sqrt{2}}$$

$$=\frac{3+2\sqrt{2}}{(3-2\sqrt{2})(3+2\sqrt{2})}$$

$$=\frac{3+2\sqrt{2}}{3^2-(2\sqrt{2})^2}=\frac{3+2\sqrt{2}}{9-8}=3+2\sqrt{2}$$

v) The conjugate surd of $(2+\sqrt{3})$ is $(2-\sqrt{3})$, therefore multiply numerator and denominator by $(2-\sqrt{3})$.

$$\frac{5-\sqrt{3}}{2+\sqrt{3}}=\frac{5-\sqrt{3}}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}=\frac{10-5\sqrt{3}-2\sqrt{3}+3}{4-3}$$

$$=13-7\sqrt{3}$$

vi) Since the conjugate surd of $\sqrt{5}+\sqrt{2}$ is $\sqrt{5}-\sqrt{2}$, multiply numerator and denominator by $\sqrt{5}-\sqrt{2}$.

$$\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}=\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}\times\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}$$

$$=\frac{5-\sqrt{10}-\sqrt{10}+2}{5-2}$$

$$=\frac{7-2\sqrt{10}}{3}=\tfrac{7}{3}-\tfrac{2}{3}\sqrt{10}$$

vii) The conjugate surd of $(3-\sqrt{7})$ is $(3+\sqrt{7})$, therefore multiply numerator and denominator by $(3+\sqrt{7})$.

$$\frac{2}{3-\sqrt{7}}=\frac{2}{3-\sqrt{7}}\times\frac{3+\sqrt{7}}{3+\sqrt{7}}=\frac{6+2\sqrt{7}}{9-7}$$

$$=\frac{6+2\sqrt{7}}{2}=3+\sqrt{7}$$

viii) $(1-\sqrt{3})^2=(1-\sqrt{3})(1-\sqrt{3})$

$$=1-\sqrt{3}-\sqrt{3}+3=4-2\sqrt{3}$$

$$\therefore\ \frac{1}{(1-\sqrt{3})^2}=\frac{1}{4-2\sqrt{3}}$$

$$=\frac{1}{4-2\sqrt{3}}\times\frac{4+2\sqrt{3}}{4+2\sqrt{3}}$$

$$=\frac{4+2\sqrt{3}}{16-12}$$

$$=\frac{4+2\sqrt{3}}{4}=1+\tfrac{1}{2}\sqrt{3}$$

ix) The conjugate surd of $(\sqrt{3}-\sqrt{2})$ is $(\sqrt{3}+\sqrt{2})$, therefore multiply numerator and denominator by $(\sqrt{3}+\sqrt{2})$.

$$\frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}=\frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}\times\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\frac{3+\sqrt{6}}{3-2}$$

$$=3+\sqrt{6}$$

Note that the idea is to multiply top and bottom by the conjugate surd of the denominator.

Example 2

Simplify i) $\dfrac{2+\sqrt{3}}{\sqrt{3}-1}-\dfrac{\sqrt{3}-1}{2+\sqrt{3}}$

ii) $\dfrac{\sqrt{3}}{\sqrt{3}-1}+\dfrac{\sqrt{3}}{\sqrt{3}+1}$

Solution

i) Since the product of the denominators of the two terms does not give difference of two squares, we rationalize each term separately.

$$\frac{2+\sqrt{3}}{\sqrt{3}-1}=\frac{2+\sqrt{3}}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$=\frac{2\sqrt{3}+2+3+\sqrt{3}}{3-1}$$

$$=\frac{5+3\sqrt{3}}{2}=\tfrac{5}{2}+\tfrac{3}{2}\sqrt{3}$$

$$\frac{\sqrt{3}-1}{2+\sqrt{3}}=\frac{\sqrt{3}-1}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}$$

$$=\frac{2\sqrt{3}-3-2+\sqrt{3}}{4-3}$$

$$=\frac{-5+3\sqrt{3}}{1}=-5+3\sqrt{3}$$

$$\therefore\ \frac{2+\sqrt{3}}{\sqrt{3}-1}-\frac{\sqrt{3}-1}{2+\sqrt{3}}$$

$$=(\tfrac{5}{2}+\tfrac{3}{2}\sqrt{3})-(-5+3\sqrt{3})$$

$$=\tfrac{15}{2}-\tfrac{3}{2}\sqrt{3}$$

ii) Since the product of the denominators gives difference of two squares, we first find the LCM

$$\frac{\sqrt{3}}{\sqrt{3}-1}+\frac{\sqrt{3}}{\sqrt{3}+1}$$

$$=\frac{\sqrt{3}(\sqrt{3}+1)+\sqrt{3}(\sqrt{3}-1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$=\frac{3+\sqrt{3}+3-\sqrt{3}}{3-1}=\frac{6}{2}=3$$

Example 3

Simplify the following:

i) $\frac{1}{\sqrt{3}}$ ii) $\frac{1}{\sqrt{8}}$

iii) $\frac{2}{\sqrt{32}}$ iv) $\frac{1}{\sqrt{2}+1}$

v) $\frac{1}{\sqrt{3}-1}$ vi) $\frac{1}{2-\sqrt{3}}$

vii) $\frac{5}{2+\sqrt{5}}$ viii) $\frac{2}{2\sqrt{3}-3}$

ix) $\frac{1}{3+2\sqrt{5}}$ x) $\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{2}-1}$

xi) $\frac{1}{\sqrt{6}-\sqrt{5}}$ xii) $\frac{1}{2\sqrt{3}+\sqrt{2}}$

Solution

Note: In each case the final answer has been provided. Work out the missing steps.

i) $\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$ ii) $\frac{1}{\sqrt{8}}=\frac{\sqrt{2}}{4}$

iii) $\frac{2}{\sqrt{32}}=\frac{\sqrt{2}}{4}$

iv) $\frac{1}{\sqrt{2}+1}=\sqrt{2}-1$

v) $\frac{1}{\sqrt{3}-1}=\frac{\sqrt{3}+1}{2}$

vi) $\frac{3}{2-\sqrt{3}}=3(2+\sqrt{3})$

vii) $\frac{5}{2+\sqrt{5}}=5(\sqrt{5}-2)$

viii) $\frac{2}{2\sqrt{3}-3}=\frac{2}{3}(2\sqrt{3}+3)$

ix) $\frac{1}{3+2\sqrt{5}}=\frac{2\sqrt{5}-3}{11}$

x) $\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{2}-1}=2\sqrt{2}$

xi) $\frac{1}{\sqrt{6}-\sqrt{5}}=\sqrt{6}+\sqrt{5}$

xii) $\frac{1}{2\sqrt{3}+\sqrt{2}}=\frac{2\sqrt{3}-\sqrt{2}}{10}$

Solved examination questions

The following examples combine all the concepts you have learnt so far in this chapter.

Example 1

Express $\dfrac{2\sqrt{5}+5\sqrt{2}}{2\sqrt{5}-5\sqrt{2}}$ in the form $q+r\sqrt{s}$ where q, r and s are rational numbers.

Solution

In this example, we multiply top and bottom by $2\sqrt{5}+5\sqrt{2}$ i.e. the conjugate surd of the denominator

$$\frac{2\sqrt{5}+5\sqrt{2}}{2\sqrt{5}-5\sqrt{2}}=\frac{2\sqrt{5}+5\sqrt{2}}{2\sqrt{5}-5\sqrt{2}}\times\frac{2\sqrt{5}+5\sqrt{2}}{2\sqrt{5}+5\sqrt{2}}$$

$$=\frac{20+10\sqrt{10}+10\sqrt{10}+50}{20-50}$$

$$=\frac{70+20\sqrt{10}}{-30}=-\tfrac{7}{3}-\tfrac{2}{3}\sqrt{10}$$

$$=q+r\sqrt{s}$$

where $q=-\frac{7}{3}$, $r=-\frac{2}{3}$ and $s=10$

Example 2

Simplify $\dfrac{5}{3+\sqrt{2}}-\dfrac{2}{3-\sqrt{2}}$ leaving your answer in form $a+b\sqrt{n}$ where a, b and n are constants.

Solution

Since the product of the denominators gives difference of two squares, we first find the LCM

$$\frac{5}{3+\sqrt{2}}-\frac{2}{3-\sqrt{2}}=\frac{5(3-\sqrt{2})-2(3+\sqrt{2})}{(3+\sqrt{2})(3-\sqrt{2})}$$

$$\frac{5}{3+\sqrt{2}}-\frac{2}{3-\sqrt{2}}=\frac{15-5\sqrt{2}-6-2\sqrt{2}}{3^2-(\sqrt{2})^2}$$

$$=\frac{9-7\sqrt{2}}{9-2}=\tfrac{9}{7}-\sqrt{2}$$

$$=a+b\sqrt{n}$$

where $a=\frac{9}{7}$, $b=-1$ and $n=2$

Example 3

Express $\dfrac{8-2\sqrt{6}}{2\sqrt{3}+3\sqrt{2}}$ in the form $m\sqrt{3}+n\sqrt{2}$, where m and n are rational numbers.

Solution

$$\frac{8-2\sqrt{6}}{2\sqrt{3}+3\sqrt{2}}=\frac{8-2\sqrt{6}}{2\sqrt{3}+3\sqrt{2}}\times\frac{2\sqrt{3}-3\sqrt{2}}{2\sqrt{3}-3\sqrt{2}}$$

$$=\frac{16\sqrt{3}-24\sqrt{2}-4\sqrt{18}+6\sqrt{12}}{12-18}$$

$$=\frac{28\sqrt{3}-36\sqrt{2}}{-6}=-\frac{14}{3}\sqrt{3}+6\sqrt{2}$$

$$\therefore\ m=-\frac{14}{3}\ \text{and}\ n=6$$

Example 4

Simplify

i) $\sqrt{5}(\sqrt{5}+\dfrac{3}{2\sqrt{5}})$

ii) $\sqrt{0.0675}$

Solution

$$\text{i)}\quad \sqrt{5}(\sqrt{5}+\frac{3}{2\sqrt{5}})=(\sqrt{5})^2+\frac{3}{2\sqrt{5}}\sqrt{5}$$

$$=5+\frac{3}{2}=6\tfrac{1}{2}$$

$$\text{ii)}\quad \sqrt{0.0675}=\sqrt{\frac{675}{10000}}=\frac{\sqrt{225\times3}}{\sqrt{10000}}=\frac{15\sqrt{3}}{100}$$

$$=\tfrac{3}{20}\sqrt{3}$$

Example 5

Express $\frac{\sqrt{2}+\sqrt{5}}{\sqrt{10}}$ in the form $a\sqrt{5}+b\sqrt{2}$ where $a, b \in R$

Solution

$$\frac{\sqrt{2}+\sqrt{5}}{\sqrt{10}} = \frac{(\sqrt{2}+\sqrt{5})}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}$$

$$= \frac{\sqrt{10}(\sqrt{2}+\sqrt{5})}{(\sqrt{10})^2} = \frac{\sqrt{20}+\sqrt{50}}{10}$$

$$= \frac{2\sqrt{5}+5\sqrt{2}}{10} = \frac{2\sqrt{5}}{10} + \frac{5\sqrt{2}}{10}$$

$$= 0.2\sqrt{5} + 0.5\sqrt{2}$$

∴ $a = 0.2$ and $b = 0.5$

Example 6

Express $\sqrt{15}(\sqrt{27} - \frac{2}{\sqrt{3}})$ in the form $p\sqrt{q}$, where p and q are real numbers.

Solution

$$\sqrt{15}(\sqrt{27} - \frac{2}{\sqrt{3}}) = \sqrt{15}(3\sqrt{3} - \frac{2}{\sqrt{3}})$$

$$= 3\sqrt{45} - \frac{2\sqrt{15}}{\sqrt{3}} = 9\sqrt{5} - 2\sqrt{5}$$

$$= 7\sqrt{5} = p\sqrt{q}$$

∴ $p = 7$ and $q = 5$

Note: $\frac{2\sqrt{15}}{\sqrt{3}} = 2\sqrt{\frac{15}{3}} = 2\sqrt{5}$ and

$3\sqrt{45} = 3\sqrt{(9 \times 5)} = 9\sqrt{5}$

Example 7

Simplify $\frac{\sqrt{50}\times\sqrt{45}}{\sqrt{200}\times\sqrt{75}}$, leaving your answer in the form $a\sqrt{b}$, where a is rational and b is a positive integer.

Solution

$$\frac{\sqrt{50}\times\sqrt{45}}{\sqrt{200}\times\sqrt{75}} = \frac{(\sqrt{25\times 2})(\sqrt{9\times 5})}{(\sqrt{100\times 2})(\sqrt{25\times 3})}$$

$$= \frac{5\sqrt{2}\times 3\sqrt{5}}{10\sqrt{2}\times 5\sqrt{3}} = \frac{3\sqrt{5}}{10\sqrt{3}}$$

$$= \frac{3\sqrt{5}}{10\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{15}}{10(\sqrt{3})^2}$$

$$= \frac{3\sqrt{15}}{10\times 3} = \tfrac{1}{10}\sqrt{15}$$

$$= 0.1\sqrt{15} = a\sqrt{b}$$

∴ $a = 0.1$ and $b = 15$

Note: The final answer must always be rationalized.

Example 8

Simplify the following

i) $\sqrt{63} + \frac{1}{\sqrt{7}}$ ii) $\frac{1}{\sqrt{3}}\left(2\sqrt{3} - \frac{1}{\sqrt{3}}\right)$

Solution

i) $\sqrt{63} + \frac{1}{\sqrt{7}} = \sqrt{(9\times 7)} + \left(\frac{1}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}\right)$

$$= 3\sqrt{7} + \frac{\sqrt{7}}{(\sqrt{7})^2} = 3\sqrt{7} + \frac{\sqrt{7}}{7}$$

$$= 3\sqrt{7} + \tfrac{1}{7}\sqrt{7} = 3\tfrac{1}{7}\sqrt{7} = \tfrac{22}{7}\sqrt{7}$$

ii) $\frac{1}{\sqrt{3}}\left(2\sqrt{3} - \frac{1}{\sqrt{3}}\right) = \frac{2\sqrt{3}}{\sqrt{3}} - \frac{1}{(\sqrt{3})^2}$

$$= 2 - \tfrac{1}{3} = \tfrac{5}{3}$$

Example 9

i) Simplify $(1-\sqrt{3})(\frac{1}{\sqrt{3}} + \sqrt{3})$, giving your answer in the form $p + q\sqrt{3}$.

ii) Given that $\sqrt{2} = 1.41214$, calculate without using tables or calculator, the value of $3-\dfrac{3}{\sqrt{2}}$ correct to five significant figures.

Solution

i) $(1-\sqrt{3})(\tfrac{1}{\sqrt{3}}+\sqrt{3}) = \dfrac{1}{\sqrt{3}}+\sqrt{3}-\dfrac{\sqrt{3}}{\sqrt{3}}-(\sqrt{3})^2$

$$= \frac{1}{\sqrt{3}}+\sqrt{3}-1-3$$

$$= \tfrac{1}{3}\sqrt{3}+\sqrt{3}-4$$

$$= \tfrac{4}{3}\sqrt{3}-4$$

$$= -4+\tfrac{4}{3}\sqrt{3}$$

$$= p+q\sqrt{3}$$

where $p=-4$ and $q=\tfrac{4}{3}\sqrt{3}$

Note: $\frac{1}{\sqrt{3}}=\frac{1}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{3}}{3}=\frac{1}{3}\sqrt{3}$

ii) $3-\dfrac{3}{\sqrt{2}}=3-\left(\dfrac{3}{\sqrt{2}}\times\dfrac{\sqrt{2}}{\sqrt{2}}\right)$

$$=3-\frac{3\sqrt{2}}{2}=3(1-\frac{\sqrt{2}}{2})$$

But $\sqrt{2}=1.41214$

$$\therefore\ 3-\frac{3}{\sqrt{2}}=3(1-\frac{1.41214}{2})$$

$$=3(1-0.70607)$$

$$=3\times 0.29393$$

$$=0.88179\text{(to 5 s.f.)}$$

Note: First simplify before substituting the value of the surd given.

Example 10

Given that $\sqrt{5}=2.236$, calculate without using tables or calculator, the value of $2\sqrt{5}(6-2\sqrt{5})$.

Solution

$2\sqrt{5}(6-2\sqrt{5})=12\sqrt{5}-4(\sqrt{5})^2$

$$=12\sqrt{5}-20=12(2.236)-20$$

$$=26.832-20=6.832$$

Note: First simplify before substituting the value of the surd given.

Example 11

Evaluate $3\sqrt{7}(7-2\sqrt{7})$ if $\sqrt{7}=2.646$.

Solution

$3\sqrt{7}(7-2\sqrt{7})=21\sqrt{7}-6(\sqrt{7})^2$

$$=21\sqrt{7}-(6\times 7)\ =21\sqrt{7}-42$$

$$=21(2.646)-42\ =55.566-42$$

$$=13.566$$

Note: First simplify before substituting the value of the given surd.

Example 12

If $p*q=p^2+q^2+2pq$, find

i) $\sqrt{3}*\dfrac{1}{\sqrt{12}}$ ii) $\sqrt{3}*\sqrt{12}$

iii) $\sqrt{5}*\sqrt{20}$

Solution

i) Given that $p*q=p^2+q^2+2pq$

$$\sqrt{3}*\frac{1}{\sqrt{12}}=(\sqrt{3})^2+\left(\frac{1}{\sqrt{12}}\right)^2+2(\sqrt{3})\left(\frac{1}{\sqrt{12}}\right)$$

$$=3+\frac{1}{12}+\frac{2\sqrt{3}}{\sqrt{12}}=3+\frac{1}{12}+2\sqrt{\frac{3}{12}}$$

$$=3+\frac{1}{12}+2\sqrt{\frac{1}{4}}=3+\frac{1}{12}+2(\tfrac{1}{2})$$

$$=4+\tfrac{1}{12}=\tfrac{49}{12}$$

Note: Refer to Chapter 3 for notes on Binary Operations.

ii) $\sqrt{3} * \sqrt{12} = (\sqrt{3})^2 + (\sqrt{12})^2 + 2(\sqrt{3})(\sqrt{12})$

$$= 3 + 12 + 2\sqrt{36}$$

$$= 3 + 12 + 2(6) = 27$$

iii) $\sqrt{5} * \sqrt{20} = (\sqrt{5})^2 + (\sqrt{20})^2 + 2(\sqrt{5})(\sqrt{20})$

$$= 5 + 20 + 2\sqrt{100}$$

$$= 5 + 20 + 2(10) = 45$$

Example 13

Evaluate the following, leaving your answer in the surd form

i) $5\sqrt{75} - 3\sqrt{48} + 2\sqrt{27}$

ii) $3\sqrt{5}(\frac{3}{\sqrt{5}} + \sqrt{5})$

iii) $\sqrt{7}(3\sqrt{7} + \frac{6}{\sqrt{7}})$

iv) $(5 + 2\sqrt{5})(3 - \frac{2}{\sqrt{5}})$

v) $(1 - \sqrt{2})^2$

vi) $\frac{1}{\sqrt{3}}(\sqrt{6} + \sqrt{12})$

Solution

i) $5\sqrt{75} - 3\sqrt{48} + 2\sqrt{27}$

$$= 5\sqrt{(25 \times 3)} - 3\sqrt{(16 \times 3)} + 2\sqrt{(9 \times 3)}$$

$$= 25\sqrt{3} - 12\sqrt{3} + 6\sqrt{3} = 19\sqrt{3}$$

ii) $3\sqrt{5}(\frac{3}{\sqrt{5}} + \sqrt{5}) = \frac{9\sqrt{5}}{\sqrt{5}} + 3(\sqrt{5})^2$

$$= 9 + 15 = 24$$

iii) $\sqrt{7}(3\sqrt{7} + \frac{6}{\sqrt{7}}) = 3(\sqrt{7})^2 + \frac{6\sqrt{7}}{\sqrt{7}}$

$$= 21 + 6 = 27$$

iv) $(5 + 2\sqrt{5})(3 - \frac{2}{\sqrt{5}})$

$$= 15 - \frac{10}{\sqrt{5}} + 6\sqrt{5} - \frac{4\sqrt{5}}{\sqrt{5}}$$

$$= 15 - \left(\frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}\right) + 6\sqrt{5} - 4$$

$$= 11 - \left(\frac{10\sqrt{5}}{(\sqrt{5})^2}\right) + 6\sqrt{5}$$

$$= 11 - \frac{10\sqrt{5}}{5} + 6\sqrt{5}$$

$$= 11 - 2\sqrt{5} + 6\sqrt{5} = 11 + 4\sqrt{5}$$

v) $(1 - \sqrt{2})^2 = (1 - \sqrt{2})(1 - \sqrt{2})$

$$= 1 - \sqrt{2} - \sqrt{2} + (\sqrt{2})^2$$

$$= 1 - 2\sqrt{2} + 2 = 3 - 2\sqrt{2}$$

vi) $\frac{1}{\sqrt{3}}(\sqrt{6} + \sqrt{12}) = \frac{\sqrt{6}}{\sqrt{3}} + \frac{\sqrt{12}}{\sqrt{3}}$

$$= \sqrt{\frac{6}{3}} + \sqrt{\frac{12}{3}}$$

$$= \sqrt{2} + \sqrt{4} = \sqrt{2} + 2$$

Example 14

Express the following in the form $q + r\sqrt{s}$ where q, r and s are rational numbers.

i) $\frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}}$

ii) $\frac{\sqrt{5} + 1}{\sqrt{5} - \sqrt{3}}$

iii) $\frac{2\sqrt{2} - \sqrt{3}}{\sqrt{2} + \sqrt{3}}$

iv) $\frac{\sqrt{2} + 2\sqrt{5}}{\sqrt{5} - \sqrt{2}}$

v) $\frac{\sqrt{6} + \sqrt{3}}{\sqrt{6} - \sqrt{3}}$

vi) $\frac{\sqrt{10} + 2\sqrt{5}}{\sqrt{10} + \sqrt{5}}$

vii) $\frac{2\sqrt{3} + 3\sqrt{2}}{3\sqrt{2} - 2\sqrt{3}}$

viii) $\frac{6 + 2\sqrt{5}}{6 - 3\sqrt{6}}$

ix) $\dfrac{\sqrt{2}}{\sqrt{3}-\sqrt{2}}-\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ x) $\dfrac{1}{3\sqrt{2}-2\sqrt{3}}$

xi) $\dfrac{1}{\sqrt{5}+\sqrt{3}}-\dfrac{1}{\sqrt{5}-\sqrt{3}}$ xii) $\dfrac{2\sqrt{3}}{3\sqrt{2}-\sqrt{5}}$

xiii) $\dfrac{3}{3-\sqrt{2}}+\dfrac{1}{3+\sqrt{2}}$ xiv) $\dfrac{5\sqrt{7}-7\sqrt{5}}{\sqrt{7}-\sqrt{5}}$

Solution

Work out the solution and compare your answers.

Equality of surds

Two surd expressions $a+\sqrt{b}$ and $c+\sqrt{d}$ can only be **equal** if $a=c$ and $b=d$.

We can use this to find the square root of a surd expression.

Note that: $(\sqrt{a}+\sqrt{b})^2=(a+b)+2\sqrt{ab}$

and $(\sqrt{a}-\sqrt{b})^2=(a+b)-2\sqrt{ab}$

Therefore the square root of say $3+2\sqrt{2}$ must be of the form $\sqrt{a}+\sqrt{b}$ and the square root of say $24-16\sqrt{2}$ must be of the form $\sqrt{a}-\sqrt{b}$.

Illustrative examples

Example 1

Find the positive square root of $49-12\sqrt{5}$.

Solution

The square root of $49-12\sqrt{5}$ must be of the form $\sqrt{a}-\sqrt{b}$, where $a>b$ as the root must be positive.

$$\sqrt{49-12\sqrt{5}}=\sqrt{a}-\sqrt{b}$$

$$49-12\sqrt{5}=(\sqrt{a}-\sqrt{b})^2$$

[i.e. squaring both sides]

$$49-12\sqrt{5}=(a+b)-2\sqrt{ab}$$

Since the two surd expressions are equal

$$49=a+b \;\ldots\ldots(1) \text{ and}$$

$$12\sqrt{5}=2\sqrt{ab} \Rightarrow 6\sqrt{5}=\sqrt{ab}$$

$$180=ab\ldots\ldots\ldots\ldots(2)$$

We now solve equations (1) and (2) for a and b.

From eqn(1), $a=49-b$. Therefore eqn(2) becomes $180=b(49-b)$

$$b^2-49b+180=0$$

$$(b-4)(b-45)=0$$

$$b=4 \text{ or } b=45$$

Hence $a=45$ or 4.

Since $a>b$, we select $a=45$ and $b=4$.

Thus, the square root of $49-12\sqrt{5}$ is $\sqrt{45}-\sqrt{4}=3\sqrt{5}-2$

Example 2

If $(3+4\sqrt{3})(2-a\sqrt{3})=-18+2\sqrt{3}$, find the value of a.

Solution

$$(3+4\sqrt{3})(2-a\sqrt{3})=-18+2\sqrt{3}$$

$$(6-12a)+(-3a+8)\sqrt{3}=-18+2\sqrt{3}$$

i.e. expanding the LHS and rearranging like terms.

Equating corresponding terms

$$6-12a=-18$$

$a=2$ or

$$-3a+8=2 \Rightarrow a=2$$

Radical equations

The following steps are used for solving equations involving surds

1. Group terms containing the surds
2. Square both sides and simplify
3. Square again if necessary, simplify and solve the resulting equation
4. Check if any or all the roots satisfy the equation.

Illustrative examples

Example 1

Solve $\sqrt{(3x+4)}-\sqrt{(x+5)}=1$

Solution

$$\sqrt{(3x+4)}-\sqrt{(x+5)}=1$$

$$\sqrt{3x+4}=1+\sqrt{x+5}$$

$$3x+4=1+x+5+2\sqrt{x+5}$$

$$2\sqrt{x+5}=2x-2$$

$$4(x+5)=4-8x+4x^2$$

$$x^2-3x-4=0$$

$$(x-4)(x+1)=0$$

$$x=-1 \text{ or } 4$$

$x=4$ is a solution since it satisfies the original equation. $x=-1$ is not a solution since it does not satisfy the original equation.

Example 2

Solve the following equations

i) $\sqrt{(3x+4)}-\sqrt{(x-3)}=3$

ii) $\sqrt{(x^2-7)}=3$

iii) $x+\sqrt{x+1}=5$

iv) $\sqrt{(x+7)}-\sqrt{(3x-2)}=1$

Solution

i) Squaring both sides (note that the terms are already grouped)

$$\sqrt{(3x+4)}-\sqrt{(x-3)}=3$$

$$(3x+4)-2\left(\sqrt{(3x+4)}\right)\left(\sqrt{(x-3)}\right)+(x-3)=9$$

$$(4x+1)-2\left(\sqrt{(3x+4)(x-3)}\right)=9$$

$$(4x-8)=2\left(\sqrt{(3x+4)(x-3)}\right)$$

$$(2x-4)=\left(\sqrt{(3x^2-5x-12}\right)$$

Squaring both sides again, we have

$$(2x-4)^2=3x^2-5x-12$$

$$4x^2-16x+16=3x^2-5x-12$$

$$x^2-11x+28=0$$

$$(x-7)(x-4)=0$$

$$x=4 \text{ or } 7$$

Since they all satisfy the original equation.

ii) $\sqrt{(x^2-7)}=3$

$$x^2-7=9 \qquad \Rightarrow \quad x^2-16=0$$

$$(x-4)(x+4)=0 \quad \Rightarrow \quad x=\pm 4$$

iii) $x+\sqrt{x+1}=5 \quad \Rightarrow \quad \sqrt{x+1}=5-x$

Squaring both sides, we have

$$x+1=(5-x)^2$$

$$x+1=25-10x+x^2$$

$$x^2-11x+24=0$$

$$(x-3)(x-8)=0$$

$$x=3 \text{ or } 8$$

$x=3$ is a solution since it satisfies the original equation. $x=8$ is not a solution since it does not satisfy the original equation.

iv) Squaring both sides (note that the terms are already grouped)

$$\sqrt{(x+7)}-\sqrt{(3x-2)}=1$$

$$(x+7)-2\left(\sqrt{(x+7)}\right)\left(\sqrt{(3x-2)}\right)+(3x-2)=1$$

$$(4x+4)=2\left(\sqrt{(x+7)(3x-2)}\right)$$

$$2(2x+2)=2\sqrt{(3x^2+19x-14}$$

$$(2x+2)=\sqrt{(3x^2+19x-14}$$

Squaring both sides again, we have

$$(2x+2)^2=3x^2+19x-14$$

$$4x^2+8x+4=3x^2+19x-14$$

$$x^2-11x+18=0$$

$$(x-2)(x-9)=0$$

$$x=2 \text{ or } 9$$

Since they all satisfy the original equation.

Example 3
Solve the following equations

i) $\sqrt{x-3}+1=\sqrt{x}$
ii) $\sqrt{x+5}=1+\sqrt{x}$
iii) $\sqrt{4x-3}-\sqrt{3x-5}=1$
iv) $\sqrt{5x+1)}-\sqrt{3x-5)}=2$
v) $\sqrt{5x-9)}+3=\sqrt{x}$
vi) $\sqrt{7x+4}-\sqrt{x+1}=3$
vii) $\sqrt{x+3}=1+\sqrt{x-2}$
viii) $\sqrt{5x-6}=1+\sqrt{3x-5}$
ix) $\sqrt{2x-1}-1=\sqrt{x-1}$
x) $\sqrt{4x+1}-\sqrt{3x-2}=1$
xi) $\sqrt{2x+3}+\sqrt{x-2}=2$
xii) $\sqrt{5x}-\sqrt{x+4}=2$

Solution
Work out the solution and compare your answers.

Review Exercise 2

1. Simplify each of the following, leaving your answer as a surd.
   i) $\sqrt{27}$ ii) $\sqrt{32}$
   iii) $\sqrt{8}$ iv) $\sqrt{128}$
   v) $\sqrt{128}-\sqrt{50}+\frac{2}{3}\sqrt{162}$
   vi) $\frac{4}{\sqrt{5}}+3\sqrt{125}$ vii) $\sqrt{72}$
   viii) $\sqrt{162}$ ix) $\sqrt{288}$
   x) $\sqrt{500}$ xi) $\sqrt{450}$
   xii) $\sqrt{48}$

2. Express $\frac{1+\sqrt{2}}{\sqrt{2}}$ in the form $p+q\sqrt{2}$

3. Evaluate $\sqrt{125}-\sqrt{180}+\sqrt{45}$

4. Simplify the following, leaving your answer as a surd.
   i) $\sqrt{50}-\sqrt{2}-2\sqrt{18}+\sqrt{8}$
   ii) $\sqrt{28}+\sqrt{175}-\sqrt{63}$
   iii) $\sqrt{24}-3\sqrt{6}-\sqrt{216}$
   iv) $\sqrt{75}+2\sqrt{12}-\sqrt{27}$
   v) $(1-\sqrt{2})(3+2\sqrt{2})$
   vi) $\sqrt{80}+\sqrt{20}-\sqrt{45}$
   vii) $\sqrt{28}+\sqrt{63}$
   viii) $\sqrt{112}-\sqrt{28}$
   ix) $3\sqrt{8}+\sqrt{50}$
   x) $\frac{1}{\sqrt{2}}+\frac{1}{3\sqrt{2}}$

5. If $\sqrt{3}=1.732$, find without using tables $\frac{2}{\sqrt{3}}$ correct to three significant figures

6. Express $\frac{1}{\sqrt{5}+\sqrt{3}}$ in the form $p\sqrt{5}+q\sqrt{3}$ where p and q are rational numbers

7. Express $\frac{4\sqrt{3}+3\sqrt{2}}{2\sqrt{3}+3\sqrt{2}}$ in the form $p+q\sqrt{r}$, where p, q and r are rational numbers.

8. Simplify
   i) $\frac{2\sqrt{3}+3\sqrt{2}}{2\sqrt{3}-3\sqrt{2}}$
   ii) $\frac{1}{\sqrt{5}-\sqrt{3}}-\frac{1}{\sqrt{5}+\sqrt{3}}$
   iii) $\frac{1}{(\sqrt{5}-\sqrt{3})^2}$

9. Simplify without using tables

i) $\dfrac{1}{3\sqrt{7}-5}-\dfrac{1}{3\sqrt{7}+5}$

ii) $\dfrac{\sqrt{2}}{2\sqrt{2}-\sqrt{3}}-\dfrac{\sqrt{3}}{2\sqrt{2}+\sqrt{3}}$

10. Without using tables, evaluate

$\dfrac{3\sqrt{7}+5}{3\sqrt{7}-5}+\dfrac{3\sqrt{7}-5}{3\sqrt{7}+5}$

11. Express $\dfrac{2}{3-\sqrt{7}}$ in the form $a+\sqrt{b}$, where a and b are constants.

12. Given that $\sqrt{2}=1.4142$, find the values, correct to 3 decimal places of

i) $\dfrac{3}{\sqrt{2}-1}$ ii) $\dfrac{3\sqrt{2}-4}{3-\sqrt{2}}$

13. Simplify the following:

i) $\dfrac{\sqrt{3}+\sqrt{12}+\sqrt{108}-\sqrt{75}}{\sqrt{6}-\sqrt{96}+\sqrt{150}}$

ii) $\dfrac{\sqrt{8}+5\sqrt{32}-4\sqrt{50}}{1+\sqrt{18}}$

iii) $\dfrac{21}{\sqrt{3}}-2\sqrt{162}+\sqrt{8}$

14. Find the positive square roots of the following:

i) $3+2\sqrt{2}$ ii) $28-10\sqrt{3}$

iii) $24-16\sqrt{2}$ iv) $10+2\sqrt{21}$

15. Solve the following:

i) $\sqrt{(x+7)}-\sqrt{(x-5)}=2$

ii) $2\sqrt{(3x+4)}+x=36$

iii) $\sqrt{(x+8)}-\sqrt{(x+1)}=7$

iv) $2\sqrt{(3x+1)}+3\sqrt{4x+5}=23$

v) $3\sqrt{(x+2)}-x=4$

vi) $2\sqrt{x}-\sqrt{4x-11}=1$

vii) $\sqrt{(x+7)}-\sqrt{(3x-2)}=1$

viii) $\sqrt{(5x-4)}-\sqrt{(x+8)}=2$

16. Expand and simplify the following surds

i) $\sqrt{3}(2-\sqrt{3})$

ii) $\sqrt{2}(5+4\sqrt{2})$

iii) $\sqrt{5}(2+\sqrt{75})$

iv) $(\sqrt{2}(\sqrt{32}+\sqrt{8})$

v) $(\sqrt{3}+1)(\sqrt{2}-1)$

vi) $(\sqrt{3}+2)(\sqrt{3}+5)$

vii) $(\sqrt{5}-1)(\sqrt{5}+1)$

viii) $(2\sqrt{2}-1)(\sqrt{2}-1)$

ix) $(\sqrt{5}-3)(2\sqrt{5}-4)$

x) $(4+\sqrt{7})(4-\sqrt{7})$

xi) $(\sqrt{6}-2)^2$

xii) $(2+3\sqrt{3})^2$

17. Rationalise and simplify the following:

i) $\dfrac{3}{\sqrt{2}}$ ii) $\dfrac{1}{\sqrt{7}}$

iii) $\dfrac{2}{\sqrt{11}}$ iv) $\dfrac{3\sqrt{2}}{\sqrt{5}}$

v) $\dfrac{1}{\sqrt{27}}$ vi) $\dfrac{\sqrt{5}}{\sqrt{10}}$

vii) $\dfrac{1}{\sqrt{2}-1}$ viii) $\dfrac{3\sqrt{2}}{5+\sqrt{2}}$

ix) $\dfrac{2}{2\sqrt{3}-3}$ x) $\dfrac{5}{2-\sqrt{5}}$

xi) $\frac{1}{\sqrt{7}-\sqrt{3}}$ xii) $\frac{4\sqrt{3}}{2\sqrt{3}-3}$

xiii) $\frac{3-\sqrt{5}}{\sqrt{5}+1}$ xiv) $\frac{2\sqrt{3}-1}{4-\sqrt{3}}$

xv) $\frac{\sqrt{5}-1}{\sqrt{5}+2}$ xvi) $\frac{3}{\sqrt{3}-\sqrt{2}}$

xvii) $\frac{3\sqrt{5}}{2\sqrt{5}+1}$ xviii) $\frac{\sqrt{2}+1}{\sqrt{2}-1}$

xix) $\frac{2\sqrt{7}}{\sqrt{7}+2}$ xx) $\frac{\sqrt{5}-1}{3-\sqrt{5}}$

xxi) $\frac{1}{\sqrt{11}-\sqrt{7}}$ xxii) $\frac{4-\sqrt{3}}{3-\sqrt{3}}$

xxiii) $\frac{1-3\sqrt{2}}{3\sqrt{2}+2}$ xxiv) $\frac{1}{3\sqrt{2}-2\sqrt{3}}$

xxv) $\frac{\sqrt{3}}{\sqrt{2}(\sqrt{6}-\sqrt{3})}$

xxvi) $\frac{1}{\sqrt{3}(\sqrt{21}+\sqrt{7})}$

xxvii) $\frac{\sqrt{2}}{\sqrt{3}(\sqrt{5}-\sqrt{2})}$

xxviii) $\frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}}-\frac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$

xxix) $\frac{1+\sqrt{2}}{2+\sqrt{3})}-\frac{1-\sqrt{2}}{2-\sqrt{3}}$

xxx) $\frac{\sqrt{2}}{2\sqrt{2}-\sqrt{3}}-\frac{\sqrt{3}}{2\sqrt{2}+\sqrt{3}}$

18. Simplify the following:

i) $\sqrt{18}+\sqrt{162}$

ii) $(5\sqrt{2}+3\sqrt{3})(2\sqrt{3}-\sqrt{2})$

iii) $\frac{3+2\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

iv) $\sqrt{180}+\sqrt{20}$

v) $(7\sqrt{2}+4\sqrt{7})(2\sqrt{7}-\sqrt{2})$

vi) $\frac{6\sqrt{2}-3\sqrt{3}}{5\sqrt{3}-2\sqrt{2}}$

vii) $(3+\sqrt{2})(4-2\sqrt{2})$

viii) $(\sqrt{5}-\sqrt{2})^2$

ix) $\sqrt{3}(7-2\sqrt{3})$

x) $\sqrt{2}(2\sqrt{2}+\sqrt{8})$

xi) $\frac{3}{\sqrt{13}-2}$

xii) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

CHAPTER 3

BINARY OPERATIONS

Definition

A binary operation * on a non-empty set R of real numbers is a rule which combines any two numbers a and $b \in R$ to produce another number.

The *simple operations* in mathematics are addition (+), subtraction (–), division (÷), multiplication ($\times$), intersection ($\cap$) and union ($\cup$). These operations may be defined on the set of real numbers. They are called **binary operations** because they combine two numbers to give another number.

However, if the operation is not one of the simple operations, then the operation will be defined for you. Any two real numbers can therefore be combined by strictly following the definition of the operation. For example, suppose the operation * is defined on the set of real numbers R by: $a * b = a + b + ab$; then by definition,

i) $2*3 = 2 + 3 + 2(3) = 11$ which is a number

ii) $3 * 4 = 3 + 4 + 3(4) = 19$ which is also a number.

Properties of binary operations

Closure

If $a * b$ produces a number which belongs to the set R, for all $a, b \in R$, then we say the set R is **closed** under the operation *.

For example, if the operation * is defined over the set R of real numbers by $a * b = a + b + ab$; for all $a, b \in R$, then $2 * 3 = 2 + 3 + 2(3) = 11$ which is also a real number. Therefore since when we combine two real numbers under the operation * we obtain a real number we say the set of real numbers R is closed under the operation *.

Now if the operation * is defined over the set $S = \{1, 2, 3, 4\}$, then $1 * 2 = 1 + 2 + 1(2) = 5$ which does not belong to S. Therefore the set S is not closed under the operation *.

Illustrative examples

Example 1

The operation * is defined on the set $S = (a, b, c)$ by the table below.

Determine whether or not S is closed under *

*	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Solution

The set S is closed with respect to the operation * if and only if any two elements in S when combined, will result in an element in set S. From the table, it is clear that whenever any two elements in S are combined under * the result is an element in S. i.e. $a * b = b \in S$;

$b * c = a \in S$,

$c * c = b \in S$ etc.

Hence the set S is closed with respect to *

Example 2

A binary operation * is defined on the set $S = \{2, 3, 5, 7\}$ by $a * b = a + b - 2$, where $a, b \in S$.

a) Copy and complete the table below.

*	2	3	5	7
2	2	3	5	7
3	3	4		8
5	5		8	
7	7	8	10	

b) i) Show whether or not S is closed under *.

ii) Evaluate $3 * (2 * 5)$.

Solution

a)

*	2	3	5	7
2	2	3	5	7
3	3	4	6	8
5	5	6	8	10
7	7	8	10	12

b) i) S is **not** closed under * because $6, 10, 12 \notin S$.

ii) $3 * (2 * 5) = 3 * 5 = 6$

Commutative operations

A binary operation * defined on a set R of real numbers is **commutative** if:

$$a * b = b * a \text{ for all } a, b \in R.$$

For example, $a + b = b + a$ and $a \times b = b \times a$. Therefore addition (+) and multiplication (×) of real numbers are commutative. Note that subtraction (–) and division (÷) of real numbers are not commutative since $a - b \neq b - a$ and $a \div b \neq b \div a$.

For any two sets A and B, $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

Therefore union ($\cup$) and intersection ($\cap$) of sets are commutative.

Illustrative examples

Example 1

The operation * is defined on the set of real numbers R by $a * b = a + b + ab$.

a) Find i) $3 * 4$ ii) $4 * 5$

iii) $\frac{1}{4} * \frac{3}{4}$

b) Show that * is commutative.

Solution

a) Following the definition of the operation

i) $3 * 4 = 3 + 4 + 3(4) = 7 + 12 = 19$

ii) $4 * 5 = 4 + 5 + 4(5) = 9 + 20 = 29$

iii) $\frac{1}{4} * \frac{3}{4} = \frac{1}{4} + \frac{3}{4} + (\frac{1}{4})(\frac{3}{4}) = 1 + \frac{3}{16} = 1\frac{3}{16}$

b) The operation * is commutative if $a * b = b * a$; for all $a, b \in R$

Now, $a * b = a + b + ab$ .... (1)

and $b * a = b + a + ab$

$= a + b + ab$ .... (2)

(Since + and × of real numbers are both commutative)

$\therefore$ From (1) and (2), $a * b = b * a$

Therefore the operation * is commutative.

Example 2

The operation * defined on the set $S = (a, b, c)$ by the table below

*	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Determine whether or not the operation * is commutative.

Solution

From the table,

$$a * b = b * a = b; \quad a * c = c * a = c;$$
$$b * c = c * b = a$$

Hence the operation * is commutative, since for any two elements $x, y \in S$, $x * y = y * x$

Associative operations

A *closed* binary operation * defined on a set R of real numbers is **associative** if:

$(a * b) * c = a * (b * c)$; for all $a, b, c \in R$.

For example,

$(ab)c = a(bc)$ and

$(a + b) + c = a + (b + c)$.

Therefore addition (+) and multiplication (×) of real numbers are associative.

Also for any three sets A, B and C,

$$A \cap (B \cap C) = (A \cap B) \cap C \text{ and } (A \cup B) \cup C = A \cup (B \cup C)$$

Therefore union ($\cup$) and intersection ($\cap$) of sets are associative.

Illustrative example

The operation * is defined over the set R of real numbers by $a * b = a + b + ab$.

a) Find:
   i) $3 * (4 * 1)$
   ii) $(2 * 4) * 5$

b) Show that * is associative.

Solution

a) Following the definition of the operation

i) $3 * (4 * 1) = 3 * [4 + 1 + 4(1)]$
$= 3 * 9 = 3 + 9 + 3(9)$
$= 12 + 27 = 39$

[Work out the operation in the bracket first]

ii) $(2 * 4) * 5 = [2 + 4 + 2(4)] * 5$
$= 14 * 5 = 14 + 5 + 14(5)$
$= 19 + 70 = 89$

b) The operation * is associative if $(a * b) * c = a * (b * c)$ for all $a, b, c \in R$.

Now,

$(a * b) * c$
$= (a + b + ab) * c$ (Bracket first)
$= (a + b + ab) + c + (a + b + ab)c$
$= a + b + c + ab + ac + bc + abc$..... (1)

Also,

$a * (b * c)$
$= a * (b + c + bc)$ (Bracket first)
$= a + (b + c + bc) + a(b + c + bc)$
$= a + b + c + bc + ac + ab + abc$
$= a + b + c + ab + ac + bc + abc$....(2)

From (1) and (2), it can be seen that $(a * b) * c = a * (b * c)$

Therefore the operation * is associative.

Distributive operations

Consider the two closed operations multiplication and addition defined on the set of real numbers. We can show, for example, that $2(3 + 4) = (2 \times 3) + (2 \times 4)$.

In fact, if a b and c are real numbers, then $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$. We can therefore say that multiplication ($\times$) is **distributive** over addition (+).

In general, when two closed binary operations * and Δ defined on a set R of real numbers are such that: $a * (b \,\Delta\, c) = (a * b) \,\Delta\, (a * c)$; for all $a, b, c \in R$ we say that the operation * is **left distributive** over the operation Δ.

Similarly, if $(b \,\Delta\, c) * a = (b * a) \,\Delta\, (c * a)$; for all $a, b, c \in R$, we say that the operation * is **right distributive** over the operation Δ. When the two results above are true then we say that the operation * is distributive over the operation Δ.

Note that the two operations, for example, * and Δ, will be defined in the question for you.

Illustrative example

The operation * is defined by

$$a * b = \tfrac{1}{2}ab$$

If $a = 4$, $b = 6$ and $c = 8$ evaluate
   i) $a * (b + c)$
   ii) $(a * b) + (a * c)$
   iii) What conclusion can you draw about * from your answers in (b) i) and ii).

Solution

b) Given that $a = 4$, $b = 6$ and $c = 8$
   i) $a * (b + c) = 4 * (6 + 8)$
$= 4 * 14$
$= \tfrac{1}{2}(4 \times 14) = 28$

ii) $(a * b) + (a * c) = (4 * 6) + (4 * 8)$

$$= \tfrac{1}{2}(4\times 6) + \tfrac{1}{2}(4\times 8)$$
$$= 12 + 16 = 28$$

iii) Since $4 * (6 + 8) = (4 * 6) + (4 * 8)$ from (b) (i) and (ii) above, it follows that the operation * is distributive over the operation +.

Identity or neutral elements

If $e \in R$ is the **identity or neutral** element under a binary operation * on a set R of real numbers, then;

$$a * e = e * a = a;\ \text{for } a \in R.$$

That is when we combine any number with the identity element under a given operation we get the number. The identity element e is unique i.e. we have only one identity element under each operation.

For example, zero (0) is the identity element under addition of real numbers since $1 + 0 = 1,\ 2 + 0 = 2$ etc.

Similarly, one (1) is the identity element under multiplication of real numbers since $2 \times 1 = 2;\ 3 \times 1 = 3$ etc.

Illustrative examples

Example 1

The operation * is defined over the set R of real numbers by $a * b = a + b + ab$. Find the identity element under the operation *.

Solution

Let $e \in R$ be the identity element under the operation *, then

$a * e = e * a = a$; for $a \in R$

But $a * e = a + e + ae = a$.

Solving for e, we have

$$a + e + ae = a$$
$$e + ae = a - a = 0$$
$$e(1 + a) = 0$$
$$e = \frac{0}{1 + a} = 0$$

$e = 0$ is the identity element under the operation *.

Example 2

The operation * defined on the set $S = (a, b, c)$ by the table below.

*	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Determine the identity element for the operation *

Solution

The identity element is such that when it is combined with any other element it gives the same element. From the table, $a * a = a$; $b * a = b$; and $c * a = c$.

Hence a is the identity element under *.

Inverse elements

An element $a^{-1} \in R$ is called the **inverse** of $a \in R$ under the operation *, if

$$a * a^{-1} = a^{-1} * a = e,$$

where e is the identity element under the binary operation *.

That is when we combine any number and its inverse we obtain the identity element.

Illustrative examples

Example 1

The operation * is defined over the set R of real numbers by $a * b = a + b + ab$; for all $a, b \in R$. Find the inverse under * of a general element $a \in R$ and state which element has no inverse. Find the inverse of 1.

Solution

If $a^{-1} \in R$ be the inverse of $a \in R$ under the operation *, then $a * a^{-1} = a^{-1} * a = e$. Note that we must first determine the identity element, e under the operation *. This has been determined as $e = 0$ in example 1 of page 49.

$\therefore \; a * a^{-1} = a^{-1} * a = e = 0;$

But $a * a^{-1} = a + a^{-1} + a\,a^{-1}$

$$a + a^{-1} + a\,a^{-1} = 0$$

Solving for a^{-1}, we have

$$a^{-1} + aa^{-1} = -a$$

$$a^{-1}(1 + a) = -a$$

$a^{-1} = \dfrac{-a}{1+a}$ is the inverse of a.

The expression for a^{-1} is undefined if the denominator is zero. Therefore to find the element which has no inverse, equate the denominator to zero.

i.e. $1 + a = 0 \;\Rightarrow\; a = -1$.

Therefore the real number -1 has no inverse under the operation *.

To find the inverse of 1, substitute this value (i.e. 1) into the expression for a^{-1}.

i.e. the inverse of $1 = \dfrac{-1}{1+1} = -\tfrac{1}{2}$.

Therefore the inverse of 1 is $-\tfrac{1}{2}$.

Example 2

The operation * defined on the set $S = (a, b, c)$ by the table below.

*	*a*	*b*	*c*
a	*a*	*b*	*c*
b	*b*	*c*	*a*
c	*c*	*a*	*b*

Determine the inverse of each element of S under *

Solution

The inverse element is such that when it is combined with its element it gives the identity element. The identity element is a.

From the table, $a * a = a$

$\therefore$ the inverse of a is a.

Also $b * c = a \;\; \therefore$ the inverse of b is c.

Similarly, $c * b = a \;\; \therefore$ the inverse of c is b.

Solved examination questions

The following examples combine all the concepts you have learnt, so far, in this chapter.

Example 1

An operation * is defined on the set R of real numbers by $a * b = a + b - ab$, where $a, b \in R$.

a) Calculate $8 * (-4)$ and $(-4) * 8$.

b) Find the identity element e of R under the operation *.

c) Determine the inverse under * of a general element $a \in R$ and state which element of R has no inverse. Find the inverse of 2.

Solution

a) From the definition, $a * b = a + b - ab$

$$\therefore \; 8 * (-4) = 8 + (-4) - 8(-4)$$
$$= 8 - 4 + 32 = 36$$

and

$$(-4) * 8 = (-4) + 8 - (-4)8$$
$$= -4 + 8 + 32 = 36$$

b) Since $e \in R$ is the identity element, for any element $a \in R$

$$a * e = e * a = a$$

But $a * e = a + e - ae$

$$a + e - ae = a$$
$$e - ae = a - a = 0$$
$$e(1 - a) = 0$$
$$e = \frac{0}{1-a} = 0.$$

Therefore the identity element $e = 0$.

c) Let $a^{-1} \in R$ be the inverse of $a \in R$ under the operation *, then $a * a^{-1} = a^{-1} * a = e$. But $a * a^{-1} = a + a^{-1} - aa^{-1} = e$; where $e = 0$ as determined in (b) above

$$a + a^{-1} - aa^{-1} = 0$$
$$a^{-1} - aa^{-1} = -a$$
$$a^{-1}(1 - a) = -a$$
$$a^{-1} = \frac{-a}{1-a} \text{ is the inverse of } a \in R \text{ under } *$$

The inverse does not exist if the denominator of a^{-1} is zero, i.e. if $1 - a = 0$ or $a = 1$. Therefore the element of R that does not have an inverse is 1.

To find the inverse of 2, substitute 2 into the expression for a^{-1} i.e. $\frac{-2}{1-2} = \frac{-2}{-1} = 2$.

Therefore the inverse of 2 is 2.

Example 2

A binary operation Δ is defined on the set R of real numbers by $x \Delta y = 2x + 3y - xy$, where $x, y \in R$.

i) Determine whether or not Δ is
   a) commutative b) associative

ii) Find y such that $x \Delta y = x$, stating the value of x for which there is no such value of y.

Solution

i) a) The binary operation Δ is commutative if $x \Delta y = y \Delta x$ for all $x, y \in R$

$$x \Delta y = 2x + 3y - xy \ldots\ldots\ldots\ldots(1)$$

and $y \Delta x = 2y + 3x - yx$

$$= 3x + 2y - xy \ldots\ldots\ldots\ldots(2)$$

From (1) and (2), $x \Delta y \neq y \Delta x$

Hence the operation Δ is not commutative.

b) The operation Δ is associative if $(x \Delta y) \Delta z = x \Delta (y \Delta z)$ for all $x, y, z \in R$.

Now,

$$(x \Delta y) \Delta z = (2x + 3y - xy) \Delta z$$
$$= 2(2x + 3y - xy) + 3z - (2x + 3y - xy)z$$
$$= 4x + 6y - 2xy + 3z - 2xz - 3yz + xyz$$
$$= 4x + 6y + 3z - 2xy - 2xz - 3yz + xyz \ldots(1)$$

Also

$$x \Delta (y \Delta z) = x \Delta (2y + 3z - yz)$$
$$= 2x + 3(2y + 3z - yz) - x(2y + 3z - yz)$$
$$= 2x + 6y + 9z - 3yz - 2xy - 3xz + xyz \ldots(2)$$

Comparing (1) and (2), we have

$$(x \Delta y) \Delta z \neq x \Delta (y \Delta z)$$

Therefore the operation Δ is not associative.

ii) If $x \Delta y = x$ $\quad 2x + 3y - xy = x$.

Solving for y,

$$3y - xy = -x$$
$$y(3 - x) = -x$$
$$\therefore \quad y = \frac{-x}{3-x}$$

y does not exist if the denominator of the expression for y is zero i.e. $3 - x = 0$ or $x = 3$.

Therefore the value of x for which there is no such value of y is 3.

Example 3

A binary operation is defined on the set R of real numbers by $a \Delta b = a + b + 2ab$, where $a, b \in R$.

i) Calculate $(2 \Delta 3) \Delta 5$.

ii) Find the truth set of the equation $(a \Delta 7) = (a \Delta 5) + (a \Delta 2)$.

iii) If $a \Delta b = 0$, express b in terms of a and state the largest domain of b.

Solution

i) By definition, $a \Delta b = a + b + 2ab$

$$\therefore \quad (2 \Delta 3) \Delta 5 = [2 + 3 + 2(2)(3)] \Delta 5$$
$$= [(5 + 12)] \Delta 5$$
$$= 17 \Delta 5$$
$$= 17 + 5 + 2(17)(5)$$
$$= 22 + 170 = 192$$

ii) $a \Delta 7 = (a \Delta 5) + (a \Delta 2)$

$$a + 7 + 2(a)(7)$$
$$= [a + 5 + 2(a)(5)] + [a + 2 + 2(a)(2)]$$

$a + 7 + 14a$
$= [a + 5 + 10a] + [a + 2 + 4a]$
$15a + 7 = [11a + 5] + [5a + 2]$
$15a + 7 = 16a + 7$
$a = 0$

Hence the truth set of the equation is $\{a : a = 0\}$

iii) If $a \Delta b = 0$ then $a + b + 2ab = 0$.
Solving for b, we have
$b + 2ab = -a$
$b(1 + 2a) = -a$

$$\therefore \quad b = \frac{-a}{1 + 2a}$$

The largest domain of b is $\{a : a \in R, \text{ except } -\tfrac{1}{2}\}$

Example 4

A binary operation * is defined on the set R of real numbers by

$$a * b = \frac{a}{b} - \frac{b}{a} \quad \text{where } a, b \in R \text{ and } a, b \neq 0$$

a) i) Calculate $(3\sqrt{2} - 2\sqrt{3}) * (\sqrt{3} - \sqrt{2})$.

ii) Deduce the value of $(\sqrt{3} - \sqrt{2}) * (3\sqrt{2} - 2\sqrt{3})$.

b) i) If $a * b = 1$, show that $b = \frac{1}{2}(-a \pm a\sqrt{3})$.

ii) Hence find the values of b for which $(4 - 2\sqrt{3}) * b = 1$.

Give your answer in the form $p + q\sqrt{r}$ where p, q and r are rational numbers.

Solution

a) By the definition, $a * b = \frac{a}{b} - \frac{b}{a}$

i) $(3\sqrt{2} - 2\sqrt{3}) * (\sqrt{3} - \sqrt{2})$

$$= \frac{3\sqrt{2} - 2\sqrt{3}}{\sqrt{3} - \sqrt{2}} - \frac{\sqrt{3} - \sqrt{2}}{3\sqrt{2} - 2\sqrt{3}}$$

Rationalizing the denominators, we have

$$\frac{3\sqrt{2} - 2\sqrt{3}}{\sqrt{3} - \sqrt{2}} = \frac{(3\sqrt{2} - 2\sqrt{3})(\sqrt{3} + \sqrt{2})}{(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})}$$

$$= \frac{3\sqrt{6} + 3\sqrt{4} - 2\sqrt{9} - 2\sqrt{6}}{(\sqrt{3})^2 - (\sqrt{2})^2}$$

$$= \frac{\sqrt{6} + 6 - 6}{3 - 2} = \frac{\sqrt{6}}{1} = \sqrt{6}$$

Also

$$\frac{\sqrt{3} - \sqrt{2}}{3\sqrt{2} - 2\sqrt{3}} = \frac{(\sqrt{3} - \sqrt{2})(3\sqrt{2} + 2\sqrt{3})}{(3\sqrt{2} - 2\sqrt{3})(3\sqrt{2} + 2\sqrt{3})}$$

$$= \frac{3\sqrt{6} + 6 - 6 - 2\sqrt{6}}{(3\sqrt{2})^2 - (2\sqrt{3})^2}$$

$$= \frac{\sqrt{6}}{18 - 12} = \frac{\sqrt{6}}{6}$$

$$\therefore \quad (3\sqrt{2} - 2\sqrt{3}) * (\sqrt{3} - \sqrt{2})$$

$$= \sqrt{6} - \frac{\sqrt{6}}{6} = \tfrac{5}{6}\sqrt{6}$$

ii) To deduce the value of $(\sqrt{3} - \sqrt{2}) * (3\sqrt{2} - 2\sqrt{3})$ we see from (i) that the positions of $(\sqrt{3} - \sqrt{2})$ and $(3\sqrt{2} - 2\sqrt{3})$ have been interchanged

i.e. from $a * b = \frac{a}{b} - \frac{b}{a}$.

$$b * a = \frac{b}{a} - \frac{a}{b} = -\left(\frac{a}{b} - \frac{b}{a}\right) = -(a * b)$$

Hence

$$(\sqrt{3} - \sqrt{2}) * (3\sqrt{2} - 2\sqrt{3})$$

$$= -[(3\sqrt{2} - 2\sqrt{3}) * (\sqrt{3} - \sqrt{2})]$$

$$= -\tfrac{5}{6}\sqrt{6}$$

Since $(3\sqrt{2} - 2\sqrt{3}) * (\sqrt{3} - \sqrt{2}) = \frac{5}{6}\sqrt{6}$ from a)(i) above.

b) i) Given that $a * b = 1$

$$\frac{a}{b} - \frac{b}{a} = 1,$$

Solving for b, we have

$$\frac{a^2 - b^2}{ab} = 1$$

$$a^2 - b^2 = ab$$

$$b^2 + ab - a^2 = 0 \text{ ....(1)}$$

Equation (1) is a quadratic equation in b. To solve for b using the general formula, we have

$$b = \frac{-a \pm \sqrt{a^2 - 4(1)(-a)^2}}{2(1)}$$

$$= \frac{-a \pm \sqrt{a^2 + 4a^2}}{2}$$

$$= \frac{-a \pm \sqrt{5a^2}}{2} = \tfrac{1}{2}(-a \pm a\sqrt{5})$$

ii) Comparing $(4 - 2\sqrt{5})*b = 1$ with $a* b = 1$ we have $a = 4 - 2\sqrt{5}$.

From (i) above, $b = \frac{1}{2}(-a \pm a\sqrt{5})$

$$b = \tfrac{1}{2}\{-(4 - 2\sqrt{5}) \pm (4 - 2\sqrt{5})\sqrt{5}\}$$

$$b = \tfrac{1}{2}\{-4 + 2\sqrt{5}) \pm (4\sqrt{5} - 10)\}$$

The two values of b are given by each sign.

i.e. $b_1 = \frac{1}{2}\{(-4 + 2\sqrt{5}) + (4\sqrt{5} - 10)\}$

$$= \tfrac{1}{2}(-14 - 6\sqrt{5}) = -7 + 3\sqrt{5}$$

and $b_2 = \frac{1}{2}\{(-4 + 2\sqrt{5}) - (4\sqrt{5} - 10)\}$

$$= \tfrac{1}{2}\{6 - 2\sqrt{5}\} = 3 - \sqrt{5}$$

$\therefore \; b = -7 + 3\sqrt{5}$ or $3 - \sqrt{5}$

Example 5

An operation $\odot$ is defined over the set R of real numbers by $x \odot y = x + y + xy$.

a) Determine whether or not the operation $\odot$ is commutative.

b) Find the neutral element, e under the operation.

c) Determine whether or not, to each real number x, there corresponds a real number y, such that $y \odot x = e$.

d) Find y such that $(x \odot y) \odot x = 3$. Stating the values of x for which there is no such y.

Solution

By definition $x \odot y = x + y + xy$

a) The operation $\odot$ is commutative if

$$x \odot y = y \odot x$$

Now $x \odot y = x + y + xy$ and

$$y \odot x = y + x + yx$$

$$= x + y + xy$$

(Since both addition (+) and multiplication (×) are commutative)

$\therefore$ since $x \odot y = y \odot x$ it implies the operation $\odot$ is commutative.

b) If e is the neutral element under $\odot$, then

$$x \odot e = e \odot x = x$$

But $x \odot e = x + e + xe = x$

Solving for e

$$e + xe = x - x = 0$$

$$e(1 + x) = 0$$

$$e = \frac{0}{1 + x} = 0$$

$\therefore$ the neutral element $e = 0$.

c) If $y \odot x = e$ then

$$y + x + yx = e$$

but $e = 0$ from b) above

$$y + x + yx = 0$$

$$y + yx = -x$$

$$y(1 + x) = -x$$

$$y = \frac{-x}{1 + x}$$

$\therefore$ To each real number x there corresponds a real number $y = \dfrac{-x}{1 + x}$, except $x = -1$.

d) $(x \odot y) \odot x = 3$

$(x + y + xy) \odot x = 3$

$(x + y + xy) + x + (x + y + xy)x = 3$

$2x + y + xy + x^2 + xy + x^2y = 3$

$2x + 2xy + y + x^2 + x^2y = 3$

$2xy + y + x^2y = 3 - 2x - x^2$

$y(2x + 1 + x^2) = 3 - 2x - x^2$

$$y = \frac{3 - 2x - x^2}{x^2 + 2x + 1} \quad \Rightarrow y = \frac{(3 + x)(1 - x)}{(x + 1)^2}$$

∴ y does not exist if $(x + 1)^2 = 0$ or $x = -1$ (twice).

Example 6

A binary operation * is defined on the set P of non-negative real numbers by

$x * y = (\sqrt{x} + \sqrt{y})^2$, where $x, y \in P$

a) Calculate:
   i) $24 * 18$, leaving your answer as a surd in its simplest form;
   ii) $r * r$, where $r \in P$;

b) Show that the operation * is associative;

c) Find the identity element in P under *;

d) Determine the subset of P consisting of the elements with inverse.

Solution

a) By definition $x * y = (\sqrt{x} + \sqrt{y})^2$

i) $24 * 18$

$= (\sqrt{24} + \sqrt{18})^2$

$= (2\sqrt{6} + 3\sqrt{2})^2$

$= (2\sqrt{6} + 3\sqrt{2})(2\sqrt{6} + 3\sqrt{2})$

$= 24 + 6\sqrt{12} + 6\sqrt{12} + 18$

$= 42 + 12\sqrt{12} \quad = 42 + 24\sqrt{3}$

ii) $r * r = (\sqrt{r} + \sqrt{r})^2$

$= (\sqrt{r} + \sqrt{r})(\sqrt{r} + \sqrt{r})$

$= r + r + r + r = 4r$

b) If the operation * is associative then $(x * y) * z = x * (y * z)$ for all $x, y, z \in P$

Now $(x * y) * z = (\sqrt{x} + \sqrt{y})^2 * z$

$= \{\sqrt{(\sqrt{x} + \sqrt{y})^2} + \sqrt{z}\}^2$

$= \{\sqrt{x} + \sqrt{y} + \sqrt{z}\}^2$ ------(1)

Also $x * (y * z) = x * (\sqrt{y} + \sqrt{z})^2$

$= [\sqrt{x} + (\sqrt{(\sqrt{y} + \sqrt{z})^2}]^2$

$= \{\sqrt{x} + \sqrt{y} + \sqrt{z}\}^2$ ------(2)

Note

that $\sqrt{x^2} = x$ and $\sqrt{(\sqrt{x} + \sqrt{y})^2} = \sqrt{x} + \sqrt{y}$

∴ Since eqns (1) and (2) are equal

$(x * y) * z = x * (y * z)$

Therefore the operation * is associative.

c) Let the identity element under * be e, then for any element $x \in P$ we have

$x * e = x \Rightarrow (\sqrt{x} + \sqrt{e})^2 = x.$

Taking the positive square root,

$\sqrt{x} + \sqrt{e} = \sqrt{x}$

$\sqrt{e} = \sqrt{x} - \sqrt{x} = 0$

$e = 0$

∴ The identity element under * is $e = 0$

d) Let $y \in P$ be the inverse of the general element $x \in P$, then

$x * y = y * x = e$ but

$x * y = (\sqrt{x} + \sqrt{y})^2 = e = 0.$

Solving for y gives

$(\sqrt{x} + \sqrt{y})^2 = 0 \quad \Rightarrow \sqrt{x} + \sqrt{y} = 0.$

Since $\sqrt{x} + \sqrt{y}$ can be zero only if $x = 0$ and $y = 0$ for positive square roots.

The only element in P which has an inverse is $x = 0$ and its inverse is $y = 0$.

∴ The subset of P required is $\{0\}$

Example 7

A binary operation * is defined on a set $T = \{p, q, r, s\}$ by the following table.

*	p	q	r	s
p	q	s	p	r
q	s	r	q	p
r	p	q	r	s
s	r	p	s	q

a) Show from the table whether or not;
   i) T is closed with respect to *
   ii) There is an identity element.
   iii) The operation * is commutative

b) Find, where possible, the inverse of the elements p, q, r and s

c) Find m, if $(s * p) * (r * q) = m$

Solution

a) i) The set T is closed with respect to the operation * if and only if any two elements in T when combined, will result in an element in set T. From the table, it is clear that whenever any two elements in T are combined under * the result is an element in T.
i.e. $p * q = s \in T$; $r * p = p \in T$,
$q * q = r \in T$ etc.
Hence T is closed with respect to *

ii) The identity element is such that when it is combined with any other element it gives the same element. From the table, $p * r = p$; $q * r = q$; $r * r = r$, and $s * r = s$.

Hence r is the identity element under *.

iii) From the table, $p * q = q * p = s$;
$p * r = r * p = p$; $p * s = s * p = r$ etc.

Hence the operation * is commutative, since for any two elements $x, y \in T$, $x * y = y * x$

b) The inverse element is such that when it is combined with its element it gives the identity element. The identity element from a(ii) is r. From the table,
$p * s = r \Rightarrow$ the inverse of p is s.
Similarly the inverses of q, r and s are q, r and p respectively.

c) $(s * p) * (r * q) = m$

From the table, $s * p = r$ and $r * q = q$

$\therefore (s * p) * (r * q) = r * q$ but $r * q = q$

$\therefore \quad m = q$

Example 8

A binary operation $\oplus$ on the set G of ordered pairs (p, q) of real numbers is given by

$$(p, q) \oplus (r, s) = (p + r,\ qs).$$

Show that $\oplus$ is

a) commutative

b) associative

Solution

a) If $\oplus$ is commutative then
$(p, q) \oplus (r, s) = (r, s) \oplus (p, q)$ for all $(p, q),\ (r, s) \in G$

Now $(p, q) \oplus (r, s) = (p + r, qs)$

and $\quad (r, s) \oplus (p, q) = (r + p, sq)$
$\qquad\qquad\qquad\qquad = (p + r, qs)$

(Since addition and multiplication are commutative)

$\therefore \quad (p, q) \oplus (r, s) = (r, s) \oplus (p, q)$

Therefore the operation $\oplus$ is commutative.

b) If the operation $\oplus$ is associative then
$[(p,q) \oplus (r, s)] \oplus (t, u) = (p, q) \oplus [(r, s) \oplus (t, u)]$
for all (p, q), (r, s) and $(t, u) \in G$

Now $[(p, q) \oplus (r, s)] \oplus (t, u)$

$$= (p + r, qs) \oplus (t, u)$$
$$= [(p + r) + t, qsu]$$
$$= (p + r + t, qsu) \text{ --------(1)}$$

Also $(p, q) \oplus [(r, s) \oplus (t, u)]$

$$= (p, q) \oplus (r + t, su)$$
$$= [p + (r + t), qus]$$
$$= (p + r + t, qsu) \text{ ----------(2)}$$

$\therefore$ Comparing (1) and (2)

$[(p, q) \oplus (r, s)] \oplus (t, u)$

$= (p, q) \oplus [(r, s) \oplus (t, u)]$

$\therefore$ The operation $\oplus$ is associative.

Example 9

A binary operation Δ is defined on the set R of real numbers by

$$a \,\Delta\, b = \frac{ab}{a+b} \quad \text{where } a, b \in R$$

a) Determine whether or not, the operation Δ is commutative.

b) Calculate i) $a \,\Delta\, (b \,\Delta\, c)$ ii) $(a \,\Delta\, b)\, c$ and hence determine whether or not the operation Δ is associative.

c) Find, correct to 2 decimal places, the truth set of the equation $(x \,\Delta\, 1) + (x \,\Delta\, 2) = 1 \,\Delta\, 2$.

Solution

a) If the operation Δ is commutative then $a \,\Delta\, b = b \,\Delta\, a$ for all $a, b \in R$

$$a \,\Delta\, b = \frac{ab}{a+b} \text{ ....(1)}$$

$$b \,\Delta\, a = \frac{ba}{b+a} = \frac{ab}{a+b} \text{ ....(2)}$$

(Since addition (+) and multiplication (×) are commutative).

From (1) and (2), $a \,\Delta\, b = b \,\Delta\, a$

Therefore the operation Δ is commutative.

b) i) $a \,\Delta\, (b \,\Delta\, c)$

$$= a \,\Delta \left(\frac{bc}{b+c}\right) = \frac{a\left(\frac{bc}{b+c}\right)}{a + \left(\frac{bc}{b+c}\right)}$$

$$= \frac{\frac{abc}{b+c}}{\frac{ab+ac+bc}{b+c}} = \frac{abc}{ab+ac+bc}$$

ii) $(a \,\Delta\, b) \,\Delta\, c = \dfrac{ab}{a+b} \,\Delta\, c$

$$= \frac{\left(\frac{ab}{a+b}\right)c}{\left(\frac{ab}{a+b}\right)+c}$$

$$= \frac{\frac{abc}{a+b}}{\frac{ab+ac+bc}{a+b}} = \frac{abc}{ab+ac+bc}$$

From (i) and (ii) above, $a\Delta(b\Delta\, c) = (a\Delta b)\Delta c$

Hence the operation Δ is associative.

c) $(x \,\Delta\, 1) + (x \,\Delta\, 2) = 1 \,\Delta\, 2$

$$\frac{x(1)}{x+1} + \frac{x(2)}{x+2} = \frac{1(2)}{1+2}$$

$$\frac{x}{x+1} + \frac{2x}{x+2} = \frac{2}{3}$$

$$\frac{x(x+2) + 2x(x+1)}{(x+1)(x+2)} = \frac{2}{3}$$

$$\frac{x^2 + 2x + 2x^2 + 2x}{x^2 + 3x + 2} = \frac{2}{3}$$

$$\frac{3x^2 + 4x}{x^2 + 3x + 2} = \frac{2}{3}$$

$$9x^2 + 12x = 2x^2 + 6x + 4$$

$$7x^2 + 6x - 4 = 0$$

This equation is a quadratic equation in x. To solve for x using the general formula, we have

$$x = \frac{-6 \pm \sqrt{36 - 4(7)(-4)}}{2(7)}$$

$$= \frac{-6 \pm \sqrt{36 + 112}}{14} = \frac{-6 \pm \sqrt{148}}{14}$$

$$x=\frac{-6\pm 12.17}{14}=\frac{-6-12.17}{14}=-1.30$$

$$\text{or}\quad x=\frac{-6+12.17}{14}=0.44\ \ (2\text{ d.p})$$

Example 10

A binary operation * is defined on the set R of real numbers by $x * y = x + y + x^2y$ where $x, y \in R$. Determine whether or not the operation * is commutative.

Solution

$x * y = x + y + x^2y$

The operation * is commutative if

$x * y = y * x$ for all $x, y \in R$

$x * y = x + y + x^2y$ .... (1) and

$y * x = y + x + y^2x$

$\quad = x + y + xy^2$ .... (2)

Comparing (1) and (2), $x * y \neq y * x$

Hence the operation * is not commutative.

Example 11

A binary operation * is defined on the set R of real numbers by $p * q = p + q + \sqrt{2}$ where $p, q \in R$.

a) Evaluate $(\sqrt{2} - 1) * (\sqrt{2} + 1)$

b) Find
  i) the neutral element, e of R under *
  ii) the inverse p' of p
  iii) the inverse of $2\sqrt{2}$

Solution

a) From $p * q = p + q + \sqrt{2}$

$(\sqrt{2} - 1) * (\sqrt{2} + 1)$

$= (\sqrt{2} - 1) + (\sqrt{2} + 1) + \sqrt{2}$

$= \sqrt{2} - 1 + \sqrt{2} + 1 + \sqrt{2}$

$= 3\sqrt{2}$

b) i) If e is the neutral element under * then $e * p = p$

$e + p + \sqrt{2} = p$

$e + \sqrt{2} = p - p = 0$

$e = -\sqrt{2}$

$\therefore$ The neutral element $e = -\sqrt{2}$

ii) If p' is the inverse of p under * then $p' * p = e$ where $e = -\sqrt{2}$ from (i) above.

$p' + p + \sqrt{2} = -\sqrt{2}$

$p' = -p - \sqrt{2} - \sqrt{2}$

$p' = -p - 2\sqrt{2}$

$p' = -(p + 2\sqrt{2})$

iii) For the inverse of $2\sqrt{2}$, put $p = 2\sqrt{2}$ in the expression for p'

i.e. $-(2\sqrt{2} + 2\sqrt{2}) = -4\sqrt{2}$

$\therefore$ The inverse of $2\sqrt{2}$ is $-4\sqrt{2}$

Example 12

A binary operation * is defined on the set R of real numbers by $m * n = m + n + 10$. Find

a) the identity element;

b) the inverses of 2 and –5 under *.

Solution

a) Let $e \in R$ be the identity element under *

$\therefore$ $m * e = m$

$m + e + 10 = m$

$e = m - m - 10$

$e = -10$

Hence the identity element is –10.

b) Let the inverse of 2 be x then

$2 * x = e$ But $e = -10$

$2 + x + 10 = -10$

$x = -22$

Hence the inverse of 2 is –22.

Also let the inverse of –5 be y, then

$-5 * y = e$ But $e = -10$

$-5 + y + 10 = -10$

$y = -15$

Hence the inverse of –5 is –15.

Example 13

A binary operation * on the set R, of real numbers is defined by $p * q = p + q - 2$

a) Show that * is associative

b) Find

  i) the identity (neutral) element;

  ii) the inverse p^{-1} of p;

  iii) the inverse of 4

Solution

The operation * is defined by

$p * q = p + q - 2$

a) The operation * is associative if $(p * q) * r = p * (q * r)$ for all $p, q, r \in R$.

$$(p * q) * r = (p + q - 2) * r$$
$$= (p + q - 2) + r - 2$$
$$= p + q + r - 4 \text{ ....(1)}$$

$$p * (q * r) = p * (q + r - 2)$$
$$= p + (q + r - 2) - 2$$
$$= p + q + r - 4 \text{ ....(2)}$$

From (1) and (2), $(p * q) * r = p * (q * r)$.
Hence the operation * is associative.

b) i) Let $e \in R$ be the identity (neutral) element under *.

$\therefore$ $p * e = p$

$p + e - 2 = p$

$e = 2$

$\therefore$ The identity element is 2

ii) If p^{-1} is the inverse of p then

$p^{-1} * p = e$ ($e = 2$ from (i) above)

$p^{-1} * p = 2$

$p^{-1} + p - 2 = 2$

$p^{-1} = 4 - p$

iii) To find the inverse of 4, put $p = 4$ in the expression for p^{-1} above.

i.e. $p^{-1} = 4 - 4 = 0$

$\therefore$ The inverse of 4 is 0.

Example 14

A binary operation Δ is defined on the set $T = \{0, 2, 3, 4, 5, 6\}$ by $a \Delta b = a + b - ab$ where $a, b \in T$. Find:

a) whether Δ is closed under T,

b) the identity element e of T

c) the inverse of the element 2.

Solution

The operation Δ is defined by

$a \Delta b = a + b - ab$

a) The operation Δ is closed under T if any two numbers in the set T when combined gives a number which belongs to T.

i.e. $a \Delta b \in T$ for all $a, b \in T$.

But $5 \Delta 6 = 5 + 6 - 5(6) = -19$ which does not belong to T. Therefore the set T is not closed under Δ.

b) If e is the identity element then $a \Delta e = a$

$a + e - ae = a$

$e(1 - a) = 0$

$e = 0$

$\therefore$ The identity element is 0.

c) Let x be the inverse of 2, then

$2 * x = e$ But $e = 0$

$2 + x - 2x = 0$

$2 - x = 0$

$x = 2$

$\therefore$ The inverse of 2 is 2

Example 15

For any two elements A and B of a universal set U the operation A is defined by

$$A * B = (A \cap B') \cup (B \cap A'),$$

where A' and B' denote the complements of A and B respectively. Show that:

a) the operation * is commutative;

b) the empty set $\varnothing$ is the identity element under *.

Solution

a) The operation * is commutative if
$A * B = B * A$ for all $A, B \in U$
$A * B = (A \cap B') \cup (B \cap A')$ and
$B * A = (B \cap A') \cup (A \cap B')$
$= (A \cap B') \cup (B \cap A')$
since $\cup$ is commutative
$\therefore$ since $A * B = B * A$ it implies the operation * is commutative.

b) If the empty set $\varnothing$ is the identity element under * then
$A * \varnothing = (A \cap \varnothing') \cup (\varnothing \cap A')$
$= (A \cap U) \cup \varnothing$
(Since $\varnothing' = U$ and $\varnothing \cap A' = \varnothing$)
$A * \varnothing = A \cup \varnothing = A$ (since $A \cap U = A$)
Therefore since $A * \varnothing = A \Rightarrow \varnothing$ is the identity element under *.

Example 16

Given that $m * p = m^2 + p^2 - mp$, evaluate 4 * 3.

Solution

$4 * 3 = 4^2 + 3^2 - 4(3) = 16 + 9 - 12 = 13$

Example 17

A binary operation Δ is defined on the set $T = \{2, 3, 4, 5\}$ by $a \Delta b = 3a + 2b - ab$, where $a, b \in T$.

a) Copy and complete the table below.

Δ	2	3	4	5
2	6	6	6	6
3		6		4
4			4	2
5	9	6	3	0

b) Show whether or **not** T is closed under Δ

c) Evaluate (3 Δ 4) Δ (3 Δ 5).

Solution

a) $a \Delta b = 3a + 2b - ab$
$3 \Delta 2 = 3(3) + 2(2) - (3)(2) = 9 + 4 - 6 = 7$
$3 \Delta 4 = 3(3) + 2(4) - (3)(4) = 9 + 8 - 12 = 5$
$4 \Delta 2 = 3(4) + 2(2) - (4)(2) = 12 + 4 - 8 = 8$
$4 \Delta 3 = 3(4) + 2(3) - (4)(3) = 12 + 6 - 12 = 6$
The completed table is shown below:

Δ	2	3	4	5
2	6	6	6	6
3	**7**	6	**5**	4
4	**8**	**6**	4	2
5	9	6	3	0

b) T is **not** closed under Δ since there are elements in the table which do not belong to T. i.e. $6, 7, 8, 9 \notin T$

c) From the table,
$3 \Delta 4 = 5$ and $3 \Delta 5 = 4$
$(3 \Delta 4) \Delta (3 \Delta 5) = 5 \Delta 4 = 3$

Example 18

The binary operations * and Δ are defined on the set R, of real numbers by $a * b = 2a + ab$ and $x \Delta y = x + y - xy$ respectively, where $a, b, x, y \in R$.

a) Determine whether or not Δ is commutative.

b) Evaluate (3 * 2) Δ (5 Δ 3).

Solution

a) $x \Delta y = x + y - xy$
$y \Delta x = y + x - yx$
$\therefore\ x \Delta y = y \Delta x$
Hence Δ is commutative

b) $(3 * 2) = 2(3) + 3(2) = 12$
$(5 \Delta 3) = 5 + 3 - 5(3) = -7$
$12 \Delta -7 = 12 - 7 - 12(-7) = 5 + 84 = 89$
$\therefore (3 * 2) \Delta (5 \Delta 3) = 89$

Example 19

The binary operation Δ, is defined on the set R, of real numbers by $x \Delta y = \sqrt{x} + y$, for all $x, y \in R$, where $x \geq 0$.

i) Evaluate $(3 \Delta 2)(3 \Delta 4)$.

ii) Find n such that $n \Delta 5 = 7$.

iii) Determine **whether** or **not** Δ is commutative.

Solution

i) $$(3\Delta 2)(3\Delta 4) = \left(\sqrt{3}+2\right)\left(\sqrt{3}+4\right)$$
$$= 3 + 4\sqrt{3} + 2\sqrt{3} + 8$$
$$= 11 + 6\sqrt{3}$$

ii) $$n \Delta 5 = 7$$
$$\sqrt{n} + 5 = 7$$
$$\sqrt{n} = 2$$
$$n = 4$$

iii) If Δ is commutative then

$$a \Delta b = b \Delta a$$
$$a \Delta b = \sqrt{a} + b \text{ and}$$
$$b \Delta a = \sqrt{b} + a$$

But $\sqrt{a} + b \neq \sqrt{b} + a$ except $a = b$,

∴ The operation Δ is **not** commutative.

Review exercise 3

1. An operation * is defined on the set R of real numbers by $x * y = x + y - 2xy$.
   a) Determine whether or not * is commutative.
   b) Find the identity element e of R under the operation *
   c) Determine the inverse under * of an element $x \in R$ stating which element of x for which no inverse exists. Find also the inverse of 1.

2. An operation Δ is defined by $a \Delta b = a + b + 3ab$ where $a, b \in R$
   a) Calculate
      (i) $5 \Delta (-5)$
      (ii) $(-5) \Delta 5$
   b) Find the identity element e of R under the operation Δ.
   c) Determine the inverse under Δ of the general element $a \in R$ and state which element of $a \in R$ has no inverse.

3. A binary operation is defined on the set R of real numbers by
   $a * b = a^2 + \dfrac{a}{b}$, where $a, b \in R$ and $b \neq 0$.
   Calculate
   i) $(3 - 2\sqrt{5}) * (3 + 2\sqrt{5})$
   ii) $(3 - 2\sqrt{5}) * (1 + 3\sqrt{5})$
   iii) The value of b such that $(3 - 2\sqrt{5}) * b = 25 - 9\sqrt{5}$

4. A binary operation Δ is defined on the set R of real numbers by
   $a \Delta b = a + b - 2ab$, where $a, b \in R$.
   i) Calculate $(5 - 3\sqrt{2}) \Delta (6 + 5\sqrt{2})$, giving your answer in the form $m + n\sqrt{2}$ where m and n are rational numbers.
   ii) Determine whether or not the operation Δ is commutative.
   iii) Find the identity element e of R under the operation Δ.
   iv) Find the inverse of a general element $a \in R$, and state which element of R has no inverse.

5. The table below defines the operation $\otimes$ on the set $Y = (r, s, t, u)$.

$\otimes$	r	s	t	u
r	s	u	r	t
s	u	t	s	r
t	r	s	t	u
u	t	r	u	s

a) Find, giving reasons whether the following are correct
  i) Y is closed with respect to $\otimes$
  ii) $\otimes$ is commutative
  iii) There is an identity element

b) Find where possible the inverse of the elements r, s, t and u.

c) Find p and q if
  i) $(u \otimes r) \otimes (t \otimes s) = p$
  ii) $(q \otimes r) \otimes (t \otimes u) = s$

6. The operation * is defined over the set of real numbers $p * q = 2pq$

a) Determine whether or not * is commutative

b) Determine whether or not * is a associative

c) Find the identity element for the operation *

d) Find the inverse of a real number p under *, stating the value of p for which no inverse exists.

e) Show that $p * (q + r) = (p * q) + (p * r)$ (Note : this is the distributive law)

7. The operation $\odot$ is defined over the set of real numbers by $p \odot q = p + q + pq$

a) Show that $\odot$ is commutative and associative

b) Find the identity element for the operation $\odot$

c) Find the inverse (under $\odot$) of the real number p, stating the value of p for which no inverse exists.

d) Find the inverse –2

e) Determine whether or not $p \odot (q + r) = (p \odot q) + (p \odot r)$ for all p, q and $r \in R$

8. The operation * is defined over the set of ordered pairs by $(a, b) * (c, d) = (ac - bd, ad + bc)$ for all $(a, b), (c, d) \in R$.

a) Determine whether or not this operation is commutative

b) Find the identity element for *

c) Given that the inverse of (a, b) is (c, d) find an expression for c and d in terms of a and b.
Hence find the inverse of $(2, 1)$

9. The operation * defined on the set $S = (a, b, c)$ by the table below

*	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

a) Determine whether or not
  i) S is closed under *
  ii) * is commutative

b) Determine the identity element for the operation *

c) Determine the inverse (under *) of each element of S.

10. An operation * is defined on the set $T = \{1, 2, 3, 4\}$ by $a * b = a + b - ab$

a) i) Copy and complete the table
  ii) Is T closed with respect to *.

*	1	2	3	4
1	1	1	1	1
2	1		-1	-2
3	1	-1		
4	1	-2		-8

b) Evaluate $(2 * 3) * 4$

11. The binary operation * is defined by $m * n = \frac{1}{2}(m - n)$.

a) Form a table of the operation * on the set $T = \{2, 4, 6\}$

b) State with reason whether or not * is

i) closed with respect to T

ii) commutative under T.

12. A binary operation Δ is defined on the set R of real numbers by

$a \Delta b = a^2 + b^2 - 2ab$, where $a, b \in R$.

i) Determine whether or not the operation Δ is commutative.

ii) Calculate $\sqrt{3} \Delta \frac{1}{\sqrt{12}}$ and $\frac{1}{\sqrt{12}} \Delta \sqrt{3}$

iii) Use your result in (i) above to calculate:

$$\left(\sqrt{3} \Delta \frac{1}{\sqrt{12}}\right) - \left(\frac{1}{\sqrt{12}} \Delta \sqrt{3}\right)$$

iv) Find the truth set of $2 \Delta a = 9$

13. A binary operation * is defined on the set R of real numbers by

$x * y = 3x + 2y + xy$, where $x, y \in R$.

i) Determine whether or not * is

a) commutative b) associative

ii) Find y such that $x * y = x$, stating the value of x for which there is no such value of y.

14. An operation * is defined on the set R of real numbers by

$a * b = a + b - 2ab$, where $a, b \in R$..

a) Calculate $5 * (-2)$ and $(-2) * 5$.

b) Find the identity element e of R under the operation *.

c) Determine the inverse under * of a general element $a \in R$ and state which element of R has no inverse. Find the inverse of 1.

15. A binary operation * on the set R, of real numbers is defined by $p * q = p + q + 4$

a) Show that * is associative

b) Find

i) the identity (neutral) element;

ii) the inverse p^{-1} of p;

iii) the inverse of 2

16. A binary operation Δ is defined on the set $T = \{0, 1, 2, 3, 4, 5, 6\}$ by

$a \Delta b = a + b - 2ab$, where $a, b \in T$.

Find:

a) whether Δ is closed under T,

b) the identity element e of T

c) the inverse of the element 1.

17. An operation * is defined on the set R of real numbers by

$a * b = a + b - 6ab$, where $a, b \in R$..

a) Determine whether or not * is commutative

b) Find the identity element e of R under the operation *.

c) Determine the inverse under * of a general element $a \in R$ and state which element of R has no inverse. Find the inverse of 1.

18. i) The operation Δ is defined on the set of non-zero numbers by $a \Delta b = \frac{a + b}{ab}$ determine whether Δ is

a) commutative

b) associative

ii) Calculate a) $3 \Delta (-2)$ b) $2 \Delta 1$

19. i) The operation Δ is defined on the set of non-zero numbers by : $a \Delta b = a^2 + b^2 + ab$ determine whether Δ is
   a) commutative
   b) associative

ii) Calculate
   a) $2 \Delta 3$
   b) $(2 \Delta 3) \Delta 4$
   c) $2 \Delta (3 \Delta 4)$

iii) Solve the equation $a \Delta 6 = 27$

20. The operation * is defined on the set $T = \{0, 1, 2, 3\}$ by $a * b = |a - b|$, where $|a - b|$ denotes the magnitude of $a - b$.

State with reasons whether or not
   i) the set T is closed under the operation *
   ii) the operation * is commutative
   iii) there is an identity element
   vi) each element has an inverse.

21. An operation * is defined on the set Z of integers by $x * y = x + y + 3xy$
   i) Construct a table for this operation on the set $S = \{-2, -1, 0, 1, 2\}$.
   ii) Find, from your table, a number $b \in S$, such that $c * b = c$ for all $c \in S$.

22. An operation * is defined on the set of real numbers by $x * y = x + y - 5xy$.
   a) Determine whether or not * is commutative.
   b) Find the identity element e of R under the operation *.
   c) Determine the inverse under * of an element $x \in R$, stating the element of x for which no inverse exists.

23. An operation Δ is defined on the set R of real numbers by $x \Delta y = x + y - 3xy$
   a) Determine whether or not the operation Δ is commutative.
   b) Find the neutral element e under the operation
   c) Determine whether or not, to each real number x, there corresponds a real number y, such that $y \Delta x = e$
   d) Find y such that $(x \Delta y) \Delta x = -6xy - 18x^2y$.

24. An operation Δ is defined on the set of real numbers by $x \Delta y = x + y - 5xy$.
   a) Determine whether or not Δ is commutative.
   b) Find the identity element e of R under the operation Δ
   c) Determine the inverse under Δ of an element $x \in R$, stating which element of x for which no inverse exists.

25. The operation Δ is defined on the set R of positive real numbers by

$$x \Delta y = \frac{x^2 + y^2}{x + y}.$$

Determine whether or not the operation Δ is
   i) commutative
   ii) associative

26. The operation * is defined on R, the set real numbers by $x * y = 3xy$
   a) Determine whether or not the operation * is associative
   b) Find the identity element, e of R under the operation *.
   c) Find the inverse of $x \in R$ under the operation * stating the value of x for which no inverse exists.

CHAPTER 4

RELATIONS AND FUNCTIONS

Relations

In many situations of ordinary life, we are concerned with relations which exist between the elements of two different sets.

Let us consider the following sentences:

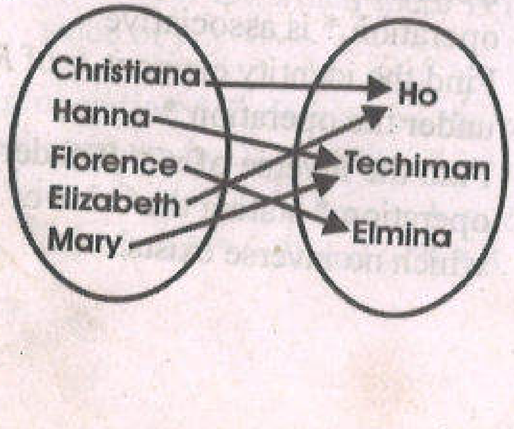
1. Kwasi Asiedu is the father of Akua
2. Esinam is the mother of Atsu.
3. Asibi is the wife of Musa.

All these sentences express relationship between two objects. In each case the first element is related to the second element by the relation underlined. The two elements named are from two separate sets.

Again, consider the table below which shows part of a form list showing the hometowns of students.

Name of student	Home town
Christiana	Ho
Hanna	Techiman
Florence	Elmina
Elizabeth	Ho
Mary	Techiman

The relation defined here is between the set of students {Christiana, Hanna, Florence, Elizabeth, Mary} and the set of hometowns {Ho, Techiman, Elmina}. A way of showing the relation, apart from the list already given, is to draw an *arrow diagram* as shown below.



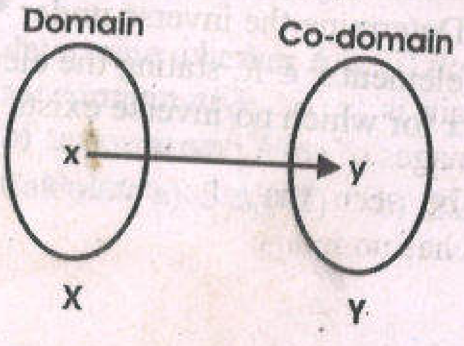
In this form the relation is represented by drawing arrows from the first set to the second set. The arrows show which student comes from which hometown.

i.e. Student ⟶ Hometown

A **relation** therefore associates the elements of one set to the elements of another set. Thus a relation is a connection between two sets. The diagram above (fig. 7.1) illustrates what is called a relation. The set of all students {Christiana, Hanna, Florence, Elizabeth, Mary} is called the **domain** of the relation. The set of hometowns {Ho, Techiman, Elmina} is called the **co-domain**. The relation is 'comes from'.

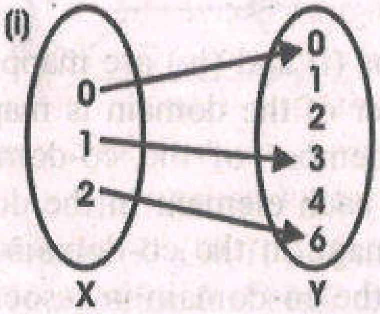
We can also show the relation by means of **ordered pairs** as follows:
{(Christiana, Ho), (Hanna, Techiman) (Florence, Elmina), (Elizabeth, Ho), (Mary, Techiman)}. In the ordered pair (Christiana, Ho), Christiana is the first component and Ho is the second component.

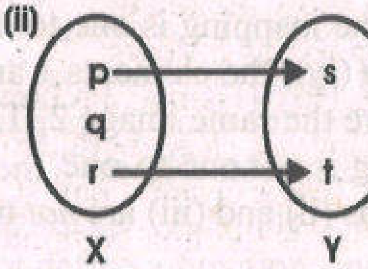
The first component in all cases is the student and the second component her hometown. The order if changed shall alter the sense of the pair also. In mathematics, the quantities connected by the relation are often numbers. In general, the relation between two sets can be illustrated by an arrow diagram.



From the arrow diagram, if an element x of set X (*called the domain*) is related to an element y of set Y (*called the co-domain*), then we say that y is the **image** of x under the given relation. The set of all possible images of the domain is called the **range.** Therefore the range is a subset of the co-domain.

Note: A relation may exist between two sets but not all the elements of the domain may be associated with elements of the co-domain. Also the co-domain may contain members which are not images of members of the domain as in the relations below.





In relation (i) above, the domain is $\{0, 1, 2\}$ and the co-domain is $\{0, 1, 2, 3, 4, 6\}$. The range, which is the set of images of set X (the domain) is $\{0, 3, 6\}$.
It can be seen that the elements $\{1, 2, 4\}$ of the co-domain have no elements of the domain associated with them.

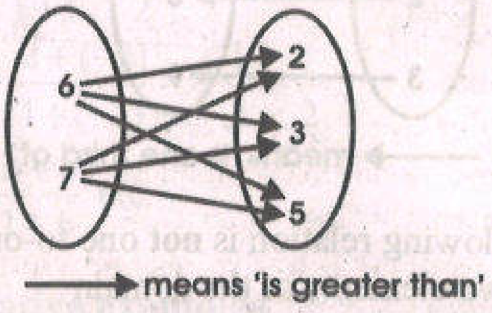
In relation (ii), the domain is $\{p, q, r\}$ and the co-domain is $\{s, t\}$. The range, which is the set of images of set X (the domain) is $\{s, t\}$.
It can be seen that the element q of the domain has no image.

Types of relations

Below are **arrow diagrams** illustrating the *four different types of relations.*

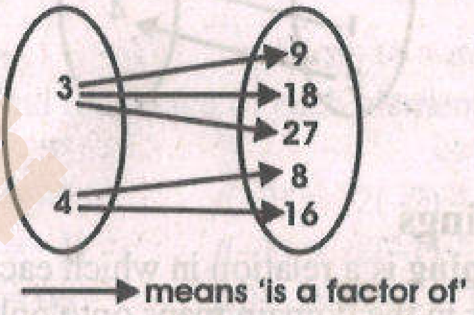
Many–to–many relation

In this relation, *several* elements in the domain have *many* elements in the co-domain and *several* elements in the co-domain are associated with *many* elements in the domain.



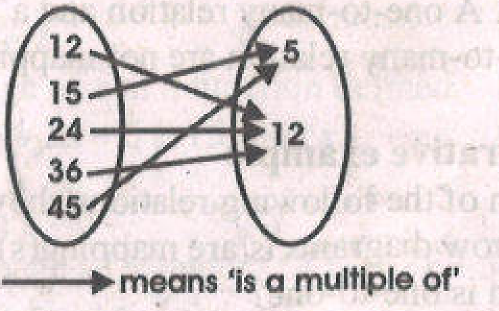
One–to–many relation

In this relation, *one* element in the domain has many images in the co-domain.



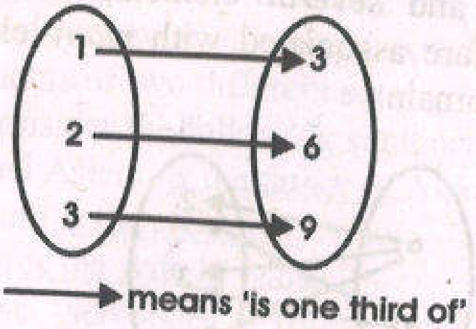
Many–to–one relation

In this relation, *several* elements in the domain have *one* image in the co-domain.

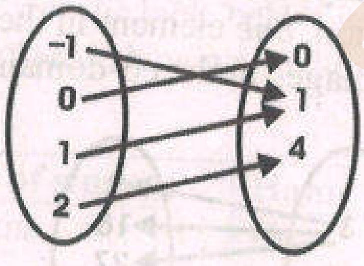


One–to–one relation

In this relation, each element in the domain has only *one* image in the co-domain and each element in the co-domain is associated with only one element in the domain.
In particular, each element of the domain has a unique image.



The following relation is **not** one-to-one since two elements of the domain {i.e. –1 and 1} have the same image (i.e. 1).



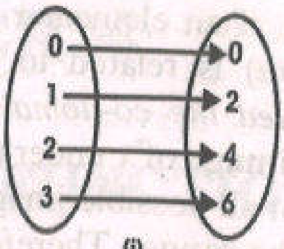
Mappings

A **mapping** is a relation in which each member in the domain maps onto only one member in the co-domain. From this definition, a one-to-one relation and a many-to-one relation are mappings.

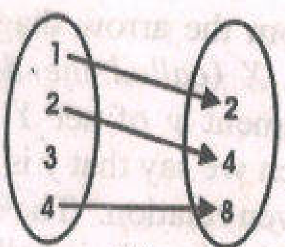
Note: A one-to-many relation and a many-to-many relation are not mappings.

Illustrative example

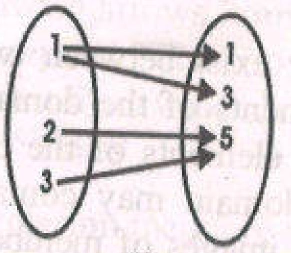
Which of the following relations shown in the arrow diagrams is/are mapping(s)? Which is one-to-one?



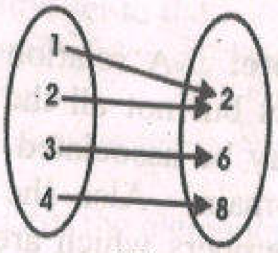
(i)



(ii)



(iii)



(iv)

Solution

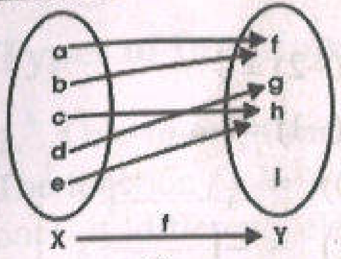
The relations (i) and (iv) are mappings since each member of the domain is mapped onto only one member of the co-domain. For relation (i), each element in the domain has only *one* image in the co-domain and each element in the co-domain is associated with only one element in the domain.
Therefore the mapping is one-to-one.
For relation (iv), the elements 1 and 2 in the domain have the same image 2. Therefore the mapping is *not* one-to-one.
The relations (ii) and (iii) are *not* mappings.

Functions

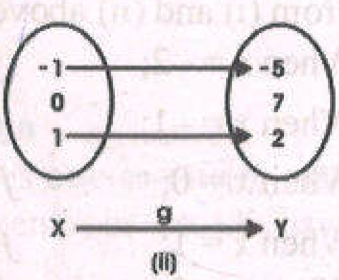
Let X and Y be two sets. A very important type of relation between the two sets X and Y arises when each member of the set X is related to one and only one member of the set Y. This relation is known as a **function** from X to Y written $f : X \rightarrow Y$ and read as "the function f from set X to set Y, or by the equation $Y = f(x)$. $f(x)$ is the *image* of x, under f or the image set of the function. The set X is called the **domain** of f and the set Y is called the **co-domain** or **image set** of f. It is not necessary for all the elements of a set Y to be the image of each member of set X

under f. The **range** of f is that *subset* of the co-domain which consists all the possible images of all members of the domain X.

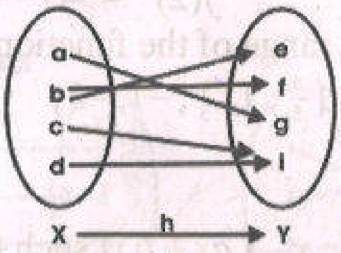
Illustrations



(i)



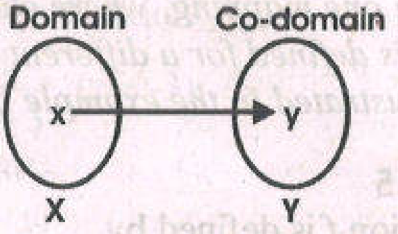
(ii)



(iii)

The *domain* of the relation f is $\{a, b, c\ d, e\}$, the *co-domain* is $\{f, g, h, i\}$ and the *range* is $\{f, g, h\}$. In this case it is clear that the range is a subset of the co-domain.
The relation f above is a function since each element in X has only one image in Y.
The relation g in (ii) *does not* define a function from X into Y since an element $0 \in X$ has no image. Also the relation h in (iii) does not define a function since an element $b \in X$ has more than one image in Y.

In general, the function between two sets can be illustrated by an arrow diagram below.



The functional notation
Given that $f(x) = 2x + 3$ then by definition
$f(1) = 2(1) + 3 = 2 + 3 = 5$
(i.e. replace x by 1)
$f(2) = 2(2) + 3 = 4 + 3 = 7$
(i.e. replace x by 2)
$f(-3) = 2(-3) + 3 = -6 + 3 = -3$
$f(a) = 2a + 3$ (i.e. replace x by a)

Note that $f(1)$ means the image of 1 under the function f.

Also if $h(x) = \dfrac{2+x}{1+x}$ then:

$$h(1) = \frac{2+1}{1+1} = \frac{3}{2}\ ; \qquad h(2) = \frac{2+2}{1+2} = \frac{4}{3}\ ;$$

$$h(-3) = \frac{2+(-3)}{1+(-3)} = \frac{-1}{-2} = \frac{1}{2}\ ;$$

$$h(b) = \frac{2+b}{1+b} \text{ etc.}$$

Illustrative examples

Example 1
A function is defined by $f : x \to 2x - 3$ on the domain $\{-2, -1, 0, 1, 2\}$, find the range of the function.

Solution
Note: Find the image of each element in the domain.
When $x = -2$; $f(-2) = 2(-2) - 3 = -7$
When $x = -1$; $f(-1) = 2(-1) - 3 = -5$
When $x = 0$; $f(0) = 2(0) - 3 = -3$
When $x = 1$; $f(1) = 2(1) - 3 = -1$
When $x = 2$; $f(2) = 2(2) - 3 = 1$
Therefore the range of the function is the set $\{-7, -5, -3, -1, 1\}$.

Example 2
The image of x of a function defined by $f : x \to 2x + 5$ is 15, find x.

Solution
If the image of x under $f : x \to 2x + 5$ is 15 then $2x + 5 = 15$
$2x = 15 - 5 = 10$
$x = 5$

Example 3

A function is defined by $f : x \rightarrow \dfrac{3x-1}{x-3}$ on the domain $\{-2, -1, 0, 1, 2\}$.

i) Find the images of the following elements of the domain.
   a) 0 b) 1 c) 2

ii) What elements of the domain correspond to the following images a) $\frac{7}{5}$ b) 1

iii) What value of x makes the function undefined?

iv) Determine the range of the function.

Solution

i) Given that $f(x) = \dfrac{3x-1}{x-3}$

a) The image of 0 is given by

$$f(0) = \frac{3(0)-1}{0-3} = \frac{-1}{-3} = \frac{1}{3}$$

b) The image of 1 is given by

$$f(1) = \frac{3(1)-1}{1-3} = \frac{2}{-2} = -1$$

c) The image of 2 is given by

$$f(2) = \frac{3(2)-1}{2-3} = \frac{5}{-1} = -5$$

ii) a) $f(x) = \frac{7}{5}$

$$\frac{3x-1}{x-3} = \frac{7}{5}$$

$$5(3x-1) = 7(x-3)$$

$$15x - 5 = 7x - 21 \quad \Rightarrow \quad 8x = -16$$

$$x = -2$$

Therefore the element −2 of the domain corresponds to the image $\frac{7}{5}$.

b) $f(x) = 1$

$$\frac{3x-1}{x-3} = 1 \quad \Rightarrow \quad 3x - 1 = x - 3$$

$$3x - x = -3 + 1 \quad \Rightarrow \quad 2x = -2$$

$$x = -1$$

Therefore the element −1 of the domain corresponds to the image 1.

iii) The function is undefined if $x - 3 = 0$
i.e. if $x = 3$

iv) Note: Find the image of each element in the domain

From (i) and (ii) above,

When $x = -2$; $f(-2) = \frac{7}{5}$

When $x = -1$; $f(-1) = 1$

When $x = 0$; $f(0) = \frac{1}{3}$

When $x = 1$; $f(1) = -1$

When $x = 2$; $f(2) = -5$

Therefore the range of the function is the set of the images $\{\frac{7}{5}, 1, \frac{1}{3}, -1, -5\}$

Example 4

A function $f : x \rightarrow ax + b$ is such that $f(1) = 9$ and $f(2) = 14$. Find

i) the values of a and b

ii) $f(-1)$ iii) $f(0)$ iv) $f(-2)$

Solution

i) $f(x) = ax + b$

$f(1) = a + b = 9$ ……….(1) and

$f(2) = 2a + b = 14$ ……..(2)

Solving equations (1) and (2) simultaneously gives $a = 5$ and $b = 4$.

Therefore $f(x) = 5x + 4$

ii) $f(-1) = -5 + 4 = -1$

iii) $f(0) = 0 + 4 = 4$

iv) $f(-2) = -10 + 4 = -6$

Sometimes a function can be made up of more than one mapping, where each mapping is defined for a different domain. This is illustrated in the example below.

Example 5

The function f is defined by

$f(x) = x^2$ for $x \leq 0$ and

$f(x) = x$ for $x > 0$

Find $f(4)$ and $f(-4)$

Solution

For $x > 0$, $f(x) = x \Rightarrow f(4) = 4$

For $x \leq 0$, $f(x) = x^2$

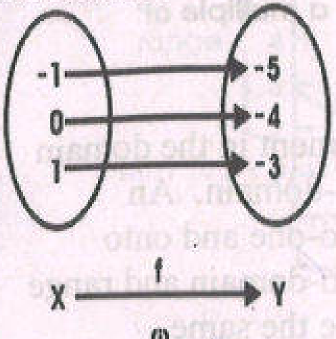
$$f(-4) = (-4)^2 = 16$$

Types of functions

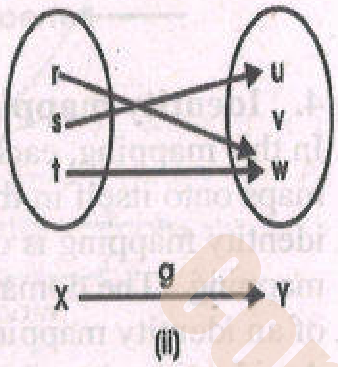
1. One-to-one function

The function $f : X \rightarrow Y$ is **one-to-one**(or one-one) if different elements in set X have distinct images in set Y.

Illustrations



(i)



(ii)

The function f in (i) above is one-to-one since all the elements in its domain $\{-1, 0, 1\}$ have distinct (different) images.

The function g in (ii) above is not one-to-one since r and t, two distinct elements in the domain of the function g have the same image w.

Mathematically, we say a function is one-to-one if : $f(a) = f(b) \Rightarrow a = b$.

Illustrative examples

Example 1

Show that the function

$$f : x \rightarrow \frac{1}{x+3} \quad (x \neq -3)$$

is one-to-one.

Solution

To show that the function f is one-to-one, we first equate $f(a)$ to $f(b)$ and then show whether $a = b$.

Thus $f(a) = f(b) \Rightarrow a = b$

$$f(a) = f(b) \Rightarrow \frac{1}{a+3} = \frac{1}{b+3}$$

$$a + 3 = b + 3 \Rightarrow a = b$$

Since $f(a) = f(b) \Rightarrow a = b$; it follows that the function f is one-to-one.

Example 2

Show whether or not the function g defined by $g : x \rightarrow \dfrac{1}{x^2+3}$ is one-to-one.

Solution

The function g is one-to-one if

$g(a) = g(b) \Rightarrow a = b$

Now $g(a) = g(b)$

$$\frac{1}{a^2+3} = \frac{1}{b^2+3}$$

$$a^2 + 3 = b^2 + 3$$

$$a^2 = b^2 \Rightarrow a = \sqrt{b^2} \Rightarrow a = \pm b$$

i.e. $a = b$ or $a = -b$

$\therefore$ $g(a) = g(b) \Rightarrow a \neq b$ hence g is not a one-to-one function.

Example 3

Show that the function $f(x) = \dfrac{x^2 - 9}{x^2}$ $(x > 0)$ is one-to-one

Solution

To show that f is one-to-one function, we use the condition: $f(a) = f(b) \Rightarrow a = b$.

Now $f(a) = f(b) \Rightarrow \dfrac{a^2 - 9}{a^2} = \dfrac{b^2 - 9}{b^2}$

$$b^2(a^2 - 9) = a^2(b^2 - 9)$$

$$a^2b^2 - 9b^2 = a^2b^2 - 9a^2$$

$$-9b^2 = -9a^2$$

$$9a^2 = 9b^2 \Rightarrow a^2 = b^2$$

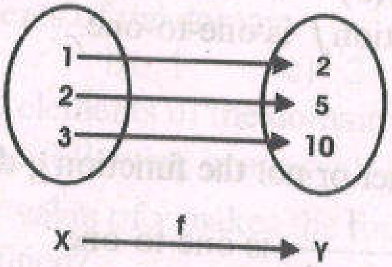
$$a = b \text{ since } a > 0.$$

Hence the function f is one-to-one.

Example 4

A function $f: x \to x^2 + 1$ is defined on the domain $\{1, 2, 3\}$. Use arrow diagram to show whether or not f is one-to-one.

Solution

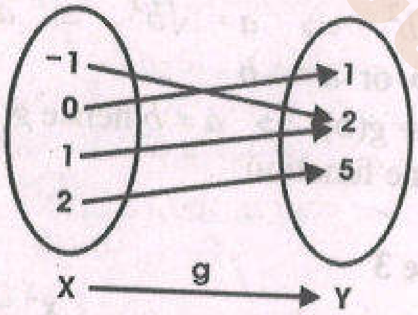


Since each member of the domain $\{1, 2, 3\}$ has distinct image in the co-domain, the function f is one-to-one

Example 5

A function $g: x \to x^2 + 1$ is defined on the domain $\{-1, 0, 1, 2\}$. Use arrow diagram to show whether or not g is one-to-one.

Solution

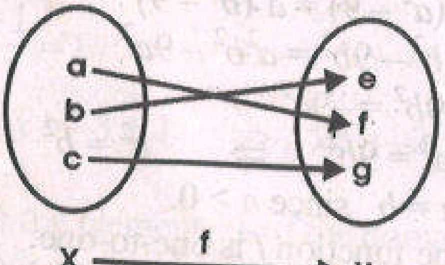


Since the members of the domain -1 and 1 have the same image in the co-domain, the function f is **not** one-to-one.

2. Onto functions

A function f is said to be an **onto function** if and only if the range of f is equal to the co-domain.

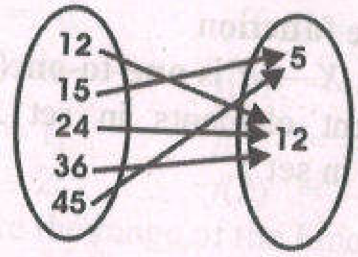
Illustration



From the diagram, f is an onto function since the range of f is $\{e, f, g\}$ which is the same as the co-domain of f which is $\{e, f, g\}$

3. Many–to–one function

In this relation, *several* elements in the domain have *one* image in the co-domain.



⟶ means 'is a multiple of'

4. Identity mapping

In this mapping, each element in the domain maps onto itself in the co-domain. An identity mapping is one-to-one and onto mapping. The domain, co-domain and range of an identity mapping are the same.

An identity mapping is: $f(x) = x$

5. Constant mapping

A constant mapping is a mapping in which every member in the domain maps onto *only one* member in the co-domain. Thus the range consist of only one member. A constant mapping is: $f(x) = k$, where k is a constant

6. Even functions

If a function f is such that $f(x) = f(-x)$ then f is an **even function**.

For example, $f(x) = x^2 + 3$ is an even function.

7. Odd functions

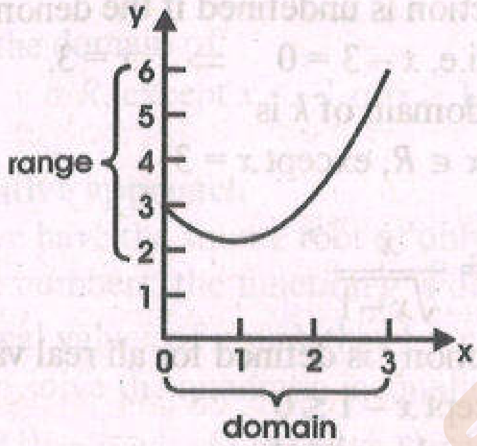
If a function f is such that $f(x) = -f(-x)$, then f is an **odd function**.

For example, $f(x) = x^3$ is an odd function.

Graphs of functions

The graph of a function is usually represented using rectangular Cartesian coordinates and plotting $f(x)$ or y against x. It is important to identify the function as linear, quadratic, cubic or logarithmic. We shall meet these functions in the chapters that follow.

The graph of the function $f(x) = x^2 - 2x + 3$, where x is real and $0 \le x \le 3$ is shown below.

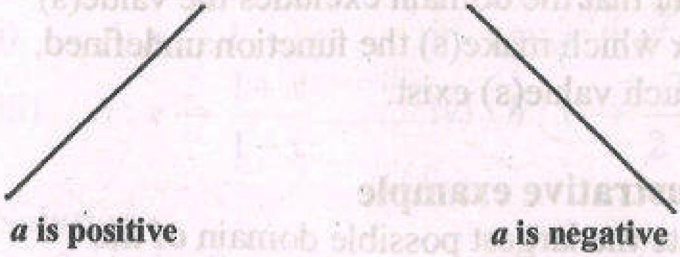


Note: To find the range when the domain and function are given, it is safer to sketch the graph of the function over its domain. Note that the end points of the domain do not necessarily give the end points of the range. From the graph it is clear that the range of $f(x)$ is $\{y : 2 \le y \le 6)$. The range would be different if you attempt to substitute the values of the end points of the domain into the expression for the function.

To investigate the nature of graphs of a function $f(x)$, first find the points of intersection of the curve on the x–axis if there are any (i.e. put $f(x) = 0$). Next, find the intercepts on the y–axis by putting $x = 0$ i.e. find $f(0)$.

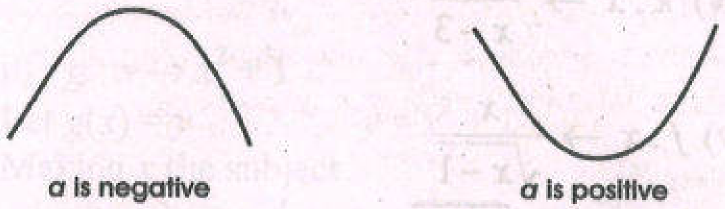
At this stage the above information together with the following is enough to sketch the curve of a linear, quadratic or cubic function.

1. If the function is linear i.e. $f(x) = ax + b$, then we have a straight line The two shapes depending on the value of a are shown below.

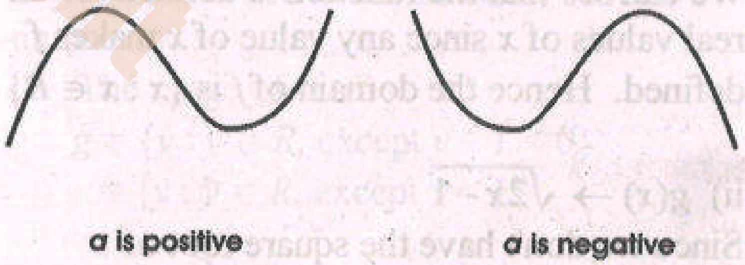


2. If the function is quadratic i.e. $f(x) = ax^2 + bx + c$, then the curve is a parabola.

The two shapes depending on the value of a are shown below.



3. If the function is cubic i.e. $f(x) = ax^3 + bx^2 + cx + d$, we have two shapes of the curve depending on the value of a as shown below.



Note: Use graphic calculators and computers (MS EXCEL) to investigate nature of graphs of functions.

To show **graphically** that a relation is a function, draw any vertical line to intersect the graph. If the line cuts the graph at *exactly one point* then the relation is a function.

Largest Domain of a function

The **largest domain** of a function is the set of possible values of x which make the function defined.

Note that the domain excludes the value(s) of x which make(s) the function undefined, if such value(s) exist.

Illustrative example

State the largest possible domain of the following functions

i) $f: x \to 1 - 2x$

ii) $g: x \to \sqrt{2x-1}$

iii) $h: x \to x^2 + 1$

iv) $k: x \to \dfrac{x+5}{x-3}$

v) $f: x \to \dfrac{x}{\sqrt{x-1}}$

vi) $g: x \to \sqrt{1-x^2}$

Solution

i) $f(x) \to 1 - 2x$

We can see that the function is defined for all real values of x since any value of x makes f defined. Hence the domain of f is $\{x : x \in R\}$

ii) $g(x) \to \sqrt{2x-1}$

Since we don't have the square root of negative numbers, the function g is defined for all real values of x, except $2x - 1 < 0$. (Note that any number less than zero is negative).

Hence the domain of

$g = \{x : x \in R, \text{ except } 2x - 1 < 0\}$

$= \{x : x \in R, \text{ except } x < \frac{1}{2}\}$

Alternative approach

Since we have the square root of only positive numbers, the function g is defined for all real values of x such that $2x - 1 \geq 0$.

We then solve the inequality:

$2x - 1 \geq 0 \quad \Rightarrow \quad x \geq \frac{1}{2}$

Hence the domain of $g = \{x : x \in R : x \geq \frac{1}{2}\}$

iii) $h(x) = x^2 + 1$, is defined for all real values of x. Hence the domain of h is $\{x : x \in R\}$.

iv) $k(x) = \dfrac{x+5}{x-3}$

The function is undefined if the denominator is zero. i.e. $x - 3 = 0 \quad \Rightarrow \quad x = 3$.

$\therefore$ The domain of k is

$\{x : x \in R, \text{ except } x = 3\}$.

v) $f(x) = \dfrac{x}{\sqrt{x-1}}$

The function f is defined for all real values of x, except $x - 1 \leq 0$

Hence the domain of

$f = \{x : x \in R, \text{ except } x - 1 \leq 0\}$

$= \{x : x \in R, \text{ except } x \leq 1\}$.

Note: In this case the denominator is an expression in a square root. Therefore it is not only zero that makes the expression in the square root undefined but also negative numbers.

vi) $g: x \to \sqrt{1-x^2}$

Since we do not have the square root of negative numbers, the function g is defined for all real values of x, except $1 - x^2 < 0$. (Note that any number less than zero is negative).

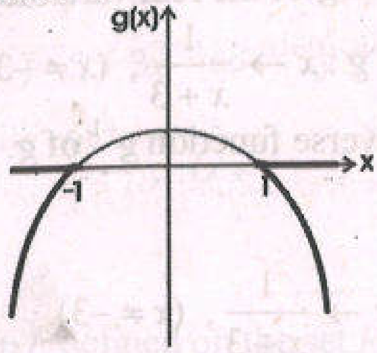
We then solve the quadratic inequality:

$1 - x^2 < 0$

$(1 - x)(1 + x) < 0.$

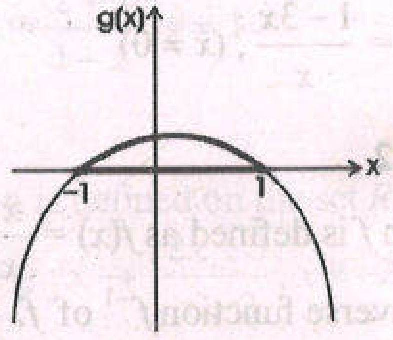
Sketch the graph of the function $g(x) = (1 - x)(1 + x)$

The inequality symbol $<$ means we consider the part of the graph below the x-axis.
i.e. $x < -1$ or $x > 1$



Hence the domain of
$g = \{x : x \in R, \text{ except } x < -1 \text{ or } x > 1\}$

Alternative approach
Since we have the square root of only positive numbers, the function g is defined for all real values of x such that $1 - x^2 \geq 0$.
We then solve the quadratic inequality:
$1 - x^2 \geq 0 \quad \Rightarrow \quad (1 - x)(1 + x) \geq 0.$
Sketch the graph of the function
$y = (1 - x)(1 + x)$
The inequality symbol $\geq$ means we consider the part of the graph above the x-axis.
i.e. $\quad -1 \leq x \leq 1$



Hence the domain of $g = \{x: x \in R: -1 \leq x \leq 1\}$

Range of a function

The **range** of a function is the se of values of the dependent variable, y that can result from the substitutions for the independent variable, x
It can also be defined as the set of values of the dependent variable, y which make the independent variable, x defined. Therefore to find the range, first make x the subject and find the values of y which make x defined.

Illustrative example
State the range of the functions
i) $\quad f : x \to 1 - 2x \qquad$ ii) $\quad g : x \to x^2 + 1$
iii) $\quad h : x \to \dfrac{1+x}{1-x} \qquad$ iv) $\quad h : x \to \dfrac{2x}{2-x}$

Solution
i) $f(x) = 1 - 2x$
Step 1 : $\quad$ Let $f(x) = y \quad \Rightarrow \quad y = 1 - 2x$
Step 2 : $\quad$ Make x the subject
$$2x = 1 - y \qquad \Rightarrow \quad x = \tfrac{1}{2}(1 - y)$$
Since x is defined for all real values of y, the range of $f = \{y : y \in R\}$

ii) $\; g : x \to x^2 + 1$
Let $g(x) = y \quad \therefore \quad y = x^2 + 1$
Making x the subject
$$x^2 = y - 1 \quad \Rightarrow x = \sqrt{y-1}$$
$\therefore\;$ x is defined for all real values of y, except
$\quad y - 1 < 0$
(Since we do not have the square root of negative numbers under the set R of real numbers)
$\therefore\;$ The range of
$\quad g = \{y : y \in R, \text{ except } y - 1 < 0\}$
$\quad\;\; = \{y : y \in R, \text{ except } y < 1\}$

iii) $\; h(x) = \dfrac{1+x}{1-x}$

Let $\; h(x) = y \quad \Rightarrow \quad y = \dfrac{1+x}{1-x}$

Making x the subject
$$y(1 - x) = 1 + x \quad \Rightarrow \quad yx + x = y - 1$$
$$x(y + 1) = y - 1 \quad \Rightarrow \quad x = \frac{y-1}{y+1}$$
Now since x is defined for all real values of y, except $y = -1$,
$\therefore\;$ The range of h is $\{y : y \in R, \text{ except } -1\}$

iv) $h(x) = \dfrac{2x}{2-x}$

Let $h(x) = y \quad \therefore \quad y = \dfrac{2x}{2-x}$

Making x the subject

$y(2-x) = 2x \quad \Rightarrow \quad 2y - xy = 2x$

$2y = 2x + xy = x(2+y)$

$x = \dfrac{2y}{2+y}$. Now since x is defined for all real values of y, except $y = -2$,

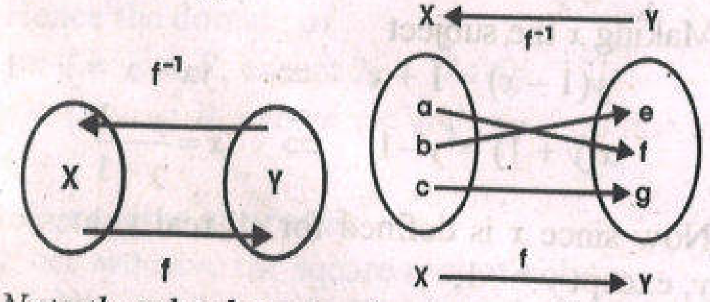
$\therefore$ The range of h is $\{y : y \in R, \text{ except } -2\}$

The inverse function

If a function f is *one-to-one* and maps an element x in the domain to an element y in the range, then the function that maps y back to x is called the **inverse** of the function f i.e. f^{-1}. Consider the function f, defined by $f = \{(a, f), (b, e), (c, g)\}$. If the elements of each ordered pair are interchanged we shall obtain a new set of ordered pairs $\{(f, a), (e, b), (g, c)\}$.

This new set of ordered pairs is called the **inverse** of f and is denoted by the symbol f^{-1} (read "f-inverse"). In general, f^{-1} needs not be a function. If a function f is *one-to-one* and at the same time *onto*, then f^{-1} is a function called the inverse function. Note that the domain of an inverse function is the range of the original function. Note also that the inverse of a function f is a function if and only if f is both **one-to-one** and **onto**.

The *inverse* of an ***inverse function***, f^{-1} gives the function, f.



Note that the domain of an inverse function is the range of the original function.

Illustrative examples

Example 1

The function g of the real variable x is defined as $g : x \rightarrow \dfrac{1}{x+3}$, $(x \neq -3)$

Find the inverse function g^{-1} of g.

Solution

Given $g : x \rightarrow \dfrac{1}{x+3}$, $(x \neq -3)$.

Step 1 Let $y = g(x)$ then

$$y = \frac{1}{x+3} \text{ ....(1)}$$

Step 2 Interchange the positions of variables x and y in equation (1)

i.e. $\quad x = \dfrac{1}{y+3}$ ....(2)

Step 3 Make y the subject of the relation

$x = \dfrac{1}{y+3} \quad \Rightarrow \quad x(y+3) = 1$

$xy + 3x = 1$

$xy = 1 - 3x \quad \Rightarrow \quad y = \dfrac{1-3x}{x}$

$\therefore$ The inverse function is

$$g^{-1}(x) = \frac{1-3x}{x}, \ (x \neq 0)$$

Example 2

A function f is defined as $f(x) = \dfrac{x+5}{x-3}$, $x \neq 3$ find the inverse function f^{-1} of f.

Solution

Given $f(x) = \dfrac{x+5}{x-3}$, $(x \neq 3)$.

Let $y = f(x) \ \therefore \quad y = \dfrac{x+5}{x-3}$

Interchanging the positions of x and y,

$$x = \frac{y+5}{y-3} \text{ ....(1)}$$

Making y the subject from (1)

$$x(y-3) = y+5$$

$$xy - 3x = y + 5 \quad \Rightarrow \quad xy - y = 5 + 3x$$

$$y(x-1) = 5 + 3x \quad \Rightarrow \quad y = \frac{5+x}{x-1}$$

$$\therefore \quad f^{-1}(x) = \frac{5+x}{x-1}, \quad (x \neq 1)$$

Example 3

The function f, defined on the set R of real numbers, is given by: $f : x \rightarrow \dfrac{x-2}{x+1}$ $(x \neq -1)$.

Find the inverse f^{-1} of f

Solution

$f(x) = \dfrac{x-2}{x+1}$ Let $y = f(x)$ $\therefore$ $y = \dfrac{x-2}{x+1}$

Interchanging the variables x and y, and making y the subject

$$x = \frac{y-2}{y+1} \quad \Rightarrow \quad x(y+1) = y-2$$

$$xy + x = y - 2 \quad \Rightarrow \quad y - xy = 2 + x$$

$$y(1-x) = 2 + x \quad \Rightarrow \quad y = \frac{2+x}{1-x}$$

$$\therefore \quad f^{-1}(x) = \frac{2+x}{1-x}, x \neq 1$$

Example 4

The function is defined on the set R of real numbers by $g : x \rightarrow \dfrac{2x+3}{x+2}, x \neq -2$.

Find its inverse.

Solution

Given that $g(x) = \dfrac{2x+3}{x+2}$, let $y = f(x)$

$$y = \frac{2x+3}{x+2}$$

Interchanging the variables x and y, and making y the subject.

$$x = \frac{2y+3}{y+2} \quad \Rightarrow \quad x(y+2) = 2y+3$$

$$xy + 2x = 2y + 3 \quad \Rightarrow \quad xy - 2y = 3 - 2x$$

$$y(x-2) = 3 - 2x \quad \Rightarrow \quad y = \frac{3-2x}{x-2}$$

$$\therefore \quad g^{-1}(x) = \frac{3-2x}{x-2}, \quad (x \neq 2)$$

Example 5

The function h is defined on the set R of real numbers by $h : x \rightarrow \dfrac{1}{1+x}$, $x \neq -1$

Find h^{-1} and the value of x for which h^{-1} is not defined.

Solution

$$h : x \rightarrow \frac{1}{1+x} \quad \Rightarrow \quad h(x) = \frac{1}{1+x}.$$

Let $y = h(x)$ $\Rightarrow$ $y = \dfrac{1}{1+x}$

Interchanging the variables x and y, and making y the subject.

$$x = \frac{1}{1+y} \quad \Rightarrow \quad x(1+y) = 1$$

$$x + xy = 1 \quad \Rightarrow \quad xy = 1 - x$$

$$y = \frac{1-x}{x}$$

$$\therefore \quad h^{-1}(x) = \frac{1-x}{x}, \; x \neq 0.$$

The value of x for which h^{-1} is not defined is 0.

Example 6

The function f is defined on the set R of real numbers by $f : x \rightarrow \dfrac{3x-2}{3-x}$.

Find f^{-1} and state the largest domain of f and f^{-1}.

Solution

$$f : x \to \frac{3x-2}{3-x} \quad \Rightarrow \quad f(x) = \frac{3x-2}{3-x}.$$

Let $y = f(x) \quad \Rightarrow \quad y = \dfrac{3x-2}{3-x}$

Interchanging the variables x and y, and making y the subject

$$x = \frac{3y-2}{3-y} \quad \Rightarrow \quad x(3-y) = 3y-2$$

$$3x - xy = 3y - 2 \quad \Rightarrow \quad 3y + xy = 3x + 2$$

$$y(3+x) = 3x+2 \quad \Rightarrow y = \frac{3x+2}{3+x}$$

$$\therefore \; f^{1}(x) = \frac{3x+2}{3+x}.$$

The domain of f
$= \{x : x \in R, \text{ except } 3 - x = 0\}$
$= \{x : x \in R, \text{ except } x = 3\}.$

The domain of f^{1}
$= \{x : x \in R, \text{ except } 3 + x = 0\}$
$= \{x : x \in R, \text{ except } x = -3\}.$

Example 7

The function f is defined on the set R of real numbers by $f : x \to \dfrac{3x+2}{4-x}$.

Find f^{1} and state the largest domain of f and f^{1}.

Solution

$$f : x \to \frac{3x+2}{4-x} \quad \Rightarrow \quad f(x) = \frac{3x+2}{4-x}.$$

Let $y = f(x) \quad \Rightarrow \quad y = \dfrac{3x+2}{4-x}$

Interchanging the variables x and y, and making y the subject

$$x = \frac{3y+2}{4-y} \quad \Rightarrow x(4-y) = 3y+2$$

$$4x - xy = 3y + 2 \quad \Rightarrow 3y + xy = 4x - 2$$

$$y(3+x) = 4x - 2$$

$$y = \frac{4x-2}{3+x}$$

$$\therefore \; f^{1}(x) = \frac{4x-2}{3+x}$$

The domain of
$f = \{x : x \in R, \text{ except } 4 - x = 0\}$
$= \{x : x \in R, \text{ except } x = 4\}.$

The domain of f^{1}
$= \{x : x \in R, \text{ except } 3 + x = 0\}$
$= \{x : x \in R, \text{ except } x = -3\}.$

Finding the function when its inverse is given

If the inverse function, $f^{1}(x)$ is given, we can find the function, f if we find the inverse of the inverse function.

Illustrative examples

Example 1

The inverse of a function f is given by

$f^{1}(x) = \dfrac{2x}{1-x}$ $x \neq 1$. Find the function f.

Solution

To find the function find the inverse of the inverse function

$$f^{1}(x) = \frac{2x}{1-x}.$$

Let $y = f^{1}(x) \quad \Rightarrow y = \dfrac{2x}{1-x}$

Interchanging the variables x and y, and making y the subject

$$x = \frac{2y}{1-y} \quad \Rightarrow x(1-y) = 2y$$

$$x - xy = 2y$$

$$2y + xy = x \quad \Rightarrow y(2+x) = x$$

$$y = \frac{x}{2+x} \quad \therefore \; f(x) = \frac{x}{2+x}, x \neq -2$$

Example 2

The inverse of a function g is given by

$$g^{-1}(x) = \frac{4x-2}{3+x}, \quad x \neq -3.$$

Find the function g.

Solution

To find the function find the inverse of the inverse function

$$g^{-1}(x) = \frac{4x-2}{3+x}.$$

Let $y = g^{-1}(x) \quad \Rightarrow \quad y = \dfrac{4x-2}{3+x}$

Interchanging the variables x and y, and making y the subject

$$x = \frac{4y-2}{3+y} \qquad \Rightarrow \quad x(3+y) = 4y-2$$

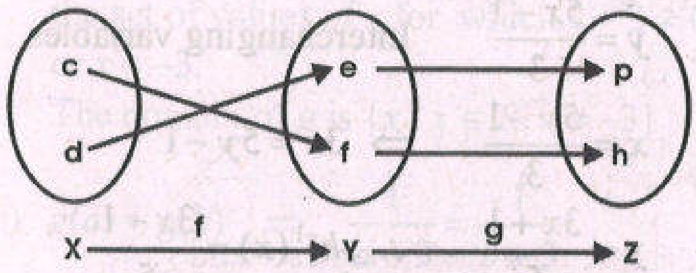
$$3x + xy = 4y - 2 \quad \Rightarrow \quad 4y - xy = 3x + 2$$

$$y(4-x) = 3x + 2 \quad \Rightarrow \quad y = \frac{3x+2}{4-x}$$

$$\therefore \quad g(x) = \frac{3x+2}{4-x}, \quad x \neq 4$$

Composition of functions

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ represent two functions as shown below.



The function which assigns each element of X into Z is called a **composite function,** denoted by $g \circ f$ or gf (i.e. f followed by g). Thus $(g \circ f)(x) = g(f(x))$. The function g operates on the result of f. Note that the use of the notation gf has nothing to do with multiplication.

Multiplication of functions is denoted by: $f(x).g(x) = f.g$.

Now from the diagram, we can compute $gf : X \rightarrow Z$.

i.e. $gf(c) = g(f(c)) = g(f) = h$ and
$gf(d) = g(f(d)) = g(e) = p$

Using *real life* situations, suppose
$h(x)$ = husband of x and
$m(x)$ = mother of x, then,
$h \circ m(x)$ = the husband of the mother of x
= the father or step father of x
$m \circ h(x)$ = the mother of the husband of x
= the mother-in-law of x

In general, to find the **composite function**, $fg(x)$, of two functions f and g acting on suitably defined sets: First find the image of x under g i.e. $g(x)$, then find the image of $g(x)$ under f.

Note that the order of composite functions is important. $f \circ g \neq g \circ f$

Note very carefully:

The inverse of fg is $(fg)^{-1} = g^{-1}f^{-1}$.

Illustrative examples

Example 1

If $f(x) = 2x$ and $g(x) = x^2 + 4$, find

i) $g \circ f$ ii) $f \circ g$ iii) $f \circ g(2)$

Solution

i) $(g \circ f)(x) = g(f(x)) = g(2x)$
$= (2x)^2 + 4 = 4x^2 + 4$

ii) $(f \circ g)(x) = f(g(x)) = f(x^2 + 4)$
$= 2(x^2 + 4) = 2x^2 + 8$

Note that $f.g = 2x(x^2 + 4) = 2x^3 + 8x \neq f \circ g$

iii) $f \circ g(2) = f(g(2)) = f(2^2 + 4)$
$= f(8) = 2(8) = 16$

Example 2

If $f(x) = x^3$ and $g(x) = x^{\frac{1}{3}}$, find $g \circ f$

Solution

$(g \circ f)(x) = gf(x) = g(x^3) = (x^3)^{\frac{1}{3}} = x$

Example 3

If $f(x) = 5x + 4$ and $g(x) = 3x - 2$, find

i) $f \circ g$ ii) $g \circ f$

Solution

i) $(f \circ g)(x) = f(g(x)) = f(3x - 2)$
$= 5(3x-2) + 4 = 15x - 10 + 4$
$= 15x - 6$

ii) $(g \circ f)(x) = g(f(x)) = g(5x + 4)$
$= 3(5x + 4) - 2 = 15x + 10$

$\therefore\ f \circ g \neq g \circ f$

Example 4

Given that $f : x \rightarrow kx + m$ and $g : x \rightarrow cx + d$, prove that $fg = gf \Leftrightarrow d(k-1) = m(c-1)$.

Solution

$f : x \rightarrow kx + m$ and $g : x \rightarrow cx + d$.

Now $fg = f\{g(x)\} = f(cx + d) = k(cx + d) + m$
and $gf = g\{f(x)\} = g(kx + m) = c(kx + m) + d$

$\therefore\ fg = gf \Leftrightarrow k(cx + d) + m = c(kx + m) + d$
$\Leftrightarrow\ kcx + kd + m = kcx + cm + d$
$\Leftrightarrow\ kd + m = cm + d \Leftrightarrow\ kd - d = cm - m$
$\Leftrightarrow\ d(k-1) = m(c-1)$ as required.

Example 5

The functions f and g are defined by $f : x \rightarrow x^2 - 1$ and $g : x \rightarrow 3x + 2$, where x is a real number. Simplify

i) $2f - 3g$ ii) $g \circ f$ iii) $g^{-1} \circ f$
iv) $g \circ f(2)$ v) $f.g$

Solution

i) $2f - 3g = 2(x^2 - 1) - 3(3x + 2)$
$= 2x^2 - 2 - 9x - 6 = 2x^2 - 9x - 8$

ii) $g \circ f = g[f(x)] = g(x^2 - 1) = 3(x^2 - 1) + 2$
$= 3x^2 - 3 + 2 = 3x^2 - 1$

iii) To find $g^{-1} \circ f = g^{-1}[f(x)]$, first find $g^{-1}(x)$
i.e. $g^{-1}(x) = \frac{1}{3}(x - 2)$

$\therefore\ g^{-1}[f(x)] = g^{-1}[x^2 - 1]$
$= \frac{1}{3}(x^2 - 1 - 2) = \frac{1}{3}(x^2 - 3)$

iv) $g \circ f(2) = g[f(2)] = g[2^2 - 1] = g(3)$
$= 3(3) + 2 = 11$

v) $f \cdot g = (x^2 - 1)(3x + 2)$
$= 3x^3 + 2x^2 - 3x - 2$

(Note that $f \cdot g \neq fg$ or $f \circ g$).

Example 6

Two functions g and h, are defined on the set R, of real numbers by

$$g : x \rightarrow \frac{1}{2}x + 1 \text{ and } h : x \rightarrow \frac{5x - 1}{3}$$

Find

a) h^{-1}, the inverse of h;

b) $h^{-1} \circ g$

Solution

a) Let $h(x) = y$

$y = \frac{5x - 1}{3}$ Interchanging variables

$x = \frac{5y - 1}{3} \Rightarrow 3x = 5y - 1$

$y = \frac{3x + 1}{5} \quad \therefore\ h^{-1}(x) = \frac{3x + 1}{5}$

b) $h^{-1} \circ g = \frac{3(\frac{1}{2}x + 1) + 1}{5} = \frac{\frac{3}{2}x + 4}{5}$
$= \frac{3}{10}x + \frac{4}{5} = \frac{3x + 8}{10}$

Solved examination questions

The following examples combine all the concepts you have learnt so far in this chapter.

Example 1

The functions f and g of the real variable x are defined as

$$f: x \to 1 - 2x \text{ and } g: x \to \frac{1}{x+3}$$

i) State the domain of each function

ii) Show that $g(a) = g(b) \Rightarrow a = b$.
What conclusion can you draw from this result?

iii) Find the inverse function g^{-1} of g and hence the values of x for which $f(g(x)) = g^{-1}(x)$, leaving your answers in surd form.

Solution

$$f: x \to 1 - 2x \text{ and } g: x \to \frac{1}{x+3}$$

i) For the function $f(x) = 1 - 2x$, we can see that any value of x will make $f(x)$ defined. Hence the domain of $f = \{x : x \in R\}$.

Also for $g(x) = \dfrac{1}{x+3}$, the domain of g is the set of values of x for which $x + 3 \neq 0$ or $x \neq -3$.

$\therefore$ The domain of g is $\{x : x \in R, x \neq -3\}$

ii) $g(a) = g(b) \Rightarrow \dfrac{1}{a+3} = \dfrac{1}{b+3}$

$$a + 3 = b + 3 \Rightarrow a = b$$

The conclusion that can be drawn here is that g is a one-to-one function.

iii) Let $y = g(x) \Rightarrow y = \dfrac{1}{x+3}$

Interchanging x and y $\Rightarrow x = \dfrac{1}{y+3}$

Making y the subject

$$x(y+3) = 1 \Rightarrow xy + 3x = 1$$

$$xy = 1 - 3x \Rightarrow y = \frac{1-3x}{x}$$

$$\therefore g^{-1}(x) = \frac{1-3x}{x}, \; x \neq 0$$

Also $f(g(x)) = f\left(\dfrac{1}{x+3}\right)$

$$= 1 - 2\left(\frac{1}{x+3}\right) = 1 - \left(\frac{2}{x+3}\right)$$

$$= \frac{x+3-2}{x+3} = \frac{x+1}{x+3}$$

$$\therefore f(g(x)) = g^{-1}(x) \Rightarrow \frac{x+1}{x+3} = \frac{1-3x}{x}$$

$$x(x+1) = (x+3)(1-3x)$$

$$x^2 + x = x - 3x^2 + 3 - 9x$$

$$x^2 + x = 3 - 8x - 3x^2$$

$$4x^2 + 9x - 3 = 0$$

$$x = \frac{-9 \pm \sqrt{81 - (4 \times 4 \times -3)}}{2 \times 4}$$

$$= \frac{-9 \pm \sqrt{81+48}}{8} = \frac{-9 \pm \sqrt{129}}{8}$$

$x = -\frac{9}{8} + \frac{\sqrt{129}}{8}$ or $-\frac{9}{8} - \frac{\sqrt{128}}{8}$ in surd form

Example 2

The functions g and h are defined by

$g: x \to x^2 - 4$ and $h: x \to \dfrac{x}{2-x}$, where x is a real number.

i) State the largest possible domain of each of the two functions.

ii) Find the range of g

iii) a) Determine whether or not g is a one-to-one function.

b) The functions f and p on the set R of real numbers are defined by
$f: x \to ax + b$ and $p: x \to cx - d$
where a, b, c and d are constants.
Show that:
$f \circ p = p \circ f \Leftrightarrow (c-1)b = (a-1)d$

Solution

i) For the function $g(x) = x^2 - 4$, it can be seen that any value of x will make g defined. Hence the domain of $g = \{x : x \in R\}$.

Also for $h(x) = \dfrac{x}{2-x}$, the domain of h is the set of values of x for which $2 - x \neq 0$ or $x \neq 2$

∴ The domain of $h = \{x : x \in R, x \neq 2\}$

ii) Let $y = g(x) \quad \Rightarrow \quad y = x^2 - 4$.
Making x the subject

$y = x^2 - 4 \quad \Rightarrow \quad x^2 = y + 4$

$x = \sqrt{y+4}$

Since the square root of a negative number under the set R of real numbers does not exist, the expression for x i.e. $\sqrt{y+4}$ is real for all values of y except $y + 4 < 0$ or $y < -4$.

∴ Range of $g = \{y : y \in R \text{ except } y < -4\}$.

iii) a) The function g is one-to-one if and only if, $g(a) = g(b) \quad \Rightarrow \quad a = b$.

Now $g(a) = g(b) \quad \Rightarrow a^2 - 4 = b^2 - 4$

$a^2 = b^2 \quad \Rightarrow a = \sqrt{b^2}$

$a = \pm b \quad \Rightarrow a = b \text{ or } a = -b$

Thus $g(a) = g(b) \Rightarrow a \neq b$.

Hence g is not a one-to-one function

b) $f : x \rightarrow ax + b$ and $p : x \rightarrow cx + d$.

To show that

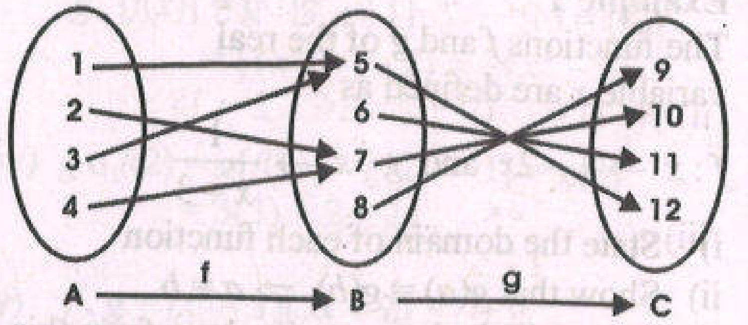
$f \circ p = f(p(x)) = f(cx + d)$
$= a(cx + d) + b$
$= acx + ad + b$ ....(1)

$p \circ f = p(f(x)) = p(ax + b)$
$= c(ax + b) + d$
$= acx + cd + d$ ....(2)

∴ $f \circ p = p \circ f$
$acx + ad + b = acx + cb + d$
$ad - d = cb - b$
$d(a - 1) = b(c - 1)$
$(c - 1)b = (a - 1)d$ as required.

Example 3

a) In the diagram below $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7, 8\}$, $C = \{9, 10, 11, 12\}$, and the mappings $f : A \rightarrow B$ and $g : B \rightarrow C$ are defined as shown.



Determine the images of the elements of:

i) B under g

ii) A under f

iii) A under $g \circ f$ (i.e. f followed by g).

Which of the mappings has an inverse? Give reasons.

b) The function f and g are defined over R as follows:

$f : x \rightarrow x^2 + 2$ and $g : x \rightarrow \dfrac{2x}{1-x}$ $(x \neq 1)$

Find

i) $f(4)$ ii) $f(-4)$ iii) $g(3)$ iv) $f \circ g(3)$ (where $f \circ g$ means g followed by f)

Solution

i) The images of B under g are ;
$g(5) = 12, \; g(6) = 11, \; g(7) = 10, \; g(8) = 9$
Thus $g = \{(5, 12), (6, 11), (7, 10), (8, 9)\}$

ii) The images of A under f are;
$f(1) = 5, \quad f(2) = 7, \quad f(3) = 5, \quad f(4) = 7$
Thus $f = \{(1, 5), (2, 7), (3, 5), (4, 7)\}$

iii) A under $g \circ f$

$gof(1) = g(f(1)) = g(5) = 12$
$gof(2) = g(f(2)) = g(7) = 10$
$gof(3) = g(f(3)) = g(5) = 12$
$gof(4) = g(f(4)) = g(7) = 10$

Thus $gf = \{1, 12), (2, 10), (3, 12), (4, 10)\}$
The mapping g has an inverse since g is a one-to-one function and at the same time an onto function.

b) i) $f(x) = x^2 + 2$

$f(4) = (4)^2 + 2 = 16 + 2 = 18$

ii) $f(-4) = (-4)^2 + 2 = 16 + 2 = 18$

iii) $g(x) = \dfrac{2x}{1-x}$ and so

$g(3) = \dfrac{2(3)}{1-3} = \dfrac{6}{-2} = -3$

iv) $f \circ g(3) = f(g(3)) = f(-3)$
$= (-3)^2 + 2 = 9 + 2 = 11$

Example 4

The functions f and g are $f : x \rightarrow 3x + 2$ and $g : x \rightarrow x^2 - 1$, where x is a real number.

a) Find the composite function $g \circ f$ (i.e. f followed by g).

b) A function $h : x \rightarrow px + q$, where p and q are positive constants, such that $[f \circ g \circ h](x) = 3x^2 + 11x + 11$.

Find: i) The values of p and q.
ii) The truth set of $[f \circ g \circ h](x) = 2 + [g \circ f](x)$.

Solution

Given the two function
$f : x \rightarrow 3x + 2$ and $g : x \rightarrow x^2 - 1$

a) The composite function

$$g \circ f = g(f(x)) = g(3x + 2)$$
$$= (3x + 2)^2 - 1$$
$$= 9x^2 + 12x + 4 - 1$$
$$= 9x^2 + 12x + 3$$

b) i) $h : x \rightarrow px + q$ where p and q are positive constants. Now

$$[f \circ g \circ h](x) = f \circ g[h(x)] = f \circ g(px + q)$$
$$= f[(px + q)^2 - 1]$$
$$= 3[(px + q)^2 - 1] + 2$$
$$= 3[p^2x^2 + 2pqx + q^2 - 1] + 2$$
$$= 3p^2x^2 + 6pqx + 3q^2 - 3 + 2$$
$$= 3p^2x^2 + 6pqx + 3q^2 - 1$$

But $[f \circ g \circ h](x) = 3x^2 + 12x + 11$

$3p^2x^2 + 6pqx + 3q^2 - 1 = 3x^2 + 12x + 11.$

Comparing the coefficient of x^2, we have $3p^2 = 3 \Rightarrow p = 1$, since p is a positive constant.

Also, comparing the coefficient of x, we have

$6pq = 12 \Rightarrow 6(1)q = 12$

$6q = 12 \Rightarrow q = 2$

Hence the values of p and q are 1 and 2 respectively.

ii) To find the truth set of
$[f \circ g \circ h](x) = 2 + [g \circ f](x)$ we have
$[f \circ g \circ h](x) = 3x^2 + 12x + 11$ and
$2 + [g \circ f](x) = 2 + (9x^2 + 12x + 3)$

$\therefore$ $[f \circ g \circ h](x) = 2 + [g \circ f](x)$

$$3x^2 + 12x + 11 = 2 + 9x^2 + 12x + 3.$$
$$3x^2 + 12x + 11 = 9x^2 + 12x + 5$$
$$9x^2 + 12x + 5 - 3x^2 - 12x - 11 = 0$$
$$6x^2 - 6 = 0 \Rightarrow 6(x^2 - 1) = 0$$
$$6(x + 1)(x - 1) = 0$$
$$(x + 1)(x - 1) = 0$$
$$(x + 1) = 0 \text{ or } (x - 1) = 0$$
$$x = -1 \text{ or } x = +1$$

Hence the truth set of
$[f \circ g \circ h](x) = 2 + [g \circ f](x)$ is
$\{x : x = -1 \text{ or } 1\}$

Example 5

a) State the zeros and the largest possible domain of each of the following functions:

$f : x \rightarrow \dfrac{x + 5}{2x - 3}$ and $g : x \rightarrow x^2 - 2$,

where $x \in R$ and R is the set of real numbers.

b) Find: i) f^{-1} (the inverse of f)
ii) $g \circ f^{-1}$ (f^{-1} followed by g).

Solution

a) The largest domain of

$f = \{x : x \in R, 2x - 3 \neq 0\}$

$= \{x : x \in R,\ x \neq \frac{3}{2}\}$

The largest domain of $g = \{x : x \in R\}$, since any value of x makes g defined.

The zeros of f are such that $f(x) = 0$

$$\frac{x+5}{2x-3} = 0$$

$$x + 5 = 0$$

$$x = -5.$$

Therefore the zero of f is -5

Also $g(x) = 0 \Rightarrow x^2 - 2 = 0$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

$\therefore$ The zeros of g are $\sqrt{2}$ and $-\sqrt{2}$.

b) i) Let $f(x) = y \Rightarrow y = \dfrac{x+5}{2x-3}$

Interchanging the positions of x and y and making y the subject, we have

$$x = \frac{y+5}{2y-3}$$

$$x(2y-3) = y + 5$$

$$2xy - 3x = y + 5$$

$$2xy - y = 5 + 3x$$

$$y(2x-1) = 5 + 3x$$

$$y = \frac{5+3x}{2x-1}$$

$$\therefore f^{-1}(x) = \frac{5+3x}{2x-1},\ x \neq \tfrac{1}{2}$$

ii) $g \circ f^{-1} = g[f^{-1}(x)]$

$$= g\left(\tfrac{5+3x}{2x-1}\right) = \left(\tfrac{5+3x}{2x-1}\right)^2 - 2$$

$$= \frac{(25 + 30x + 9x^2) + (8x^2 - 8x + 2)}{(2x-1)^2}$$

$$= \frac{17x^2 + 22x + 27}{(2x-1)^2}$$

Example 6

i) State the largest possible domain of each of the functions $f : x \rightarrow \dfrac{x-2}{x+4}$ and $g : x \rightarrow x^2 + 2$, where x is a real number

ii) Find:

a) the range of g when $-2 \leq x \leq 5$.

b) the inverse function f^{-1} of f, stating its domain.

c) the composite function $g \circ f^{-1}$ [i.e. f^{-1} followed by g] [expressing your result as a single fraction].

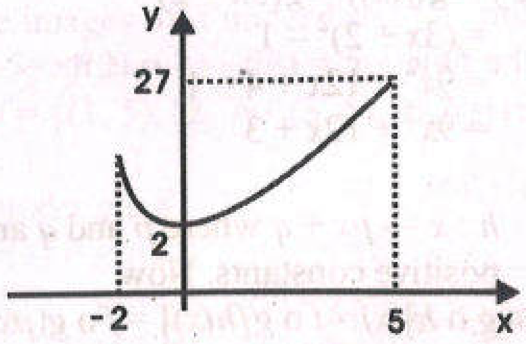
Solution

Given $f(x) = \dfrac{x-2}{x+4}$ and $g(x) = x^2 + 2$

i) The largest possible domain of $f = \{x : x \in R, x \neq -4\}$

The largest possible domain of $g = \{x : x \in R\}$.

ii) a) Note that to find the range when the domain and function are given, it is safer to sketch the graph of the function over its domain. Note that the end points of the domain do not necessarily give the end points of the range. From the graph below, it is clear that the range of $f(x)$ is $\{y : 2 \leq y \leq 27)$



b) Let $f(x) = y \Rightarrow y = \dfrac{x-2}{x+4}$

Interchanging the positions of x and y, and making y the subject we have:

$$x = \frac{y-2}{y+4} \quad \Rightarrow x(y+4) = y-2$$

$$xy_y + 4x = y - 2 \quad \Rightarrow y - xy = 4x + 2$$

$$y(1-x) = 4x + 2 \quad \Rightarrow y = \frac{4x+2}{1-x}$$

$$\therefore \; f^{-1}(x) = \frac{4x+2}{1-x}$$

The domain of f^{-1} is $\{x : x \in R, x \neq 1\}$.

c) $g \circ f^{-1} = g[f^{-1}(x)]$

$$= g\left(\tfrac{4x+2}{1-x}\right) = \left(\tfrac{4x+2}{1-x}\right)^2 + 2$$

$$g \circ f^{-1} = \frac{(16x^2 + 16x + 4) + (2 - 4x + 2x^2)}{(1-x)^2}$$

$$= \frac{18x^2 + 12x + 6}{(1-x)^2} = \frac{6(3x^2 + 2x + 1)}{(1-x)^2}$$

Example 7

The functions f and g are defined by

$$f : x \to \frac{1}{1-x}, x \neq 1 \quad \text{and} \quad g : x \to \frac{x-1}{x}, x \neq 0,$$

where x is real.

a) Show that f is one-to-one.

b) Find
   - i) g^{-1}, the inverse of g;
   - ii) the composite function $f \circ g$ (g followed by f).

Solution

a) If the function f is one-to-one then

$$f(a) = f(b) \;\Rightarrow\; a = b$$

$$f(a) = f(b) \;\Rightarrow\; \frac{1}{1-a} = \frac{1}{1-b}$$

$$1 - a = 1 - b \;\Rightarrow\; a = b$$

$\therefore$ Since $f(a) = f(b) \Rightarrow a = b$, the function f is one-to-one.

b) i) Let $g(x) = y \quad \Rightarrow \quad y = \dfrac{x-1}{x}$

Now interchanging the positions of x and y and making y the subject

$$x = \frac{y-1}{y} \qquad \Rightarrow xy = y - 1$$

$$xy - y = -1 \qquad \Rightarrow y(x-1) = -1$$

$$y = \frac{-1}{x-1} = \frac{1}{1-x}$$

$$\therefore \; g^{-1}(x) = \frac{1}{1-x}, \; x \neq 1$$

ii) $f \circ g = f[g(x)] = f\left(\tfrac{x-1}{x}\right)$

$$= \frac{1}{1 + \frac{x-1}{x}} = \frac{1}{\frac{x+x-1}{x}} = \frac{x}{2x-1}$$

Example 8

Given that $f : x \to x^2 + 1$ and

$$g : x \to \frac{3x+1}{x-1}, \; x \neq 1$$

Find i) $g \circ f$ ii) $g \circ f(2)$

Solution

i) $g \circ f = g[f(x)] = g(x^2 + 1)$

$$= \frac{3(x^2+1)+1}{(x^2+1)-1} = \frac{3x^2+4}{x^2}$$

ii) $g \circ f(2) = \dfrac{3(2)^2 + 4}{2^2} = \dfrac{16}{4} = 4$

Example 9

The functions f and g are defined by

$$f : x \to \frac{x}{x+4}, x \neq -4 \text{ and } g : x \to \frac{2x}{x-1}, x \neq 1,$$

where x is real.

a) Show that f is one-to-one.

b) Find
   - i) g^{-1}, the inverse of g;
   - ii) the image of -1 under the composite function $f \circ g$ (g followed by f).

Solution

a). If the function f is one-to-one then

$$f(a) = f(b) \Rightarrow a = b$$

$$f(a) = f(b) \Rightarrow \frac{a}{a+4} = \frac{b}{b+4}$$

$$a(b+4) = b(a+4)$$

$$ab + 4a = ab + 4b$$

$$4a = 4b \Rightarrow a = b$$

$\therefore$ Since $f(a) = f(b) \Rightarrow a = b$, the function f is one-to-one.

b) i) Let $g(x) = y \Rightarrow y = \dfrac{2x}{x-1}$

Now interchanging the positions of x and y and making y the subject

$$x = \frac{2y}{y-1} \Rightarrow x(y-1) = 2y$$

$$xy - x = 2y \Rightarrow xy - 2y = x$$

$$y(x-2) = x \Rightarrow y = \frac{x}{x-2}$$

$$\therefore \quad g^{-1}(x) = \frac{x}{x-2}, x \neq 2$$

ii) $f \circ g = f[g(x)] = f\left(\frac{2x}{x-1}\right) = \dfrac{\left(\frac{2x}{x-1}\right) - 3}{\left(\frac{2x}{x-1}\right) + 4}$

$$= \frac{2x - 3(x-1)}{2x + 4(x-1)}$$

$$= \frac{2x - 3x + 3}{2x + 4x - 4} = \frac{-x+3}{6x-4}$$

The image of -1 under the composite function $f \circ g$ is given by:

$$f \circ g(-1) = \frac{-(-1)+3}{6(-1)-4} = \frac{4}{-6-4} = -\frac{4}{10} = -\frac{2}{5}$$

Example 10

The functions f and g are defined on the set of real numbers such that

$$f : x \rightarrow \frac{2x-3}{x} \quad \text{and} \quad g : x \rightarrow 1 - \frac{3}{2x}$$

a) Find the inverse functions f^{-1} and g^{-1}, stating the largest domain of each

b) Find the image of

i) x under $g \circ f^{-1}$ ii) 4 under $f^{-1} \circ g^{-1}$

Solution

Note: In each case the final answer has been provided. Work out the missing steps.

a) $f^{-1}(x) = \dfrac{3}{2-x}$,

Domain of $f^{-1} = \{x : x \in R, x \neq 2\}$

$g^{-1}(x) = \dfrac{3}{2(1-x)}$,

Domain of $g^{-1} = \{x : x \in R, x \neq 1\}$

b) i) $g \circ f^{-1}(x) = \dfrac{x}{2}$

ii) $f^{-1} \circ g^{-1}(4) = \dfrac{6}{5}$

Example 11

The functions f and g are defined on the set of real numbers by

$f : x \rightarrow 3x + 4$ and

$g : x \rightarrow \dfrac{2x+1}{x+3}$, $x \neq -3$.

a) Find i) $g \circ f$

ii) the inverse of g, g^{-1}

b) i) Express $g(x)$ in the form

$g(x) = A + \dfrac{B}{x+3}$, $x \neq -3$,

where A and B are constants

ii) Hence find $\lim\limits_{x \to \infty} g(x)$

Solution

a) i) $g \circ f = g[f(x)] = g(3x+4)$

$$= \frac{2(3x-4)+1}{(3x+4)+3}$$

$$= \frac{6x-8+1}{3x+4+3} = \frac{6x-7}{3x+7}$$

ii) Let $g(x) = y \Rightarrow y = \dfrac{2x+1}{x+3}$

Now interchanging the positions of x and y and making y the subject

$$x = \frac{2y+1}{y+3} \Rightarrow x(y+3) = 2y+1$$

$$xy+3x = 2y+1 \Rightarrow xy-2y = 1-3x$$

$$y(x-2) = 1-3x \Rightarrow y = \frac{1-3x}{x-2}$$

$$\therefore \quad g^{-1}(x) = \frac{1-3x}{x-2}, x \neq 2$$

b) i) $g(x) = \dfrac{2x+1}{x+3} = \dfrac{2(x+3)-5}{x+3}$

$$= \frac{2(x+3)}{x+3} + \frac{-5}{x+3}$$

$$= 2 + \frac{-5}{x+3} = A + \frac{B}{x+3}$$

where $A = 2$ and $B = -5$

ii) $\lim\limits_{x\to\infty} g(x) = \lim\limits_{x\to\infty}\left(\dfrac{2x+1}{x+3}\right)$

$$= \lim_{x\to\infty}\left(2 + \frac{-5}{x+3}\right)$$

$$= 2 + 0 = 2$$

Example 12

Two functions f and g are defined on the set R, of real numbers by:

$f : x \to x^2 + 1$ and $g : x \to x - 2$.

Find

a) $f \circ g$;

b) the values of x such that $f \circ g\,(x) = 5$.

Solution

a) $fog = f(g(x)) = (x-2)^2 + 1$

$$= x^2 - 4x + 4 + 1$$

$$= x^2 - 4x + 5$$

b) $f \circ g = x^2 - 4x + 5 = 5$

$$x^2 - 4x + 5 - 5 = 0 \Rightarrow x^2 - 4x = 0$$

$$x(x-4) = 0 \Rightarrow x = 0, x = 4$$

Example 13

Two functions f and g are defined on the set R, of real numbers by

$$f : x \to \frac{2x-3}{x}, x \neq 0$$

$$g : x \to 1 - \tfrac{3}{2}x$$

i) Find f^{-1}, the inverse of f.

ii) Find the largest possible domain of $g \circ f$.

Solution

i) $f^{-1} : x \to \dfrac{3}{2-x},\ x \neq 2$

ii) $\{x : x \in R,\ x \neq 0\}$

Example 14

Two functions f and g are defined on the set R, of real numbers by $f : x \to x^2 - 1$ and

$g : x \to \dfrac{3x-1}{4}$.

Find a) $g^{-1}(x)$ b) $fog^{-1}(2)$

Solution

a) Let $y = g(x) \Rightarrow y = \dfrac{3x-1}{4}$

Interchanging x and y

$x = \dfrac{3y-1}{4}$ Making y the subject

$$4x = 3y - 1 \Rightarrow 3y = 4x + 1$$

$$y = \frac{4x+1}{3} \quad \therefore \quad g^{-1}(x) = \frac{4x+1}{3}$$

b) $fog^{-1}(x) = \left(\dfrac{3x-1}{4}\right)^2 - 1$

$$= \frac{9x^2 - 6x + 1}{16} - 1$$

$$fog^{-1}(2) = \frac{16(2)^2 + 8(2) + 1}{9} - 1$$

$$= \frac{81}{9} - 1 = 9 - 1 = 8$$

Example 15

A function is defined by $g : x \rightarrow ax^2 + bx + 2$, where a and b are constants. If $g(-2) = 6$ and $g(3) = 11$, find the values of a and b.

Solution

$g : x \rightarrow ax^2 + bx + 2$

$g(-2) = 6 \Rightarrow a(-2)^2 + b(-2) + 2 = 6$

$4a - 2b + 2 = 6$

$2a - b = 2$ ………………..(1)

Also $g(3) = 11 \Rightarrow a(3)^2 + b(3) + 2 = 11$

$9a + 3b = 9$

$3a + b = 3$ ………………..(2)

From (1), $b = 2a - 2$

$3a + 2a - 2 = 3$

$5a = 5 \Rightarrow a = 1$

$b = 2(1) - 2 = 0 = 0$

$\therefore$ $a = 1$ and $b = 0$

Example 16

Two functions, f and g are defined by $f(x) = x^2 + 3$ and $g(x) = \sqrt{x-3}$, where $x \in R$, the set of real numbers.

a) State the domain of g.

b) Find $f \circ g(x)$.

c) Determine whether or not f is one-to-one

Solution

a) $g(x) = \sqrt{x-3}$

$x - 3 \geq 0 \Rightarrow x \geq 3$

Domain of $g = \{x : x \in R, x \geq 3\}$ **or**
$g = \{x : x$ is not less than $3\}$ **or**
$g = \{x : x \in R$, except $x < 3\}$

b) $f \circ g(x) = f\left(\sqrt{x-3}\right)$

$= \left(\sqrt{x-3}\right)^2 + 3 = x$

c) $f(y) = f(x)$

$x^2 + 3 = y^2 + 3$

$x^2 = y^2$

$x = \pm y$

Hence f is **not** one-to-one.

Example 17

a) Given that $p(x) = 2x$ and $q(x) = x^2 - 1$, find the range of $p(q(x))$ for the domain $\{-2, -1, 0, 1, 2\}$.

b) Two functions f and g are defined on R, the set of real numbers by
$f : x \rightarrow x^2 + 3,\ g : x \rightarrow x - 2$. Find

i) $g^{-1}(f^{-1}(3))$

ii) the value of x for which $f(g(x)) = g(f(x))$

Solution

a) Given $p(x) = 2x$ and $q(x) = x^2 - 1$

$p[q(x)] = 2(x^2 - 1)$

For the domain $\{-2, -1, 0, 1, 2\}$;

Range = $\{-2, 0, 6\}$

b) i) Let $f(x) = y \Rightarrow y = x^2 + 3$

Interchanging positions of x and y, we have

$x = y^2 + 3$

$y = \sqrt{x-3}$

$\therefore$ $f^{-1} : x \rightarrow \sqrt{x-3}$

Let $g(x) = y \Rightarrow y = x - 2$

Interchanging positions of x and y, we have

$x = y - 2$

$y = x + 2 \ \therefore\ g^{-1} : x \rightarrow x + 2$

Hence $g^{-1}[f^{-1}(3)] = (3-3)^{\frac{1}{2}} + 2 = 2$

ii) $f[g(x)] = g[f(x)]$

$$(x-2)^2 + 3 = (x^2+3) - 2$$
$$x^2 - 4x + 4 + 3 = x^2 + 1$$
$$4x = 6 \Rightarrow \ x = 1.5$$

Example 18

Two functions g and h are defined on the set R, of real numbers by

$g: x \rightarrow x^2 - 2$ and $h: x \rightarrow \dfrac{1}{x+2}$, $x \neq -2$

Find i) h^{-1}, the inverse of h;

ii) $g\,o\,h\,(-\tfrac{1}{2})$.

Solution

i) Given that $h: x \rightarrow \dfrac{1}{x+2}$, let $y = \dfrac{1}{x+2}$

Interchanging variables $\Rightarrow \ x = \dfrac{1}{y+2}$

$$xy + 2x = 1 \quad \Rightarrow xy = 1 - 2x$$

$$y = \frac{1-2x}{x} \quad \Rightarrow h^{-1}(x) = \frac{1-2x}{x},\ x \neq 0$$

ii) $$goh = g\left(\frac{1}{x+2}\right) = \left(\frac{1}{x+2}\right)^2 - 2$$

$$= \frac{1 - 2(x^2+4x+4)}{x^2+4x+4}$$

$$= \frac{-2x^2 - 8x - 7}{x^2+4x+4}$$

$$goh(-\tfrac{1}{2}) = \frac{-2(\frac{1}{4}) - 8(-\frac{1}{2}) - 7}{\frac{1}{4} + 4(-\frac{1}{2}) + 4}$$

$$= \frac{-2+16-28}{1-8+16} = -\frac{14}{9} = -1\tfrac{5}{9}$$

Alternatively

$$goh(-\tfrac{1}{2}) = g\left(\frac{1}{(-\frac{1}{2})+2}\right)$$

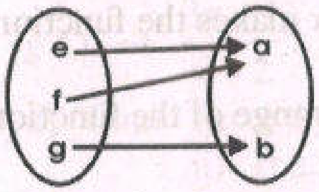
$$= g(\tfrac{2}{3}) = (\tfrac{2}{3})^2 - 2$$

$$= \tfrac{4}{9} - 2 = -\tfrac{14}{9} = -1\tfrac{5}{9}.$$

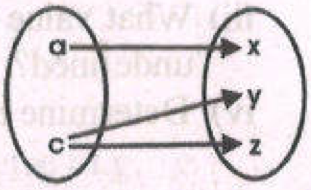
Review exercise 4

1. Which the following mappings are functions.

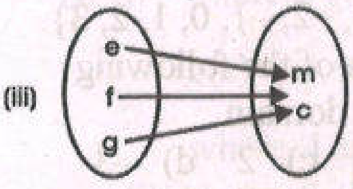
(i)



(ii)



(iii)



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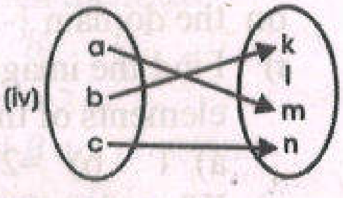
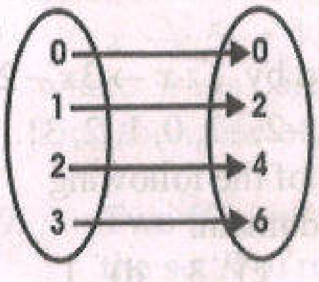
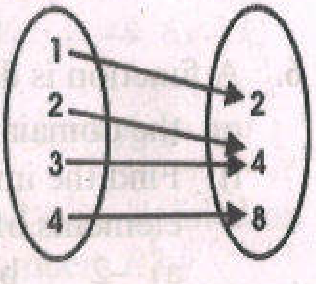


2. Which of the following relations is/are functions(s)? Which is one-to-one?

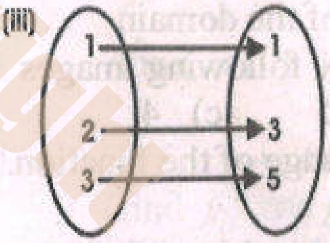
(i)



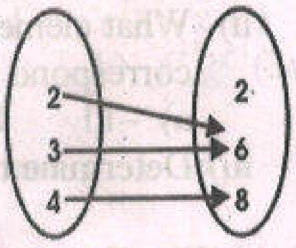
(ii)



(iii)



(iv)



3. Draw arrow diagrams for the mappings defined by each of the following rules on the domain $\{-3, -2, -1, 0, 1, 2, 3\}$.
   i) $x \rightarrow 4x + 3$
   ii) $x \rightarrow x^2 - 1$

   Which of these mappings is a one-to-one function.

4. A function is defined by $f: x \rightarrow \dfrac{3x-2}{x+4}$ on the domain $\{-3, -2, -1, 0, 1, 2, 3\}$.
   i) Find the images of the following elements of the domain.
      a) 0 b) −1 c) 3 d) −3

ii) What elements of the domain correspond to the following images
a) $\frac{1}{5}$ b) $\frac{2}{3}$ c) -4

iii) What value of x makes the function undefined?

iv) Determine the range of the function.

5. A function is defined by $f : x \rightarrow 2x + 1$ on the domain $\{-3, -2, -1, 0, 1, 2, 3\}$.

i) Find the images of the following elements of the domain.
a) 1 b) -2 c) 2 d) -3

ii) What elements of the domain correspond to the following images
a) 1 b) -1 c) 7

iii) Determine the range of the function.

6. A function is defined by $f : x \rightarrow 3x - 2$ on the domain $\{-3, -2, -1, 0, 1, 2, 3\}$.

i) Find the images of the following elements of the domain.
a) -2 b) 0 c) 3 d) 1

ii) What elements of the domain correspond to the following images
a) -11 b) -5 c) 4

iii) Determine the range of the function.

7. A function $f : x \rightarrow ax + b$ is such that $f(0) = -1$ and $f(2) = 5$. Find;

i) the values of a and b

ii) $f(-1)$

iii) $f(1)$

iv) $f(3)$

8. Draw mapping tables for the mappings defined by each of the following rules on the domain $\{-3, -2, -1, 0, 1, 2, 3\}$.

i) $x \rightarrow 2x + 1$

ii) $x \rightarrow x^2 - 2$

9. Two functions f and g are defined over the set R of real numbers as follows.

$$f : x \rightarrow 2x^2 - x \ ; \ g : x \rightarrow \frac{7x - 6}{x}, (x \neq 0)$$

a) If $f(a) = g(a)$, find the values of a.

b) Find:

i) The inverse function of g.

ii) $f \circ g(2)$ where $f \circ g$ means g followed by f.

10. f and g are functions on the set of real numbers such that

$$f : x \rightarrow \frac{2x - 3}{x}, \ (x \neq 0) \text{ and}$$

$$g : x \rightarrow 1 - \tfrac{3}{2}x$$

a) Find the inverse functions, f^{-1} and g^{-1}, stating the largest domain of each.

b) Find the image of

i) x under $g \circ f^{-1}$

ii) 4 under $f^{-1} \circ g^{-1}$

11. A function f, defined on the set R of real numbers, is given by

$$f : x \rightarrow \frac{x - 2}{x + 1}, \ x \neq -1$$

a) Show that f is a one-to-one function.

b) Find $f(1)$, $f(3)$ and $f(-2)$.

12. The functions f and g are such that

$$f : x \rightarrow \frac{5x}{x + 3}, x \neq -3 \text{ and } g : x \rightarrow 2x - 1$$

where $x \in R$. Find

i) $f.g$

ii) $f \circ g$

13. Two functions f and g are defined on the set R of real numbers R by

$$f : x \rightarrow 2x + 1 \text{ and } g : x \rightarrow \frac{2x + 3}{x + 2}, x \neq -2$$

Find

a) the inverse of g, g^{-1}

b) $f \circ g$

14. The functions g and h are defined on the set R of real number by:

$g: x \to x^2 - 2$ and $h: x \to \dfrac{1}{x+1}$, $x \neq -1$

Find

i) h^{-1}, the inverse of function of h, stating the value of x for which h^{-1} is not defined.

ii) $g \circ h$, when $x = -\frac{1}{2}$.

15. The functions f and g are defined by:

$f: x \to \dfrac{3}{x+4}, x \neq -4$

$g: x \to \dfrac{2x}{x-1}, x \neq -1$, where x is real.

Find i) g^{-1}, the inverse of g;

ii) the image of -1 under $f \circ g$

16. State the domain and find the inverses of the following functions:

i) $f: x \to \dfrac{3x-2}{x+4}$

ii) $f: x \to \dfrac{5x+7}{3x+2}$

iii) $g: x \to \dfrac{2x}{1-x}$

iv) $g: x \to \dfrac{3x+1}{2}$

v) $h: x \to 1 - x$

vi) $h: x \to \dfrac{1}{3x+1}$

vii) $f: x \to 4x^2 + 5$

viii) $f: x \to \dfrac{3x-1}{x-2}$

ix) $f: x \to \dfrac{x-3}{2x+1}$

x) $f: x \to \dfrac{ax+b}{cx+d}$

17. Two functions f and g are defined on the set R of real numbers R by

$f: x \to 7x + 4$ and

$g: x \to \dfrac{3x-1}{x-2}$, $x \neq 2$

a) Find i) $g \circ f$

ii) the inverse of g, g^{-1}

b) i) Express $g(x)$ in the form

$$g(x) = A + \frac{B}{x-2},\ x \neq 2,$$

where A and B are constants

ii) Hence find $\lim\limits_{x \to \infty} g(x)$

18. The functions f and g are defined on the set R of real numbers R by

$f: x \to 5x + 4$ and $g: x \to 6x - k$,

where k is a constant.

Find the value of k for which $fg = gf$

19. i) Two functions f and g are defined on the set R of real numbers R by

$f: x \to 3x - 1$ and $g: x \to x + 1$

Find a) $f^{-1}g^{-1}(3)$ b) $f^{-1}g^{-1}(3x)$

ii) If $f: x \to 4x + 1$ and $g: x \to x - 1$, find f^{-1} and g^{-1}

What is $g^{-1}f^{-1}(3)$?

20. The function f is defined on the set R of real numbers R by

$f: x \to \dfrac{1}{2-x}$, $x \neq 2$

a) Find the inverse function f^{-1} and state its domain

b) Evaluate i) $f(3)$ ii) $f^{-1}(3)$

21. The function $f: x \to x^2 - 2$ has domain $\{-2, -1, 0, 1, 2\}$. State the range.

22. The functions f and g are defined on the set R of real numbers R by

$$f : x \to \frac{1}{x},\ x \neq 0 \text{ and } g : x \to 1 - x$$

Find

a) $fg(x)$ b) $gf(x)$

23. a) The function f is defined on the set R of real numbers R by

$$f : x \to \frac{2x}{x-4},\ x \neq 4, \text{ find } f^2 \text{ and } f^{-1}.$$

Hence find the non-zero value of x for which $f^2 = f^{-1}$.

b) The function g is defined by

$g : x \to ax^2 + bx$.

Given that $g(3) = 9$ and $gf(2) = 14$, find a and b.

24. Find the domain of the function

$$f : x \to \frac{x+2}{\sqrt{2x-1}}$$

25. The functions f and g are defined on the set R of real number by

$$f : x \to x^2 - 9 \text{ and } g : x \to \frac{x-1}{2-x},\ x \neq 1$$

i) State the largest domain of each of the two functions

ii) State the range of f and g.

iii) Show that the function g is one to one

iv) Find $g \circ f(1)$

26. a) The functions f and g are defined on the set R of real number by

$$f : x \to x^2 + 1 \text{ and } g : x \to \frac{x}{1-x},\ x \neq 1$$

i) State the largest domain of each of the two functions

ii) State the range of f and g.

iii) Show that the function g is one to one

b) Given that the function $f(x) = kx + m$ and $g(x) = cx + d$, prove that

$$f \circ g = g \circ f \Leftrightarrow d(k-1) = m(c-1)$$

27. The function f is defined on the set R of real number by $f : x \to 5x - 9$. Find the inverse of f.

28. The function h is defined on the set R of real numbers by $h : x \to \frac{4-3x}{x-5},\ x \neq 5$

Find the inverse of h.

29. If $f(x) = \frac{x+1}{x}$, find

i) $f(3)$ ii) $f(\frac{1}{2})$ iii) $f(-4)$

30. The functions f and g are defined on the set R of real numbers by

$$f : x \to \frac{1}{x-1} \text{ and } g : x \to \frac{x-1}{x},\ x \neq 1$$

i) State the largest domain of each of the two functions

ii) State the range of f and g.

iii) Show that the functions f and g are one to one

iv) Find the composite functions

a) $f \circ g$ b) $g \circ f^{-1}$

31. The inverse of a function f is given by

$$f^{-1}(x) = \frac{4x-2}{3+x}\ x \neq -3.$$

Find the function f.

32. The inverse of a function g is given by

$$g^{-1}(x) = \frac{1}{1+x},\ x \neq -1.$$

Find the function g.

33. State the largest possible domain of the following functions

i) $f : x \rightarrow 1 - 5x$ ii) $g : x = \sqrt{3x - 2}$

iii) $h : x \rightarrow x^2 + 3$ iv) $k : x \rightarrow \dfrac{x+3}{x-1}$,

v) $f : x \rightarrow \dfrac{x}{\sqrt{2x-3}}$ vi) $g : x = \sqrt{4 - x^2}$

34. State the range of the functions

i) $f : x \rightarrow 2 - 3x$

ii) $g : x \rightarrow x^2 - 2$

iii) $h : x \rightarrow \dfrac{2+x}{3+x}$

iv) $h : x \rightarrow \dfrac{3x}{3-x}$

35. i) State the largest possible domain of each of the functions

$f : x \rightarrow \dfrac{x-1}{x+5}$ and $g : x \rightarrow x^2 - 4$,

where x is a real number

ii) Find:

a) the range of g when $-4 \le x \le 5$

b) the inverse function f^{-1} of f, stating its domain

c) the composite function $g \circ f^{-1}$ [i.e. f^{-1} followed by g]

d) show whether or not f is a one-to-one function.

36. The functions f and g are defined as follows:

$f : x \rightarrow x^2 + 2$ and $g : x \rightarrow \dfrac{2x}{1-x}$, find

i) $f(4)$ ii) $f(-4)$

iii) $g(3)$ iv) $f \circ g(3)$

37. Two functions f and g are defined on the set R, of real numbers by $f : x \rightarrow 1 - x$ and g: $x \rightarrow 2 - x^2$. Find $g\{f(-1)\}$.

38. A function, f defined on the set R of real numbers, is given by

$f : x \rightarrow \dfrac{x-2}{x+1}, x \neq -1$

a) Show that f is a one-to-one function

b) Find the inverse f^{-1} of f.

39. f and g are functions on the set R of real numbers such that

$f : x \rightarrow 3x - 4$ and $g : x \rightarrow \dfrac{4x+1}{2x-3}$

a) Find the inverse functions, f^{-1} and g^{-1}, stating the largest domain of each.

b) Find the image of :

i) x under $g \circ f^{-1}$ ii) 4 under $f^{-1} \circ g^{-1}$

40. If the functions f and g are defined as $f : x \rightarrow 2x - 1$ and g:$x \rightarrow 3x - 5$, find $(f \circ g)^{-1}(x)$

41. Two functions $f(x)$ and $g(x)$ are defined on the set of real numbers by $f(x) = 3x^2 - 2$ and $g(x) = x + 3$. Find

a) $f(-2)$; b) $g^{-1}(\frac{-3}{4})$

c) the value of x for which $f(g(x)) = g(f(x))$;

d) $f^{-1}(g(4))$

42. A function is defined by

$f : x \rightarrow \dfrac{x+1}{x-3}, x \neq 3$. Find

i) $f(2)$ ii) f^{-1}, the inverse of f.

43. Two functions g and h, are defined on the set R, of real numbers by

$g : x \rightarrow \frac{1}{2}x + 1$ and $h : x \rightarrow \dfrac{5x-1}{3}$

Find a) h^{-1}, the inverse of h;

b) $h^{-1} \circ g$

44. A function f is defined on the set of real numbers by $f : x \rightarrow \frac{x}{2} + 11$. Find $f^{-1}(15)$

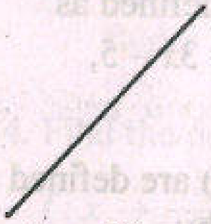
CHAPTER 5

POLYNOMIAL FUNCTIONS

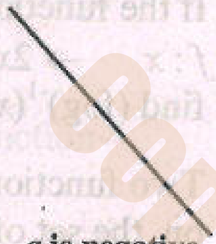
Linear functions

Any function of the form $f(x) = ax + b$, where a and b are constants is called a **linear function**. For example, $f(x) = 2x + 1$, $f(x) = 4x - 1$, and $f(x) = x$ are linear functions. The graph of any linear function is a *straight line*. To sketch a linear function, find the intercepts on the x and y axes.

The two shapes of the graph of a straight line depending on the value of 'a' are shown below.



***a* is positive**



***a* is negative**

Quadratic functions

The general **quadratic function** is of the form $f(x) = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$. The function has a *minimum value* when $\boldsymbol{a > 0}$ or a is positive. For example, $f(x) = 2x^2 + 3x - 7$ has a minimum value. The function has a *maximum value* when $\boldsymbol{a < 0}$ or a is negative. For example, $f(x) = 5 - 3x - x^2$ has a maximum value.

Graphs of a quadratic function

By completing the square of the general quadratic function $f(x) = ax^2 + bx + c$,

$f(x)=a\left(x^2+\dfrac{b}{a}x+\dfrac{c}{a}\right)$ i.e. making the coefficient of x unity (1) by factorising a out.

$$f(x)=a\left[x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right]$$

i.e. adding and subtracting the square of half the coefficient of x. [i.e. $\left(\frac{b}{2a}\right)^2$].

Simplifying gives:

$$f(x)=a\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a}\right)$$

$$= a(x+d)^2+k$$

where $d=\dfrac{b}{2a}$ and $k=-\left(\dfrac{b^2-4ac}{4a}\right)$

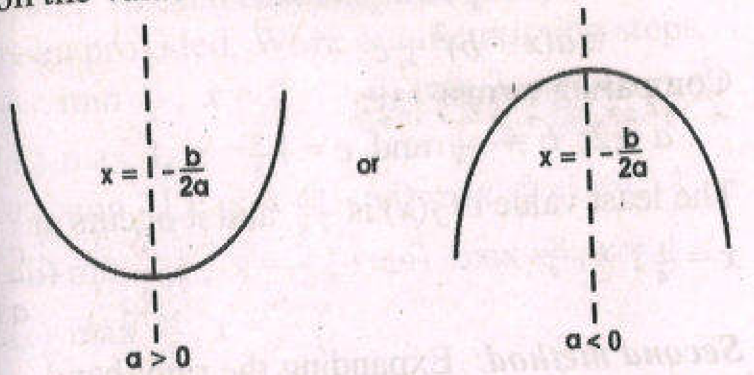
The graph of $f(x) = ax^2 + bx + c$, is a **parabola**.

Its **axis of symmetry** is $x=-\dfrac{b}{2a}$ and its **greatest** or **least** value is $-\left(\dfrac{b^2-4ac}{4a}\right)$.

If $\boldsymbol{a > 0}$ i.e. if a is positive, then $f(x)$ has a minimum or least value of $-\left(\dfrac{b^2-4ac}{4a}\right)$, occurring at $x=-\dfrac{b}{2a}$.

If $\boldsymbol{a < 0}$, i.e. if a is negative, then $f(x)$ has a maximum or greatest value of $-\left(\dfrac{b^2-4ac}{4a}\right)$, occurring at $x=-\dfrac{b}{2a}$.

The two shapes of the graph depending on the value of a are shown below.



Finding the maximum and minimum values of quadratic functions analytically

The **maximum** or **minimum** value of a quadratic function can be found by the method of completing the square. An alternative method, using calculus, will be considered under differentiation. Using the *method of completing the square,* if any quadratic function can be put in the form $f(x) = a(x \pm b)^2 + c$, then c is the maximum or minimum value of f and this value occurs at $x \pm b = 0$. Note that $x \pm b = 0$ is also the **equation of the axis of symmetry**.

Illustrative examples

Example 1

What is the minimum value of $f(x) = 3x^2 - 2x + 1$ and what value of x does it occur?

Solution

First make sure the coefficient of x^2 is 1 if it is not before completing the square on the RHS.

i.e. $f(x) = 3x^2 - 2x + 1$

$f(x) = 3(x^2 - \frac{2}{3}x + \frac{1}{3})$

(i.e. factorizing 3 out)

Next add and subtract the square of half the coefficient of x {i.e. $(-\frac{2}{3})$, half of it is $(-\frac{1}{3})$ and the square of this is $(-\frac{1}{3})^2$} from the expression in bracket.

$$f(x) = 3[x^2 - \tfrac{2}{3}x + (-\tfrac{1}{3})^2 - (-\tfrac{1}{3})^2 + \tfrac{1}{3})$$

$$f(x) = 3[(x - \tfrac{1}{3})^2 - \tfrac{1}{9} + \tfrac{1}{3}]$$

$$f(x) = 3[(x - \tfrac{1}{3})^2 + \tfrac{2}{9}]$$

$$f(x) = 3(x - \tfrac{1}{3})^2 + \tfrac{2}{3}$$

Hence the minimum value of f is $\frac{2}{3}$ and it occurs at $x - \frac{1}{3} = 0$ i.e. $x = \frac{1}{3}$.

Alternative method

Comparing $f(x) = 3x^2 - 2x + 1$ with the general quadratic function $f(x) = ax^2 + bx + c$

$a = 3$, $b = -2$ and $c = 1$

The minimum value is $k = -\left(\frac{b^2 - 4ac}{4a}\right)$

$$= -\left(\frac{(-2)^2 - 4(3)(1)}{4(3)}\right) = -\frac{4-12}{12}$$

$$= -\tfrac{-8}{12} = -\tfrac{-2}{3} = \tfrac{2}{3}$$

This occurs at $x = -\frac{b}{2a} = \frac{1}{3}$ *Refer to page 92.*

Example 2

Find the maximum value of the function $f(x) = 5 - x - 2x^2$ and the value of x it occurs.

Solution

$$f(x) = 5 - x - 2x^2 = -2x^2 - x + 5$$

$$f(x) = -2(x^2 + \tfrac{1}{2}x - \tfrac{5}{2})$$

Now add and subtract $(\frac{1}{4})^2$

$$f(x) = -2[x^2 + \tfrac{1}{2}x + (\tfrac{1}{4})^2 - (\tfrac{1}{4})^2 - \tfrac{5}{2}]$$

$$f(x) = -2[(x + \tfrac{1}{4})^2 - \tfrac{1}{16} - \tfrac{5}{2}]$$

$$f(x) = -2[(x + \tfrac{1}{4})^2 - \tfrac{41}{16}]$$

$$f(x) = -2(x + \tfrac{1}{4})^2 + \tfrac{41}{8}$$

Hence the maximum value of f is $\frac{41}{8}$ and it occurs when $x + \frac{1}{4} = 0$ i.e. $x = -\frac{1}{4}$.

Alternative method

Comparing $f(x) = 5 - x - 2x^2$ with the general quadratic function $f(x) = ax^2 + bx + c$

$a = -2$, $b = -1$ and $c = 5$

The maximum value is $k = -\left(\dfrac{b^2 - 4ac}{4a}\right)$

$$= -\left(\frac{(-1)^2 - 4(-2)(5)}{4(-2)}\right) = -\frac{1+40}{-8}$$

$$= -\tfrac{41}{-8} = \tfrac{41}{8}$$

This occurs at $x = -\dfrac{b}{2a} = -\tfrac{1}{4}$

Example 3

Find the minimum value of the function $f(x) = x^2 + 2x - 3$ and the value of x it occurs.

Solution

Here the co-efficient of x^2 is 1

$$f(x) = x^2 + 2x - 3$$
$$f(x) = x^2 + 2x + (1)^2 - (1)^2 - 3$$
$$f(x) = (x+1)^2 - 4$$

Hence the minimum value of $f(x)$ is -4 and it occurs when $x + 1 = 0$ i.e. $x = -1$.

Use the alternative method to arrive at the same answer.

Example 4

Express the following functions in the form $a(x+b)^2 + c$. Hence find the least or greatest value of functions and the value of x at which they occur.

i) $f(x) = 2x^2 - x - 1$
ii) $f(x) = 3x^2 - x - 6$
iii) $f(x) = 2 - 4x - x^2$

Solution

i) ***First method:*** Using the method of completing the square, we have

$$f(x) = 2x^2 - x - 1$$
$$= 2[x^2 - \tfrac{1}{2}x - \tfrac{1}{2}]$$
$$= 2[x^2 - \tfrac{1}{2}x + (-\tfrac{1}{4})^2 - (-\tfrac{1}{4})^2 - \tfrac{1}{2}]$$
$$= 2[(x - \tfrac{1}{4})^2 - \tfrac{1}{16} - \tfrac{1}{2}]$$
$$= 2[(x - \tfrac{1}{4})^2 - \tfrac{9}{16}] = 2(x - \tfrac{1}{4})^2 - \tfrac{9}{8}$$

$$\therefore f(x) = 2x^2 - x - 1$$
$$= 2(x - \tfrac{1}{4})^2 - \tfrac{9}{8}$$
$$= a(x+b)^2 + c$$

Comparing terms:

$a = 2$; $b = -\tfrac{1}{4}$ and $c = -\tfrac{9}{8}$

The least value of $f(x)$ is $-\tfrac{9}{8}$ and it occurs at $x = \tfrac{1}{4}$.

Second method: Expanding the right hand side and comparing terms

$$f(x) = 2x^2 - x - 1 = a(x+b)^2 + c$$
$$= a(x^2 + b^2 + 2bx) + c$$
$$= ax^2 + 2abx + (c + ab^2)$$

Comparing terms: $a = 2$ and $2ab = -1$

$b = -\tfrac{1}{4}$ and also $(c + ab^2) = -1$

$c = -\tfrac{9}{8}$

Note: In (ii) and (iii), the final answer has been provided. Work out the missing steps.

ii) $f(x) = 3x^2 - x - 6 = 3(x - \tfrac{1}{6})^2 - \tfrac{73}{12}$) ;

The least value of $f(x)$ is $-\tfrac{73}{12}$ and it occurs at $x = \tfrac{1}{6}$.

iii) $f(x) = 2 - 4x - x^2 = 6 - (x+2)^2$;

The least value of $f(x)$ is 6 and it occurs at $x = -2$

Example 5

Find the maximum or minimum value of the following functions and the values of x at which they occur.

i) $f(x) = x^2 - 3x + 5$
ii) $f(x) = 2x^2 - 4x + 5$
iii) $f(x) = 3 - 2x - x^2$
iv) $f(x) = x^2 - 2x + 5$
v) $f(x) = x^2 + 4x - 8$
vi) $f(x) = 2 - 5x - 3x^2$
vii) $f(x) = 2x^2 - 6x + 3$
viii) $f(x) = 7 + x - x^2$
ix) $f(x) = 4 - 7x - x^2$

Solution

Note: In each case the final answer has been provided. Work out the missing steps.

i) min $\frac{11}{4}$, $x = \frac{3}{2}$ ii) min 3, $x = 1$

iii) max 4, $x = -1$ iv) min 4, $x = 1$

v) min -12, $x = -2$ vi) max $\frac{49}{12}$, $x = -\frac{5}{6}$

vii) min $-\frac{3}{2}$, $x = \frac{3}{2}$ viii) max $\frac{29}{4}$, $x = \frac{1}{2}$

ix) max $\frac{65}{4}$, $x = -\frac{7}{2}$

Finding the features of a quadratic curve and sketching it

The *maximum* or *minimum* value of a quadratic function occurs at $x = -\dfrac{b}{2a}$

Therefore for any quadratic function $f(x) = ax^2 + bx + c$.

$$f(x)\text{ has}\begin{cases}\text{a least value of } f\left(-\dfrac{b}{2a}\right) \text{ if } a > 0\\ \text{a greatest value of } f\left(-\dfrac{b}{2a}\right) \text{ if } a < 0\end{cases}$$

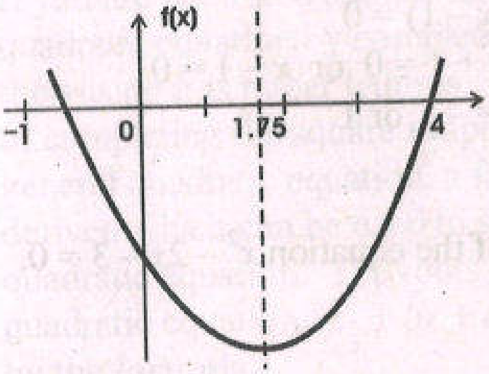
and is symmetrical in shape about the line $x = -\dfrac{b}{2a}$, which is called the axis of symmetry.

From the two shapes of the curve;

1. when $a > 0$, the curve **increases** when $x > -\dfrac{b}{2a}$ and **decreases** when $x < -\dfrac{b}{2a}$
2. when $a < 0$, the curve **decreases** when $x > -\dfrac{b}{2a}$ and **increases** when $x < -\dfrac{b}{2a}$

Finding the intercepts on the x–axis together with the above features the curve representing any quadratic function can be sketched.

A sketch of the quadratic function $f(x) = 2x^2 - 7x - 4$ is shown below.



Exercise

Sketch the following quadratic functions

i) $f(x) = x^2 - 2x + 1$

ii) $f(x) = 2 + x - 3x^2$

iii) $f(x) = 18 + 9x - 2x^2$

iv) $f(x) = 5x^2 + 3x + 2$

v) $f(x) = 4 - 3x - x^2$

vi) $f(x) = 9 - 6x - x^2$

vii) $f(x) = x^2 - 2x - 3$

viii) $f(x) = 1 - x - 2x^2$

Solving quadratic equations

The general **quadratic equation** is of the form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$. The two values of x which make $ax^2 + bx + c = 0$ are called the **roots** of the equation. The roots of quadratic equations can be found using any of the three methods described below.

1. Method of factorization

This method is used to solve quadratic equations which do **factorize.**

Illustrative examples

Example 1

Find the roots of the equation $2x^2 - x - 1 = 0$

Solution

$2x^2 - x - 1 = 0$

The quadratic expression on the LHS can be factorised.

i.e. $2x^2 - 2x + x - 1 = 0$

$2x(x-1) + 1(x-1) = 0$

$(2x+1)(x-1) = 0$

either $2x + 1 = 0$ or $x - 1 = 0$

either $x = -\frac{1}{2}$ or 1.

Example 2

Find the roots of the equation $x^2 - 2x - 3 = 0$.

Solution

$x^2 - 2x - 3 = 0$

The quadratic expression on the LHS can be factorised

i.e. $(x+1)(x-3) = 0$

$x + 1 = 0$ or $x - 3 = 0 \Rightarrow x = -1$ or 3.

Always use this method first. If it fails, use one of the following methods.

2. Solution by completing the square

This is used when the method of factors fails, i.e. when there are no obvious factors or it is used for solving quadratic equations which **do not factorize.**

Illustrative examples

Example 1

Solve the equation $2x^2 - 3x - 1 = 0$

Solution

First make sure the coefficient of x^2 is 1 if it is not before completing the square.

i.e. $2x^2 - 3x - 1 = 0$

Dividing through by the coefficient of x^2

$$x^2 - \tfrac{3}{2}x = \tfrac{1}{2}$$

Next add the square of half the coefficient of x {i.e. the coefficient of x is $-\frac{3}{2}$, half of it is $-\frac{3}{4}$ and the square of this is $(-\frac{3}{4})^2$} to both sides of the equation to make the LHS a perfect square.

$$x^2 - \tfrac{3}{2}x + (-\tfrac{3}{4})^2 = \tfrac{1}{2} + (-\tfrac{3}{4})^2$$

Note that the LHS is a perfect square

$$(x - \tfrac{3}{4})^2 = \tfrac{1}{2} + \tfrac{9}{16}$$

$$(x - \tfrac{3}{4})^2 = \tfrac{17}{16}$$

$$x - \tfrac{3}{4} = \pm\sqrt{\tfrac{17}{16}} = \pm\tfrac{1}{4}\sqrt{17}$$

$$x = \tfrac{3}{4} \pm \tfrac{1}{4}\sqrt{17}$$

$$x = \frac{3}{4} \pm \frac{4.123}{4}$$

(Both signs must be taken)

$$x = \frac{3}{4} + \frac{4.123}{4} \text{ or } x = \frac{3}{4} - \frac{4.123}{4}$$

$$x = \tfrac{7.123}{4} = 1.78 \text{ or } \tfrac{-1.123}{4} = -0.28$$

Example 2

Using the method of completing the square, find the roots of the equation $x^2 - 6x + 7 = 0$.

Solution

The coefficient of x^2 is 1

Therefore $x^2 - 6x + 7 = 0$

$x^2 - 6x = -7$

Next add the square of half the coefficient of x to each side of the equation {i.e. the coefficient of x is -6, half of it is -3 and the square of this is $(-3)^2$}.

$$x^2 - 6x + (-3)^2 = -7 + (-3)^2$$

$$(x-3)^2 = -7 + 9$$

$$(x-3)^2 = 2$$

$$x - 3 = \pm\sqrt{2} = \pm 1.414$$

$$x = 3 \pm 1.414$$

$$x = 3 + 1.414 = 4.414 \text{ or}$$

$$x = 3 - 1.141 = 1.586$$

Example 3

Find in the surd form the roots of the equation $2x^2 - 3x - 3 = 0$

Solution

First make sure the coefficient of x^2 is 1 if it is not before completing the square.

i.e. $2x^2 - 3x - 3 = 0$

Dividing through by 2

$$x^2 - \tfrac{3}{2}x = \tfrac{3}{2}$$

Next add the square of half the coefficient of x {i.e. the coefficient of x is $-\frac{3}{2}$, half of it is $-\frac{3}{4}$ and the square of this is $(-\frac{3}{4})^2$} to both sides of the equation to make the LHS a perfect square.

$$x^2 - \tfrac{3}{2}x + (-\tfrac{3}{4})^2 = \tfrac{3}{2} + (-\tfrac{3}{4})^2$$

Note that the LHS is a perfect square

$$(x - \tfrac{3}{4})^2 = \tfrac{3}{2} + \tfrac{9}{16}$$

$$(x - \tfrac{3}{4})^2 = \tfrac{33}{16}$$

$$x - \tfrac{3}{4} = \pm\sqrt{\tfrac{33}{16}} = \pm\tfrac{1}{4}\sqrt{33}$$

$$x = \tfrac{3}{4} \pm \tfrac{1}{4}\sqrt{33}$$

$$x = \tfrac{1}{4}(3+\sqrt{33}) \text{ and } \tfrac{1}{4}(3-\sqrt{33})$$

Example 4

Solve the following equations by completing the square.

i) $2x^2 - 6x + 4 = 0$
ii) $x^2 + 4x - 8 = 0$
iii) $2x^2 + 7x + 3 = 0$
iv) $2x^2 - 5x + 1 = 0$
v) $x^2 + 8x = 1$
vi) $x^2 - 2x - 2 = 0$
vii) $x^2 + x - 1 = 0$
viii) $2x^2 + 2x = 1$
ix) $x^2 + 3x + 1 = 0$

Solution

Note: In each case the final answer has been provided. Work out the missing steps.

i) $2x^2 - 6x + 4 = 0 \Rightarrow x = 1$ or 2
ii) $x^2 + 4x - 8 = 0 \Rightarrow x = -5.46$ or 1.46
iii) $2x^2 + 7x + 3 = 0 \Rightarrow x = -3$ or $-\frac{1}{2}$
iv) $2x^2 - 5x + 1 = 0 \Rightarrow x = 0.22$ or 2.28
v) $x^2 + 8x = 1 \Rightarrow x = -4 \pm \sqrt{17}$
vi) $x^2 - 2x - 2 = 0 \Rightarrow x = 1 \pm \sqrt{3}$
vii) $x^2 + x - 1 = 0 \Rightarrow x = \frac{1}{2}(1 \pm \sqrt{5})$
viii) $2x^2 + 2x = 1 \Rightarrow x = -\frac{1}{2}(1 \pm \sqrt{3})$
ix) $x^2 + 3x + 1 = 0 \Rightarrow x = -\frac{1}{2}(3 \pm \sqrt{5})$

3. The Formula for solving a quadratic equation

If you are not asked to specifically solve a quadratic equation by completing the square, then using it is rather tedious. If the method of completing the square is applied to a general quadratic equation, a formula can be derived which can be used to solve any quadratic equation. The roots of the general quadratic equation $ax^2 + bx + c = 0$ is given by the **formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This formula gives two roots because each sign gives one value of x (± means + or –). This method is quicker than the method of completing the square.

This formula was obtained by completing the square of the general quadratic equation.

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \sqrt{\frac{(b^2 - 4ac)}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Illustrative examples

Example 1

Find the roots of the equations

i) $3x^2 - 4x - 1 = 0$
ii) $x^2 - 6x + 7 = 0$

Solution

i) Comparing $3x^2-4x-1=0$ with $ax^2+bx+c=0 \Rightarrow a=3,\ b=-4$ and $c=-1$.

$$\therefore\ x=\frac{-(-4)\pm\sqrt{(-4)^2-4(3)(-1)}}{2(3)}$$

$$=\frac{4\pm\sqrt{16+12}}{6}=\frac{4\pm\sqrt{28}}{6}$$

$$x=\frac{4\pm 5.292}{6}=\frac{9.292}{6}.$$

$$\text{or }\frac{-1.292}{6}=1.55\text{ or }0.22\text{ to 2 d.p}$$

ii) Comparing $x^2-6x+7=0$ with $ax^2+bx+c=0 \Rightarrow a=1,\ b=-6$ and $c=7$.

$$\therefore\ x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(7)}}{2(1)}$$

$$=\frac{6\pm\sqrt{36-28}}{2}$$

$$=\frac{6\pm\sqrt{8}}{2}=\frac{6\pm2\sqrt{2}}{2}=3\pm\sqrt{2}$$

$$x=3\pm1.414=4.414\text{ or }1.586$$

Example 2

Solve the following equations using the formula

i) $x^2+4x+2=0$ ii) $2x^2+x-2=0$
iii) $5x^2+9x+2=0$ iv) $2x^2-7x+4=0$

Solution

Note: In each case the final answer has been provided. Work out the missing steps.

i) $x^2+4x+2=0$
$x=-2\pm\sqrt{2}$
[Leaving your answer in the surd form]

ii) $2x^2+x-2=0$
$x=\frac{1}{4}(-1\pm\sqrt{17})$
[Leaving your answer in the surd form]

iii) $5x^2+9x+2=0$
$x=-0.260$ or -1.54 [correct to 2 d.p.]

iv) $2x^2-7x+4=0$
$x=0.719$ or 2.781 [correct to 3 d.p.]

Simultaneous linear and quadratic equations

The solution of two simultaneous equations, one *linear* and the other *quadratic* or *non-linear* can be solved by analytical and graphical methods.

Solving simultaneous equations using analytical method

The method used here is purely substitution.

Illustrative examples

Example 1

Solve the equations

$x+y=5$ --------(1)
$x^2+y^2=13$ -----(2)

Solution

Equation (1) is a linear equation but eqn(2) is a quadratic equation in x and y. To solve such an equation, always make one of the variables the subject of the linear equation and substitute into the quadratic equation. We shall then obtain a quadratic equation in the other variable.

i.e. from eqn(1),
$x=5-y$ ------(3)

Substitute this into eqn(2).

i.e. $(5-y)^2+y^2=13$
$25-10y+y^2=13$
$2y^2-10y+12=0$
$y^2-5y+6=0$
$(y-3)(y-2)=0$
$y=3$ or 2.

Substitute these values of y into eqn(3) to get the values of x.

i.e. for $y = 3$, $x = 2$ and for $y = 2$, $x = 3$

$\therefore$ The solution set are

$\{x, y : x = 2, y = 3 \text{ and } x = 3, y = 2\}$.

Note that this is the general method of solving such type of simultaneous equations.

Example 2

Solve the equations

$x + y = 5$ ------ -(1)

$xy = 6$ ---------(2)

Solution

From the linear equation (1), $y = 5 - x$ ---(3)

Substitute eqn(3) into eqn(2) i.e. the non-linear equation.

$$x(5 - x) = 6$$
$$5x - x^2 = 6$$
$$x^2 - 5x + 6 = 0$$
$$(x - 3)(x - 2) = 0 \Rightarrow x = 2 \text{ or } 3.$$

Substitute these values of x into eqn(3) to get the values of y

i.e. $x = 2 \Rightarrow y = 3$ and $x = 3 \Rightarrow y = 2$

Example 3

Find the truth set of the following pairs of equations:

i) $x^2 + y^2 = 5$ and $y - x = 1$
ii) $y^2 - x^2 = 8$ and $x + y = 2$
iii) $3x^2 - y^2 = 3$ and $2x - y = 1$
iv) $y = 4x^2$ and $y + 2x = 2$
v) $xy + y^2 = 3$ and $2x + y = 1$
vi) $x^2 - xy = 14$ and $x + y = 3$
vii) $xy = 2$ and $x + y = 3$
viii) $x^2 - y = 5$ and $2x - y = 2$
ix) $y^2 - 5x^2 = 20$ and $y - x = 4$
x) $y^2 + x = 10$ and $x - 2y = 2$
xi) $4x^2 + y = 0$ and $4x + y = 1$
xii) $3xy - x = 0$ and $x + 3y = 2$

Solution

Note: In each case the final answer has been provided. Work out the missing steps.

i) $\{x, y : x = -2, y = -1 \text{ and } x = 1, y = 2\}$
ii) $\{x, y : x = -1, y = 3\}$
iii) $\{x, y : x = 2, y = 3\}$
iv) $\{x, y : x = -1, y = 4 \text{ and } x = \frac{1}{2}, y = 1\}$
v) $\{x, y : x = 2, y = -3 \text{ and } x = -\frac{1}{2}, y = 2\}$
vi) $\{x, y : x = \frac{7}{2}, y = -\frac{1}{2} \text{ and } x = -2, y = 5\}$
vii) $\{x, y : x = 1, y = 2 \text{ and } x = 2, y = 1\}$
viii) $\{x, y : x = -1, y = -4 \text{ and } x = 3, y = 4\}$
ix) $\{x, y : x = 1, y = 5\}$
x) $\{x, y : x = 6, y = 2 \text{ and } x = -6, y = -4\}$
xi) $\{x, y : x = \frac{1}{2}, y = -1\}$
xii) $\{x, y : x = 1, y = \frac{1}{3} \text{ and } x = 0, y = \frac{2}{3}\}$

Roots of quadratic equations

Nature of the roots of quadratic equations

In the general quadratic formula for finding the roots of the quadratic equation

$$ax^2 + bx + c = 0 \quad (\text{i.e. } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}),$$

$b^2 - 4ac = \Delta$ is called the **discriminant** since it determines the nature of the roots.

1. If $b^2 - 4ac \geq 0$, then the equation will have two **real** roots.
2. If $b^2 - 4ac > 0$, then the equation will have two **real** and **different** (or distinct) roots.
3. If $b^2 - 4ac = 0$, then the equation will have two **real** and **equal** roots
4. If $b^2 - 4ac < 0$, then the equation will have **no real** roots (i.e. the equation has **complex roots**).

Illustrative examples

Example 1

Describe the nature of the roots of the following equations.

i) $2x^2 - x - 1 = 0$ ii) $x^2 + 5x - 1 = 0$
iii) $3x^2 - 2x + 4 = 0$ iv) $2x^2 - x + 3 = 0$
v) $2x^2 - 3x - 1 = 0$ vi) $4x^2 - 12x + 9 = 0$

Solution

i) Using $b^2 - 4ac$ with $a = 2,\ b = -1$ and $c = -1$

$$b^2 - 4ac = (-1)^2 - 4(2)(-1) = 9 > 0$$

$\therefore$ The equation has two real and different roots.

ii) Using $b^2 - 4ac$ with $a = 1,\ b = 5$ and $c = -1$

$$b^2 - 4ac = (5)^2 - 4(1)(-1) = 29 > 0$$

$\therefore$ The equation has two real and different roots.

iii) Using $b^2 - 4ac$ with $a = 3,\ b = -2$ and $c = 4$

$$b^2 - 4ac = (-2)^2 - 4(3)(4) = -44 < 0$$

$\therefore$ The equation has complex roots.

iv) Using $b^2 - 4ac$ with $a = 2,\ b = -1$ and $c = 3$

$$b^2 - 4ac = (-1)^2 - 4(2)(3) = -23 < 0$$

$\therefore$ The equation has complex roots.

v) Using $b^2 - 4ac$ with $a = -1,\ b = -3$ and $c = 2$

$$b^2 - 4ac = (-3)^2 - 4(-1)(2) = 17 > 0$$

$\therefore$ The equation has two real and different roots.

vi) Using $b^2 - 4ac$ with $a = 4,\ b = -12$ and $c = 9$

$$b^2 - 4ac = (-12)^2 - 4(4)(9) = 0$$

$\therefore$ The equation has two real and equal roots.

Example 2

Describe the nature of the roots of the following equations.

i) $x^2 - 6x + 9 = 0$
ii) $x^2 - 6x + 10 = 0$
iii) $2x^2 - 5x + 3 = 0$
iv) $3x^2 + 4x + 2 = 0$
v) $4x^2 - 12x + 9 = 0$
vi) $4x^2 - 12x - 9 = 0$

Solution

Note: In each case the final answer has been provided. Work out the missing steps.

i) $x^2 - 6x + 9 = 0$; equal
ii) $x^2 - 6x + 10 = 0$; not real
iii) $2x^2 - 5x + 3 = 0$; real and different
iv) $3x^2 + 4x + 2 = 0$; not real
v) $4x^2 - 12x + 9 = 0$; equal
vi) $4x^2 - 12x - 9 = 0$; real and different

Example 3

Find the value of k for which the quadratic equation $kx^2 + 12x + 9 = 0$ has equal roots.

Solution

If $kx^2 + 12x + 9 = 0$ has equal roots, then $b^2 - 4ac = 0$, where $a = k,\ b = 12$ and $c = 4$

i.e. $(12)^2 - 4(k)(9) = 0$

$$144 - 36k = 0$$
$$36k = 144$$
$$k = 4$$

Example 4

Find the value of k for which the quadratic equation $2x^2 - kx + 8 = 0$ has equal roots.

Solution

If $2x^2 - kx + 8 = 0$ has equal roots, then $b^2 - 4ac = 0$, where $a = 2,\ b = -k$ and $c = 8$

i.e. $(-k)^2 - 4(2)(8) = 0$

$$k^2 - 64 = 0$$
$$k^2 = 64$$
$$k = \pm 8$$

Example 5

Find the value of k for which the quadratic equation $x^2 - 2kx + (k+2) = 0$ has equal roots.

Solution

If $x^2 - 2kx + (k+2) = 0$ has equal roots, then $b^2 - 4ac = 0$, where

$a = 1,\ b = -2k$ and $c = (k+2)$

i.e. $(-2k)^2 - 4(1)(k+2) = 0$

$$4k^2 - 4(k+2) = 0$$

$$4k^2 - 4k - 8 = 0$$

$$(k-2)(k+1) = 0$$

$$k = -1 \text{ or } k = 2$$

Example 6

The roots of the quadratic equation $x^2 - 2(3k+1)x + 7(2k+3) = 0$, where k is a constant, are equal. Find the values of k.

Solution

If $x^2 - 2(3k+1)x + 7(2k+3) = 0$, has equal roots, then $b^2 - 4ac = 0$, where

$a = 1, b = -2(3k+1)$ and $c = 7(2k+3)$

i.e. $4(3k+1)^2 - 28(2k+3) = 0$

$$9k^2 - 8k - 20 = 0$$

$$(9k+10)(k-2) = 0$$

$$k = -\frac{10}{9} \text{ or } 2$$

Forming the quadratic equation when the roots are given

Given the roots of any quadratic equation, we can write the equation as :

x^2 –**(sum of roots)**x + **(product of roots)** = 0

Illustrative examples

Example 1

If the roots of a quadratic equation are 3 and –4, find the equation.

Solution

Sum of roots $= 3 + (-4) = -1$

Product of roots $= 3(-4) = -12$

Therefore the equation is

$x^2 - (-1)x + (-12) = 0 \Rightarrow x^2 + x - 12 = 0$

Example 2

Find the quadratic equation whose roots are 3 and $\frac{2}{3}$.

Solution

Sum of roots $= 3 + \frac{2}{3} = \frac{11}{3}$

Product of roots $= 3 \times \frac{2}{3} = 2$

Therefore the equation is $x^2 - (\frac{11}{3})x + (2) = 0$

Multiply through by 3.

$$3x^2 - 11x + 6 = 0$$

Relationships between the roots and coefficients of a quadratic equation

If α and β are the roots of the general quadratic equation $ax^2 + bx + c = 0$, then:

$$(x-\alpha)(x-\beta) = 0$$

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0 \quad \text{.........(1)}$$

Also $\quad ax^2 + bx + c = 0$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \text{.............(2)}$$

Comparing equations (1) and (2), we can write the following relationships connecting the coefficients of the equation and the roots:

Sum of roots: $\quad \alpha + \beta = -\frac{b}{a}$

Products of roots: $\quad \alpha\beta = \frac{c}{a}$

The equation may be written as :

x^2–**(sum of roots)**x + **(product of roots)** = 0.

For example, if $2x^2 - 3x + 6 = 0$ has roots α and β, then

The sum of its roots: $\alpha + \beta = -\left(\frac{-3}{2}\right) = \frac{3}{2}$

The product of its roots: $\quad \alpha\beta = \frac{6}{2} = 3$

Also if a quadratic equation has roots whose sum is 7 and whose product is 10, the equation can be written as:
$x^2 - 7x + 10 = 0$.

Symmetric functions

Knowing the values of $\alpha + \beta$ and $\alpha\beta$ for a given equation, we can calculate the values of other **symmetric functions** of α and β.
A symmetric function of α and β is one in which, if α and β are interchanged, the function is the same or multiplied by –1.
For example, $\alpha^2 + \beta^2$ is a symmetric function since when α and β are interchanged the function remains the same.
Also $\alpha^2 - \beta^2$ is a symmetric function since when α and β are interchanged $\alpha^2 - \beta^2$ becomes $-(\beta^2 - \alpha^2)$. But $2\alpha^2 + \beta^2$ **is not** a symmetric function.

Note the following useful relationships

1. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
2. $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$
3. $\alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$
4. $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$
5. $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
6. $\alpha^3 - \beta^3 = (\alpha - \beta)[(\alpha + \beta)^2 - \alpha\beta]$
7. $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\alpha + \beta}{\alpha\beta}$
8. $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} = \dfrac{\alpha^2 + \beta^2}{\alpha^2\beta^2} = \dfrac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha^2\beta^2}$
9. $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$

Note that the quadratic equation may be written as:
x^2– **(sum of roots)**x + **(product of roots)** = 0
This can be used to find a **new** quadratic equation whose roots are functions of α and β.

Illustrative examples

Example 1
If α and β are the roots of the equation $3x^2 - 4x - 1 = 0$, find the values of $\alpha + \beta$ and $\alpha\beta$.

Solution
Comparing the given equation $3x^2 - 4x - 1 = 0$ with the general equation $ax^2 + bx + c = 0$

$$a = 3,\ b = -4 \text{ and } c = -1$$

$$\therefore\ \alpha + \beta = -\frac{b}{a} = -\left(\frac{-4}{3}\right) = \tfrac{4}{3} \text{ and}$$

$$\alpha\beta = \frac{c}{a} = \frac{-1}{3} = -\tfrac{1}{3}$$

Example 2
If the quadratic equation $x^2 + 5x + c = 0$ has roots α and $\alpha - 1$. Find the value of c.

Solution
If the roots are α and $\alpha - 1$, then we can write the equation as
x^2 – (sum of roots)x + (product of roots) = 0.
Sum of roots
$= \alpha + \alpha - 1 = 2\alpha - 1 = -5$ ....(1)
Product of roots
$= \alpha(\alpha - 1) = \alpha^2 - \alpha = c$ ......(2)
From (1), $2\alpha - 1 = -5 \quad \Rightarrow \quad \alpha = -2$.
Substituting into (2), we have
$(-2)^2 - (-2) = c \qquad \Rightarrow \quad c = 4 + 2 = 6$

Example 3
If α and β are the roots of $2x^2 - x - 2 = 0$, find the values of the following:

i) $\alpha^2 + \beta^2$	ii) $\alpha^3 + \beta^3$
iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$	iv) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$
v) $(\alpha - \beta)^2$	vi) $\alpha - \beta$

Solution

Comparing the given equation $2x^2 - x - 2 = 0$ with the general equation $ax^2 + bx + c = 0$

$a = 2,\; b = -1$ and $c = -2$

$\therefore\; \alpha + \beta = -\frac{b}{a} = \frac{1}{2}$ and $\alpha\beta = \frac{c}{a} = -1$

i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (\frac{1}{2})^2 - 2(-1) = \frac{1}{4} + 2 = \frac{9}{4}$

ii) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= (\frac{1}{2})^3 - 3(-1)(\frac{1}{2}) = \frac{1}{8} + \frac{3}{2} = \frac{13}{8}$

iii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{\frac{1}{2}}{-1} = -\frac{1}{2}$

iv) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2}$

$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$

$= \frac{(\frac{1}{2})^2 - 2(-1)}{(-1)^2} = \frac{9}{4}$

v) $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$= (\frac{1}{2})^2 - 4(-1) = \frac{1}{4} + 4 = \frac{17}{4}$

vi) $\alpha - \beta = \sqrt{(\alpha - \beta)^2} = \sqrt{\frac{17}{4}} = \pm\frac{\sqrt{17}}{2}$

The trick is to express the given expression into *sums* and *products* of α and β before substituting their values. If you have fractions, first find the LCM.

Example 4

If α and β are the roots of $3x^2 + 5x - 1 = 0$, find the equation whose roots are :

i) $\frac{1}{\alpha}, \frac{1}{\beta}$ ii) α^2, β^2

iii) $\alpha + 1,\; \beta + 1$ iv) $\alpha + \frac{1}{\beta},\; \beta + \frac{1}{\alpha}$

Solution

The equation is given by the expression

$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

From the given equation $3x^2 + 5x - 1 = 0$;

$\alpha + \beta = -\frac{5}{3}$ and $\alpha\beta = -\frac{1}{3}$

i) Sum of new roots $= \frac{1}{\alpha} + \frac{1}{\beta}$

$= \frac{\alpha + \beta}{\alpha\beta} = \frac{-\frac{5}{3}}{-\frac{1}{3}} = 5$

Product of new roots $= \frac{1}{\alpha} \cdot \frac{1}{\beta}$

$= \frac{1}{\alpha\beta} = \frac{1}{-\frac{1}{3}} = -3$

Hence the new equation is $x^2 - 5x - 3 = 0$

ii) Sum of new roots

$= \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (-\frac{5}{3})^2 - 2(-\frac{1}{3})$

$= \frac{25}{9} + \frac{2}{3} = \frac{31}{9}$

Product of new roots $= \alpha^2\beta^2 = (\alpha\beta)^2$

$= (-\frac{1}{3})^2 = \frac{1}{9}$

Hence the new equation is

$x^2 - \frac{31}{9}x + \frac{1}{9} = 0$ or $9x^2 - 31x + 1 = 0$

iii) Sum of new roots $= (\alpha + 1) + (\beta + 1)$

$= (\alpha + \beta) + 2$

$= -\frac{5}{3} + 2 = -\frac{1}{3}$

Product of new roots

$= (\alpha + 1)(\beta + 1)$

$= \alpha\beta + (\alpha + \beta) + 1$

$= (-\frac{1}{3}) + (-\frac{5}{3}) + 1 = -1$

Hence the new equation is

$x^2 + \frac{1}{3}x - 1 = 0$ or $3x^2 + x - 3 = 0$

iv) Sum of new roots

$$= (\alpha + \frac{1}{\beta}) + (\beta + \frac{1}{\alpha})$$

$$= (\alpha + \beta) + (\frac{1}{\beta} + \frac{1}{\alpha})$$

$$= (\alpha + \beta) + (\frac{1}{\alpha} + \frac{1}{\beta})$$

$$= (\alpha + \beta) + \frac{\alpha + \beta}{\alpha\beta}$$

$$= (-\tfrac{5}{3}) + \frac{-\frac{5}{3}}{-\frac{1}{3}} = \tfrac{10}{3}$$

Product of new roots

$$= (\alpha + \frac{1}{\beta})(\beta + \frac{1}{\alpha})$$

$$= \alpha\beta + 1 + 1 + \frac{1}{\alpha\beta}$$

$$= \alpha\beta + 2 + \frac{1}{\alpha\beta}$$

$$= (-\tfrac{1}{3}) + 2 + (\frac{1}{-\frac{1}{3}}) = -\tfrac{4}{3}$$

Hence the new equation is $x^2 - \frac{10}{3}x - \frac{4}{3} = 0$
or $3x^2 - 10x - 4 = 0$

Example 5

The roots of the quadratic equation $3x^2 - x - 1 = 0$ are α and β

Find the value of $\alpha^2 + \beta^2 + \frac{2}{9}$.

Solution

From the given equation $3x^2 - x - 1 = 0$;
$\alpha + \beta = \frac{1}{3}$ and $\alpha\beta = -\frac{1}{3}$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= (\tfrac{1}{3})^2 - 2(-\tfrac{1}{3}) = \tfrac{1}{9} + \tfrac{2}{3} = \tfrac{7}{9}$$

$$\therefore \quad \alpha^2 + \beta^2 + \tfrac{2}{9} = \tfrac{7}{9} + \tfrac{2}{9} = 1$$

Example 6

If α and β are the roots of $3x^2 - 6x + 8 = 0$, find the equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

Solution

The equation is given by the expression
$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

From the given equation $3x^2 - 6x + 8 = 0$;
$\alpha + \beta = 2$ and $\alpha\beta = \frac{8}{3}$

Sum of new roots $= \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{2}{\frac{8}{3}} = \frac{3}{4}$

Product of new roots $= \frac{1}{\alpha} . \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{\frac{8}{3}} = \frac{3}{8}$

Hence the new equation is $x^2 - \frac{3}{4}x + \frac{3}{8} = 0$
or $8x^2 - 6x + 3 = 0$

Example 7

The roots of the quadratic equation $2x^2 - 7x + 10 = 0$ are α and β

Find the value of $\frac{1}{\alpha} + \frac{1}{\beta}$ and $\frac{1}{\alpha\beta}$.

Hence write down the equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

Solution

From the given equation $2x^2 - 7x + 10 = 0$;
we see that $\alpha + \beta = \frac{7}{2}$ and $\alpha\beta = 5$

$$\therefore \quad \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{\frac{7}{2}}{5} = \frac{7}{10} \text{ and}$$

$$\frac{1}{\alpha} . \frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{5}$$

Therefore the required equation has roots whose sum, $\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$ is $\frac{7}{10}$ and whose product, $\frac{1}{\alpha\beta}$ is $\frac{1}{5}$.

So the equation is $x^2 - \frac{7}{10}x + \frac{1}{5} = 0$
or $10x^2 - 7x + 2 = 0$

Example 8

Find the value of k if the roots of the quadratic equation $x^2 - 2x - k = 0$ differ by one.

Solution

Let one root of the equation be α,

then the other is $\alpha + 1$.

Sum of new roots $= \alpha + (\alpha + 1) = -(-2)$

$$2\alpha + 1 = 2 \quad \Rightarrow \quad 2\alpha = 1$$

$$\alpha = \tfrac{1}{2}$$

Product of new roots $= \alpha(\alpha + 1) = -k$

$$\tfrac{1}{2}(\tfrac{1}{2} + 1) = -k \quad \Rightarrow \quad k = -\tfrac{3}{4}$$

Example 9

Find the value of a if one root of the quadratic equation $x^2 - ax + 8 = 0$ is the square of the other.

Solution

Let one root of the equation be α,

then the other is α^2

Sum of new roots $= \alpha + \alpha^2 = -(-a)$

$$\alpha + \alpha^2 = a \text{..........(1)}$$

Product of new roots $= \alpha \times \alpha^2 = 8$

$$\alpha^3 = 8 \quad \Rightarrow \quad \alpha = 2$$

Therefore from (1),

$$2 + 2^2 = a \quad \Rightarrow \quad a = 6$$

Example 10

If α and β are the roots of $2x^2 - 4x + 5 = 0$, find the values of the following:

i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ ii) $(\alpha + 1)(\beta + 1)$

iii) $\alpha^2 + \beta^2$ iv) $\alpha^2\beta + \alpha\beta^2$

v) $(\alpha - \beta)^2$ vi) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

vii) $\dfrac{1}{\alpha + 1} + \dfrac{1}{\beta + 1}$ viii) $\dfrac{1}{2\alpha + \beta} + \dfrac{1}{\alpha + 2\beta}$

ix) $\dfrac{1}{\alpha^2 + 1} + \dfrac{1}{\beta^2 + 1}$

Solution

Note: In each case the final answer has been provided. Work out the missing steps.

i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{4}{5}$ ii) $(\alpha + 1)(\beta + 1) = 5\tfrac{1}{2}$

iii) $\alpha^2 + \beta^2 = -1$ iv) $\alpha^2\beta + \alpha\beta^2 = 5$

v) $(\alpha - \beta)^2 = -6$ vi) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = -\dfrac{2}{5}$

vii) $\dfrac{1}{\alpha + 1} + \dfrac{1}{\beta + 1} = \dfrac{8}{11}$

viii) $\dfrac{1}{2\alpha + \beta} + \dfrac{1}{\alpha + 2\beta} = \dfrac{4}{7}$

ix) $\dfrac{1}{\alpha^2 + 1} + \dfrac{1}{\beta^2 + 1} = \dfrac{4}{25}$

Example 11

If α and β are the roots of $x^2 - 2x + 3 = 0$, find the equation whose roots are:

i) $\alpha + 2,\ \beta + 2$

ii) $\dfrac{1}{\alpha},\ \dfrac{1}{\beta}$

iii) $\alpha^2,\ \beta^2$

iv) $\dfrac{\alpha}{\beta},\ \dfrac{\beta}{\alpha}$

v) $\alpha - \beta,\ \beta - \alpha$

Solution

Note: In each case the final answer has been provided. Work out the missing steps.

i) $x^2 - 6x + 11 = 0$

ii) $3x^2 - 2x + 1 = 0$

iii) $x^2 + 2x + 9 = 0$

iv) $3x^2 + 2x + 3 = 0$

v) $x^2 + 8 = 0$

Solved examination questions

The following examples will boost your understanding on what you have learnt so far.

Example 1

i) Find the constants a, b and c such that $2x^2 - 6x + 5 = a(x + b)^2 + c$.
Deduce the least value of $2x^2 - 6x + 5$.

ii) Given the equation $4(x - 2)^2 = k(x - 1)$, show that sum of the roots is equal to their product. Find the values of the constant k for which the difference between the roots is $3\frac{3}{4}$.

Solution

i) ***First method:*** Using the method of completing the square, we have

$$2x^2 - 6x + 5 = 2[x^2 - 3x + \tfrac{5}{2}]$$
$$= 2[x^2 - 3x + (-\tfrac{3}{2})^2 - (-\tfrac{3}{2})^2 + \tfrac{5}{2}]$$
$$= 2[(x - \tfrac{3}{2})^2 - \tfrac{9}{4} + \tfrac{5}{2}]$$
$$= 2[(x - \tfrac{3}{2})^2 + \tfrac{1}{4}] = 2(x - \tfrac{3}{2})^2 + \tfrac{1}{2}$$
$$\therefore\ 2x^2 - 6x + 5 = 2(x - \tfrac{3}{2})^2 + \tfrac{1}{2}$$
$$= a(x + b)^2 + c$$

Comparing terms

$$a = 2;\ b = -\tfrac{3}{2} \text{ and } c = \tfrac{1}{2}$$

The least value of $2x^2 - 6x + 5$ is $\frac{1}{2}$

Second method: Expanding the right hand side and comparing terms

$$2x^2 - 6x + 5 = a(x + b)^2 + c$$
$$= a(x^2 + b^2 + 2bx) + c$$
$$= ax^2 + 2abx + (c + ab^2)$$

Comparing terms:

$a = 2$;

$2ab = -6 \Rightarrow b = -\frac{3}{2}$ and

$(c + ab^2) = 5 \quad \Rightarrow c = \frac{1}{2}$

The least value of $2x^2 - 6x + 5$ is $\frac{1}{2}$

ii) $$4(x - 2)^2 = k(x - 1)$$
$$4x^2 - 16x + 16 = kx - k$$
$$4x^2 - (16 + k)x + (16 + k) = 0$$

$\therefore$ Comparing this equation with the general equation $ax^2 + bx + c = 0$, we have

$a = 4,\ b = -(16 + k)$ and $c = 16 + k$

Sum of roots $= -\dfrac{b}{a} = \dfrac{16 + k}{4}$ ---------(1)

Product of roots $= \dfrac{c}{a} = \dfrac{16 + k}{4}$ --------(2)

$\therefore$ From (1) and 2),

sum of roots = product of roots

Let α and β be the roots of the equation, then from the question

$\alpha - \beta = 3\frac{3}{4} = \frac{15}{4}$

Squaring both sides

$$(\alpha - \beta)^2 = \tfrac{225}{16}$$
$$\alpha^2 + \beta^2 - 2\alpha\beta = \tfrac{225}{16}$$
$$(\alpha + \beta)^2 - 2\alpha\beta - 2\alpha\beta = \tfrac{225}{16}$$
$$(\alpha + \beta)^2 - 4\alpha\beta = \tfrac{225}{16}$$

Substituting $\alpha + \beta = \dfrac{16 + k}{4}$ and

$$\alpha\beta = \frac{16 + k}{4}$$

$$\left(\frac{16 + k}{4}\right)^2 - 4\left(\frac{16 + k}{4}\right) = \frac{225}{16}$$
$$(16 + k)^2 - 16(16 + k) = 225$$
$$16^2 + 32k + k^2 - 16^2 - 16k = 225$$
$$k^2 + 16k - 225 = 0$$
$$(k + 25)(k - 9) = 0$$
$$k = -25 \text{ or } 9$$

Example 2

If $p(x) = 3x^2 + 4x + 5$, write $p(x)$ in the form $a(x + b)^2 + c$ where a, b and c are constants. Hence, find the least value of $p(x)$ and the value of x for which this occurs.

Solution

By completing the square,

$$p(x) = 3x^2 + 4x + 5 = 3[x^2 + \tfrac{4}{3}x + \tfrac{5}{3}]$$

$$p(x) = 3[x^2 + \tfrac{4}{3}x + (\tfrac{2}{3})^2 - (\tfrac{2}{3})^2 + \tfrac{5}{3}]$$

$$= 3[(x + \tfrac{2}{3})^2 - \tfrac{4}{9} + \tfrac{5}{3}]$$

$$p(x) = 3[x + \tfrac{2}{3})^2 + \tfrac{11}{9}] = 3(x + \tfrac{2}{3})^2 + \tfrac{11}{3}$$

$$= a(x+b)^2 + c$$

where $a = 3,\ b = \frac{2}{3}$ and $c = \frac{11}{3}$

Hence the least value of p is $\frac{11}{3}$ and it occurs at $x + \frac{2}{3} = 0$ i.e. $x = -\frac{2}{3}$

Alternative method:

Expanding the right hand side and comparing terms

$$3x^2 + 4x + 5 = a(x+b)^2 + c$$

$$= a(x^2 + b^2 + 2bx) + c$$

$$= ax^2 + 2abx + (c + ab^2)$$

Comparing terms:

$a = 3;\ 2ab = 4 \Rightarrow b = \frac{2}{3}$ and

$(c + ab^2) = 5 \Rightarrow c = \frac{11}{3}$

Hence the least value of p is $\frac{11}{3}$ and it occurs at $x + \frac{2}{3} = 0$ i.e. $x = -\frac{2}{3}$

Example 3

If α and β are the roots of the equation $2x^2 - 3x - 7 = 0$. Find the equation whose roots are $\dfrac{1}{\alpha} + 1$ and $\dfrac{1}{\beta} + 1$.

Solution

From the given equation $2x^2 - 3x - 7 = 0$;

$\alpha + \beta = \frac{3}{2}$ and $\alpha\beta = -\frac{7}{2}$

$$\text{Sum of new roots} = (\frac{1}{\alpha} + 1) + (\frac{1}{\beta} + 1)$$

$$= \frac{1}{\alpha} + \frac{1}{\beta} + 2 = \frac{\alpha + \beta}{\alpha\beta} + 2$$

$$= \frac{3/2}{-7/2} + 2 = \frac{11}{7}$$

$$\text{Product of roots} = (\frac{1}{\alpha} + 1)(\frac{1}{\beta} + 1)$$

$$= \frac{1}{\alpha\beta} + \frac{1}{\alpha} + \frac{1}{\beta} + 1$$

$$= \frac{1}{\alpha\beta} + \frac{\alpha + \beta}{\alpha\beta} + 1$$

$$= \frac{1}{-7/2} + \frac{3/2}{-7/2} + 1 = \frac{2}{7}$$

Hence the new equation is $x^2 - \frac{11}{7}x + \frac{2}{7} = 0$ or $7x^2 - 11x + 2 = 0$

Example 4

Find the truth set of the equation $5x^2 - 7x + 2 = 0$ and hence write down the values of y which satisfy the equation $5(y+1)^2 - 7(y+1) + 2 = 0$.

Solution

$$5x^2 - 7x + 2 = 0$$

$$(x-1)(5x-2) = 0$$

$$(x : x = \tfrac{2}{5} \text{ or } 1)$$

By comparing the two equations:

$5x^2 - 7x + 2 = 0$ ………………(1) and

$5(y+1)^2 - 7(y+1) + 2 = 0$……(2), we have

$x = y + 1 \Rightarrow y = x - 1$

When $x = \frac{2}{5} \Rightarrow y = \frac{2}{5} - 1 = -\frac{3}{5}$ and

When $x = 1 \Rightarrow y = 0$

$\therefore$ The solution set of the equation is $(y : y = -\frac{3}{5}$ or $0)$

Example 5

Solve the simultaneous equation

$2y - x = 4$ ………………(1)

$x^2 + 3xy - 2y^2 = 4$ ……. (2)

Solution

From eqn(1), $x = 2y - 4$ -------(3)

Substituting (3) into eqn(2)

$$(2y-4)^2 + 3(2y-4)y - 2y^2 = 4$$

$$4y^2 - 16y + 16 + 6y^2 - 12y - 2y^2 = 4$$

$$8y^2 - 28y + 12 = 0$$
$$2y^2 - 7y + 3 = 0$$
$$(2y-1)(y-3) = 0$$
$$y = 3 \text{ or } \tfrac{1}{2}$$

Substituting these values into eqn(3)

$$y = 3 \Rightarrow x = 6 - 4 = 2 \text{ and}$$
$$y = \tfrac{1}{2} \Rightarrow x = 1 - 4 = -3$$
$$\therefore \quad x = 2,\ y = 3 \text{ or } x = -3,\ y = \tfrac{1}{2}.$$

Example 6

Express $y = 5 - 2x - 4x^2$ in the form $y = A - B(x + C)^2$, where A, B and C constants. Hence state the maximum value of the function $f : x \to 5 - 2x - 4x^2$.

Solution

By completing the square,

$$y = 5 - 2x - 4x^2$$
$$= -4x^2 - 2x + 5$$
$$= -4[x^2 + \tfrac{1}{2}x - \tfrac{5}{4}]$$
$$= -4[x^2 + \tfrac{1}{2}x + (\tfrac{1}{4})^2 - (\tfrac{1}{4})^2 - \tfrac{5}{4}]$$
$$= -4[(x + \tfrac{1}{4})^2 - \tfrac{1}{16} - \tfrac{5}{4}] = -4[(x + \tfrac{1}{4})^2 - \tfrac{21}{16}]$$
$$y = \tfrac{21}{4} - 4(x + \tfrac{1}{4})^2 = A - B(x + C)^2,$$

where $A = \frac{21}{4}$, $B = 4$ and $C = \frac{1}{4}$

Hence the maximum value of $f : x \to 5 - 2x - 4x^2$ is $\frac{21}{4}$

Alternative method

Expanding the right hand side and comparing terms

$$5 - 2x - 4x^2 = A - B(x + C)^2$$
$$= A - B(x^2 + C^2 + 2Cx)$$
$$= (A - BC^2) - 2CBx - Bx^2$$

Comparing terms:

$$B = 4,$$
$$2CB = 2 \Rightarrow C = \tfrac{1}{4} \text{ and}$$
$$(A - BC^2) = 5 \Rightarrow A = \tfrac{21}{4}$$

Hence the maximum value of $f : x \to 5 - 2x - 4x^2$ is $\frac{21}{4}$.

Example 7

If α and β are the roots of the equation $2x^2 - 10x + 5 = 0$, find the value of

$$\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^2$$

Solution

From the given equation $2x^2 - 10x + 5 = 0$;

$$\alpha + \beta = 5 \text{ and } \alpha\beta = \tfrac{5}{2}$$
$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{5}{\frac{5}{2}} = 2$$
$$\therefore \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^2 = 2^2 = 4$$

Example 8

Find the equations of the curve with k and -2 as zeros.

Solution

a) Given the roots of any quadratic equation, we can write the equation as : $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$.
$\therefore\ x^2 - (k - 2)x - 2k = 0$ is the equation with zeros or roots k and -2.

Example 9

If α and β are the roots of the equation $3x^2 - x - 1 = 0$, find the equation whose roots are $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$.

Solution

Given the equation $3x^2 - x - 1 = 0$;

$$(\alpha + \beta) = -\frac{b}{a} = \frac{1}{3}$$
$$\alpha\beta = \frac{c}{a} = -\frac{1}{3}$$

Given new roots as $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$,

$$\text{Sum of roots} = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha+\beta}{\alpha\beta}$$

$$= \frac{\frac{1}{3}}{-\frac{1}{3}} = -1$$

$$\text{Product of roots} = \frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1$$

The equation is given as

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

$$x^2 - (-1)x + 1 = 0$$

$$x^2 + x + 1 = 0$$

Example 10

If $2x^2 - x - 3 = A(x+B)^2 + C$, find the values of A, B and C. Hence, find the minimum value of $2x^2 - x - 3$.

Solution

$$2x^2 - x - 3 = 2\left(x^2 - \frac{x}{2} - \frac{3}{2}\right)$$

$$= 2\left(x - \frac{1}{4}\right)^2 - \frac{50}{16}$$

$$= 2\left(x - \frac{1}{4}\right)^2 - \frac{25}{8}$$

$$A = 2, B = -\tfrac{1}{4}, C = -\tfrac{25}{8}$$

Hence minimum value of $2x^2 - x - 3$ is $-\frac{25}{8}$.

Use an alternative method to arrive at the answer.

Example 11

If α and β are the roots of the equation $3x^2 - 5x + 1 = 0$, find the value of $\alpha^2 - \alpha\beta + \beta^2$.

Solution

$\alpha^2 - \alpha\beta + \beta^2 = \frac{16}{9}$ [Work it out]

Polynomial functions

A **polynomial** function in x, is an expression of the form :

$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_o$, where n is a positive integer, x is a variable, and $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_o$ are constant coefficients some of which may be zero.

If $a_n \neq 0$, then n is called the **degree** of the polynomial i.e. the highest power of x.

The **constant** term is a_o.

For example, $f(x) = 3x^3 + 4x^2 - 2x + 8$ is a polynomial function of degree 3, with a constant term 8. We shall limit ourselves to polynomials of degree not exceeding 4.

Types of polynomial functions

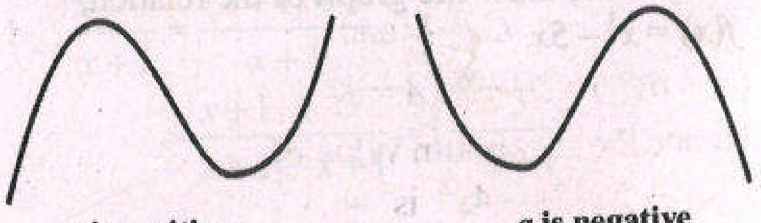
Polynomials of low degree have special names. The three types of polynomial functions which will be considered at this level are:

1. Linear functions ($y = f(x) = 3x + 4$)
   [Degree = 1]
2. Quadratic functions
   ($y = f(x) = 3x^2 + 2x - 1$)
   [Degree = 2]
3. Cubic functions
   ($y = f(x) = 2x^3 + 3x^2 + 5x + 2$)
   [Degree = 3]

We shall meet these functions in later work.

Drawing graphs of cubic functions in a given range

If a polynomial function is cubic, then it is of the form $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are constants some of which may be zero and $a \neq 0$. We have two shapes of the curve depending on the value of a as shown below.



***a* is positive** ***a* is negative**

Method 1

To sketch the graph of $f(x)$, first find the zeros of $f(x)$. This gives the points of intersection of the curve on the x–axis. Next, find the intercepts on the y–axis by putting $x = 0$ i.e. find $f(0)$. The above information together with the sign of the value of a is enough to sketch the curve.

Method 2

To draw the graph of a cubic function such as $f(x) = x^3 - 5x$ in the range $-2 \le x \le 2$ for example, first calculate the value of $f(x)$ for each integral value of x in the given range as shown below.

Given $y = x^3 - 5x$

i) for $x = -2$
$y = (-2)^3 - 5(-2)$
$= -8 + 10 = 2$

ii) when $x = -1$
$y = (-1)^3 - 5(-1)$
$= -1 + 5 = 4$

iii) When $x = 0$
$y = (0)^3 - 5(0)$
$= 0$

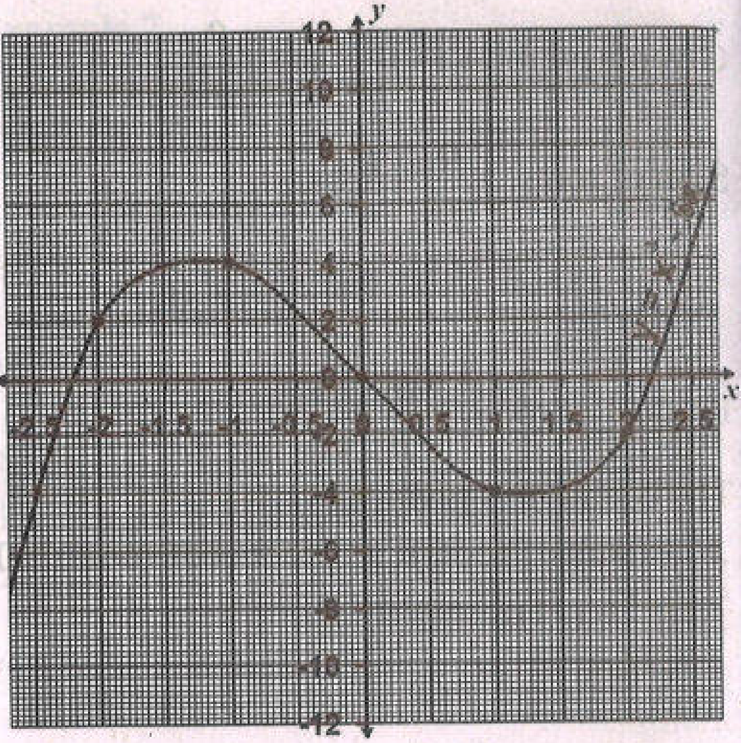
iv) When $x = 1$
$y = (1)^3 - 5(1)$
$= 1 - 5 = -4$

v) When $x = 2$
$y = (2)^3 - 5(2)$
$= 8 - 10 = -2$

The pair of values of x and y are shown in the table below.

x	-2	-1	0	1	2
y	2	4	0	-4	-2

Using an appropriate scale, such as 2 cm to 1 unit on the x-axis and 2 cm to 2 units on the y-axis, draw the graph of the relation, $f(x) = x^3 - 5x$.



Algebra of polynomials

Addition of polynomials

Add corresponding terms (powers of x).

Illustrative example

Find $f(x) + g(x)$ if

i) $f(x) = 3x^3 + 4x^2 - 2x + 8$ and
$g(x) = 2x^3 + 3x^2 + 5x + 2$,

ii) $f(x) = 3x^3 + 4$ and $g(x) = 2x^2 + 3x + 5$

Solution

i)
$$\begin{array}{rl} f(x) = & 3x^3 + 4x^2 - 2x + 8 \\ g(x) = & 2x^3 + 3x^2 + 5x + 2 \\ \hline f(x) + g(x) = & 5x^3 + 7x^2 + 3x + 10 \\ \hline \end{array}$$

ii)
$$\begin{array}{rl} f(x) = & 3x^3 \qquad\qquad\quad + 4 \\ g(x) = & \qquad 2x^2 + 3x + 5 \\ \hline f(x) + g(x) = & 3x^3 + 2x^2 + 3x + 9 \\ \hline \end{array}$$

Subtraction of polynomials

Subtract corresponding terms.

Illustrative example

i) If $h(x) = 5x^3 + 2x^2 + x + 5$ and
$p(x) = 2x^3 + 4x + 8$, find $h(x) - p(x)$

ii) If $f(x) = 7x^3 + x - 4$ and $g(x) = 4x^3 - 3x^2 + 4x + 3$, find $f(x) - g(x)$

Solution

i)
$$\begin{array}{rl} h(x) = & 5x^3 + 2x^2 + x + 5 \\ p(x) = & 2x^3 \qquad\quad + 4x + 8 \\ \hline h(x) - p(x) = & 3x^3 + 2x^2 - 3x - 3 \\ \hline \end{array}$$

ii)
$$\begin{array}{rl} f(x) = & 7x^3 \qquad\quad + x - 4 \\ g(x) = & 4x^3 - 3x^2 + 4x + 3 \\ \hline f(x) - g(x) = & 3x^3 + 3x^2 - 3x - 7 \\ \hline \end{array}$$

Multiplication of polynomials

This can be done by using the distributive property. i.e. each term in one bracket multiplies each term in another bracket and the results are added.

Illustrative example

i) If $f(x) = 2x^2 + 4x + 2$ and $g(x) = 3x^2 + 5x + 4$, find $f(x) \cdot g(x)$

ii) If $p(x) = 2x + 4$ and $q(x) = x + 3$, find $p(x) \cdot q(x)$

Solution

i) $f(x).g(x)$
$$\begin{aligned} &= (2x^2 + 4x + 2)(3x^2 + 5x + 4) \\ &= 2x^2(3x^2 + 5x + 4) + 4x(3x^2 + 5x + 4) + \\ &\quad 2(3x^2 + 5x + 4) \\ &= 6x^4 + 22x^3 + 34x^2 + 26x + 8 \end{aligned}$$

ii) $p(x) \cdot q(x) = (2x + 4)(x + 3)$
$$\begin{aligned} &= 2x^2 + 6x + 4x + 12 \\ &= 2x^2 + 10x + 12 \end{aligned}$$

Division of polynomials

Division of polynomials is done using the 'long division' method as shown in the following examples. If $x^3 + 4x^2 - 7$ is divided by $x^2 - 3$, the number over the division line is called the *quotient*, and what is left is called the *remainder*. $x^3 + 4x^2 - 7$ is called the *dividend* and by $x^2 - 3$ is called the *divisor*. It is helpful to indicate the missing terms in the dividend with zero coefficient as in the example below.

$$\begin{array}{r} x + 4 \\ x^2 - 3\overline{\smash{)}\,x^3 + 4x^2 + 0x - 7} \\ \underline{x^3 + 0x^2 - 3x} \\ 4x^2 + 3x - 7 \\ \underline{4x^2 + 0x - 12} \\ + 3x + 5 \end{array}$$

The relationship between the divisor, the dividend, the quotient and the remainder can be written as:

Dividend = (quotient)×(divisor) + remainder

$\therefore \quad x^3 + 4x^2 - 7 = (x + 4)(x^2 - 3) + 3x + 5$

Note that **improper fractions** can also be expressed in a form which makes them proper fractions by dividing the numerator by the denominator using the long division method.

i.e. $\dfrac{x^3 + 4x^2 - 7}{x^2 - 3} = (x + 4) + \dfrac{3x + 5}{x^2 - 3}$

$$\text{Improper Fraction} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}$$

An improper fraction is any fraction in which the highest power of the numerator is greater than or *equal* to the highest power of the denominator.

The long division method is not always necessary if the highest power of the numerator is *equal* to the highest power of the denominator. It is often simpler to rearrange the numerator by adding and subtracting appropriate terms.

For example:

1. $\dfrac{x + 4}{x + 1} = \dfrac{(x + 1) + 3}{x + 1}$

$\quad = \dfrac{x + 1}{x + 1} + \dfrac{3}{x + 1} = 1 + \dfrac{3}{x + 1}$

2. $$\frac{x^2+3}{x^2-1}=\frac{(x^2-1)+4}{x^2-1}$$
$$=\frac{x^2-1}{x^2-1}+\frac{4}{x^2-1}=1+\frac{4}{x^2-1}$$

3. $$\frac{x^3+5x^2-3x}{x^3+1}=\frac{(x^3+1)+5x^2-3x-1}{x^3+1}$$
$$=\frac{x^3+1}{x^3+1}+\frac{5x^2-3x-1}{x^3+1}$$
$$=1+\frac{5x^2-3x-1}{x^3+1}$$

Illustrative examples

Example 1

Divide

i) x^2+2x-8 by $x+4$
ii) $x^3+6x-20$ by $x-2$
iii) $2x^3-3x^2-5x+6$ by $x-1$
iv) x^2+2x-8 by $x+3$

Solution

i)
$$\begin{array}{r} x-2 \\ x+4\,\overline{)\,x^2+2x-8} \\ \underline{x^2+4x} \\ -2x-8 \\ \underline{2x-8} \\ 0 \end{array}$$

$$\therefore\ \frac{x^2+2x-8}{x+4}=x-2$$

ii)
$$\begin{array}{r} x^2+2x+10 \\ x-2\,\overline{)\,x^3+0x^2+6x-20} \\ \underline{x^3-2x^2} \\ 2x^2+6x-20 \\ \underline{2x^2-4x} \\ 10x-20 \\ \underline{10x-20} \\ 0 \end{array}$$

$$\therefore\ \frac{x^3+6x-20}{x-2}=x^2+2x+10$$

iii)
$$\begin{array}{r} 2x^2-x-6 \\ x-1\,\overline{)\,2x^3-3x^2-5x+6} \\ \underline{2x^3-2x^2} \\ -x^2-5x+6 \\ \underline{-x^2+x} \\ -6x+6 \\ \underline{-6x+6} \\ 0 \end{array}$$

$$\therefore\ \frac{2x^3-3x^2-5x+6}{x-1}=2x^2-x-6$$

iv)
$$\begin{array}{r} x-1 \\ x+3\,\overline{)\,x^2+2x-8} \\ \underline{x^2+3x} \\ -x-8 \\ \underline{-x-3} \\ -5 \end{array}$$

$$\therefore\ \frac{x^2+2x-8}{x+3}=(x-1)+\frac{-5}{x^2+2x-8}$$

Example 2

Find the quotient and the remainder in each of the following divisions.

i) $(x^3+x^2-3x+6)\div(x-1)$
ii) $(x^4-5x^2+2)\div(x+1)$
iii) $(2x^3-4x^2+3x-1)\div(x^2-1)$
iv) $(4x^3-9x+1)\div(2x-1)$

Solution

Note: In each case the final answer has been provided. Use the long division method to obtain the answers.

i) Quotient $=x^2+2x-1$, Remainder $=5$
ii) Quotient $=x^3-x^2-4x+4$, Remainder $=-2$
iii) Quotient $=2x-4$, Remainder $=5x-5$
iv) Quotient $=2x^2+x-4$, Remainder $=-3$

Exercise

i) Divide $(2x^3 - 3x^2 - x + 2)$ by $(x - 2)$

ii) Divide $(6x^3 + 5x + 3)$ by $(3x - 2)$

iii) Divide $(4x^3 - x^2 + 5x + 4)$ by $(4x + 3)$

iv) Divide $(x^3 + 2x - 1)$ by $(2x^2 + 4)$

v) Divide $(x^3 + 27)$ by $(x + 3)$

vi) Divide $(3x^3 - 4x^2 - 2x - 5)$ by $(x - 3)$

vii) Divide $(3x^3 + 5x^2 - 3x + 2)$ by $(x + 2)$

viii) Divide $(x^3 + 4x - 13)$ by $(x - 3)$

Functional notation

Given that $f(x) = 3x^3 - 2x^2 + 5x - 1$ we can evaluate $f(1)$, $f(2)$ etc.

Thus $f(1) = 3(1)^3 - 2(1)^2 + 5(1) - 1$
$= 3 - 2 + 5 - 1 = 5$ and
$f(2) = 3(8) - 2(4) + 5(2) - 1 = 25$

Illustrative example

Given $f(x) = x^2 + 2x - 8$.
Evaluate $f(1)$, $f(2)$ and $f(-1)$

Solution

$f(1) = (1)^2 + 2(1) - 8$
$= 1 + 2 - 8 = -5$

$f(2) = (2)^2 + 2(2) - 8$
$= 4 + 4 - 8 = 0$

$f(-1) = (-1)^2 + 2(-1) - 8$
$= 1 - 2 - 8 = -9$

The remainder theorem

When $f(x) = x^3 - 7x^2 + 6x - 2$ is divided by $(x - 2)$, we get a quotient and a remainder. The relationship between these quantities can be written as:

$f(x) = x^3 - 7x^2 + 6x - 2$
$= (\text{quotient})(x - 2) + \text{remainder}$

Now if we substitute 2 for x, the term containing the quotient is eliminated, giving
$f(2) = \text{remainder}$

Therefore, in general, if a polynomial $f(x)$ is divided by $(x - a)$ then
$f(x) = (\text{quotient})(x - a) + \text{remainder}$
$f(a) = \text{remainder}$

This result is called the **remainder theorem** and can be summarised as follows:
When a polynomial $f(x)$ is divided by $(x - a)$, where a is a constant, the **remainder, R,** is $f(a)$. Similarly, when $f(x)$ is divided by $(x + a)$, the remainder is $f(-a)$.

Illustrative examples

Example 1

If $f(x) = x^3 - 3x^2 + x - 5$, what is the remainder when $f(x)$ is divided by $x - 1$?

Solution

Here the divisor is $(x - 1)$. Hence from the remainder theorem, $f(1)$ is the remainder.

i.e. $f(1) = (1)^3 - 3(1)^2 + (1) - 5 = -6$.
Hence -6 is the remainder.

Example 2

Find the remainder when $x^2 + 2x - 8$ is divided by $x + 3$.

Solution

$x + 3 = 0 \Rightarrow x = -3$
$\therefore f(-3) = (-3)^2 + 2(-3) - 8$
$= 9 - 6 - 8 = -5$
Hence the remainder is -5.

Example 3

Let $f(x) = x^2 + 2x - 8$. Find the remainder when $f(x)$ is divided by $x - 2$.

Solution

$x - 2 = 0 \Rightarrow x = 2$
$\therefore f(2) = (2)^2 + 2(2) - 8 = 0$
Hence the remainder is 0.

Example 4

Let $f(x) = 6x^2 - 7x + 2$. Find the remainder when $f(x)$ is divided by $2x - 1$.

Solution

$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

$\therefore f(\tfrac{1}{2}) = 6(\tfrac{1}{2})^2 - 7(\tfrac{1}{2}) + 2$

$= \frac{6}{4} - \frac{7}{2} + 2 = 0$

Hence the remainder is 0.

Example 5

Find the remainder when

i) $f(x) = x^3 - 2x + 4$ is divided by $x - 1$

ii) $f(x) = x^3 + 3x^2 - 6x + 2$ is divided by $x + 2$

iii) $f(x) = x^3 - x^2 + 2$ is divided by $x - 3$

iv) $f(x) = 4x^4 - 3x^3 + 5x$ is divided by $2x - 1$

Solution

i) When $f(x) = x^3 - 2x + 4$ is divided by $x - 1$, the remainder is

$f(1) = (1)^3 - 2(1) + 4 = 1 - 2 + 4 = 3$

ii) When $f(x) = x^3 + 3x^2 - 6x + 2$ is divided by $x + 2$, the remainder is

$f(-2) = (-2)^3 + 3(-2)^2 - 6(-2) + 2$

$= -8 + 12 + 12 + 2 = 18$

iii) When $f(x) = x^3 - x^2 + 2$ is divided by $x - 3$, the remainder is 47.

iv) When $f(x) = 4x^4 - 3x^3 + 5x$ is divided by $2x - 1$, the remainder is $\frac{35}{16}$.

Example 6

If $f(x) = x^2 - 7x + a$ has a remainder 1 when divided by $x + 1$, find a.

Solution

Since $f(x) = x^2 - 7x + a$ leaves a remainder 1 when divided by $x + 1$ $\Rightarrow$ $f(-1) = 1$

But $f(-1) = (-1)^2 - 7(-1) + a$

$\Rightarrow \quad 1 + 7 + a = 1 \quad \Rightarrow \quad a = -7$

The factor theorem

This is a special case of the remainder theorem because if $(x - a)$ is a **factor** of a polynomial $f(x)$ then the remainder when $f(x)$ is divided by $(x - a)$ is zero. i.e. $f(a) = 0$.

This result which is called the **factor theorem**, states that:

If $(x - a)$ is a **factor** of the polynomial $f(x)$, then $f(a) = 0$.

Also, if $(x + a)$ is a **factor** of the polynomial $f(x)$, then $f(-a) = 0$.

Conversely, if $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

Similarly, if $f(-a) = 0$ then $(x + a)$ is a factor of $f(x)$.

The factor theorem is very useful when factorising polynomials with higher degrees.

Illustrative examples

Example 1

Factorize completely

i) $x^3 - 2x^2 - 5x + 6$

ii) $x^3 - x^2 + 2x - 8$

Solution

i) Let $f(x) = x^3 - 2x^2 - 5x + 6$

Step 1 : Test all the possible factors of the constant term 6 i.e. $\{\pm1, \pm2, \pm3, \pm6\}$

$f(-1) = (-1)^3 - 2(-1)^2 - 5(-1) + 6 = 8$

Therefore $(x - 1)$ is **not** a factor of $f(x)$.

$f(1) = (1)^3 - 2(1)^2 - 5(1) + 6 = 0$.

Hence by the factor theorem, $(x - 1)$ is a factor of $f(x)$.

Step 2: Divide $x^3 - 2x^2 - 5x + 6$ by the factor $(x - 1)$ to find the other factor which is a quadratic expression. Use the long division method.

$$\begin{array}{rl}
 & x^2-x-6 \\
x-1 & \overline{)\,x^3-2x^2-5x+6} \\
 & \underline{x^3-x^2} \\
 & \quad -x^2-5x \\
 & \quad \underline{-x^2+x} \\
 & \qquad -6x+6 \\
 & \qquad \underline{-6x+6} \\
 & \qquad\quad 0
\end{array}$$

Hence $f(x) = (x-1)(x^2-x-6)$

But x^2-x-6 is a quadratic expression which can be factorised easily using the method of factors

i.e. $x^2-x-6 = (x-3)(x+2)$

$\therefore \quad f(x) = (x-1)(x-3)(x+2)$

ii) Let $f(x) = x^3-x^2+2x-8$

Step 1 : Test all the possible factors of the constant term 8 i.e. $\{\pm1, \pm2, \pm4, \pm8\}$

$f(1) = (1)^3-(1)^2+2(1)-8 \neq 0$,

so $x-1$ is not a factor of $f(x)$

$f(-1) = (-1)^3-(-1)^2+2(-1)-8 \neq 0$,

so $x+1$ is **not** a factor of $f(x)$

$f(2) = (2)^3-(2)^2+2(2)-8 = 0$,

Hence by the factor theorem, $(x-2)$ is a factor of $f(x)$.

Step 2: Divide x^3-x^2+2x-8 by the factor $x-2$ to find the other factor which is a quadratic expression. Use the long division method.

$$\begin{array}{rl}
 & x^2+x+4 \\
x-2 & \overline{)\,x^3-x^2+2x-8} \\
 & \underline{x^3-2x^2} \\
 & \quad x^2+2x \\
 & \quad \underline{x^2-2x} \\
 & \qquad 4x-8 \\
 & \qquad \underline{4x-8} \\
 & \qquad\quad 0
\end{array}$$

Hence $f(x) = (x-2)(x^2+x+4)$

But x^2+x+4 is a quadratic expression which **cannot** be factorised.

$\therefore \quad x^3-x^2+2x-8 = (x-2)(x^2+x+4)$

Example 2

Factorise the following polynomials.

i) $f(x) = x^3+2x^2-x-2$

ii) $f(x) = x^3-x^2-x-2$

iii) $f(x) = x^3+3x^2+3x+2$

iv) $f(x) = 2x^3-x^2+2x-1$

Solution

Note: In each case the final answer has been provided. Work out the missing steps.

i) $f(x) = x^3+2x^2-x-2$
$\quad = (x-1)(x+2)(x+1)$

ii) $f(x) = x^3-x^2-x-2$
$\quad = (x-2)(x^2+x+1)$

iii) $f(x) = x^3+3x^2+3x+2$
$\quad = (x+2)(x^2+x+1)$

iv) $f(x) = 2x^3-x^2+2x-1$
$\quad = (2x-1)(x^2+1)$

Example 3

Determine whether:

i) $x-1$ is a factor of $f(x) = x^3-7x+6$

ii) $x+1$ is a factor of $f(x) = 2x^3+3x-4$

iii) $x-2$ is a factor of $f(x) = x^3-6x^2+6x-2$

iv) $x-3$ is a factor of $f(x) = x^3-27$

Solution

i) $f(1) = (1)^3-7(1)+6 = 1-7+6 = 0$

When $f(x) = x^3-7x+6$ is divided by $x-1$, the remainder is 0, so $x-1$ is a factor of x^3-7x+6

ii) $f(-1) = 2(-1)^3+3(-1)^2-4$
$\quad = -2+3-4 = -3 \neq 0$

When $f(x) = 2x^3+3x-4$ is divided by $x+1$, the remainder is not 0, so $x+1$ is **not** a factor of $2x^3+3x-4$.

iii) $x-2$ is not a factor of $f(x) = x^3-6x^2+6x-2$

iv) $x-3$ is a factor of $f(x) = x^3-27$.

Example 4

If $x-2$ is a factor of $f(x) = ax^2 - 12x + 4$, find a.

Solution

Since $x-2$ is a factor of $f(x) = ax^2 - 12x + 4$

$$f(2) = 0$$
$$f(2) = a(2)^2 - 12(2) + 4 = 0$$
$$4a - 24 + 4 = 0$$
$$4a = 20 \quad \therefore \quad a = 5$$

Example 5

A function f is defined by:

$f(x) = 5x^3 - px^2 + x - q$

When $f(x)$ is divided by $(x-2)$, the remainder is 3. Given that $(x-1)$ is a factor of $f(x)$, find p and q.

Solution

From the question, $f(2) = 3$ and $f(1) = 0$

$$f(2) = 5(2)^3 - p(2)^2 + 2 - q = 3$$
$$-4p - q = -39 \text{ ....(1)}$$
$$f(1) = 5(1)^3 - p(1)^2 + 1 - q = 0$$
$$-p - q = -6 \text{ ....(2)}$$

Solving equations (1) and (2) simultaneously

$$p = 11 \text{ and } q = -5$$

Example 6

The polynomial $f(x) = x^3 + ax^2 - 5x + b$ has $(x+3)$ as a factor and when it is divided by $(x+2)$ the remainder is 4. Find the values of a and b.

Solution

$(x+3)$ as a factor $\Rightarrow$ $x = -3$

$$f(-3) = (-3)^3 + a(-3)^2 - 5(-3) + b = 0$$
$$-27 + 9a + 15 + b = 0$$
$$9a + b = 12 \text{.................(1)}$$

When $f(x)$ is divided by $(x+2)$ the remainder is 4.

$$f(-2) = (-2)^3 + a(-2)^2 - 5(-2) + b = 4$$
$$-8 + 4a + 10 + b = 4$$
$$4a + b = 2 \text{.................(2)}$$

From (1), $b = 12 - 9a$

Substitute b into (2)

$$4a + 12 - 9a = 2$$
$$-5a = -10 \quad \Rightarrow \quad a = 2$$

Putting $a = 2$ into $b = 12 - 9a$

$$b = 12 - 9(2) = -6$$
$$\therefore \ f(x) = x^3 + 2x^2 - 5x - 6$$

The factors of $a^3 - b^3$ and $a^3 + b^3$

1. $a^3 - b^3 \equiv (a-b)(a^2 + ab + b^2)$
2. $x^3 - 1 \equiv (x-1)(x^2 + x + 1)$
3. $a^3 + b^3 \equiv (a+b)(a^2 - ab + b^2)$
4. $x^3 + 1 \equiv (x+1)(x^2 - x + 1)$

Zeros of a polynomial function

The **zeros** of a polynomial $f(x)$ are the values of x which make $f(x) = 0$ or the zeros of $f(x)$ are the roots of the equation $f(x) = 0$. To **solve the equation** $f(x) = 0$, first factorize the polynomial.

Illustrative examples

Example 1

If $f(x) = x^2 + 2x - 8$, find the zeros of $f(x)$.

Solution

$$f(x) = 0 \quad \Rightarrow \quad x^2 + 2x - 8 = 0$$
$$(x+4)(x-2) = 0 \quad \Rightarrow \quad x = -4 \text{ and } 2$$

$\therefore$ The zeros of $f(x)$ are -4 and 2.

Example 2

Find the zeros of the polynomials

i) $f(x) = x^3 - 2x^2 - x + 2$

ii) $f(x) = 2x^3 - x^2 - 2x + 1$

Solution

i) The zeros of $f(x)$ are 1, -1 and 2.

ii) The zeros of $f(x)$ are -1, $\frac{1}{2}$ and 1.

[Work out the solution]

Solved examination questions

The following questions will boost your understanding on what you have learnt so far under polynomials.

Example 1

If $(x-3)$ and $(x+7)$ are factors of $ax^2+12x+b$, find the values of a and b.

Solution

Let $f(x)=ax^2+12x+b$

Since $(x-3)$ is a factor of $f(x)$, it follows by the factor theorem that $f(3)=0$.

$f(3)= a(3)^2+12(3)+b=0$

$9a+36+b=0$

$9a+b=-36$ ---------(1)

Also since $(x+7)$ is a factor of $f(x)$, it follows by the factor theorem that $f(-7)=0$.

$f(-7)=0 \Rightarrow a(-7)^2+12(-7)+b=0$

$49a-84+b=0$

$49a+b=84$ ---------(2)

Solving eqns (1) and (2) simultaneously, we can obtain the values of a and b.

$9a+b=-36$ --------(1)

$49a+b=84$ --------(2)

Eqn (2) – eqn (1) $\Rightarrow$ $40a=120$

$a=3$

Now putting $a=3$ in (1)

$27+b=-36 \quad \Rightarrow \quad b=-63$

Hence $a=3$ and $b=-63$

Example 2

If $(x+1)$ and $(x-2)$ are factors of x^3+ax^2-5x+b, find the values of a and b and hence find the third factor.

Solution

Let $f(x)=x^3+ax^2-5x+b$

Since $(x+1)$ is a factor of $f(x)$, it follows by the factor theorem that $f(-1)=0$.

$f(-1)=(-1)^3+a(-1)^2-5(-1)+b=0$

$-1+a+5+b=0$

$a+b=-4$ --------(1)

Also since $(x-2)$ is a factor of $f(x)$, it follows by the factor theorem that $f(2)=0$.

$f(2)=0 \Rightarrow (2)^3+a(2)^2-5(2)+b=0$

$8+4a-10+b=0$

$4a+b=2$ --------(2)

Solving eqns (1) and (2) simultaneously, we can find the values of a and b.

$a+b=-4$ --------(1)

$4a+b=2$ --------(2)

Eqn (2) – eqn (1) $\Rightarrow$ $3a=6$ $\Rightarrow$ $a=2$

Now putting $a=2$ in (1)

$2+b=-4 \Rightarrow b=-6$

Hence $a=2$ and $b=-6$ and

$f(x)=x^3+2x^2-5x-6$

To find the third factor, divide x^3+2x^2-5x-6 by the product of the factors $(x+1)$ and $(x-2)$

$(x+1)(x-2)=x^2-x-2$

$$\begin{array}{r} x+3 \\ x^2-x-2\,\overline{)\,x^3+2x^2-5x-6} \\ \underline{x^3-x^2-2x} \\ 3x^2-3x-6 \\ \underline{3x^2-3x-6} \\ 0 \end{array}$$

Hence the third factor is $(x+3)$.

Example 3

i) Find the values of the constants a and b such that the remainder of $f(x)=2x^3+ax^2+bx-6$ is 12 when $f(x)$ is divided by $(x-1)$ and -18 when it is divided by $(x+4)$.

ii) Factorize $f(x)$ completely and state its zeros.

Solution

Given that the remainder is 12 when $f(x)$ is divided by $(x-1)$

$f(1) = 12$, from the remainder theorem.

$$2(1)^3 + a(1)^2 + b(1) - 6 = 12$$
$$2 + a + b - 6 = 12$$
$$a + b = 16 \text{ -------(1)}$$

Also the remainder is -18 when $f(x)$ is divided by $(x + 4)$

$$f(-4) = -18$$
$$2(-4)^3 + a(-4)^2 + b(-4) - 6 = -18$$
$$-128 + 16a - 4b - 6 = -18$$
$$16a - 4b = 116 \text{ --------(2)}$$

Solving eqns (1) and (2) simultaneously, we can find the values of a and b.

$$a + b = 16 \text{ ------------(1)}$$
$$16a - 4b = 116 \text{ -----------(2)}$$

Eqn(1) + 4 x eqn(2)

$\Rightarrow\ 20a = 180 \quad \Rightarrow\ a = 9.$

Putting $a = 9$ into eqn (1) $\Rightarrow\ b = 7$

Hence $f(x) = 2x^3 + 9x^2 + 7x - 6$

ii) To factorize $f(x) = 2x^3 + 9x^2 + 7x - 6$, we have to test for factors of the constant (6) i.e. $\{\pm1, \pm2, \pm3, \pm6\}$.

$$f(-1) = 2(-1)^3 + 9(-1)^2 + 7(-1) - 6 = -6$$
$$f(1) = 2(1)^3 + 9(1)^2 + 7(1) - 6 = 12$$
$$f(-2) = 2(-2)^3 + 9(-2) + 7(-2) - 6 = 0$$

By the factor theorem $(x + 2)$ is a factor of $f(x)$.

Now divide $f(x)$ by $(x + 2)$ to obtain the other factor (By the long division method)

$$\begin{array}{r|l}
 & 2x^2 + 5x - 3 \\ \hline
x+2 & 2x^3 + 9x^2 + 7x - 6 \\
 & \underline{2x^3 + 4x^2} \\
 & \quad 5x^2 + 7x \\
 & \quad \underline{5x^2 + 10x} \\
 & \qquad -3x - 6 \\
 & \qquad \underline{-3x - 6} \\
 & \qquad\quad 0
\end{array}$$

i.e. $f(x) = (x + 2)(2x^2 + 5x - 3)$

but $2x^2 + 5x - 3 = (2x - 1)(x + 3)$

$\therefore\ f(x) = (x + 2)(x + 3)(2x - 1)$

The zeros of $f(x)$ are the values of x for which $f(x) = 0$.

$$(x + 2)(x + 3)(2x - 1) = 0$$
$$x + 2 = 0,\ x + 3 = 0 \text{ or } 2x - 1 = 0$$
$$\therefore\ x = -2,\ x = -3 \text{ or } x = \tfrac{1}{2}.$$

Hence the zeros of $f(x)$ are -3, -2 and $\frac{1}{2}$.

Example 4

a) If $x^2 + 2x - 3$ is a factor of the polynomial $f(x) = 2x^3 + 9x^2 + ax - b$ where a and b are constants, find the values of the constants a and b.

b) Hence
   - i) Factorize $f(x)$ completely.
   - ii) State the zeros of $f(x)$.
   - iii) Find the remainder when $f(x)$ is divided by $(x - 5)$.

Solution

If $x^2 + 2x - 3 = (x + 3)(x - 1)$ is a factor of $2x^3 + 9x^2 + ax - b$, then $(x + 3)$ and $(x - 1)$ are factors of $f(x)$. By the factor theorem, it follows that $f(-3) = 0$ and $f(1) = 0$

Now $f(-3) = 0$

$$2(-3)^3 + 9(-3)^2 + a(-3) - b = 0$$
$$27 - 3a - b = 0 \ \Rightarrow\ -3a - b = -27$$
$$3a + b = 27 \text{ ----------(1)}$$

Also $f(1) = 0 \ \Rightarrow\ 2(1)^3 + 9(1)^2 + a(1) - b = 0$

$$2 + 9 + a - b = 0 \ \Rightarrow\ 11 + a - b = 0$$
$$a - b = -11 \text{ -----------(2)}$$

Now solving (1) and (2) simultaneously, we can find the values of a and b

$$3a + b = 27 \text{ ---------(1)}$$
$$a - b = -11 \text{ --------(2)}$$

Eqn (1) + eqn (2) = $4a = 16 \ \Rightarrow\ a = 4$

Putting $a = 4$ in eqn (1)

$$3(4) + b = 27 \quad \Rightarrow \quad b = 15$$

Hence $f(x) = 2x^3 + 9x^2 + 4x - 15$

To factorize $f(x)$ completely, divide $f(x)$ by $x^2 + 2x - 3$ using the long division method.

$$\begin{array}{r} 2x+5 \\ x^2+2x-3\overline{)2x^3+9x^2+4x-15} \\ \underline{2x^3+4x^2-6x} \\ 5x^2+10x-15 \\ \underline{5x^2+10x-15} \\ 0 \end{array}$$

Hence $f(x) = (x+3)(x-1)(2x+5)$
since $x^2 + 2x - 3 = (x+3)(x-1)$

ii) For the zeros of $f(x)$, we find the values of x for which $f(x) = 0$

$f(x) = 0 \Rightarrow (x+3)(x-1)(2x+5) = 0$

$x + 3 = 0$ or $x - 1 = 0$ or $2x + 5 = 0$

$x = -3$ or $x = 1$ or $x = -\frac{5}{2}$.

Hence the zeros of $f(x)$ are $-3, -\frac{5}{2}$ and 1.

iii) By the remainder theorem, the remainder when $f(x)$ is divided by $(x-5)$ is $f(5)$.

$f(5) = 2(5)^3 + 9(5)^2 + 4(5) - 15 = 480$

$\therefore$ The remainder is 480.

Example 5

a) Factorize $2x^2 + 3x - 2$

b) When the polynomial $p(x) = ax^3 + 5x^2 + bx - 2$, where a and b are constants, is divided by $2x^2 + 3x - 2$, the remainder is $9x - 6$

i) Find the values of a and b.

ii) Using these values of a and b, factorize $p(x)$ completely.

iii) Find the remainder when $p(x)$ is divided by $3x + 1$.

Solution

a) Since $2x^2 + 3x - 2$ is a quadratic expression we use the method of factors to factorize it.

i.e. $2x^2 + 3x - 2 = 2x^2 + 4x - x - 2$

$= 2x(x+2) - 1(x+2)$

$= (2x-1)(x+2)$

b) (i) Since the remainder when $p(x) = ax^3 + 5x^2 + bx - 2$ is divided by $2x^2 + 3x - 2 = (x+2)(2x-1)$ is $9x - 6$

$p(-2) = 9(-2) - 6 = -24$ and

$p(\frac{1}{2}) = 9(\frac{1}{2}) - 6 = \frac{3}{2}$

But $p(-2) = -24$

$a(-2)^3 + 5(-2)^2 + b(-2) - 2$

$= -8a + 20 - 2b - 2 = -24$

$-8a - 2b = -42$

$8a + 2b = 42$ ------------(1)

Also $p(\frac{1}{2}) = -\frac{3}{2}$

$a(\frac{1}{2})^3 + 5(\frac{1}{2})^2 + b(\frac{1}{2}) - 2 = -\frac{3}{2}$

$\frac{1}{8}a + \frac{5}{4} + \frac{1}{2}b - 2 = -\frac{3}{2}$

Multiplying through by 8,

$a + 10 + 4b - 16 = -12$

$a + 4b = -6$ ------------(2)

Solving eqns (1) and (2) simultaneously, we can find the values of a and b

$8a + 2b = 42$ ---------(1)

$a + 4b = -6$ ----------(2)

$2 \times$ eqn (1) – eqn (2)

$15a = 90 \Rightarrow a = 6$

Now putting $a = 6$ in eqn (2)

$6 + 4b = -6 \Rightarrow 4b = -12 \Rightarrow b = -3$

Hence $a = 6$ and $b = -3$

ii) $p(x) = 6x^3 + 5x^2 - 3x - 2$

To factorize $p(x)$, we first test the values ±1, ±2 etc. to get one factor using the factor theorem i.e.

$f(-1) = 6(-1)^3 + 5(-1)^2 - 3(-1) - 2 = 0$

$(x+1)$ is a factor of $p(x)$.

Next we divide $p(x)$ by $(x+1)$ using the long division method.

$$\begin{array}{r} 6x^2-x-2 \\ x+1\overline{)6x^3+5x^2-3x-2} \\ \underline{6x^3+6x^2} \\ -x^2-3x-2 \\ \underline{-x^2-x} \\ -2x-2 \\ \underline{-2x-2} \end{array}$$

$$p(x) = (x+1)(6x^2 - x - 2)$$

But $6x^2 - x - 2 = (3x-2)(2x+1)$

$\therefore\ p(x) = (x+1)(3x-2)(2x+1)$.

iii) The remainder when $p(x)$ is divided by $3x + 1$ (i.e. $3x + 1 = 0 \Rightarrow x = -\frac{1}{3}$) is given by $p(-\frac{1}{3})$

i.e. $p(-\frac{1}{3}) = 6(-\frac{1}{3})^3 + 5(-\frac{1}{3})^2 - 3(-\frac{1}{3}) - 2$

$= -\frac{2}{3}$

Example 6

If the polynomial $f(x) = px^2 + qx + r$, where p, q and r are constants, is divided by $(x + 1)$, the remainder is 2 and if it is divided by $(x - 2)$ the remainder is 8. If $(x - 1)$ is a factor of $f(x)$, find the values of p, q, and r and hence find the zeros of $f(x)$.

Solution

When $f(x)$ is divided by $(x + 1)$ the remainder is 2 $\Rightarrow\ f(-1) = 2$

$p(-1)^2 + q(-1) + r = 2$

$p - q + r = 2$ -------------(1)

When $f(x)$ is divided by $(x - 2)$, the remainder is 8 $\Rightarrow\ f(2) = 8$

$p(2)^2 + q(2) + r = 8$

$4p + 2q + r = 8$ ------------(2)

Also since $(x - 1)$ is a factor of $f(x)$, then by the factor theorem $f(1) = 0$

$p(1)^2 + q(1) + r = 0$

$p + q + r = 0$ ------------(3)

Now solving equations (1), (2) and (3) simultaneously, we can find the values of p, q and r.

$p - q + r = 2$ ---------(1)

$4p + 2q + r = 8$ ------(2)

$p + q + r = 0$ ---------(3)

Eqn (2) – eqn(1) $\Rightarrow\ 3p + 3q = 6$ ---------(4)

Eqn (2) – eqn(3) $\Rightarrow\ 3p + q = 8$ ----------(5)

Now Eqn (4) –eqn(5)

$2q = -2 \quad \Rightarrow q = -1$

Putting $q = -1$ into equation (4)

$3p - 3 = 6 \Rightarrow\ 3p = 9 \quad \Rightarrow p = 3$

Putting $q = -1$ and $p = 3$ into (1)

$3 - (-1) + r = 2 \quad \Rightarrow\ 3 + 1 + r = 2$

$\therefore\ r = -2$

Hence the values of the constants p, q and r are respectively 3, –1 and –2.

$\therefore\ f(x) = 3x^2 - x - 2$

The zeros of $f(x)$ are the values of x for which $f(x) = 0$

$f(x) = 0 \quad \Rightarrow\ 3x^2 - x - 2 = 0$

$(3x + 2)(x - 1) = 0 \quad \Rightarrow\ x = -\frac{2}{3}$ and 1

$\therefore$ The zeros of $f(x)$ are $-\frac{2}{3}$ and 1.

Example 7

a) If $(x - k)$ is a factor of the polynomial $f(x) = 4x^3 - (3k + 2)x^2 - (k^2 - 1)x + 3$, where k is a constant, find the possible values of k.

b) A polynomial $p(x)$ is of the form $p(x) = \dfrac{ax^2 + bx + c}{mx + k}$, where a, b, c, m and k are constants.

   i) If the truth set of $p(x) = 0$ is $\{1, 2\}$ find the values of a, b and c.

   ii) If $p(0) = -\frac{1}{2}$ and $p(3) = -2$, using your results in (i) find the values of m and k.

   iii) Write down the expression for $p(x)$ and state the values of x for which it is defined.

Solution

a) If $(x - k)$ is a factor of the polynomial $f(x)$ then it follows from factor theorem that $f(k) = 0$

$4k^3 - (3k + 2)k^2 - (k^2 - 1)k + 3 = 0$

$4k^3 - 3k^3 - 2k^2 - k^3 + k + 3 = 0$

$$-2k^2+k+3=0$$
$$2k^2-k-3=0$$
$$(k+1)(2k-3)=0 \Rightarrow k=-1 \text{ or } k=\tfrac{3}{2}$$

Hence the possible values of k are -1 or $\frac{3}{2}$

b) $p(x)=\dfrac{ax^2+bx+c}{mx+k}$, where a, b, c, m, k are constants

i) $p(x)=0 \Rightarrow \dfrac{ax^2+bx+c}{mx+k}=0.$

$ax^2+bx+c=0$.

This a quadratic equation.

But the truth set of $p(x)=0$ is $\{1, 2\}$

i.e. $x=1$ and $x=2 \Rightarrow (x-1)(x-2)=0$

Since $p(x)=0$ is a quadratic equation whose roots are given as 1 and 2, we can write

$$ax^2+bx+c=(x-1)(x-2)=0$$
$$ax^2+bx+c=x^2-3x+2$$

Comparing coefficients we have,

$a=1$, $b=-3$ and $c=2$

ii) Putting the values of a, b and c into $p(x)$, we have $p(x)=\dfrac{x^2-3x+2}{mx+k}$

$\therefore \quad p(0)=-\frac{1}{2}$

$$\frac{(0)^2-3(0)+2}{m(0)+k}=-\frac{1}{2}$$

$$\frac{2}{k}=-\frac{1}{2} \Rightarrow -k=4 \quad \therefore \; k=-4$$

Also $p(3)=-2$

$$p(3)=\frac{(3)^2-3(3)+2}{m(3)-4}=-2$$

$$\frac{2}{3m-4}=-2 \Rightarrow 2=-2(3m-4)$$

$$2=-6m+8 \Rightarrow 6m=6 \quad \therefore \; m=1$$

iii) $p(x)=\dfrac{x^2-3x+2}{x-4}$

$\therefore$ $p(x)$ is defined for all values of x except 4.

Example 8

When $f(x)=px^3-3x^2+8x+36$ is divided by $x+2$ it leaves no remainder.

i) Find the value of p.

ii) Hence express $f(x)$ in the form $f(x)=(x+2)(ax^2+bx+c)$, where a, b and c are constants.

iii) State the zeros of $f(x)$.

Solution

i) If $f(x)$ leaves no remainder when divided by $(x+2)$, then $(x+2)$ is a factor of $f(x)$. By the factor theorem, it follows that $f(-2)=0$.

$$f(-2)=p(-2)^3-3(-2)^2+8(-2)+36=0$$
$$-8p-12-16+36=0$$
$$-8p=-8 \quad \Rightarrow \quad p=1$$

Therefore $f(x)=x^3-3x^2+8x+36$

ii) $f(x)=(x+2)(ax^2+bx+c)$

$x+2$ is a factor of $f(x)$

Divide using the long division method to find the other factor.

$$\begin{array}{r}
x^2-5x+18 \\
x+2\,\overline{)\,x^3-3x^2+8x+36} \\
\underline{x^3+2x^2} \\
-5x^2+8x+36 \\
\underline{-5x^2-10x} \\
18x+36 \\
\underline{18x+36} \\
0
\end{array}$$

Hence

$$f(x)=(x+2)(x^2-5x+18)$$
$$=(x+2)(ax^2+bx+c),$$

where $a=1$, $b=-5$ and $c=18$

iii) To find the zeros of $f(x)$, put $f(x)=0$

i.e. $(x+2)(x^2-5x+18)=0$

$x+2=0$ or $x^2-5x+18=0$

$x=-2$ since the quadratic equation $x^2-5x+18=0$ has no real roots.

Therefore the zeros of $f(x)$ is -2.

Example 9

When the polynomial, $f(x)$ is divided by $(x-1)$ the remainder is 7, and when it is divided by $(x-3)$ the remainder is 13. Find, by writing $f(x)=(x-1)(x-3)Q(x)+ax+b$ the remainder when $f(x)$ is divided by $(x-1)(x-3)$. If also $f(x)$ is a cubic function in which the coefficient of x^3 is unity and $f(2)=6$, determine $Q(x)$.

Solution

Division by $(x-1)$ leaves a remainder of 7

$f(1)=7$

$a+b=7$ ……..(1)

Division by $(x-3)$ leaves a remainder of 13

$f(1)=13$

$3a+b=7$ ……(2)

(2) – (1) gives $2a=6 \Rightarrow a=3$

From (1), $b=4$

$\therefore \quad f(x)=(x-1)(x-3)Q(x)+3x+4$

$$\frac{f(x)}{(x-1)(x-3)}=Q(x)+\frac{3x+4}{(x-1)(x-3)}$$

i.e. dividing through by $(x-1)(x-3)$

Thus the remainder on division by $(x-1)(x-3)$ is $3x+4$.

If $f(x)$ is a cubic function in which the coefficient of x^3 is unity then $Q(x)=(x+c)$

i.e. $f(x)=(x-1)(x-3)(x+c)+3x+4$

$f(2)=(2-1)(2-3)(2+c)+3(2)+4=6$

$-(2+c)+10=6$

$c=2$ and $Q(x)=x+2$

Review exercise 5.1

1. Solve the following equations:
   - i) $x^2-2x-1=0$
   - ii) $3x^2-2x-2=0$
   - iii) $2x^2-9x-18=0$
   - iv) $x^2+2x+1=0$
   - v) $x^2+x-4=0$
   - vi) $x^2-6x+9=0$
   - vii) $x^2-2x-3=0$
   - viii) $2x^2+x-1=0$

2. Find the minimum value of $f(x)=3x^2-x-2$ and the value of x for which it occurs.

3. Find the values of the constant k for which the equation $x^2-(k+1)x+5k-19=0$ has equal roots.

4. If the roots of the quadratic equation $3x^2+2x-1=0$ are α and β find
   - i) $\alpha^2+\beta^2$
   - ii) $\alpha^3+\beta^3$
   - iii) $\frac{1}{\alpha}+\frac{1}{\beta}$
   - iv) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$
   - v) $\alpha^2-\beta^2$
   - vi) $\alpha-\beta$
   - vii) $\frac{\beta}{\alpha}+\frac{\alpha}{\beta}$
   - viii) $(\frac{1}{\alpha}+1)+(\frac{1}{\beta}+1)$
   - ix) $\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^2$

5. Solve the following simultaneous equations.
   - i) $x+y=3,\ x^2+y^2=5$
   - ii) $x-y=1, x^2-y^2=5$
   - iii) $x-2y=1, x^2+y^2=29$
   - iv) $2x+y=1, x^2-xy=12$
   - v) $x+y=7, x^2+y^2=25$

6. Given that the roots of the quadratic equation $x^2 + px + q = 0$ are α and β, form the equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

7. a) Find the set of values of k for which the equation $x^2 + kx + 2k - 3 = 0$ has no real roots.

b) When $k = 7$, the roots of the equation $x^2 + kx + 2k - 3 = 0$ are α and β, where $\alpha > \beta$

i) Write down the values of $(\alpha + \beta)$ and $\alpha\beta$.

ii) Form an equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$

iii) Prove that $\alpha - \beta = \sqrt{5}$.

8. a) If α^2 and β^2 are the roots of the equation $x^2 - 21x + 4 = 0$ and α and β are positive, find:

i) $(\alpha + \beta)$

ii) $\alpha\beta$

iii) the equation with roots $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$

b) If $\alpha + \beta = 5$ and $\alpha\beta = 2$, calculate $\frac{1}{\alpha} + \frac{1}{\beta}$ and hence determine the values m and n such that the equation $x^2 + mx + n = 0$ has roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

9. Given that the roots of the quadratic equation $3x^2 + x + 2 = 0$ are α and β,

i) evaluate $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

ii) find the equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$

iii) show that $27\alpha^4 = 11\alpha + 10$

10. The roots of the equation $2x^2 + 6x + 3 = 0$ are α and β.

a) Show that $\alpha^2 + \beta^2 = 6$

b) The roots of the equation $2x^2 + px + q = 0$ are $2\alpha + \beta$ and $\alpha + 2\beta$. Calculate the values of p and q.

11. The roots of the equation $3x^2 + 8x + 16 = 0$ are denoted by α and β. Find the quadratic equation whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$

12. a) For what range of real values of k does the equation $kx^2 + kx - 2 = 0$ have real roots?

b) If the roots are denoted by α and β, determine the value of k for which $\alpha = -\frac{5}{\beta}$.

13. The real roots of the equation $x^2 + 6x + c = 0$ differ by $2n$, where n is a real non-zero.
Show that $n^2 = 9 - c$.
Given that the roots also have opposite signs, find the set of possible values of n.

14. Given that the roots of the quadratic equation $2x^2 + 3x + 4 = 0$ are α and β,

a) evaluate

i) $\alpha^2 + \beta^2$ ii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ iii) $\frac{1}{\alpha^2}$

b) find the equation whose roots are $2\alpha + \beta$ and $\alpha + 2\beta$

15. Given that the roots of the quadratic equation $x^2 + 3x - 2 = 0$ are α and β,

a) evaluate i) $\alpha^3 + \beta^3$ ii) $\alpha^3\beta^3$

b) find the equation whose roots are α^3 and β^3

16. Find the value of k if the roots of the quadratic equation $3x^2 + 5x - k = 0$ differ by two.

17. For what values of k are the roots of the quadratic equation $3x^2 + (k-1)x - 2 = 0$ equal and opposite.

18. If one root of $ax^2 + bx + c = 0$ is treble the other prove that $3b^2 - 16ac = 0$

19. Given that the roots of the quadratic equation $ax^2 + bx + c = 0$ are α and β, form the equation whose roots are:

i) $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$ ii) $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$

20. Given that the roots of the quadratic equation $x^2 + px + q = 0$ are α and β, form the equation whose roots are $\frac{2}{\alpha}$ and $\frac{2}{\beta}$

21. Given that the roots of the quadratic equation $x^2 + px + q = 0$ are α and β, form the equation whose roots are $\alpha^3 - p\alpha^2$ and $\beta^3 - p\beta^2$.

22. Given that $px^2 + (p-3)x + 1 = 0$, find the range of value(s) of p for which the equation has equal roots.

23. Given that the roots of the quadratic equation $2x^2 - 5x + 1 = 0$ are α and β,

a) evaluate i) $\alpha^2 + \beta^2$ ii) $\alpha^2 - \beta^2$

b) find the equation whose roots are α^3 and β^3

24. Find the truth set of the following pairs of equations:

i) $x^2 + 4y^2 = 2$ and $2y + x + 2 = 0$
ii) $2x + 3xy = 1$ and $x + 3y = 0$
iii) $6xy = 1$ and $3x - 4y = 1$
iv) $x^2 + 4y^2 = 2$ and $x + 2y = 2$
v) $xy = 9$ and $x - 2y = 3$
vi) $4x + y^2 = 8$ and $4x + y = 2$
vii) $xy + y^2 = 2$ and $2x + y = 3$
viii) $xy + x = -10$ and $2x + 5y = 8$

25. If the roots of the following equations are equal, find the value of k

i) $x^2 - kx + 12 = 0$
ii) $x^2 - 3x + k = 0$
iii) $x^2 + px + (p-1) = 0$

26. Find the relationship between p and q if the roots of the equation $px^2 + qx + 1 = 0$ are equal

27. Given the equation $4(x-2)^2 = k(x-1)$, show that the sum of the roots is equal to their product. Find the values of k for which the difference between the roots is $3\frac{3}{4}$.

28. The function f is defined by $f : x \rightarrow 4 - 2x - x^2$, where $x \in R$. Find the maximum value of $f(x)$ and the value of x for which it occurs.

29. If $x = 1$ is a root of the equation $x^3 - 2x^2 - 5x + 6 = 0$, find the other roots.

30. Find the values of the constant k for which the equation $2x^2 + (k + 3)x + 2k = 0$ has equal roots.

31. Find the values of a for which the equation $x^2 - (a-2)x + 2a + 1 = 0$ has equal roots.

32. Find the value of k so that the quadratic equation $kx^2 - 6x + 1 = 0$ has two equal roots.

33. What will be the value of p so that the quadratic equation $px^2 - 4x + 1 = 0$ has 2 equal roots?

34. If the roots of $(x - 2)^2 = k$ are equal find the value of k.

35. The sum of roots of a quadratic equation is $\frac{7}{3}$ and the product of its roots is 5. Find the quadratic equation.

36. If the roots of $x^2 - 7x + k = 0$ are m and $m - 1$, find the value of the constant k.

37. If p, q are the roots of the equation $2x^2 + 6x + b = 0$, find the value of b when
    a) $p = q$
    b) $p = q + 2$
    c) $\frac{1}{p} + \frac{1}{q} = 3$.

38. The roots of the equation $2x^2 - 3x - 3 = 0$ are $\alpha + \beta$
    a) Find the values of
        i) $\alpha^2 + \beta^2$ ii) $\frac{1}{\alpha} + \frac{1}{\beta}$
    b) Find an equation whose roots are $\alpha + \frac{1}{\beta}$ and $\frac{1}{\alpha} + \beta^2$

39. If α and β are the roots of the equation $4 + 3x - 2x^2 = 0$, find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

40. The roots of $x^2 + 5x - 7 = 0$ are α and β. Find the equation whose roots are α^2 and β^2

41. If α and β are the roots of the equation $x^2 - 2x - 5 = 0$, find the equation whose roots are $\alpha^2\beta$ and $\alpha\beta^2$. If α and β are the root of $3x^2 - 2x - 6 = 0$, find
    a) $\alpha^2 + \beta^2$
    b) $\alpha^3 + \beta^3$

Review exercise 5.2

1. One of the factors of the polynomial $x^3 + 2x^2 + px + q$, where p and q are constant is $(x + 1)$. The polynomial leaves a remainder of 12 when divided by $(x - 2)$
    i) Find the values of p and q.
    ii) With these values of p and q, factorize the polynomial completely.
    iii) Sketch the curve of $f(x)$.

2. The function f is defined by $f(x) = 2x^3 + 7x^2 + 7x + 2$, find the linear factors of $f(x)$ and sketch the curve of $f(x)$.

3. The polynomial $2x^3 + qx^2 + rx + 2$ has a factor $(x - 1)$ and leaves a remainder of 12 when divided by $(x - 2)$. Find the constants q and r and hence the other factors of the polynomial.

4. $(x - 2)$ is a factor of the polynomial $f(x) = x^3 + ax + b$, where a and b are constants. The remainder when $f(x)$ is divided by $(x + 3)$ is 25.

Find:

a) The values of the constants a and b

b) The remainder when $f(x)$ is divided by $(x-3)$

c) The truth set of the equation $f(x) = 0$ and sketch the curve of $f(x)$

5. If $(2x+1)$ is a factor of the polynomial $f(x) = 2x^3 - x^2 - px - 3$ where p is a constant, find the value of p. Hence find the zeros of $f(x)$ and sketch the curve of $f(x)$

6. When the polynomial $f(x) = x^3 - px^2 + qx + 2$ where p and q are constants is divided by $(x-2)$ the remainder is 2. When divided by $(x+1)$ the remainder is -10.

a) Find the values of p and q.

b) Hence find $f(-2)$.

7. The polynomial $2x^3 + ax^2 + bx + 2$ is exactly divisible by $(x+2)$ and leaves a remainder of 12 when divided by $(x-2)$. Find the constants a and b and factorise the polynomial completely.

8. The quadratic polynomial $p(x)$ leaves a remainder of 3 when divided by $(x-1)$, a remainder of 12 when divided by $(x-2)$ and no remainder when divided by $(x+2)$. Find $p(x)$ and solve $p(x) = 0$ and hence sketch the curve of $p(x)$

9. When the polynomial $f(x) = 2x^3 + px^2 + qx + 6$ where p and q are constants is divided by $(x-1)$ the remainder is -6. When divided by $(x+1)$ the remainder is 12.

a) Find the values of p and q.

b) Show that $f(\frac{1}{2}) = 0$ and hence write $f(x)$ as the product of three linear factors.

10. $f(x) = x^2 + ax + b$. When $f(x)$ is divided by $(x-2)$ the remainder is 8. When $f(x)$ is divided by $(x+3)$ the remainder is 18. Find the values of a and b.

11. The polynomial $2x^3 + ax^2 + bx + 6$ is exactly divisible by $(x-2)$ and leaves a remainder of -12 when divided by $(x+1)$. Find the constants a and b and factorise the polynomial completely.

12. If $f(x) = 2x^3 - 3x^2 - 8x - 3$, find the remainders when $f(x)$ is divided by

i) $(x-1)$ ii) $(x+1)$ iii) $(2x+1)$.

Deduce which of $(x-1)$, $(x+1)$ and $(2x+1)$ is a factor of $f(x)$ and find its remaining factors.

13. The cubic polynomial $6x^3 - 7x^2 - ax + b$ leaves a remainder of 9 when divided by $(x-2)$ and is exactly divisible by $(x+1)$. Find the constants a and b and show that $(2x-1)$ is a factor of the polynomial and obtain the third factor completely.

14. One of the factors of a polynomial $2x^3 + 3x^2 + ax + b$ is $(x-2)$. When the polynomial is divided by $(x-1)$ a remainder of -5 results. Find the constants a and b and factorize the polynomial completely.

15. When the polynomial $f(x) = x^3 + ax^2 - 3x + 4$ is divided by $(x-3)$ the remainder obtained is twice the remainder when the polynomial is divided by $(x-2)$. Find a.

16. If $(2x-1)$ is a factor of the polynomial $f(x) = 8x^3 + 4x^2 + kx + 15$ where k is a constant, find the value of k and factorise $f(x)$ completely when k has this value.

17. Find the remainder when

i) $x^3 - x^2 + x - 1$ is divided by $(x - 3)$

ii) $x^3 - 3x^2 - 4x + 2$ is divided by $(x - 1)$

iii) $4x^3 - 5x + 4$ is divided by $(2x - 1)$.

18. $(x + 2)$ is a factor of the polynomial $f(x) = ax^3 - 2x^2 + bx - 18$, where a and b are constants. The remainder when $f(x)$ is divided by $(x + 1)$ is 8. Find:

a) The values of the constants a and b.

b) Factorise $f(x)$ completely).

c) The zeros of $f(x)$ and sketch the curve of $f(x)$.

19. The polynomial $f(x) = x^3 + ax^2 + bx + c$ is exactly divisible by $(x + 2)$ and $(x - 1)$ and leaves a remainder of 12 when divided by $(x - 2)$. Find the constants a, b and c.

20. Solve the following equations

i) $x^3 - 2x^2 - 5x + 6 = 0$

ii) $x^3 + 9x^2 + 26x + 24 = 0$

iii) $x^3 - x^2 - 8x + 12 = 0$

iv) $x^3 + x^2 - 6x + 4 = 0$

v) $x^3 - x = 0$

vi) $2x^3 - x^2 - 2x + 1 = 0$

21. Given the polynomial,
$f(x) = 3 - 7x + 5x^2 - x^3$,
show that $3 - x$ is a factor of $f(x)$. Factorise $f(x)$ completely and hence find the zeros of $f(x)$.

22. Given the polynomial,
$f(x) = x^3 + kx^2 - 2x + 1$, and that $f(x)$ is divided $x - k$ the remainder is k, find the possible values of k.

23. When the polynomial, $f(x)$ is divided by $(x - 1)$ the remainder is 5, and when its divided by $(x - 2)$ the remainder is 7. Given that $f(x)$ may be written in the form $f(x) = (x - 1)(x - 2)Q(x) + Ax + B$ find the remainder when $f(x)$ is divided by $(x - 1)(x - 2)$

24. When the polynomial, $f(x)$ is divided by $(x - 2)$ the remainder is 4, and when its divided by $(x - 3)$ the remainder is 7. Given that $f(x)$ may be written in the form $f(x) = (x - 2)(x - 3)Q(x) + ax + b$, find the remainder when $f(x)$ is divided by $(x - 2)(x - 3)$. If also $f(x)$ is a cubic function in which the coefficient of x^3 is unity and $f(1) = 1$, determine $Q(x)$.

25. When the cubic polynomial is $f(x) = x^3 + ax^2 - 3x + 4$ is divided by $x - 3$, the remainder obtained is twice the remainder when the polynomial is divided by $x - 3$. Calculate a.

26. Prove that $(x - 2)$ is a factor of $f(x) = 2x^3 - 9x^2 + 7x + 6$. Hence or otherwise, solve the equation
$2x^3 - 9x^2 + 7x + 6 = 0$

27. If $(x - 1)$ is a factor of the polynomial $f(x) = 4x^3 - 4x^2 - x - k$, where k is a constant,

i) find the value of k.

ii) factorise $f(x)$ completely and state the zeros of $f(x)$.

28. The polynomial
$f(x) = 2x^3 + ax^2 + bx + 6$ is exactly divisible by $2x - 3$ and has the remainder -28 when divided by $(x + 3)$.

a) Find the values of a and b.

b) Solve the equation $f(x) = 0$.

29. Given the polynomial, $f(x) = x^2 + ax + 3$ is such that the remainder when divided by $x - 1$ is three times the remainder when divided by $x + 1$. Find the value of a.

30. When the polynomial is $f(x) = 5x^3 - px^2 + x - q$ is divided by $x - 2$, the remainder is 3. Given that $x - 1$ is a factor of $f(x)$, find p and q. Hence find the real roots of $f(x) = 0$.

31. The polynomial $f(x) = 6x^3 - 7x^2 + kx + 11$ has a remainder -23, when divided by $(x + 2)$. Find the value of k.

32. When the polynomial $f(x)$ is divided by $x + 2$, the quotient is $x^2 + x - 1$ and the remainder is 3. Find $f(x)$.

33. If $(x - 2)$ is a factor of the polynomial $f(x) = x^3 + x^2 - px + 8$, find the value of p.

34. i) If $(x + 1)$ and $(x - 2)$ are factors of the polynomial $f(x) = ax^3 + x^2 + b$, find the values of the constants a and b.

ii) Find the remainder when the polynomial $f(x)$ in (i) is divided by $(x - 1)$

35. $(x - 3)$ is a factor of the polynomial $x^3 + ax^2 + bx + 6$. When this polynomial is divided by $(x + 1)$ the remainder is 8.

a) Find the values of a and b.

b) With these values find the remaining factors of the polynomial.

c) Hence solve the equation $x^3 + ax^2 + bx + 6 = 0$.

36. A polynomial $f(x) = mx^3 - 11x^2 + nx - 1$ has $(2x - 1)$ as a factor. When it is divided by $(x + 1)$ the remainder is -24

a) the values of m and n

b) the roots of the polynomial.

37. $(5x - 1)$ is a factor of $5x^3 + ax^2 + bx + 6$ and the remainder when this expression is divided by $(x - 2)$, is 72.

a) Find the values of a and b

b) With these values factorize the expression completely.

c) Hence solve the equation $5x^3 + ax^2 + bx + 6 = 0$

38. $(x - 2)$ is a factor of the polynomial $x^3 - ax^2 - 31x + b$. When this polynomial is divided by $(x-3)$ the remainder is -32

a) find the values of a and b.

b) With these values find the remaining factors of the polynomial.

c) Hence solve the equation $x^3 - ax^2 - 31x + b = 0$.

39. $(3x - 2)$ is a factor of the polynomial $3x^3 + ax^2 + bx - 4$. When this polynomial is divided by $(x - 1)$ the remainder is 6.

a) find the values of a and b.

b) With these values find the remaining factors of the polynomial.

c) Hence solve the equation $3x^3 + ax^2 + bx - 4 = 0$.

40. $(x - 2)$ and $(x - 1)$ are factors of the polynomial $ax^3 + bx^2 + cx + 2$. When this polynomial is divided by $(x + 2)$ the remainder is -12.

a) Find the values of a, b and c,

b) With these values find the remaining factor.

c) Hence solve the equation $ax^3 + bx^2 + cx + 2 = 0$.

41. (x^2-4) is a factor of the polynomial ax^3+bx^2-4x+c. When this polynomial is divided by $(x+1)$ the remainder is 6.

a) Find the values of a, b and c.

b) With these values factorise the polynomial completely.

c) Hence solve the equation $ax^3+bx^2-4x+c=0$

42. When the polynomial ax^2+bx+c is divide by $(x+1)$ and $(x-2)$ the remainders are -2 and 4 respectively. If $(x+2)$ is a factor of the polynomial,

i) find the value of a, b and c

ii) factorise completely the polynomial.

43. $(2x-3)$ is a factor of $ax^3+bx^2+11x+6$ and the remainder when this expression is divided by $(x-1)$, is 4.

a) Find the values of a and b

b) With these values factorize the expression completely.

c) Hence solve the equation $ax^3+bx^2-11x+6=0$

44. Given that the polynomial $f(x)=6-x-x^2$ is a factor of the polynomial $g(x)=px^3+5x^2+qx-18$, where p and q are constants, find the

a) values of p and q;

b) remainder of $g(x)$ when it is divided by $(x+2)$;

c) values of x for which $g(x)=f(x)$

45. Find the value of the constant k if $4x^3+kx^2+7x-23$ has a remainder 7 when divided by $2x-5$

46. If $f(x)=6x^3+13x^2+2x-5$;

a) show that $f(-1)=0$

b) find the factors of $f(x)$.

47. Find the remainder when the polynomial $f(x)=x^3+2x^2-5x-6$ is divided by $(x-3)$.

CHAPTER 6

RATIONAL FUNCTIONS

Definition

A **rational function** is function of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x.

For example, $f(x) = \frac{x+1}{x^2-1}$ and $\frac{x^2+2x-8}{x+4}$ are rational functions.

If the degree of the numerator $P(x)$ is less than the degree of the denominator $Q(x)$, then $\frac{P(x)}{Q(x)}$ is a **proper fraction**.

For example, $\frac{x+1}{x^2-1}$ is a proper fraction since the degree of the numerator is 1 and that of the denominator is 2. If the degree of the numerator $P(x)$ is greater than or equal to the degree of the denominator $Q(x)$, then $\frac{P(x)}{Q(x)}$ is an **improper fraction.**

For example, $\frac{x^2+2x-8}{x+4}$ is an improper fraction since the degree of the numerator is 2 and that of the denominator is 1.

Domain of rational functions

The domain of a rational function is the set of all real numbers except those for which the denominator is zero.

If $f(x) = \frac{P(x)}{Q(x)}$, then the **domain** of $f(x)$ is $\{x : x \in R\, ,\ Q(x) \neq 0\}$.

Thus to find the domain of a rational function, set the denominator equal to zero and solve the resulting equation. The solutions of the equation are the values that must be excluded from the domain.

Illustrative examples

Example 1

If the function of the real variable x is defined as

$f : x \rightarrow \frac{x^2+2x-8}{(x-1)(x+3)}$, where x is real,

find the domain of f.

Solution

Set the denominator equal to zero

i.e. $(x-1)(x+3) = 0$

$x - 1 = 0 \quad \text{or} \quad x + 3 = 0,$

$x = 1 \text{ or } x = -3$

The domain of $f = \{x : x \in R,\ x \neq 1 \text{ or } -3\}$ **or**
The domain of f is the set of all real numbers except 1 and -3.

Example 2

The function g of the real variable x is defined as $g : x \rightarrow \frac{1}{x+3}$.

State the largest possible domain of g.

Solution

Set the denominator equal to zero

i.e. $x + 3 = 0 \quad \Rightarrow \quad x = -3$

The domain of $g = \{x : x \in R,\ x \neq -3\}$ **or**
The domain of g is the set of all real numbers except -3.

Example 3

The function h of the real variable x is defined as $h : x \rightarrow \dfrac{x}{x^2+1}$.

State the domain of h.

Solution

Set the denominator equal to zero i.e. $x^2 + 1 = 0$. Since this equation has no real number solution, the domain of $h = \{x : x \not\in R\}$ i.e. the domain of h is the set of all real numbers.

Example 4

Determine the domain of the rational function. $f(x) = \dfrac{1}{\sqrt{x-2}}$

Solution

Note that in this case the denominator is an expression in a square root. Therefore it is not only zero that makes the expression in the square root undefined but also negative numbers. The function f is defined for all real values of x, except $x - 2 \leq 0$ or $x \leq 2$. Hence the domain of $f = \{x : x \in R, \text{except } x \leq 2\}$.

Example 5

Determine the domain of each rational function.

i) $f(x) = \dfrac{2x}{x+2}$

ii) $g(x) = \dfrac{x^2+5}{2x^2+3x}$

iii) $h(x) = \dfrac{3x}{2x^2+5}$

iv) $f(x) = \dfrac{2x}{\sqrt{2x-3}}$

Solution

i) The domain of $f = \{x : x \in R, x \neq -2\}$

ii) The domain of $g = \{x : x \in R, x \neq -\frac{3}{2} \text{ or } 0\}$

iii) The domain of $h = \{x : x \in R\}$

iv) The domain of $f = \{x : x \in R, \text{except } x \leq \frac{3}{2}\}$

The range of rational functions

The **range** of a function is the values of y which make x defined. Therefore to find the range, first make x the subject.

We shall consider cases where the numerator and denominator are linear functions.

Illustrative examples

Example 1

Find the range of the function $f : x \rightarrow \dfrac{x}{1-x}$.

Solution

Let $y = f(x) \quad \Rightarrow \quad y = \dfrac{x}{1-x}$

Making x the subject

$$y(1-x) = x$$

$$y - xy = x \quad \Rightarrow \quad y = x + xy$$

$$x(1+y) = y \quad \Rightarrow \quad x = \frac{y}{1+y}$$

This expression for x is defined for all values of y except -1.

$\therefore$ The range of f is $\{y : y \in R, \text{except } -1\}$.

Example 2

If $f : x \rightarrow \dfrac{1}{2+x}$, find the range if the domain is the set $\{x : 1 \leq x \leq 5\}$.

Solution

Given $f(x) = \dfrac{1}{2+x}$

$$f(1) = \frac{1}{2+1} = \tfrac{1}{3} \text{ and } f(5) = \frac{1}{2+5} = \frac{1}{7}$$

Hence the range of f is $\{y : \frac{1}{7} \leq y \leq \frac{1}{3}\}$.

Example 3

The function f is defined on the set R of real numbers by

$$f : x \to \frac{2x+3}{x+2},\ x \neq -2.$$ Find the range.

Solution

Let $y = f(x) \Rightarrow \quad y = \dfrac{2x+3}{x+2}$.

Making x the subject

$$y(x+2) = 2x+3$$
$$xy + 2y = 2x + 3$$
$$xy - 2x = 3 - 2y$$
$$x(y-2) = 3 - 2y$$
$$x = \frac{3-2y}{y-2}$$

The expression for x is defined for all real values of y except 2.

$\therefore$ The range of $f = \{y : y \in R, \text{ except } 2\}$.

Finding the value(s) of *x* which make(s) a rational function undefined

Any rational function of the form

$f(x) = \dfrac{P(x)}{Q(x)}$ is *undefined* or *not defined* if

$Q(x) = 0$ or if the denominator is equal to zero. Thus to find the values of x which make a rational function undefined, set the denominator equal to zero and solve the resulting equation.

Illustrative examples

Example 1

Find the value(s) of x which make(s) the following functions undefined.

i) $f(x) = \dfrac{1}{x+3}$

ii) $g(x) = \dfrac{x+2}{(x-2)(x+1)}$

Solution

Any fraction is undefined if the denominator is zero. Therefore

i) $f(x)$ is undefined if $x + 3 = 0$ or $x = -3$

ii) $g(x)$ is undefined if $(x-2)(x+1) = 0$

$x = -1$ or 2

Example 2

Find the values of x which make the function

$h(x) = \dfrac{1}{x^2 + 3x + 2}$ not defined.

Solution

The function is undefined if $x^2 + 3x + 2 = 0$

$$(x+1)(x+2) = 0$$
$$x = -1 \text{ or } -2$$

Note: Equate the denominator to zero and solve for x.

Zeros of rational functions

The **zeros** of a function $f(x)$ are the roots of the equation $f(x) = 0$.

Thus the zeros of any function are found by equating the function to zero and solving for the unknown variable.

Illustrative examples

Example 1

If $f(x) = \dfrac{x^2 + 2x - 8}{x+4}$, find the zeros of $f(x)$.

Solution

To find the zeros of $f(x)$, put $f(x) = 0$

$$\frac{x^2 + 2x - 8}{x+4} = 0$$
$$x^2 + 2x - 8 = 0$$
$$(x-2)(x+4) = 0$$
$$x = 2 \text{ or } -4.$$

Hence 2 and -4 are the zeros of $f(x)$.

Example 2

If $g(x)=\dfrac{x+5}{x^2+1}$, find the zeros of $g(x)$.

Solution

To find the zeros of $g(x)$, put $g(x)=0$

$$\frac{x+5}{x^2+1}=0 \Rightarrow x+5=0 \Rightarrow x=-5$$

Hence the zero of the $g(x)$ is -5.

Example 3

Find the value(s) of x for which the following functions are not defined.

i) $f(x)=\dfrac{2x}{3x+5}$

ii) $g(x)=\dfrac{3x+2}{36x-18}$

iii) $h(x)=\dfrac{2x(5+x^2)}{(x+1)(x-2)}$

iv) $f(x)=\dfrac{2}{x^2+x-6}$

v) $g(x)=\dfrac{3-x}{x^2+2x-15}$

vi) $h(x)=\dfrac{(x+2)(x-1)}{6x^2-11x+4}$

Solution

i) $-\dfrac{5}{3}$ ii) $\dfrac{1}{2}$ iii) -1 and 2

iv) -3 and 2 v) -5 and 3 vi) $\dfrac{1}{2}$ and $\dfrac{4}{3}$

Work out the missing steps.

Operations on rational functions

Addition and subtraction of rational functions

These also follow the process of addition and subtraction of algebraic fractions. Find the LCM of the denominators.

For example,

$$\frac{1}{x-2}+\frac{3}{x+1}=\frac{(x+1)+3(x-2)}{(x-2)(x+1)}$$

$$=\frac{x+1+3x-6}{(x-2)(x+1)}=\frac{4x-5}{(x-2)(x+1)}$$

Illustrative examples

Simplify the following

i) $\dfrac{1}{x-1}+\dfrac{1}{x+1}$ ii) $\dfrac{1}{x+1}+\dfrac{1}{x-2}$

iii) $\dfrac{4}{x+2}+\dfrac{3}{x+3}$ iv) $\dfrac{1}{x^2-1}+\dfrac{1}{x+1}$

v) $\dfrac{2}{a^2-1}-\dfrac{3}{a-1}$

vi) $\dfrac{1}{x^2+2x+1}+\dfrac{1}{x+1}$

vii) $\dfrac{4}{(x+1)^2}+\dfrac{2}{x+1}$

viii) $\dfrac{3}{(x+2)^2}-\dfrac{1}{x+4}$

ix) $\dfrac{1}{2(x-1)}+\dfrac{2}{3(x+4)}$

x) $\dfrac{7}{5(x+2)}-\dfrac{2}{x+4}$

xi) $\dfrac{3}{4x^2+4x+1}-\dfrac{2}{2x+1}$

xii) $\dfrac{2}{x^2+5x+4}-\dfrac{3}{x+1}$

Solution

Note: In each case the final answer has been provided. Work out the missing steps.

i) $\dfrac{1}{x-1}+\dfrac{1}{x+1}=\dfrac{2x}{(x-1)(x+1)}$

ii) $\dfrac{1}{x+1}+\dfrac{1}{x-2}=\dfrac{2x-1}{(x+1)(x-2)}$

iii) $\dfrac{4}{x+2}+\dfrac{3}{x+3}=\dfrac{7x+18}{(x+2)(x+3)}$

iv) $\dfrac{1}{x^2-1}+\dfrac{1}{x+1}=\dfrac{x}{(x-1)(x+1)}$

v) $\dfrac{2}{a^2-1}-\dfrac{3}{a-1}=\dfrac{-1-3a}{(a-1)(a+1)}$

vi) $\dfrac{1}{x^2+2x+1}+\dfrac{1}{x+1}=\dfrac{x+2}{(x+1)^2}$

vii) $\dfrac{4}{(x+1)^2}+\dfrac{2}{x+1}=\dfrac{2x+6}{(x+1)^2}$

viii) $\dfrac{3}{(x+2)^2}-\dfrac{1}{x+4}=\dfrac{8-x-x^2}{(x+2)^2(x+4)}$

ix) $\dfrac{1}{2(x-1)}+\dfrac{2}{3(x+4)}=\dfrac{7x+8}{6(x-1)(x+4)}$

x) $\dfrac{7}{5(x+2)}-\dfrac{2}{x+4}=\dfrac{8-3x}{5(x+2)(x+4)}$

xi) $\dfrac{3}{4x^2+4x+1}-\dfrac{2}{2x+1}=\dfrac{1-4x}{(2x+1)^2}$

xii) $\dfrac{2}{x^2+5x+4}-\dfrac{3}{x+1}=\dfrac{-3x-10}{(x+1)(x+4)}$

Multiplication and division of rational functions

Multiplication and division of rational functions follow the same rules as multiplication and division of algebraic fractions. For example, if $f(x)=\dfrac{x}{x^2-1}$ and $g(x)=\dfrac{2x}{1+x}$, then

i) $f(x).g(x)=\dfrac{x}{x^2-1}.\dfrac{2x}{1+x}$

$$=\frac{2x^2}{(x^2-1)(1+x)}=\frac{2x^2}{x^3+x^2-x-1}$$

ii) $\dfrac{f(x)}{g(x)}=\dfrac{x}{x^2-1}\div\dfrac{2x}{1+x}=\dfrac{x}{x^2-1}\times\dfrac{1+x}{2x}$

$$=\frac{x}{(x+1)(x-1)}\times\frac{1+x}{2x}=\frac{1}{2(x-1)}$$

Illustrative example

Simplify the following

i) $\dfrac{1}{x^2-9}\div\dfrac{1}{x+3}$

ii) $\dfrac{4x^2-9}{(x-1)^2}\div\dfrac{2x+3}{x(x-1)}$

iii) $\dfrac{x^2+4x+3}{5}\times\dfrac{10}{x+1}$

iv) $\dfrac{2x}{x^2-1}\div\dfrac{x^2-2x}{x^2-2x+1}$

Solution

i) $\dfrac{1}{x^2-9}\div\dfrac{1}{x+3}=\dfrac{1}{(x+3)(x-3)}\times\dfrac{x-3}{1}$

$$=\frac{1}{x+3}$$

ii) $\dfrac{4x^2-9}{(x-1)^2}\div\dfrac{2x+3}{x(x-1)}$

$$=\frac{(2x+3)(2x-3)}{(x-1)(x-1)}\times\frac{x(x-1)}{(2x+3)}$$

$$=\frac{x(2x-3)}{x-1}$$

iii) $\dfrac{x^2+4x+3}{5}\times\dfrac{10}{x+1}$

$$=\frac{(x+1)(x+3)}{5}\times\frac{10}{x+1}$$

$$=2(x+3)$$

iv) $\dfrac{2x}{x^2-1}\div\dfrac{x^2-2x}{x^2-2x+1}$

$$=\frac{2x}{(x+1)(x-1)}\times\frac{(x+1)(x+1)}{x(x-2)}$$

$$=\frac{2(x-1)}{(x+1)(x-2)}$$

Identities

An **identity** is a relationship which is true for all values of the variables. For example, $x^2 - 4 \equiv (x+2)(x-2)$ is an **identity** since the LHS ≡ RHS for any value of x.
Also, $(x-1)^2 \equiv x^2 - 2x + 1$ is an identity since the LHS ≡ RHS for any value of x.

It is important to know the difference between identities and equations. In an equation, the equality is true for a restricted set of values. For example, $(x-1)^2 = 4$ is true only when $x = 3$ or when $x = -1$.
In an identity, the equality is true for *any* value of the variable. For example, $(x+3)^2 \equiv x^2 + 6x + 9$ is an identity.
The symbol ≡ means 'is identical to'.
In practice, however the symbol '=' is often used in an algebraic identity.

Illustrative examples

Example 1
Find the values of the constants A, and B such that $x - 2 \equiv A(x-4) + B(x+3)$.

Solution
Remember that this is not an equation because the RHS is just another way of expressing the LHS.

First method – cover up rule
This is the quicker method
$x - 2 \equiv A(x-4) + B(x+3)$.
Putting $x = 4 \Rightarrow 2 = 7B \Rightarrow B = \frac{2}{7}$
Putting $x = -3 \Rightarrow -5 = -7A \Rightarrow A = \frac{5}{7}$

Second Method – expanding and comparing coefficients

$x - 2 \equiv A(x-4) + B(x+3)$ Expanding
$x - 2 \equiv Ax - 4A + Bx + 3B$

Grouping like terms on the RHS
$x - 2 \equiv (A+B)x + (-4A + 3B)$

Equating coefficients of x
$1 = A + B$.........(1)
Equating constant terms
$-2 = -4A + 3B$....(2)
Solving (1) and (2) simultaneously gives
$A = \frac{5}{7}$ and $B = \frac{2}{7}$

Example 2
Find the values of the constants A, B and C such that
$5x + 3 \equiv Ax(x+3) + Bx(x-1) + C(x-1)(x+3)$

Solution
First method – cover up rule
$5x + 3 \equiv Ax(x+3) + Bx(x-1) + C(x-1)(x+3)$
Putting $x = 0$
$3 = 0 + 0 - 3C \Rightarrow C = -1$
Putting $x = -3$,
$-15 + 3 = 0 + B(-3)(-4) + 0$
$-12 = 12B \Rightarrow B = -1$
Putting $x = 1$,
$5 + 3 = A(1)(4) + 0 + 0 \Rightarrow 8 = 4A$
$A = 2$
Therefore the values of the constants are $A = 2$, $B = -1$ and $C = -1$.

Second Method – expanding and comparing coefficients
$5x + 3 \equiv Ax(x+3) + Bx(x-1) + C(x-1)(x+3)$
$5x + 3 \equiv Ax^2 + 3Ax + Bx^2 - Bx + C(x^2 + 3x - x - 3)$
$5x + 3 \equiv Ax^2 + 3Ax + Bx^2 - Bx + Cx^2 + 2Cx - 3C.$
Grouping like terms on the RHS
$5x + 3 \equiv (A + B + C)x^2 + (3A - B + 2C)x - 3C$
Equating coefficients of x^2
$0 = A + B + C$ ------------(1)
Equating coefficients of x
$5 = 3A - B + 2C$ ----------(2)
Equating constant terms
$3 = -3C$ -------------------(3)

Now from eqn (3), $C = -1$. Substituting this into eqn (1) and (2)

$$A + B = 1 \text{ ----------------}(1b)$$

$$3A - B = 7 \text{ ----------------}(2b)$$

Solving $(1b)$ and $(2b)$ simultaneously, we have

Eqn$(1b)$ + eqn$(2b)$

$$4A = 8 \Rightarrow A = 2.$$

Substituting this into equation $(1b)$

$$B = -1$$

$\therefore A = 2,\ B = -1$ and $C = -1$.

Note: The above two methods of determining the constants will be useful when resolving a rational function into partial fractions. Sometimes a combination of the two methods is faster

Example 3

Find the values of the constants A, B and C such that

$$1 \equiv A(x-1)^2 + B(x+2)(x-1) + C(x+2)$$

Solution

$$1 \equiv A(x-1)^2 + B(x+2)(x-1) + C(x+2)$$

Putting $x = -2$

$$1 = 9A + 0 + 0 \Rightarrow A = \tfrac{1}{9}$$

Putting $x = 1$

$$1 = 0 + 0 + 3C \Rightarrow C = \tfrac{1}{3}$$

Expanding the identity and grouping like terms

$$1 \equiv (A+B)x^2 + (B - 2A + C)x + A - 2B + 2C$$

Equating coefficients of x^2

$$0 = A + B \Rightarrow \tfrac{1}{9} + B = 0$$

$$B = -\tfrac{1}{9} \text{ (since } A = \tfrac{1}{9}\text{)}$$

Therefore the values of the constants are $A = \frac{1}{9}$, $B = -\frac{1}{9}$ and $C = \frac{1}{3}$.

Example 4

Find the values of the constants A, B and C in the following identities.

i) $x = A(x-3) + B(x-2)$

ii) $x^2 + 1 = (Ax+B)(x-1) + C(x^2+2)$

iii) $x - 1 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$

iv) $5x + 4 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$

Solution

Work out the missing steps in each case.

i) $A = -2,\ B = 3$

ii) $A = \frac{1}{3},\ B = \frac{1}{3},\ C = \frac{2}{3}$

iii) $A = -\frac{2}{9},\ B = \frac{2}{9},\ C = \frac{1}{3}$

iv) $A = 1,\ B = -1,\ C = 2$

Partial fractions

A rational function which may be expressed as a sum of separate fractions is said to be resolved into **partial fractions**. By algebraic addition,

$$\frac{1}{x-2} + \frac{3}{x+1} = \frac{(x+1) + 3(x-2)}{(x-2)(x+1)} = \frac{4x-5}{x^2 - x - 2}$$

The reverse process of splitting $\dfrac{4x-5}{x^2-x-2}$ into its component fractions $\dfrac{1}{x-2} + \dfrac{3}{x+1}$ is called resolving into partial fractions.

Similarly, $\dfrac{5}{x^2+x-6} = \dfrac{1}{x-2} - \dfrac{1}{x+3}$.

Note the following carefully:

1. In order to resolve a rational function into partial fractions
   - the denominator must factorise (in the above examples, $x^2 - x - 2$ factorises as $(x-2)(x+1)$ and $x^2 + x - 6$ also factorises as $(x-2)(x+3)$).
   - the rational function must be a **proper fraction** (i.e. the degree of the numerator must be less than the degree of the denominator). In the

above example, $(4x-5)$ is of degree 1 and (x^2-x-2) is of degree 2.

- if the rational function is an **improper fraction** (i.e. the degree of the numerator is equal to or greater than the degree of the denominator), divide the numerator by the denominator to obtain a *quotient* and a *proper fraction*. Then resolve the proper fraction into partial fractions as before. Do not forget to include the quotient in your final answer.

2. The method of obtaining the partial fractions depends on the type of factors in the denominator.
   Three types are considered at this stage.
   - for every **linear factor** $(ax+b)$ of the denominator, there will be a corresponding partial fraction
     $$\frac{A}{ax+b}.$$
   - for every **repeated factor** $(cx+d)^2$ of the denominator, there will be two corresponding partial fractions
     $$\frac{B}{cx+d}+\frac{C}{(cx+d)^2}.$$
   - for every **quadratic factor** (ex^2+fx+g) of the denominator, there will be a corresponding partial fraction
     $$\frac{Dx+E}{ex^2+fx+g}.$$

3. To find the constants $A, B, C, \ldots$.
   - Form an **identity** between the original fraction and the sum of the partial fractions.
   - Write all the fractions using a common denominator.
   - Compare the two numerators by:
     a) comparing corresponding coefficients on each side of the identity and/or
     b) substituting values of x which reduce individual factors to zero **(cover up rule)**.

4. Always check your answer by combining the partial fractions to form a single fraction.

Type 1–denominator with a linear factor

When the denominator of a fraction contains a **linear factor**, then the numerator of that factor is a constant (i.e. A or B etc.) as in the examples below.

Illustrative examples

Example 1

Express $\dfrac{6}{(x+3)(x-3)}$ in partial fractions.

Solution

The denominator is already factorised and the fraction is a proper fraction.
The linear factor $(x+3)$ has partial fraction $\dfrac{A}{(x+3)}$ and the linear factor $(x-3)$ has partial fraction $\dfrac{B}{(x-3)}$ where A and B are constants to be determined.

$$\therefore \quad \frac{6}{(x+3)(x-3)} \equiv \frac{A}{(x+3)}+\frac{B}{(x-3)}$$

$$\equiv \frac{A(x-3)+B(x+3)}{(x+3)(x-3)}$$

Finding the LCM i.e. $(x+3)(x-3)$
Since the denominators are the same on each side of the identity then the numerators are equal to each other.

$$6 \equiv A(x-3)+B(x+3)$$

To determine the constants A and B, values of x are chosen to make the term in A or B equal to zero. (This is the cover up rule).

Putting $x=3, \Rightarrow 6=6B \Rightarrow B=1$

Putting $x=-3, \Rightarrow 6=-6A \Rightarrow A=-1$

$$\therefore \frac{6}{(x+3)(x-3)} \equiv \frac{-1}{(x+3)} + \frac{1}{(x-3)}$$

$$\equiv \frac{1}{(x-3)} - \frac{1}{(x+3)}$$

Note that the constants can also be determined by expanding the RHS of the identity and equating corresponding coefficients as explained earlier.

Example 2

Express $\frac{3x+2}{x^2+x-2}$ as partial fractions

Solution

The denominator is factorised as $(x+2)(x-1)$ and the fraction is a proper fraction

Note 1 : Factorise the denominator if it has not been factorised

Note 2 : The numerator of each linear factor is a constant A or B.

$$\therefore \frac{3x+2}{x^2+x-2} \equiv \frac{3x+2}{(x+2)(x-1)}$$

$$\equiv \frac{A}{(x+2)} + \frac{B}{(x-1)}$$

$$\equiv \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

$$3x+2 \equiv A(x-1)+B(x+2)$$

i.e. equating the numerators.

Putting $x=1 \Rightarrow 5=3B \Rightarrow B=\frac{5}{3}$

Putting $x=-2 \Rightarrow 4=-3A \Rightarrow A=-\frac{4}{3}$

$$\therefore \frac{3x+2}{x^2+x-2} \equiv \frac{-\frac{4}{3}}{(x+2)} + \frac{\frac{5}{3}}{(x-1)}$$

$$\equiv \frac{-4}{3(x+2)} + \frac{5}{3(x-1)}$$

Example 3

Resolve $\frac{11-3x}{x^2+2x-3}$ into partial fractions.

Solution

The denominator is factorised as $(x-1)(x+3)$ and the fraction is a proper fraction.

$$\therefore \frac{11-3x}{x^2+2x-3} \equiv \frac{11-3x}{(x-1)(x+3)}$$

$$\equiv \frac{A}{(x-1)} + \frac{B}{(x+3)}$$

$$\equiv \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$

$$11-3x \equiv A(x+3)+B(x-1)$$

i.e. equating the numerators.

Putting $x=1 \Rightarrow 8=4A \Rightarrow A=2$

Putting $x=-3 \Rightarrow 20=-4B \Rightarrow B=-5$

$$\therefore \frac{11-3x}{x^2+2x-3} \equiv \frac{2}{(x-1)} + \frac{-5}{(x+3)}$$

$$\equiv \frac{2}{(x-1)} - \frac{5}{(x+3)}$$

Example 4

Express the following in partial fractions

i) $\frac{x-1}{3x^2-11x+10}$ ii) $\frac{2x-1}{x^2-8x+7}$

iii) $\frac{3}{x^2-9}$ iv) $\frac{x+1}{3x^2-x-2}$

v) $\frac{9}{2x^2+x}$ vi) $\frac{4x}{4x^2-9}$

vii) $\frac{3x}{2x^2-2x-4}$ viii) $\frac{4x-2}{x+2x}$

ix) $\frac{3x+1}{(2x-1)(x-1)}$ x) $\frac{2x-3}{(x-2)(4x-3)}$

Solution

Note: In each case the final answer has been provided. Work out the missing steps.

i) $\dfrac{x-1}{3x^2-11x+10}=\dfrac{1}{x-2}-\dfrac{2}{3x-5}$

ii) $\dfrac{2x-1}{x^2-8x+7}=\dfrac{13}{6(x-7)}-\dfrac{1}{6(x-1)}$

iii) $\dfrac{3}{x^2-9}=\dfrac{1}{2(x-3)}-\dfrac{1}{2(x+3)}$

iv) $\dfrac{x+1}{3x^2-x-2}=\dfrac{2}{5(x-1)}-\dfrac{1}{5(3x+2)}$

v) $\dfrac{9}{2x^2+x}=\dfrac{9}{x}-\dfrac{18}{2x+1}$

vi) $\dfrac{4x}{4x^2-9}=\dfrac{1}{2x-3}+\dfrac{1}{2x+3}$

vii) $\dfrac{3x}{2x^2-2x-4}=\dfrac{1}{x-2}+\dfrac{1}{2(x+1)}$

viii) $\dfrac{4x-2}{x+2x}=\dfrac{5}{x+2}-\dfrac{1}{x}$

ix) $\dfrac{3x+1}{(2x-1)(x-1)}=\dfrac{4}{x-1}-\dfrac{5}{2x-1}$

x) $\dfrac{2x-3}{(x-2)(4x-3)}=\dfrac{1}{5(x-2)}+\dfrac{6}{5(4x-3)}$

Example 5

a) If $\dfrac{10x+1}{(x-2)(x+1)}=\dfrac{k}{x-2}+\dfrac{3}{x+1}$,

Find the value of k.

b) Resolve $\dfrac{x-1}{(x^2-4)(x+1)}$ into partial fractions.

Solution

a) $k=7$

b) $\dfrac{x-1}{(x^2-4)(x+1)}$

$=\dfrac{1}{12(x-2)}-\dfrac{3}{4(x+2)}+\dfrac{2}{3(x+1)}$

Type 2–denominator with a quadratic factor

When the denominator of the fraction contains a **quadratic factor**, the numerator of that factor is $(Bx + C)$.

Illustrative examples

Example 1

Express $\dfrac{3x+1}{(x-1)(x^2+1)}$ in partial fractions.

Solution

The denominator is a combination of a linear factor and a quadratic factor *which does not factorise.*

The linear factor $(x-1)$ has partial fraction $\dfrac{A}{(x-1)}$ and the quadratic factor (x^2+1) has partial fraction $\dfrac{Bx+C}{(x^2+1)}$ where A, B and C are constants to be determined. Therefore

$$\frac{3x+1}{(x-1)(x^2+1)}\equiv\frac{A}{(x-1)}+\frac{Bx+C}{(x^2+1)}$$

$$\equiv\frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}$$

Since the denominators are the same on each side of the identity then the numerators are equal to each other.

$$3x+1\equiv A(x^2+1)+(Bx+C)(x-1).....(1)$$

Putting $x=1 \Rightarrow 4=2A+0 \Rightarrow A=2$

Putting $x=0 \Rightarrow 1=A-C \Rightarrow 1=2-C$

$\Rightarrow C=1$

Expanding the identity (1)

$$3x+1\equiv Ax^2+A+Bx^2-Bx+Cx-C$$

$$3x+1\equiv(A+B)x^2+(C-B)x+(A-C)$$

Now comparing coefficients of x^2

$0=A+B$

$0=2+B \Rightarrow B=-2$ (since $A=2$)

Note that the coefficient of x^2 at the LHS of the identity is zero since there is no x^2 term.

Hence $\dfrac{3x+1}{(x-1)(x^2+1)} \equiv \dfrac{2}{(x-1)} + \dfrac{-2x+1}{(x^2+1)}$

$$\equiv \frac{2}{(x-1)} + \frac{1-2x}{(x^2+1)}.$$

Example 2

Express $\dfrac{7x^2+5x+13}{(x^2+2)(x+1)}$ in partial fractions.

Solution

The denominator is a combination of a quadratic factor (which does not factorise) and a linear factor. The quadratic factor (x^2+2) has partial fraction $\dfrac{Ax+B}{(x^2+2)}$ and the linear factor $(x+1)$ has partial fraction $\dfrac{C}{(x+1)}$, where A, B and C are constants to be determined.

$$\therefore \frac{7x^2+5x+13}{(x^2+2)(x+1)}$$

$$\equiv \frac{Ax+B}{(x^2+2)} + \frac{C}{(x+1)}$$

$$\equiv \frac{(Ax+B)(x+1)+C(x^2+2)}{(x^2+2)(x+1)}$$

Since the denominators are the same on each side of the identity then the numerators are equal to each other.

$$7x^2+5x+13$$
$$\equiv (Ax+B)(x+1)+C(x^2+2)$$

Putting $x=-1 \quad \Rightarrow \quad 15 = 0 + 3C$
$\qquad\qquad\qquad \Rightarrow \quad C = 5$

Putting $x=0 \quad \Rightarrow \quad 13 = B + 2C$
$\qquad\qquad\qquad \Rightarrow \quad 13 = B + 10$
$\qquad\qquad\qquad \Rightarrow \quad B = 3$

Expanding the identity (1)

$$7x^2+5x+13$$
$$\equiv Ax^2+Ax+Bx+B+Cx^2+2C$$
$$3x+1 \equiv (A+C)x^2+(A+B)x+(B+2C)$$

Now comparing coefficients of x^2

$$7 = A + C$$
$$7 = A + 5 \quad \Rightarrow \quad A = 2 \quad (\text{since } C = 5)$$

Hence $\dfrac{7x^2+5x+13}{(x^2+2)(x+1)} \equiv \dfrac{2x+3}{(x^2+2)} + \dfrac{5}{(x+1)}$

Example 3

Express the following in partial fractions

i) $\dfrac{x^2+1}{(x^2+2)(x-1)}$ ii) $\dfrac{x^2+9x+6}{(x^2+1)(x+1)}$

iii) $\dfrac{1}{x(x^2+1)}$ iv) $\dfrac{x}{(x^2+1)(x-1)}$

v) $\dfrac{1-4x-x^2}{(x+2)(x^2+1)}$ vi) $\dfrac{13x}{(x-3)(x^2+x+1)}$

Solution

Note: In each case the final answer has been provided. Work out the missing steps.

i) $\dfrac{x^2+1}{(x^2+2)(x-1)} = \dfrac{x+1}{3(x^2+2)} + \dfrac{2}{3(x-1)}$

ii) $\dfrac{x^2+9x+6}{(x^2+1)(x+1)} = \dfrac{-2x-7}{x^2+1} - \dfrac{1}{x+1}$

iii) $\dfrac{1}{x(x^2+1)} = \dfrac{1}{x} - \dfrac{x}{x^2+1}$

iv) $\dfrac{x}{(x^2+1)(x-1)} = \dfrac{1-x}{2(x^2+1)} + \dfrac{1}{2(x-1)}$

v) $\dfrac{1-4x-x^2}{(x+2)(x^2+1)} = \dfrac{1}{x+2} - \dfrac{2x}{x^2+1}$

vi) $\dfrac{13x}{(x-3)(x^2+x+1)} = \dfrac{3}{x-3} + \dfrac{1-3x}{x^2+x+1}$

Example 4

If $\frac{3x^2+4x}{(x-2)(x^2+1)} = \frac{P}{x-2} + \frac{Qx+R}{x^2+1}$, find $P+Q+R$.

Solution

$P+Q+R=5$

[Work out the missing steps]

Type 3–denominator with a repeated factor

When denominator of the fraction contains a **repeated factor** $(x-a)^2$, we have two factors with denominators $(x-a)$ and $(x-a)^2$ and constant numerators.

Illustrative examples

Example 1

Express $\frac{1}{(x+2)(x-1)^2}$ in partial fractions

Solution

The denominator is a combination of a linear factor and a repeated factor. The linear factor $(x+2)$ has partial fraction $\frac{A}{(x+2)}$ and the repeated factor $(x-1)^2$ has partial fractions $\frac{B}{(x-1)}$ and $\frac{C}{(x-1)^2}$ where A, B and C are constants to be determined.

$$\therefore \frac{1}{(x+2)(x-1)^2}$$

$$\equiv \frac{A}{(x+2)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$\frac{1}{(x+2)(x-1)^2}$$

$$\equiv \frac{A(x-1)^2+B(x+2)(x-1)+C(x+2)}{(x+2)(x-1)^2}$$

Since the denominators are the same on each side of the identity then the numerators are equal to each other.

$$1 \equiv A(x-1)^2+B(x+2)(x-1)+C(x+2) \ldots\ldots(1)$$

Putting $x=-2 \Rightarrow 1 = 9A+0+0 \Rightarrow A=\frac{1}{9}$

Putting $x=1 \Rightarrow 1=0+0+3C \Rightarrow C=\frac{1}{3}$

Expanding the identity (1) above and grouping like terms

$$1 \equiv (A+B)x^2+(B-2A+C)x+A-2B+2C$$

Equating coefficients of x^2

$$0=A+B \Rightarrow \frac{1}{9}+B=0$$

$$B=-\frac{1}{9} \text{ (since } A=\tfrac{1}{9})$$

$$\therefore \frac{1}{(x+2)(x-1)^2}$$

$$\equiv \frac{\frac{1}{9}}{(x+2)} + \frac{-\frac{1}{9}}{(x-1)} + \frac{\frac{1}{3}}{(x-1)^2}$$

$$\equiv \frac{1}{9(x+2)} - \frac{1}{9(x-1)} + \frac{1}{3(x-1)^2}$$

Example 2

Express $\frac{5x^2-2x-19}{(x+3)(x-1)^2}$ in partial fractions.

Solution

The denominator is a combination of a linear factor and a repeated factor.

The linear factor $(x+3)$ has partial fraction $\frac{A}{(x+3)}$ and the repeated factor $(x-1)^2$ has partial fractions $\frac{B}{(x-1)}$ and $\frac{C}{(x-1)^2}$, where A, B and C are constants to be determined.

$$\therefore \frac{5x^2-2x-19}{(x+3)(x-1)^2}$$

$$\equiv \frac{A}{(x+3)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$\frac{5x^2-2x-19}{(x+3)(x-1)^2}$$

$$\equiv\frac{A(x-1)^2+B(x+3)(x-1)+C(x+3)}{(x+3)(x-1)^2}$$

Since the denominators are the same on each side of the identity then the numerators are equal to each other.

$$5x^2-2x-19$$

$$\equiv A(x-1)^2+B(x+3)(x-1)+C(x+3)\text{.........}(1)$$

Putting $x=-3 \Rightarrow 32=16A \Rightarrow A=2$

Putting $x=1 \Rightarrow -16=4C \Rightarrow C=-4$

Expanding the identity (1) above and grouping like terms

$$5x^2-2x-19$$

$$\equiv(A+B)x^2+(2B-2A+C)x+A-3B+3C$$

Equating coefficients of x^2

$$5=A+B \Rightarrow 2+B=5 \quad (\text{since } A=2)$$

$$B=-3$$

$$\therefore \frac{5x^2-2x-19}{(x+3)(x-1)^2}$$

$$\equiv\frac{2}{(x+3)}+\frac{3}{(x-1)}-\frac{4}{(x-1)^2}$$

Example 3

Express the following in partial fractions

i) $\frac{2}{x^2(x-1)}$ ii) $\frac{2x-1}{(x-1)^2(2x+1)}$

iii) $\frac{x^2+3x}{(x^2+1)(x+1)^2}$ iv) $\frac{x^2-5x+6}{(x+1)(x-1)^2}$

v) $\frac{3}{x(3x-1)^2}$ vi) $\frac{x^2-1}{x^2(2x+1)}$

vii) $\frac{5x+4}{(1-x)(x+2)^2}$ viii) $\frac{11x+12}{(x+2)(x-3)^2}$

Solution

Note: In each case the final answer has been provided. Work out the missing steps.

i) $\frac{2}{x^2(x-1)}=\frac{2}{x-1}-\frac{2}{x}-\frac{2}{x^2}$

ii) $\frac{2x-1}{(x-1)^2(2x+1)}=\frac{2}{9(x-1)}+\frac{1}{3(x-1)^2}-\frac{8}{9(2x+1}$

iii) $\frac{x^2+3x}{(x^2+1)(x+1)^2}$

$$=\frac{x+3}{2(x^2+1)}-\frac{1}{2(x+1)}-\frac{1}{(x+1)^2}$$

iv) $\frac{x^2-5x+6}{(x+1)(x-1)^2}=\frac{3}{x+1}-\frac{2}{x-1}+\frac{1}{(x-1)^2}$

v) $\frac{3}{x(3x-1)^2}=\frac{3}{x}-\frac{9}{3x-1}+\frac{9}{(3x-1)^2}$

vi) $\frac{x^2-1}{x^2(2x+1)}=\frac{2}{x}-\frac{1}{x^2}-\frac{3}{2x+1}$

vii) $\frac{5x+4}{(1-x)(x+2)^2}=\frac{1}{x-1}-\frac{1}{x+2}+\frac{2}{(x+2)^2}$

viii) $\frac{11x+12}{(x+2)(x-3)^2}$

$$=\frac{-2}{5(x+2)}+\frac{2}{5(x-3)}+\frac{9}{(x-3)^2}$$

Type 4 – improper fractions

An **improper fraction** is one whose numerator is of degree equal to or greater than, that of the denominator. To deal with such a fraction, we first divide the numerator by the denominator to obtain a *quotient* and a *proper fraction.* We then split the proper fraction into partial fractions using the methods described in *Types 1 to 3.*

Illustrative examples

Example 1

Express $\dfrac{2x^2+1}{(x-1)(x+2)}$ in partial fractions.

Solution

Note that since the degree of the numerator is equal to that of denominator, we first divide the numerator by the denominator.

Note: The coefficient of a missing term is zero.

$$\begin{array}{r} 2 \\ x^2+x-2\,\overline{)\,2x^2+0x+1} \\ \underline{2x^2+2x-4} \\ -2x+5 \end{array}$$

$$\therefore\quad \frac{2x^2+1}{(x-1)(x+2)} \equiv 2+\frac{-2x+5}{(x-1)(x+2)}$$

We then express $\dfrac{-2x+5}{(x-1)(x+2)}$ (a proper fraction) in partial fractions.

$$\text{i.e.}\quad \frac{-2x+5}{(x-1)(x+2)} \equiv \frac{A}{(x-1)}+\frac{B}{(x+2)}$$

$$\equiv \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

Equating the numerators

$5-2x \equiv A(x+2)+B(x-1)$

Putting $x=1 \Rightarrow 3=3A+0 \Rightarrow A=1.$

Putting $x=-2 \Rightarrow 9=-3B \Rightarrow B=-3$

$$\frac{-2x+5}{(x-1)(x+2)} \equiv \frac{1}{(x-1)}+\frac{-3}{(x+2)}$$

$$\text{Hence}\quad \frac{2x^2+1}{(x-1)(x+2)} \equiv 2+\frac{1}{(x-1)}-\frac{3}{(x+2)}$$

Example 2

Resolve $\dfrac{x^2+1}{x^2-3x+2}$ into partial fractions.

Solution

Note that since the degree of the numerator is equal to that of denominator, we first divide the numerator by the denominator.

$$\begin{array}{r} 1 \\ x^2-3x+2\,\overline{)\,x^2+0x+1} \\ \underline{x^2-3x+2} \\ 3x-1 \end{array}$$

$$\therefore\quad \frac{x^2+1}{x^2-3x+2} \equiv 1+\frac{3x-1}{x^2-3x+2}$$

$$\equiv 1+\frac{3x-1}{(x-1)(x-2)}$$

We then express $\dfrac{3x-1}{(x-1)(x-2)}$ (a proper fraction) in partial fractions.

$$\text{i.e.}\quad \frac{3x-1}{(x-1)(x-2)} \equiv \frac{A}{(x-1)}+\frac{B}{(x-2)}$$

$$\equiv \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

Equating the numerators

$3x-1 \equiv A(x-2)+B(x-1)$

Putting $x=1 \Rightarrow 2=-A+0 \Rightarrow A=-2.$

Putting $x=2 \Rightarrow 5=0+B \Rightarrow B=5$

$$\frac{3x-1}{(x-1)(x-2)} \equiv \frac{-2}{(x-1)}+\frac{5}{(x-2)}$$

$$\text{Hence}\quad \frac{x^2+1}{x^2-3x+2} \equiv 1-\frac{2}{(x-1)}+\frac{5}{(x-2)}$$

Example 3

Express the following as partial fractions.

i) $\dfrac{x+3}{(x-2)(x+4)}$ ii) $\dfrac{x^2-3}{(x-1)(x^2+1)}$

iii) $\dfrac{x-1}{(x+1)(x-2)^2}$ iv) $\dfrac{x^3}{(x+1)(x-3)}$

v) $\dfrac{3}{(x+1)(x-1)}$ vi) $\dfrac{x}{(x-4)(x-1)}$

vii) $\dfrac{x-1}{(x+2)(x-2)}$ viii) $\dfrac{2}{(2x-1)(x-2)}$

ix) $\dfrac{x+3}{x(x+1)}$ x) $\dfrac{2x-1}{(x+1)(3x+2)}$

xi) $\dfrac{3x}{(x-1)(x-2)(x-3)}$

xii) $\dfrac{x^2-2x+4}{2x(x-3)(x+1)}$

xiii) $\dfrac{2x-1}{(3x-1)(2x+1)}$

xiv) $\dfrac{5x-x^2}{x(x-1)(2x+1)}$

xv) $\dfrac{1-3x}{(2x-1)(x+2)}$

xvi) $\dfrac{x(x+1)}{(x-1)(x-2)}$

xvii) $\dfrac{x^2-4}{x(x+1)(x+3)}$

xviii) $\dfrac{6}{(x+1)(x-1)(x-4)^2}$

xix) $\dfrac{3x-1}{(x^2-9)(x^2-1)}$

xx) $\dfrac{(x-2)(2x+3)}{(x-1)(x^2-9)}$

xxi) $\dfrac{2}{(x-1)(x^2+1)}$

xxii) $\dfrac{x-3}{(x+4)(x^2-2)}$

xxiii) $\dfrac{x^2+3}{x(x^2+2)}$

xxiv) $\dfrac{2x^2+x+1}{(x-3)(2x^2+1)}$

xxv) $\dfrac{x^2}{(x-1)(x+1)}$

xxvi) $\dfrac{x^2-2}{(x+3)(x-1)}$

xxvii) $\dfrac{x^3+3}{(x+1)(x-1)}$

xxviii) $\dfrac{1}{x^2+5x+6}$

xxix) $\dfrac{2}{(1-x^2)}$

xxx) $\dfrac{1}{(x-1)(x^2-3x+6)}$

Solution

Note: In each case the final answer has been provided. Work out the missing steps.

i) $\dfrac{x+3}{(x-2)(x+4)} \equiv \dfrac{5}{6(x-2)} + \dfrac{1}{6(x+4)}$

ii) $\dfrac{x^2-3}{(x-1)(x^2+1)} \equiv \dfrac{-1}{x-1} + \dfrac{2x+2}{x^2+1}$

iii) $\dfrac{x-1}{(x+1)(x-2)^2}$

$$\equiv \frac{-2}{9(x+1)} + \frac{2}{9(x-2)} + \frac{1}{3(x-2)^2}$$

iv) $\dfrac{x^3}{(x+1)(x-3)}$

$$\equiv (x+2) + \frac{1}{4(x+1)} + \frac{27}{4(x-3)}$$

v) $\dfrac{3}{(x+1)(x-1)} \equiv \dfrac{3}{2(x-1)} - \dfrac{3}{2(x+1)}$

vi) $\dfrac{x}{(x-4)(x-1)} \equiv \dfrac{4}{3(x-4)} - \dfrac{1}{3(x-1)}$

vii) $\dfrac{x-1}{(x+2)(x-2)} \equiv \dfrac{3}{4(x+2)} + \dfrac{1}{4(x-2)}$

viii) $\dfrac{2}{(2x-1)(x-2)} \equiv \dfrac{3}{2(x-2)} - \dfrac{4}{3(2x-1)}$

ix) $\dfrac{x+3}{x(x+1)} \equiv \dfrac{3}{x} - \dfrac{2}{x+1}$

x) $\dfrac{2x-1}{(x+1)(3x+2)} \equiv \dfrac{3}{x+1} - \dfrac{7}{3x+2}$

xi) $\dfrac{3x}{(x-1)(x-2)(x-3)}$

$\equiv \dfrac{3}{2(x-1)} - \dfrac{6}{x-2} + \dfrac{9}{2(x-3)}$

xii) $\dfrac{x^2-2x+4}{2x(x-3)(x+1)}$

$\equiv \dfrac{7}{24(x-3)} + \dfrac{7}{8(x+1)} - \dfrac{2}{3x}$

xiii) $\dfrac{2x-1}{(3x-1)(2x+1)} \equiv \dfrac{4}{5(2x+1)} - \dfrac{1}{5(3x-1)}$

xiv) $\dfrac{5x-x^2}{x(x-1)(2x+1)} \equiv \dfrac{4}{3(x-1)} - \dfrac{11}{3(2x+1)}$

xv) $\dfrac{1-3x}{(2x-1)(x+2)} \equiv -\dfrac{1}{5(2x-1)} - \dfrac{7}{5(x+2)}$

xvi) $\dfrac{x(x+1)}{(x-1)(x-2)} \equiv 1 - \dfrac{2}{x-1} + \dfrac{6}{x-2}$

xvii) $\dfrac{x^2-4}{x(x+1)(x+3)}$

$\equiv \dfrac{3}{2(x+1)} + \dfrac{5}{6(x+3)} - \dfrac{4}{3x}$

xviii) $\dfrac{6}{(x+1)(x-1)(x-4)^2}$

$\equiv -\dfrac{3}{25(x+1)} + \dfrac{1}{3(x-1)} - \dfrac{16}{75(x-4)} +$

$\dfrac{2}{5(x-4)^2}$

xix) $\dfrac{3x-1}{(x^2-9)(x^2-1)}$

$\equiv \dfrac{1}{6(x-3)} + \dfrac{5}{24(x+3)} - \dfrac{1}{8(x-1)} -$

$\dfrac{1}{4(x+1)}$

xx) $\dfrac{(x-2)(2x+3)}{(x-1)(x^2-9)}$

$\equiv \dfrac{5}{8(x-1)} + \dfrac{3}{4(x-3)} + \dfrac{5}{8(x+3)}$

xxi) $\dfrac{2}{(x-1)(x^2+1)} \equiv \dfrac{1}{x-1} - \dfrac{x+1}{x^2+1}$

xxii) $\dfrac{x-3}{(x+4)(x^2-2)} \equiv \dfrac{x-2}{2(x^2-2)} - \dfrac{1}{2(x+4)}$

xxiii) $\dfrac{x^2+3}{x(x^2+2)} \equiv \dfrac{3}{2x} - \dfrac{x}{2(x^2+2)}$

xxiv) $\dfrac{2x^2+x+1}{(x-3)(2x^2+1)}$

$\equiv \dfrac{22}{19(x-3)} + \dfrac{1-6x}{19(2x^2+1)}$

xxv) $\dfrac{x^2}{(x-1)(x+1)} \equiv 1 + \dfrac{1}{2(x-1)} - \dfrac{1}{2(x+1)}$

xxvi) $\dfrac{x^2-2}{(x+3)(x-1)} \equiv 1 - \dfrac{7}{4(x+3)} - \dfrac{1}{4(x-1)}$

xxvii) $\dfrac{x^3+3}{(x+1)(x-1)} \equiv x + \dfrac{2}{x-1} - \dfrac{1}{x+1}$

xxviii) $\dfrac{1}{x^2+5x+6} = \dfrac{1}{x+2} - \dfrac{1}{x+3}$

xxix) $\dfrac{2}{1-x^2} \equiv \dfrac{1}{1+x} + \dfrac{1}{1-x}$

xxx) $\dfrac{1}{(x-1)(x^2-3x+6)}$

$\equiv \dfrac{1}{4(x-1)} + \dfrac{-x+2}{4(x^2-3x+6)}$

Solved examination questions

The following questions combine all the concepts you have studied so far in this chapter.

Example 1

a) State the largest possible domain of the rational function

$$f: x \to \frac{x^2-2x-3}{(x-1)(x^2-4)},$$ where x is a real number.

b) Find the zeros of f.

c) Find the values of the constants A, B and C such that

$$f(x) = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2-4)}.$$

Solution

a) The domain of f

$= \{x : x \in R, (x-1)(x^2-4) \neq 0\}$

$= \{x : x \in R, (x-1)(x-2)(x+2) \neq 0\}$

$= \{x : x \in R, x \neq -2, 1, 2\}$

b) To find the zeros of f, equate $f(x) = 0$ and find the value(s) of x.

$$\frac{x^2-2x-3}{(x-1)(x^2-4)} = 0 \Rightarrow x^2-2x-3=0$$

$$(x-3)(x+1) = 0 \Rightarrow x = 3 \text{ or } -1$$

Hence −1 and 3 are the zeros of $f(x)$.

c) $f(x) = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2-4)}$

$$\frac{x^2-2x-3}{(x-1)(x^2-4)}$$

$$\equiv \frac{A}{(x-1)} + \frac{Bx+C}{(x^2-4)}$$

$$\equiv \frac{A(x^2-4)+(Bx+C)(x-1)}{(x-1)(x^2-4)}$$

$x^2 - 2x - 3$

$\equiv A(x^2-4) + (Bx+C)(x-1)$ ....(1)

Putting $x = 1 \Rightarrow 1-2-3 = -3A + 0$

$\Rightarrow -4 = -3A \Rightarrow A = \frac{4}{3}$.

Expanding the identity (1) above

$$x^2 - 2x - 3 \equiv (A+B)x^2 + (C-B)x - 4A - C$$

Equating coefficients of x^2

$$A + B = 1 \Rightarrow \tfrac{4}{3} + B = 1 \Rightarrow B = -\tfrac{1}{3}$$

Equating coefficients of x

$$C - B = -2 \Rightarrow C - (-\tfrac{1}{3}) = -2$$

$$C + \tfrac{1}{3} = -2 \Rightarrow C = -\tfrac{7}{3}.$$

Hence the values of the constants A, B and C such that $f(x) = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2-4)}$ are $\frac{4}{3}$, $-\frac{1}{3}$ and $-\frac{7}{3}$ respectively.

Example 2

a) State the largest possible domain of the function $f: x \to \frac{x}{(x^2+1)(x-1)}$, where x is a real number.

b) Find the values of the constants A, B, C if the statement

$$\frac{x}{(x^2+1)(x-1)} \equiv \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)}$$ is to be true for all values of x in the domain of f.

Solution

a) The domain of $f = \{x : x \in R, (x-1) \neq 0\}$

$= \{x : x \in R, x \neq 1\}$

b) $\frac{x}{(x^2+1)(x-1)}$

$$\equiv \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)}$$

$$\equiv \frac{(Ax+B)(x-1)+C(x^2+1)}{(x^2+1)(x-1)}$$

Equating the numerators

$x \equiv (Ax+B)(x-1) + C(x^2+1)$ ....(1)

Putting $x = 1 \Rightarrow 1 = 0 + 2C \quad \therefore \quad C = \frac{1}{2}$.

Expanding the identity (1)

$$x \equiv (A+C)x^2 + (B-A)x - B + C$$

Equating coefficients of x^2

$$0 = A + C \Rightarrow 0 = A + \tfrac{1}{2} \quad \therefore A = -\tfrac{1}{2}$$

Next equating the coefficients of x

$$1 = B - A \Rightarrow 1 = B - (-\tfrac{1}{2}) \Rightarrow B = \tfrac{1}{2}$$

Hence the values of the constants A, B and C such that $\dfrac{x}{(x^2+1)(x-1)} \equiv \dfrac{Ax+B}{(x^2+1)} + \dfrac{C}{(x-1)}$ are $-\tfrac{1}{2}$, $\tfrac{1}{2}$ and $\tfrac{1}{2}$ respectively.

Example 3

a) State the largest possible domain of the function $f : x \to \dfrac{3x^2+4x+1}{(x+3)(x^2-1)}$ where x is a real number.

b) Find the zeros of $f(x)$.

c) Find the values of the constants A, B, C if the statement $\dfrac{3x^2+4x+1}{(x+3)(x^2-1)} \equiv \dfrac{A}{(x+3)} + \dfrac{Bx+C}{(x^2-1)}$ is to be true for all values of x in the domain of f.

Solution

a) The domain of f is the values of x which make f defined.

The domain of f

$= \{x : x \in R,\ (x+3)(x^2-1) \neq 0\}$

$= \{x : x \in R,\ x \neq -3, -1 \text{ and } 1\}$

b) The zeros of f are the values of x such that $f(x) = 0$

i.e. $\dfrac{3x^2+4x+1}{(x+3)(x^2-1)} = 0$

$3x^2 + 4x + 1 = 0$

$(3x+1)(x+1) = 0$

$x = -1$ and $-\tfrac{1}{3}$ are the zeros of f.

c) $\dfrac{3x^2+4x+1}{(x+3)(x^2-1)} \equiv \dfrac{A}{(x+3)} + \dfrac{Bx+C}{(x^2-1)}$

$\equiv \dfrac{(A(x^2-1)+(Bx+C)(x+3)}{(x+3)(x^2-1)}$

Equating the numerators

$3x^2 + 4x + 1$

$\equiv A(x^2-1) + (Bx+C)(x+3)$ ....(1)

Putting $x = -3$

$3(-3)^2 + 4(-3) + 1$

$\equiv A[(-3)^2 - 1] + (-3B + C)(-3+3)$

$27 - 12 + 1 = 8A \quad \Rightarrow 16 = 8A$

$A = 2$

Expanding the identity (1)

$3x^2 + 4x + 1$

$\equiv (A+B)x^2 + (3B + C)x - A + 3C$

Equating coefficients of x^2

$$3 = A + B \Rightarrow 3 = 2 + B \quad \therefore B = 1$$

Next equating the coefficients of x

$$4 = 3B + C \Rightarrow 4 = 3 + C \Rightarrow C = 1$$

Hence the values of the constants A, B and C such that $\dfrac{3x^2+4x+1}{(x+3)(x^2-1)} \equiv \dfrac{A}{(x+3)} + \dfrac{Bx+C}{(x^2-1)}$ are 2, 1 and 1 respectively.

Example 4

a) A function is defined by the equation $y = \dfrac{x^2}{(x^2+3)}$. If this equation is expressed in the form $y = q + \dfrac{r}{(x^2+3)}$, find the values of q and r.

b) Determine the values of the constants a, b and c such that

$$\frac{x^2}{(x-2)(x-1)} \equiv a + \frac{b}{(x-2)} + \frac{c}{(x-1)}$$

for all values of x except for $x = 1$ or $x = 2$.

Solution

a) Dividing the numerator by the denominator using the long division method we have,

$$\begin{array}{r} 1 \\ x^2+3\overline{)x^2+0} \\ \underline{x^2+3} \\ -3 \end{array}$$

$$\therefore \quad \frac{x^2}{(x^2+3)} = 1 + \frac{-3}{(x^2+3)}$$

Hence $q = 1$ and $r = -3$

***Note:** The coefficient of a missing term is zero*

Alternative method

$$\frac{x^2}{(x^2+3)} = \frac{(x^2+3)-3}{(x^2+3)}$$

$$= \frac{(x^2+3)}{(x^2+3)} + \frac{-3}{(x^2+3)} = 1 + \frac{-3}{(x^2+3)}$$

b) $\dfrac{x^2}{(x-2)(x-1)}$

$$\equiv a + \frac{b}{(x-2)} + \frac{c}{(x-1)}$$

$$\equiv \frac{a(x-2)(x-1) + b(x-1) + c(x-2)}{(x-2)(x-1)}$$

Equating the numerators

$$x^2 \equiv a(x-2)(x-1) + b(x-1) + c(x-2)$$

Expanding and grouping like terms

$$x^2 \equiv ax^2 + (b+c-3a)x + (2a-b-2c)$$

Equating coefficients of x^2 and x, and the constant terms, we have

$a = 1$ ------------ (1)

$b + a - 3a = 0$ ----------------(2)

$2a - b - 2c = 0$ ----------------(3)

Putting $a = 1$ in eqn (2), we have

$b + c = 3$ ---------(2b)

Putting $a = 1$ in (3) we have

$-b - 2c = -2$ ------(3b)

Adding eqn (2b) and eqn (3b)

$-c = 1 \Rightarrow c = -1$

Putting $c = -1$ in eqn (2b)

$b - 1 = 3 \Rightarrow b = 4$

$\therefore$ The values of the constants a, b and c such that $\dfrac{x^2}{(x-2)(x-1)} \equiv a + \dfrac{b}{(x-2)} + \dfrac{c}{(x-1)}$ are 1, 4 and −1 respectively.

Example 5

a) State the largest possible domain of the rational function $f(x) = \dfrac{x^2+x-12}{(2+x)(x^2-1)}$ where x is a real number.

b) Find the zeros of $f(x)$.

c) Find the values of the constant A, B, and C such the

$$f(x) = \frac{A}{2+x} + \frac{Bx+C}{x^2-1}$$

Solution

a) The domain of f is the values of x which make f defined.

The domain of f

$= \{x : x \in R,\ (2+x)(x^2-1) \neq 0\}$

$= \{x : x \in R,\ x \neq -2,\ \pm 1\}$

b) The zeros of $f(x)$ are −4 and 3

c) $A = -\frac{10}{3}$, $B = \frac{13}{3}$, $C = -\frac{23}{3}$

[Workout the missing steps]

Example 6

a) Given $f : x \rightarrow \dfrac{8x^2-x-17}{(x-1)(x^2-3)}$ where x is a real number, state the largest possible domain of the function.

b) Find the values of the components A, B, C if the statement

$$\frac{8x^2-x-17}{(x-1)(x^2-3)} = \frac{Ax+B}{x^2-3} + \frac{C}{x-1}$$

is to be true for all values of x in the domain of f.

Solution

a) The domain of f is the values of x which make f defined.
The domain of f
$= \{x : x \in R,\ (x-1)(x^2-3) \neq 0\}$
$= \{x : x \in R,\ x \neq 1,\ \pm\sqrt{3}\}$

b) $A = 3, B = 2, C = 5$
[Workout the missing steps]

Review exercise 6

1. If $x^3 - x - 1 = a + b(x-1) + c(x-1)(x-2) + d(x-1)(x-2)(x-3)$ for all values of x, find the values of the constants a, b, c and d.

2. Find the values of the constants A, B and C in the identity
$22 - 4x - 2x^2 = A(x-1)^2 + B(x-1)(x+2) + C(x+3)$.

3. Find the values of the constants A, B and C such that

$$f(x) = \frac{3x^2-21x+24}{(x+1)(x-2)(x-3)} \equiv \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

for all values of x in the domain of f.

4. a) Find the values of the constants A, B and C such that

$$f(x) = \frac{2x^2-x+3}{(x+1)(x^2+2)} \equiv \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+2)}$$

for all values of x in the domain of f.

b) State the zeros of f,

5. a) Find the values of the constants a, b, c and d such that

$$f(x) = \frac{2x^3-x-1}{(x-3)(x^2+2)} \equiv a + \frac{b}{(x-3)} + \frac{cx+d}{(x^2+2)}$$

for all values of x in the domain of f.

b) State the largest possible domain of f.

6. a) Find the values of A, B and C such that

$$f(x) = \frac{5x+4}{(x-1)(x+2)^2} \equiv \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$$

for all values of x in the domain of f

b) State the largest possible domain of f.

c) State the zeros of f.

7. Find the values of P and Q such that

$$\frac{1}{2-3x-2x^2} \equiv \frac{P}{(1-2x)} + \frac{Q}{(2+x)}$$

8. Express $\dfrac{2x+3}{(x-2)(x-3)}$ in partial fractions.

9. For what values of x is the expression $\dfrac{(x+4)(x+3)}{x^2-x-6}$ defined?

10. Find the constants A and B in the following identities.

i) $\dfrac{x+2}{x^2-1} \equiv \dfrac{A}{(x-1)} + \dfrac{B}{(x+1)}$

ii) $\dfrac{12}{x^2-9} \equiv \dfrac{A}{(x-3)} + \dfrac{B}{(x+3)}$

11. Resolve the following into partial fractions.

i) $\dfrac{2x^2-x}{(2+x)(1+x^2)}$ ii) $\dfrac{1+2x}{(1+x)(1-2x^2)}$

iii) $\dfrac{4-3x}{(1-2x)(2+x)}$ iv) $\dfrac{1}{(1-2x)(1-3x)}$

v) $\dfrac{9}{(1+2x)(1-x)^2}$ vi) $\dfrac{5x+4}{x^2-x-6}$

vii) $\dfrac{2x+12}{x^2-3x+2}$ viii) $\dfrac{3x^2-x-2}{(1+2x)(x+2)^2}$

ix) $\dfrac{3x+1}{(x+1)^2}$ x) $\dfrac{x-1}{(x+1)(x^2+1)}$

xi) $\dfrac{x^2-x-14}{x^2-2x-3}$ xii) $\dfrac{x^2+10x+6}{x^2+2x-8}$

xiii) $\dfrac{2x+3}{(x-2)^2}$ xiv) $\dfrac{2x-2}{(x+1)(x-2)^2}$

xv) $\dfrac{x^2+9x+8}{x^2+x-6}$ xvi) $\dfrac{2}{(x+7)(x+9)}$

xvii) $\dfrac{4x+5}{(x-1)(x^2+2x+4)}$

xviii) $\dfrac{3x-5}{(x-1)^2(x^2+1)}$

12. If $\dfrac{2x-3}{(x+1)(3x+2)} \equiv \dfrac{a}{x+1} + \dfrac{b}{3x+2}$, find the value of a^2+b^2.

13. For what values of x is the expression $\dfrac{x+2}{x^2-5x-6}$ not defined?

14. Given that $\dfrac{1}{x^2-4} \equiv \dfrac{P}{x+2} + \dfrac{Q}{x-2}, x \neq \pm 2$, find the value of $(P+Q)$.

15. Express $\dfrac{2x-1}{(x+1)(2x+1)}$ in partial fractions.

16. Express $\dfrac{5}{(x-2)(x+3)}$ in partial fractions.

17. Given that $\dfrac{1+3x}{2-3x-2x^2} \equiv \dfrac{P}{1-2x} + \dfrac{Q}{2+x}, x \neq -2$ or $x=\frac{1}{2}$, find the values of P and Q.

18. Given that $\dfrac{2x^2-5x-1}{x(x-1)(x+1)} \equiv \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+1}$, find the value of $(A-B+C)$.

19. Express the following in partial fractions

i) $\dfrac{9x}{(2x+1)^2(1-x)}$ ii) $\dfrac{x-2}{(x^2+1)(x-1)^2}$

iii) $\dfrac{7x+4}{(x-3)(x+2)^2}$ iv) $\dfrac{5-x}{(1+x^2)(1-x)}$

v) $\dfrac{x+7}{(2x-1)(x+2)}$ vi) $\dfrac{x+1}{x(x+1)}$

20. Convert the following into partial fractions.

i) $\dfrac{2x^2-9x-35}{(x+1)(x-2)(x+3)}$

ii) $\dfrac{x^3+4x^2+20x-7}{(x-1)^2(x^2+8)}$

21. Resolve $\dfrac{x^2+1}{x^2-3x+2}$ into partial fractions.

22. Express $\dfrac{x^3-2x^2-4x-4}{x^2+x-2}$ in partial fractions

23. Resolve $\dfrac{2x+3}{(x-2)^2}$ into partial fractions.

24. Express $\dfrac{5x^2-2x-19}{(x+3)(x-1)^2}$ as the sum of three partial fractions.

25. Resolve $\dfrac{3x^2+16x+15}{(x+3)^3}$ into partial fractions.

26. Express $\dfrac{7x^2+5x+13}{(x^2+2)(x+1)}$ in partial fractions.

27. Resolve the following into partial fractions.

i) $\dfrac{12}{x^2-9}$

ii) $\dfrac{4(x-4)}{x^2-2x-3}$

iii) $\dfrac{x^2-3x+6}{x(x-2)(x-1)}$

iv) $\dfrac{3(2x^2-8x-1)}{(x+4)(x+1)(2x-1)}$

v) $\dfrac{x^2+9x+8}{x^2+x-6}$

vi) $\dfrac{3x^3-2x^2-16x+20}{(x-2)(x+2)}$

vii) $\dfrac{4x-3}{(x+1)^2}$

viii) $\dfrac{x^2+7x+3}{x^2(x+3)}$

ix) $\dfrac{5x^2-30x+44}{(x-2)^3}$

x) $\dfrac{18+21x-x^2}{(x-5)(x+2)^2}$

xi) $\dfrac{x^2-x-13}{(x^2+7)(x-2)}$

xii) $\dfrac{6x-5}{(x-4)(x^2+3)}$

28. Simplify each of the following expressions:

a) $\dfrac{x}{x-2}-\dfrac{x-2}{x}$

b) $\dfrac{1}{x^2-1}+\dfrac{2x}{2x^2+x-3}$

29. If $\dfrac{5x-4}{(2x-1)(x-2)} \equiv \dfrac{A}{2x-1}+\dfrac{B}{x+2}$, find $(A+B)$.

30. If $\dfrac{3x^2-x-1}{(x-2)(x+1)^2}$

$$\equiv \frac{A}{x-2}+\frac{B}{x+1}+\frac{C}{(x+1)^2},$$

find $(A+B-C)$.

CHAPTER 7

INDICES AND LOGARITHMIC FUNCTIONS

1. INDICES

Index notation

The **index form** or **index notation** is a simple way for writing numbers.

For example, the number $\mathbf{2^4}$ is in the index form. The small ^{4} is the **index** and the **2** is called the **base**. An index tells you how many of the base number to multiply together. In general:

a^3 means $a \times a \times a$ {3 factors}
a^n means $a \times a \times a \ldots \times a$ {n factors},

where n is the **index** (plural indices) and a is the base. The index is also called the **power** or an **exponent**. An index can be positive or negative or a fraction.

Laws of indices

All numbers written with indices obey the same rules.

First law of indices

We know that

$$a^3 \times a^2 = (a \times a \times a) \times (a \times a) = a^5 = a^{3+2}$$

$$a^4 \times a^2 = (a \times a \times a \times a) \times (a \times a)$$
$$= a^6 = a^{4+2}$$

This gives the first law of indices also known as the *product law.*

$$a^m \times a^n = a^{m+n}$$

Illustrative example

i) $2^2 \times 2^3 = 2^{2+3} = 2^5$
ii) $5^1 \times 5^2 = 5^{1+2} = 5^3$
iii) $x^2 \times x^3 = x^5$
iv) $(2x^4y^{-3})(4x^{-2}y^6) = 8x^{4+(-2)}y^{-3+6}$
$= 8x^2y^3$

Second law of indices

$$\frac{a^5}{a^3} = \frac{a \times a \times a \times a \times a}{a \times a \times a} = a^2 = a^{5-3}$$

$$\frac{a^4}{a^2} = \frac{a \times a \times a \times a}{a \times a} = a^2 = a^{4-2}$$

This gives the second law of indices also known as the *quotient law.*

$$a^m \div a^n = a^{m-n} \quad \text{or} \quad \frac{a^m}{a^n} = a^{m-n}$$

Illustrative example

i) $2^4 \div 2^2 = 2^{4-2} = 2^2 = 4$
ii) $3^6 \div 3^3 = 3^{6-3} = 3^3 = 27$
iii) $\dfrac{x^5}{x^2} = x^{5-2} = x^3$

Third law of indices

$$(a^3)^2 = (a^3 \times a^3)$$
$$= (a \times a \times a) \times (a \times a \times a) = a^6$$

$$(a^2)^4 = (a^2 \times a^2 \times a^2 \times a^2)$$
$$= (a \times a) \times (a \times a) \times (a \times a)(a \times a) = a^8$$

This gives the third law of indices also known as the *power law.*

$$(a^m)^n = a^{mn} \quad \text{or} \quad (a^x b^y)^n = a^{xn} b^{yn}$$

Illustrative example

i) $(2^4)^5 = 2^{4 \times 5} = 2^{20}$
ii) $(3^2)^3 = 3^{2 \times 3} = 3^6$
iii) $(x^2)^4 = x^8$
iv) $(a^2 b^3)^2 = a^4 b^6$
v) $(2x^4 y^3)^{-2} = 2^{-2}x^{-8}y^{-6}$

Properties of indices

The following properties of indices should be noted carefully in addition to the basic three laws above. They are used to simplify expressions.

***Property* 1**

$a^0 = 1$ i.e. any non-zero number raised **to the power zero** is equal to 1. For example, $8^0 = 1$; $58^0 = 1$; $5^0 = 1$; $127^0 = 1$

***Property* 2 : Power of product of two numbers**

$(ab)^n = a^n b^n$.

For example,

$(2 \times 3)^2 = 2^2 \times 3^2 = 4 \times 9 = 36$

***Property* 3: Negative index**

$a^{-n} = \frac{1}{a^n}$ provided $a \neq 0$ i.e. any number raised to a *negative* index(power) is equal to the reciprocal of the number raised to the same but *positive* index. *Note:* $\frac{1}{a^{-n}} = a^n$

For example, $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$ and

$3^{-4} = \frac{1}{3^4} = \frac{1}{81}$

Note: Any number raised to the power -1 is equal to the reciprocal of the number.

For example, $a^{-1} = \frac{1}{a}$ and $2^{-1} = \frac{1}{2}$

***Property* 4 : Power of quotient of two numbers**

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ provided $b \neq 0$.

For example, $\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9}$

***Property* 5: Negative power of quotient of two numbers**

$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$ provided $a \neq 0$ and $b \neq 0$.

For example, $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{3^2}{2^2} = \frac{9}{4}$

***Property* 6 : Fractional index**

i) $a^{\frac{1}{2}} = \sqrt{a}$ i.e. any number raised to the power $\frac{1}{2}$ is equal to the square root of the number. For example $16^{\frac{1}{2}} = \sqrt{16} = 4$ and $81^{\frac{1}{2}} = \sqrt{81} = 9$

ii) $a^{\frac{1}{3}} = \sqrt[3]{a}$ i.e. any number raised to the power $\frac{1}{3}$ is equal to the cube root of the number. For example, $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$ and $125^{\frac{1}{3}} = \sqrt[3]{125} = 5$

iii) $a^{\frac{1}{4}} = \sqrt[4]{a}$ i.e. any number raised to the power $\frac{1}{4}$ is equal to the fourth root of the number. For example, $81^{\frac{1}{4}} = \sqrt[4]{81} = 3$ and $16^{\frac{1}{4}} = \sqrt[4]{16} = 2$.

In general, $a^{\frac{1}{n}} = \sqrt[n]{a}$ i.e. any number raised to the power $\frac{1}{n}$ is equal to the nth root of the number.

Also $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$

For example, $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$

Note: Any number raised to the power 1 is equal to the number.

i.e. $x^1 = x$ and $3^1 = 3$ etc.

Illustrative examples

Example 1

Without using mathematical tables, simplify the following:

i) $\left(\frac{16}{81}\right)^{-\frac{3}{4}}$ ii) $\left(\frac{64}{25}\right)^{-\frac{1}{2}}$

iii) $32^{\frac{2}{5}}$ iv) $16^{-\frac{3}{2}}$

v) $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$ vi) $100^{1\frac{1}{2}}$

Solution

i) $\left(\frac{16}{81}\right)^{-\frac{3}{4}} = \left(\frac{81}{16}\right)^{\frac{3}{4}}$

$$= \left(\frac{3^4}{2^4}\right)^{\frac{3}{4}} = \left(\frac{3^{4\times\frac{3}{4}}}{2^{4\times\frac{3}{4}}}\right) = \frac{3^3}{2^3} = \frac{27}{8}$$

Alternatively,

$$\left(\frac{16}{81}\right)^{-\frac{3}{4}} = \left(\frac{81}{16}\right)^{\frac{3}{4}}$$

$$= \left(\frac{81^{\frac{3}{4}}}{16^{\frac{3}{4}}}\right) = \left(\frac{(81^{\frac{1}{4}})^3}{(16^{\frac{1}{4}})^3}\right)$$

$$= \frac{(\sqrt[4]{81})^3}{(\sqrt[4]{16})^3} = \frac{3^3}{2^3} = \frac{27}{8}$$

ii) $\left(\frac{64}{25}\right)^{-\frac{1}{2}} = \left(\frac{25}{64}\right)^{\frac{1}{2}} = \left(\frac{5^2}{8^2}\right)^{\frac{1}{2}} = \frac{5^{2\times\frac{1}{2}}}{8^{2\times\frac{1}{2}}} = \frac{5}{8}$

Alternatively,

$$\left(\frac{64}{25}\right)^{-\frac{1}{2}} = \left(\frac{25}{64}\right)^{\frac{1}{2}} = \frac{25^{\frac{1}{2}}}{64^{\frac{1}{2}}} = \frac{\sqrt{25}}{\sqrt{64}} = \frac{5}{8}$$

iii) $32^{\frac{2}{5}} = (2^5)^{\frac{2}{5}} = 2^{5\times\frac{2}{5}} = 2^2 = 4$

Alternatively,

$$32^{\frac{2}{5}} = (32^{\frac{1}{5}})^2 = (\sqrt[5]{32})^2 = (2)^2 = 4$$

iv) $16^{-\frac{3}{2}} = \left(\frac{1}{16}\right)^{\frac{3}{2}} = \frac{1}{(4^2)^{\frac{3}{2}}} = \frac{1}{4^{2\times\frac{3}{2}}} = \frac{1}{4^3} = \frac{1}{64}$

Alternatively,

$$16^{-\frac{3}{2}} = \left(\frac{1}{16}\right)^{\frac{3}{2}} = \frac{1}{16^{\frac{3}{2}}}$$

$$= \frac{1}{(16^{\frac{1}{2}})^3} = \frac{1}{(\sqrt{16})^3} = \frac{1}{(4)^3} = \frac{1}{64}$$

v) $\left(\frac{8}{27}\right)^{-\frac{2}{3}} = \left(\frac{27}{8}\right)^{\frac{2}{3}} = \left(\frac{3^3}{2^3}\right)^{\frac{2}{3}}$

$$= \frac{3^{3\times\frac{2}{3}}}{2^{3\times\frac{2}{3}}} = \frac{3^2}{2^2} = \frac{9}{4}$$

Alternatively,

$$\left(\frac{8}{27}\right)^{-\frac{2}{3}} = \left(\frac{27}{8}\right)^{\frac{2}{3}} = \frac{27^{\frac{2}{3}}}{8^{\frac{2}{3}}}$$

$$= \frac{(27^{\frac{1}{3}})^2}{(8^{\frac{1}{3}})^2} = \frac{(\sqrt[3]{27})^2}{(\sqrt[3]{8})^2} = \frac{(3)^2}{(2)^2} = \frac{9}{4}$$

vi) $100^{1\frac{1}{2}} = 100^{\frac{3}{2}} = (\sqrt{100})^3 = 10^3 = 1000$

Example 2

Simplify the following:

i) $(27 \times 3^{-2})(8 \times 2^{-3})$ ii) $\frac{7^6 \times 7^{-3}}{7^4}$

iii) $256^{-\frac{1}{4}}(225^{\frac{1}{2}} + 58^0)$

iv) $27^{\frac{2}{3}} \times 64^{\frac{1}{3}} \div 81^{\frac{1}{4}}$

Solution

i) $(27 \times 3^{-2})(8 \times 2^{-3})$

$= (3^3 \times 3^{-2}) \times (2^3 \times 2^{-3})$

$= (3^{3+(-2)}) \times (2^{3+(-3)})$

$= (3^1) \times (2^0) = 3 \times 1 = 3$

ii) $\frac{7^6 \times 7^{-3}}{7^4} = \frac{7^{6+(-3)}}{7^4} = \frac{7^3}{7^4} = 7^{3-4} = 7^{-1} = \frac{1}{7}$

iii) $256^{-\frac{1}{4}}(225^{\frac{1}{2}}+58^{0})$

$$=\left(\frac{1}{256^{\frac{1}{4}}}\right)[(15^{2})^{\frac{1}{2}}+1]$$

$$=\frac{1}{\sqrt[4]{256}}[15^{2\times\frac{1}{2}}+1]$$

$$=\tfrac{1}{4}(15+1)=\tfrac{1}{4}\times16=4$$

iv) $27^{\frac{2}{3}}\times64^{\frac{1}{3}}\div81^{\frac{1}{4}}$

$$=(3^{3})^{\frac{2}{3}}\times(4^{3})^{\frac{1}{3}}\div(3^{4})^{\frac{1}{4}}$$

$$=3^{2}\times4\div3=3^{2-1}\times4$$

$$=3\times4=12$$

Exponential (indicial) equations

Exponential equations are equations which have the variable or unknown quantity as an index or exponent. Such equations can be transformed to;

1. linear equations
2. quadratic equations and
3. simultaneous equations

1. Exponential equations leading to linear equations

To solve such equations we always use the fact that if $a^{x}=a^{y}\Rightarrow x=y$ provided a is not –1, 0 or 1.

Illustrative examples

Example 1

If $3^{x}=9$, find the value of x.

Solution

$3^{x}=9 \;\Rightarrow\; 3^{x}=3^{2}$

Note: We write the equation in such a way that the L.H.S. and R.H.S have a common base so that we can equate the exponents. For 3^{x}, 3 is said to be the base and x is the power or index or exponent. That is since the bases are equal, we equate the exponents

$x=2$

Example 2

If $3^{x+1}=9^{4}$, find the value of x.

Solution

$3^{x+1}=9^{4} \;\Rightarrow\; 3^{x+1}=(3^{2})^{4} \;\Rightarrow\; 3^{x+1}=3^{8}$

Since the bases are equal, equate the exponents.

$x+1=8 \;\Rightarrow\; x=8-1=7$

Example 3

If $64^{x}=16$, find the value of x

Solution

$64^{x}=16 \;\Rightarrow\; (4^{3})^{x}=4^{2} \;\Rightarrow\; 4^{3x}=4^{2}$

Since the bases are equal, equate the exponents.

$3x=2 \;\Rightarrow\; x=\frac{2}{3}$

Example 4

If i) $3^{x^{2}-1}=9^{4}$ ii) $3^{\sqrt{x}}=9$, find the value of x.

Solution

i) $3^{x^{2}-1}=9^{4} \;\Rightarrow\; 3^{x^{2}-1}=(3^{2})^{4}$

$3^{x^{2}-1}=3^{8}$

Since the bases are equal, equate the exponents.

$x^{2}-1=8 \;\Rightarrow\; x^{2}=9 \;\Rightarrow\; x=\pm3$

ii) $3^{\sqrt{x}}=9 \;\Rightarrow\; 3^{\sqrt{x}}=3^{2}$

$\sqrt{x}=2 \;\Rightarrow\; x=4$ (Squaring both sides)

Example 5

Find the truth set of the equation $8^{x-2}=2^{x-8}$ where x is a real number.

Solution

$8^{x-2}=2^{x-8}$

$(2^{3})^{x-2}=2^{x-8}$

$2^{3x-6}=2^{x-8}$

Since the bases are equal, equate the exponents.

$$3x - 6 = x - 8 \Rightarrow 3x - x = -8 + 6$$
$$2x = -2 \Rightarrow x = -1.$$

Hence the truth set of the above equation is $\{x : x = -1\}$

Example 6

If $3^{x-2} = \frac{1}{81}$, find x.

Solution

$$3^{x-2} = \frac{1}{81}$$
$$3^{x-2} = (81)^{-1}$$
$$3^{x-2} = (3^4)^{-1} \Rightarrow 3^{x-2} = 3^{-4}$$

Since the bases are equal, equate the exponents

$$x - 2 = -4 \Rightarrow x = -4 + 2 \Rightarrow x = -2$$

Example 7

Find the value of x if $2^{3x-1} = 128\sqrt{2}$.

Solution

$$2^{3x-1} = 128\sqrt{2}$$
$$2^{3x-1} = 2^7.2^{\frac{1}{2}} \Rightarrow 2^{3x-1} = 2^{7\frac{1}{2}}$$

Since the bases are equal, equate the exponents.

$$3x - 1 = 7\tfrac{1}{2}$$
$$3x = 8\tfrac{1}{2} \Rightarrow x = \tfrac{17}{6}$$

Example 8

Solve for x in the following equations

i) $3^{2x} = \frac{1}{81}$ ii) $3^{x+1} = 81$

iii) $3^{x+5} = 9^4$ iv) $3^{2x-1} = \frac{1}{27}$

v) $9^x = \frac{1}{3}(27^x)$

Solution

i) $3^{2x} = \frac{1}{81} \Rightarrow 3^{2x} = \frac{1}{3^4} \Rightarrow 3^{2x} = 3^{-4}$

$2x = -4 \Rightarrow x = -2$

ii) $3^{x+1} = 81 \Rightarrow 3^{x+1} = 3^4$

$x + 1 = 4 \Rightarrow x = 3$

iii) $3^{x+5} = 9^4 \Rightarrow 3^{x+5} = (3^2)^4$

$3^{x+5} = 3^8 \Rightarrow x + 5 = 8 \Rightarrow x = 3$

iv) $3^{2x-1} = \frac{1}{27} \Rightarrow 3^{2x-1} = \frac{1}{3^3}$

$3^{2x-1} = 3^{-3} \Rightarrow 2x - 1 = -3$

$2x = -2 \Rightarrow x = -1$

v) $9^x = \frac{1}{3}(27^x) \Rightarrow (3^2)^x = 3^{-1}(3^3)^x$

$3^{2x} = 3^{-1}3^{3x}$

$3^{2x} = 3^{-1+3x} \Rightarrow 2x = -1 + 3x \Rightarrow x = 1$

Example 9

Find the value of x in the following equations

i) $2^{3x-1} = 4^{x+3}$

ii) $4^{x+2} = 8^{x-1}$

iii) $8^x = 0.25$

Solution

i) $2^{3x-1} = 4^{x+3} \Rightarrow 2^{3x-1} = (2^2)^{x+3}$

$2^{3x-1} = 2^{2(x+3)} \Rightarrow 3x - 1 = 2(x + 3)$

$3x - 1 = 2x + 6 \Rightarrow x = 7$

ii) $4^{x+2} = 8^{x-1} \Rightarrow (2^2)^{x+2} = (2^3)^{x-1}$

$2^{2(x+2)} = 2^{3(x-1)} \Rightarrow 2(x + 2) = 3(x - 1)$

$2x + 4 = 3x - 3 \Rightarrow -x = -7 \Rightarrow x = 7$

iii) $8^x = 0.25 \Rightarrow 2^{3x} = \frac{1}{4} = 2^{-2}$

$3x = -2 \Rightarrow x = -\frac{2}{3}$

Example 10

Find x if $3^x = 8$

Solution

$3^x = 8$ Note that since 8 cannot be expressed in indices with base 3, we take logarithms of both sides to base 10.

$$\therefore \log_{10} 3^x = \log_{10} 8$$

$$x \log_{10} 3 = \log_{10} 8$$

$$x = \frac{\log_{10} 8}{\log_{10} 3} = \frac{0.9031}{0.4771} = 1.892$$

2. Exponential equations leading to quadratic equations

Illustrative examples

Example 1

Solve the equation $2^{2x} - 9(2^x) + 20 = 0$

Solution

Note that $2^{2x} = (2^x)^2$

$$2^{2x} - 9(2^x) + 20 = 0$$

$$(2^x)^2 - 9(2^x) + 20 = 0.$$

If we let $y = 2^x$, we shall obtain a quadratic equation in y.

$$y^2 - 9y + 20 = 0 \quad \text{(Factorizing)}$$

$$(y-5)(y-4) = 0$$

$$y = 5 \text{ or } 4 \text{ but } y = 2^x \quad \Rightarrow \quad 2^x = 5 \text{ or } 4.$$

Now $2^x = 4 = 2^2 \quad \Rightarrow x = 2$

For $2^x = 5$, since we cannot express 5 in indices with base 2, we take logarithm of both sides to base 10.

$$\therefore \log_{10} 2^x = \log_{10} 5$$

$$x \log_{10} 2 = \log_{10} 5$$

$$x = \frac{\log_{10} 5}{\log_{10} 2} = \frac{0.699}{0.301} = 2.32$$

Hence $x = 2$ or 2.32

Example 2

Solve the equation $2^{2x+2} - 9(2^x) + 2 = 0$.

Solution

$$2^{2x+2} - 9(2^x) + 2 = 0$$

$$2^2 \times 2^{2x} - 9(2^x) + 2 = 0$$

$$4 \times 2^{2x} - 9(2^x) + 2 = 0$$

Let $p = 2^x$

$$4p^2 - 9p + 2 = 0$$

$$(4p-1)(p-2) = 0$$

$$p = \tfrac{1}{4} = 2^{-2} \text{ or } p = 2 = 2^1$$

Now $p = 2^x = 2^{-2} \Rightarrow x = -2$

and $p = 2^x = 2 \quad \Rightarrow x = 1$

Hence $x = -2$ or 1

Example 3

Solve the equation $3^{2x} - 30(3^x) + 81 = 0$.

Solution

$$3^{2x} - 30(3^x) + 81 = 0$$

$$(3^x)^2 - 30(3^x) + 81 = 0. \quad \text{Let } y = 3^x$$

$$y^2 - 30y + 81 = 0 \quad \text{(factorizing)}$$

$$(y-3)(y-27) = 0$$

$$y = 3 \text{ or } 27 \text{ but } y = 3^x$$

$$3^x = 3 \text{ or } 27$$

$$\therefore \quad 3^x = 3^1 \quad \Rightarrow \quad x = 1 \quad \text{and}$$

$$3^x = 27 = 3^3 \Rightarrow \quad x = 3.$$

Hence $x = 1$ or 3.

Example 4

If $3^{2x} - 4(3^x) + 3 = 0$, solve for x.

Solution

$$3^{2x} - 4(3^x) + 3 = 0$$

$$(3^x)^2 - 4(3^x) + 3 = 0. \quad \text{Let } y = 3^x$$

$$y^2 - 4y + 3 = 0$$

$$(y-1)(y-3) = 0$$

$$y = 1 \text{ or } 3 \text{ but } y = 3^x$$

$$3^x = 1 \text{ or } 3$$

$$\therefore \quad 3^x = 1 = 3^0 \quad \Rightarrow x = 0 \text{ and}$$

$$3^x = 3 = 3^1 \quad \Rightarrow x = 1.$$

Hence $x = 0$ or 1.

Example 5

Find correct to one decimal, the values of x for which $3^{2x} - 3^{(x+2)} + 8 = 0$.

Solution

$$3^{2x} - 3^{(x+2)} + 8 = 0$$
$$(3^x)^2 - 3^2.3^x + 8 = 0$$
$$(3^x)^2 - 9(3^x) + 8 = 0. \quad \text{Let } y = 3^x$$
$$y^2 - 9y + 8 = 0$$
$$(y-1)(y-8) = 0$$
$$y = 1 \text{ or } 8.$$

But $y = 3^x \Rightarrow 3^x = 1$ or 8.

Now $3^x = 1 = 3^0 \quad \Rightarrow \quad x = 0$.

For $3^x = 8$, since 8 cannot be expressed in indices with base 3, we take logarithms of both sides to base 10.

$$\therefore \quad \log_{10} 3^x = \log_{10} 8$$
$$x \log_{10} 3 = \log_{10} 8$$
$$x = \frac{\log_{10} 8}{\log_{10} 3} = \frac{0.9031}{0.4771} = 1.892 = 1.9 \text{ (1 d.p)}$$

Hence $x = 0$ or 1.9

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Example 6

Find the truth set of the equation
$4^x + 2 = 3(2^x)$

Solution

$$4^x + 2 = 3(2^x) \quad \Rightarrow \quad (2^2)^x + 2 = 3(2^x)$$
$$(2^x)^2 + 2 = 3(2^x). \quad \text{Let } 2^x = y$$
$$y^2 + 2 = 3y$$
$$y^2 - 3y + 2 = 0$$
$$(y-2)(y-1) = 0$$
$$y = 1 \text{ or } 2$$

But $y = 2^x \Rightarrow 2^x = 1$ or 2

$$2^x = 1 \quad \Rightarrow 2^x = 2^0 \quad \Rightarrow \quad x = 0.$$

Also $2^x = 2 \quad \Rightarrow 2^x = 2^1 \quad \Rightarrow \quad x = 1$

$\therefore$ The truth set of the equation is $\{x : x = 0 \text{ or } 1\}$

Example 7

Find the truth set of the equation $2^{x^2} = 16^{x-1}$

Solution

$$2^{x^2} = 16^{x-1}$$
$$2^{x^2} = 2^{4(x-1)} \quad (\text{Since } 2^4 = 16)$$

Equating exponents

$$x^2 = 4(x-1)$$
$$x^2 - 4x + 4 = 0$$
$$(x-2)^2 = 0$$
$$x = 2 \text{ twice}$$

$\therefore$ The truth set is $\{x : x = 2 \text{ twice}\}$

Example 8

Find the truth set of the equation
$10^x - 4(10^{-x}) = 3$

Solution

$$10^x - 4(10^{-x}) = 3$$
$$10^x - \frac{4}{10^x} = 3. \quad \text{Let } y = 10^x$$
$$y - \frac{4}{y} = 3 \qquad \Rightarrow \qquad y^2 - 4 = 3y$$
$$y^2 - 3y - 4 = 0 \quad \Rightarrow \quad (y-4)(y+1) = 0$$
$$y = -1 \text{ or } 4$$

But $y = 10^x \Rightarrow 10^x = -1$ or 4.

Note that $10^x \neq -1$

To solve $10^x = 4$, we take logarithms of both sides to base 10

$$\log_{10} 10^x = \log_{10} 4$$
$$x \log_{10} 10 = \log_{10} 4$$
$$\{x : x = \frac{\log_{10} 4}{\log_{10} 10} = 0.602 \text{ (3 s.f.)}\}$$

Hence $x = 0.602$.

Example 9

Find the truth set of the equation

$$\left\{\frac{1}{4}\right\}^{3x^2+1} = 128^x$$

Solution

$$\left\{\frac{1}{4}\right\}^{3x^2+1} = 128^x$$

$$2^{-2(3x^2+1)} = 2^{7x} \qquad \text{(Since } 2^7 = 128\text{)}$$

$$-2(3x^2+1) = 7x \qquad \text{(Equating exponents)}$$

$$-6x^2 - 2 = 7x$$

$$6x^2 + 7x + 2 = 0$$

$$(2x+1)(3x+2) = 0$$

$$x = -\tfrac{1}{2}, -\tfrac{2}{3}$$

The truth set is $\{x : x = -\frac{1}{2}, -\frac{2}{3}\}$

Example 10

Solve $3^{2(1+x)} - 3^x = 3^{x+3} - 3$.

Solution

$$3^{2(1+x)} - 3^x = 3^{x+3} - 3$$

$$9.\,3^{2x} - 3^x = 27.\,3^x - 3$$

Let $y = 3^x \Rightarrow 9y^2 - y = 27y - 3$

$$9y^2 - 28y + 3 = 0$$

$$(9y-1)(y-3) = 0$$

$y = 3$ or $\dfrac{1}{9}$. But $y = 3^x$

$\therefore\ 3^x = 3 \Rightarrow x = 1$

or $3^x = \frac{1}{9} = 3^{-2} \Rightarrow x = -2$

Hence $x = 1$ or -2.

Example 11

Find the truth set of the equations

i) $2^{2x+1} - 5(2^x) + 2 = 0$

ii) $3^{2x+3} - 4(3^{x+1}) + 1 = 0$

iii) $3^{2x} - 30(3^{x-2}) + 1 = 0$

iv) $2^{2x} + 3(2^x) - 4 = 0$

v) $2^{2x} - 5(2^x) + 2 = 0$

vi) $3^{2x+1} - 26(3^x) - 9 = 0$

vii) $4^x - 6(2^x) - 16 = 0$

viii) $2^{2x} = 5$

ix) $4^x - 2^{x+1} - 3 = 0$

x) $4(3^{2x+1}) + 17(3^x) - 7 = 0$

Solution

i) The truth set is $\{x : x = -1, 1\}$

ii) The truth set is $\{x : x = -1, -2\}$

iii) The truth set is $\{x : x = -1, 1\}$

iv) The truth set is $\{x : x = 0\}$

v) The truth set is $\{x : x = -1, 1\}$

vi) The truth set is $\{x : x = 2\}$

vii) The truth set is $\{x : x = 3\}$

viii) The truth set is $\{x : x = 1.1610\}$

ix) The truth set is $\{x : x = 1.6\}$

x) The truth set is $\{x : x = -1\}$

Note: In each case work out the missing steps.

3. Exponential equations leading to simultaneous equations

Illustrative examples

Example 1

Find the values of x and y satisfying the equations.

$$x + 6y = 3$$

$$2^{x+y} = \tfrac{1}{4}$$

Solution

$x + 6y = 3$ ....(1)

First we have to reduce the exponential equation to a linear equation,

i.e. $2^{x+y} = \frac{1}{4} \Rightarrow 2^{x+y} = \frac{1}{2^2}$

$$2^{x+y} = 2^{-2}$$

Since the bases are equal, we equate the exponents

$x + y = -2$ ....(2)

Now solving eqn1 and eqn2 simultaneously, we have

$x + 6y = 3$ ....(1)

$x + y = -2$ ....(2)

Eqn1 – eqn2 $\Rightarrow 5y = 5 \Rightarrow y = 1$

Substituting $y = 1$ into eqn2

$x + (1) = -2 \Rightarrow x = -3$

Hence the values of x and y satisfying the above equation are –3 and 1 respectively.

Example 2

Solve the simultaneous equations

$$3^x - 2^{y+2} = 10 \text{ and}$$

$$2^y - 3^{x-2} = 2$$

Giving your answers correct to three significant figures.

Solution

$3^x - 2^{y+2} = 10$

$3^x - 4.2^y = 10 \ldots\ldots\ldots(1)$

$2^y - 3^{x-2} = 2$

$-3^{-2}(3^x) + 2^y = 2$

$-\frac{1}{9}(3^x) + 2^y = 2 \ldots\ldots(2)$

Putting $a = 3^x$ and $b = 2^y$ in (1) and (2), we have

$a - 4b = 10 \quad \Rightarrow \quad a - 4b = 10 \ldots\ldots(3)$

$-\frac{1}{9}a + b = 2 \quad \Rightarrow \quad -a + 9b = 18 \ldots\ldots(4)$

Eqn(3) + eqn(4) $\Rightarrow$ $b = \frac{28}{5}$

Substituting $b = \frac{28}{5}$ into (3)

$a - 4(\frac{28}{5}) = 10 \quad \Rightarrow a = \frac{162}{5}$

But $a = 3^x = \frac{162}{5}$

$\therefore$ Taking logarithms of both sides to base 10

$\log_{10} 3^x = \log_{10} \frac{162}{5} = \log_{10} 162 - \log_{10} 5$

$x \log_{10} 3 = \log_{10} 162 - \log_{10} 5$

$$x = \frac{\log_{10} 162 - \log_{10} 5}{\log_{10} 3}$$

$$= \frac{2.210 - 0.699}{0.4771} = 3.17 \text{ (to 3 s.f)}$$

Also $b = 2^y = \frac{28}{5}$

$\log_{10} 2^y = \log_{10} \frac{28}{5} = \log_{10} 28 - \log_{10} 5$

$y \log_{10} 2 = \log_{10} 28 - \log_{10} 5$

$$y = \frac{\log_{10} 28 - \log_{10} 5}{\log_{10} 2}$$

$$= \frac{1.477 - 0.699}{0.3010} = 2.49 \text{ (to 3 s.f.)}$$

Hence $x = 3.17$ and $y = 2.49$ (to 3 s.f)

Example 3

Solve the simultaneous equations

$$4^{2x+y} = \tfrac{1}{256},$$

$$2^x \bullet 4^y = 8$$

Solution

$4^{2x+y} = \frac{1}{4^4} \quad \Rightarrow \quad 4^{2x+y} = 4^{-4}$

and $2^x \bullet 2^{2y} = 2^3 \Rightarrow 2^{x+2y} = 2^3$

Equating exponents of both sides, we have

$2x + y = -4 \ldots\ldots\ldots\ldots\ldots\ldots(1)$

and $x + 2y = 3 \ldots\ldots\ldots\ldots\ldots\ldots(2)$

From (1), $y = -4 - 2x$

Putting $y = -4 - 2x$ into (2)

$x + 2(-4 - 2x) = 3$

$x - 8 - 4x = 3$

$-3x = 11$

$x = -\frac{11}{3}$

Putting $x = -\frac{11}{3}$ into $y = -4 - 2x$

$y = -4 - 2 \times (-\frac{11}{3})$

$= 4 + \frac{22}{3} = \frac{10}{3}$

$\therefore \quad x = -\frac{11}{3}$ and $y = \frac{10}{3}$

Example 4

Solve the following simultaneous equations

i) $2^{x-y} = 8$ and
$2^{3x-y} = 128$

ii) $8^{4-x} = 2^{1-y}$ and
$3^{x+1} = 9^{-y}$

Solution

i) $x = 2$ and $y = -1$

ii) $x = 3$ and $y = -2$

In each case work out the missing steps.

2. LOGARITHMS

The **logarithm** of any number b to the base a (written $\log_a b$) is the index or power to which the base must be raised to give the number. From the definition,

$\log_a b = c \quad \Rightarrow \quad b = a^c$ or

$b = a^c \quad \Rightarrow \quad \log_a b = c$

provided $a \neq 1$ and a is positive.

For example,

If $16 = 2^4 \quad \Rightarrow \quad \log_2 16 = 4$

Note: The base of logarithms can be written in any positive number except 1.
The logarithm of a negative number does not exist. When the base of a logarithm is not written it means it is in base 10.

Illustrative example

Using the definition of logarithms, find

i) $\log_2 32$ ii) $\log_9 27$

iii) $\log_8 64$ iv) $\log_2 0.25$

Solution

i) Let $\log_2 32 = x$

$2^x = 32 \quad \Rightarrow \quad 2^x = 2^5$

$x = 5 \quad \therefore \quad \log_2 32 = 5$

ii) Let $\log_9 27 = x$

$9^x = 27 \quad \Rightarrow \quad 3^{2x} = 3^3$

$2x = 3 \quad \Rightarrow \quad x = \frac{3}{2} \quad \therefore \quad \log_9 27 = \frac{3}{2}$

iii) Let $\log_8 64 = y$

$8^y = 64 \quad \Rightarrow \quad 8^y = 8^2$

$y = 2 \quad \therefore \quad \log_8 64 = 2$

iv) Let $\log_2 0.25 = y$

$2^y = 0.25 \Rightarrow 2^y = \frac{1}{4}$

$2^y = 2^{-2} \Rightarrow y = -2 \quad \therefore \log_2 0.25$

Laws of logarithms

1. $\log_a x + \log_a y = \log_a (xy)$

E.g. $\log_2 3 + \log_2 4 = \log_2 12$

2. $\log_a x - \log_a y = \log_a \left(\dfrac{x}{y}\right)$

E.g. $\log_2 21 - \log_2 7 = \log_2 (\frac{21}{7}) = \log_2 3$

3. $\log_a x^n = n \log_a x$

E.g. $\log_3 5^2 = 2 \log_3 5$

4. $\log_a x = \dfrac{\log_b x}{\log_b a}$ (change of base)

E.g. $\log_2 8 = \dfrac{\log_5 8}{\log_5 2}$

Note: x and y are any positive numbers.

Properties of logarithms

The following two special properties of logarithms should be noted.

***Property* 1**

$\log_a a = 1$ i.e. the logarithm of any number to the base of the same number is 1.

For example, $\log_{10} 10 = 1$ *and* $\log_2 2 = 1$ etc

***Property* 2**

$\log_a 1 = 0$ i.e. the logarithm of one(1) to any base is zero(0).

For example, $\log_{10} 1 = 0$ *and* $\log_2 1 = 0$

Proof of laws

Law 1 $\log_a x + \log_a y = \log_a (xy)$

Let $\log_a x = b \quad \Rightarrow \quad a^b = x$ ----------(1)

$\log_a y = c \quad \Rightarrow \quad a^c = y$ ----------(2)

Now eqn (1) × eqn (2)

$$a^b \times a^c = xy$$

$$a^{(b+c)} = xy \quad \text{(By the 1}^{\text{st}} \text{ law of indices)}$$

Taking logarithms of both sides to base a, we have

$$\log_a a^{(b+c)} = \log_a xy$$

$$(b+c)\log_a a = \log_a xy$$

But $\log_a a = 1 \Rightarrow b + c = \log_a xy$

Since $b = \log_a x$ and $c = \log_a y$

$$\log_a x + \log_a y = \log_a xy$$

Law 2 $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$

Let $\log_a x = b \Rightarrow a^b = x$ -----------(1)

$\log_a y = c \Rightarrow a^c = y$ -----------(2)

Now eqn (1) ÷ eqn (2)

$$a^b \div a^c = xy$$

$$a^{(b-c)} = \frac{x}{y} \quad \text{(By the 2}^{\text{nd}} \text{ law of indices)}$$

Taking logarithms of both sides to base a, we have

$$\log_a a^{(b-c)} = \log_a\left(\frac{x}{y}\right)$$

$$(b-c)\log_a a = \log_a\left(\frac{x}{y}\right)$$

But $\log_a a = 1 \Rightarrow b - c = \log_a\left(\frac{x}{y}\right)$

Since $b = \log_a x$ and $c = \log_a y$

$$\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$$

Law 3 $\log_a x^n = n\log_a x$

Let $\log_a x = b \Rightarrow a^b = x$

$a^{bn} = x^n$ ---------(1)

Taking logarithms of both sides of eqn(1) to base a, we have

$$\log_a a^{bn} = \log_a x^n$$

$$bn\log_a a = n\log_a x$$

But $\log_a a = 1 \Rightarrow bn = n\log_a x$

Since $b = \log_a x \Rightarrow n\log_a x = \log_a x^n$

$$\log_a x^n = n\log_a x$$

Law 4

$$\log_a x = \frac{\log_b x}{\log_b a} \quad \textbf{(change of base)}$$

Let $\log_a x = c \Rightarrow a^c = x$ ---------(1)

Taking logarithms of both sides of eqn(1) to a new base b, we have

$$\log_b a^c = \log_b x$$

$$c\log_b a = \log_b x$$

$$c = \frac{\log_b x}{\log_b a} \quad \text{but } c = \log_a x$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

In the special case where $x = b$, this identity becomes:

$$\log_a b = \frac{\log_b b}{\log_b a} \quad \text{or} \quad \log_a b = \frac{1}{\log_b a}$$

Illustrative examples

Example 1

Simplify

i) $\log 3 + \log 4$ ii) $\log 6 - \log 2$

iii) $\log 2 + \log 6 - \log 4$

iv) $2\log 3 + \log 2$ v) $\frac{1}{2}\log 4 - \log 6$

vi) $2 - 2\log 5$ vii) $\frac{1}{2}\log 9 + 1$

viii) $\frac{\log 9}{\log 3}$ ix) $\frac{1}{2}\log 25 - 2\log 3 + 2\log 6$

x) $\log(x+1) - \log(x^2 - 1)$

xi) $2\log_a 5 + \log_a 4 - 2\log_a 10$

xii) $\frac{\log 125}{\log 5}$

Solution

i) $\log 3 + \log 4 = \log 12$

ii) $\log 6 - \log 2 = 3$

iii) $\log 2 + \log 6 - \log 4 = \log 3$

iv) $2\log 3 + \log 2 = \log 18$

v) $\frac{1}{2}\log 4 - \log 6 = \log \frac{1}{3}$

vi) $2 - 2\log 5 = \log 10^2 - \log 5^2 = \log 4$

vii) $\frac{1}{2}\log 9 + 1 = \log 30$

viii) $\dfrac{\log 9}{\log 3} = 2$

ix) $\frac{1}{2}\log 25 - 2\log 3 + 2\log 6 = \log 20$

x) $\log(x+1) - \log(x^2 - 1) = -\log(x-1)$

xi) $2\log_a 5 + \log_a 4 - 2\log_a 10 = 0$

xii) $\dfrac{\log 125}{\log 5} = 3$

Note: In each case the final answer has been provided. Work out the missing steps.

Example 2

Simplify

i) $\log_5 12 + \log_5 10$

ii) $2\log_7 9 - \log_7 81$

iii) $\log_3 24 + \log_3 15 - \log_3 10$

iv) $\log_7 98 - \log_7 30 + \log_7 15$

Solution

i) $\log_5 12 + \log_5 10 = \log_5 120$

ii) $2\log_7 9 - \log_7 81 = 0$

iii) $\log_3 24 + \log_3 15 - \log_3 10 = 2\log_3 6$

iv) $\log_7 98 - \log_7 30 + \log_7 15 = 2$

Note: In each case the final answer has been provided. Work out the missing steps.

Example 3

Given that $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4770$, find without using tables, the values of;

i) $\log_{10} 5$ ii) $\log_{10} 3\frac{1}{3}$

iii) $\log_{10} 0.4$

Solution

i) $\log_{10} 5 = \log_{10}\left(\frac{10}{2}\right)$
$= \log_{10} 10 - \log_{10} 2$
$= 1 - 0.3010 = 0.6990$

ii) $\log_{10} 3\frac{1}{3} = \log_{10}\left(\frac{10}{3}\right)$
$= \log_{10} 10 - \log_{10} 3$
$= 1 - 0.4770 = 0.5230$

iii) $\log_{10} 0.4 = \log_{10}\left(\frac{4}{10}\right) = \log_{10} 4 - \log_{10} 10$
$= \log_{10} 2^2 - \log_{10} 10$
$= 2\log_{10} 2 - \log_{10} 10$
$= -\log_{10} 10 + 2\log_{10} 2$
$= -1 + 2(0.3010)$
$= -1 + 0.6020 = \bar{1}.6020$

Example 4

Given that $\log_5 2 = 0.431$ and $\log_5 3 = 0.682$, find the values of

i) $\log_5 6$ ii) $\log_5 1.5$

iii) $\log_5 27$ iv) $\log_5\left(\frac{2}{3}\right)$

v) $\log_5 8$ vi) $\log_5 10$

vii) $\log_5 \sqrt{3}$ viii) $\log_5 12$

ix) $\log_5 13.5$ x) $\log_5\left(\frac{1}{4}\right)$

xi) $\log_5\left(\frac{3}{8}\right) + 2\log_5\left(\frac{4}{3}\right) - \log_5\left(\frac{2}{9}\right)$

Solution

i) $\log_5 6 = 1.113$

ii) $\log_5 1.5 = 0.251$

iii) $\log_5 27 = 2.046$

iv) $\log_5\left(\frac{2}{3}\right) = -0.251$

v) $\log_5 8 = 1.293$

vi) $\log_5 10 = 1.431$

vii) $\log_5 \sqrt{3} = 0.341$

viii) $\log_5 12 = 1.544$

ix) $\log_5 13.5 = 1.615$

x) $\log_5\left(\frac{1}{4}\right) = -0.862$

xi) $\log_5\left(\frac{3}{8}\right) + 2\log_5\left(\frac{4}{3}\right) - \log_5\left(\frac{2}{9}\right) = -0.401$

Note: In each case the final answer has been provided. Work out the missing steps.

Example 5

Evaluate $\log_{\sqrt{3}} 27\sqrt{3}$

Solution

Let $\log_{\sqrt{3}} 27\sqrt{3} = x$

By the definition of logarithms

$$(\sqrt{3})^x = 27\sqrt{3}$$

$$(3^{\frac{1}{2}})^x = 3^3 . 3^{\frac{1}{2}}$$

$$3^{\frac{1}{2}x} = 3^{3\frac{1}{2}} = 3^{\frac{7}{2}}$$

$$\frac{x}{2} = \frac{7}{2} \quad \Rightarrow x = 7$$

$$\therefore \quad \log_{\sqrt{3}} 27\sqrt{3} = 7$$

Example 6

Given that $\log_2 a = 3$, $\log_2 b = -2$ and $\log_2 c = 5$, evaluate $\log_2 8\sqrt{\frac{a^5 b}{c^3}}$

Solution

$$\log_2 8\sqrt{\frac{a^5 b}{c^3}} = \log_2 8 - \log_2 \sqrt{\frac{a^5 b}{c^3}}$$

$= \log_2 2^3 - \log_2\left(\frac{a^5 b}{c^3}\right)^{\frac{1}{2}}$ [1st law of logarithm]

$$= 3\log_2 2 - \tfrac{1}{2}\log_2 \frac{a^5 b}{c^3}$$

$$= 3\log_2 2 - \tfrac{1}{2}(\log_2 a^5 b - \log_2 c^3)$$

$$= 3\log_2 2 - \tfrac{1}{2}(\log_2 a^5 + \log_2 b - \log_2 c^3)$$

$$= 3\log_2 2 - \tfrac{1}{2}(5\log_2 a + \log_2 b - 3\log_2 c)$$

$$= 3 + \tfrac{1}{2}[(5 \times 3) + (-2) - (3 \times 5)]$$

$$= 3 + \tfrac{1}{2}[15 - 2 - 15]$$

$$= 3 + \tfrac{1}{2}(-2) = 3 - 1 = 2$$

$$\log_2 8\sqrt{\frac{a^5 b}{c^3}} = 2$$

Alternatively, $\log_2 a = 3 \Rightarrow a = 2^3$; $\log_2 b = -2 \Rightarrow b = 2^{-2}$ and $\log_2 c = 5 \Rightarrow c = 2^5$.

Substitute these values into the given expression and simplify to arrive at the same answer. i.e.

$$\log_2 8\sqrt{\frac{a^5 b}{c^3}} = \log_2 8\sqrt{\frac{2^{15}2^{-2}}{2^{15}}}$$

$$= \log_2 8\sqrt{2^{-2}}$$

$$= \log_2 2^3 2^{-1} = \log_2 2^2 = 2$$

Example 7

Find the value of $\dfrac{\log_x 64}{\log_x 4}$

Solution

$$\frac{\log_x 64}{\log_x 4} = \frac{\log_x 4^3}{\log_x 4} = \frac{3\log_x 4}{\log_x 4} = 3$$

Example 8

Simplify $\dfrac{(\log 16)(\log 27)}{(\log 9)(\log 64)}$

Solution

$$\frac{(\log 16)(\log 27)}{(\log 9)(\log 64)} = \frac{(\log 4^2)(\log 3^3)}{(\log 3^2)(\log 4^3)}$$

$$= \frac{(2\log 4)(3\log 3)}{(2\log 3)(3\log 4)} = 1$$

Example 9

Find the value of $\log_4 15$.

Solution

Let $\log_4 15 = x \quad \Rightarrow \quad 4^x = 15$

Taking logarithms of both sides to base 10

$$\log_{10} 4^x = \log_{10} 15$$

$$x\log_{10} 4 = \log_{10} 15$$

$$x = \frac{\log_{10} 15}{\log_{10} 4} = \frac{1.761}{0.6021} = 1.954$$

Example 10

Without using tables, evaluate

$\frac{2}{3}\log 64 + 4\log 3 + \frac{1}{3}\log 27 + 3\log \frac{5}{3} - \log 18 - 3\log 10$

Solution

$\frac{2}{3}\log 64 + 4\log 3 + \frac{1}{3}\log 27 + 3\log\frac{5}{3} - \log 18 - 3\log 10$

$= \log(64)^{\frac{2}{3}} + \log 3^4 + \log 27^{\frac{1}{3}} + \log(\frac{5}{3})^3 - \log 18 - \log 10^3$

$= \log 16 + \log 81 + \log 3 + \log(\frac{125}{27}) - (\log 18 + \log 1000)$

$= \log(16 \times 81 \times 3 \times \frac{125}{27}) - \log 18000$

$= \log(\frac{16 \times 81 \times 3 \times 125}{18000 \times 27}) = \log 1 = 0$

Equations involving logarithms

Such equations can be transformed to linear or quadratic equations or simultaneous equations. Some logarithmic equations also require the use of change of base.

1. Logarithmic equations leading to linear or quadratic equations

Illustrative examples

Example 1

Solve the following equations

i) $\log_{10}(x^2+1) - 2\log_{10} x = 1$

ii) $3\log_{10} x + \log_{10} 3 = \log_{10} 81$

iii) $\log_{10}(5x+6) = 2\log_{10}(5x-6)$

iv) $3^x = 8$

v) $2^x = 5$

Solution

i) $\log_{10}(x^2+1) - 2\log_{10} x = 1$

$\log_{10}(x^2+1) - \log_{10} x^2 = 1$

[3rd law of Logarithm]

$\log_{10}\left(\frac{x^2+1}{x^2}\right) = \log_{10} 10$

[2nd law of logarithm ; $\log_{10} 10 = 1$]

$\frac{x^2+1}{x^2} = 10$

[Taking antilog of both sides]

$x^2 + 1 = 10x^2 \quad \Rightarrow \quad 9x^2 = 1$

$x^2 = \frac{1}{9} \quad \Rightarrow \quad x = \pm\frac{1}{3}$

ii) $3\log_{10} x + \log_{10} 3 = \log_{10} 81$

[3rd law of logarithm]

$\log_{10} 3x^3 = \log_{10} 81$

$\log_{10} x^3 + \log_{10} 3 = \log_{10} 81$

[1st law of logarithm]

$3x^3 = 81$

[Taking antilog of both sides]

$x^3 = \frac{81}{3} = 27 \quad \Rightarrow x = 3$

iii) $\log_{10}(5x+6) = 2\log_{10}(5x-6)$

$\log_{10}(5x+6) = \log_{10}(5x-6)^2$

[3rd law of log]

$5x + 6 = (5x-6)^2$

[Taking antilog of both sides]

$5x + 6 = 25x^2 - 60x + 36$

$25x^2 - 65x + 30$

$5x^2 - 13x + 6 = 0$

$(5x-3)(x-2) = 0 \quad \Rightarrow \quad x = \frac{3}{5}$ or 2

iv) $3^x = 8$

For this type of indicial equation, 8 cannot be expressed in indices with base 3. Therefore we take logarithms of both sides to base 10.

$3^x = 8$

$\log_{10} 3^x = \log_{10} 8$

$$x\log_{10}3=\log_{10}8$$

$$x=\frac{\log_{10}8}{\log_{10}3}=\frac{0.9031}{0.4771}=1.892$$

v) $2^x=5$

5 cannot be expressed in indices with base 2. Therefore we take logarithms of both sides to base 10.

$$2^x=5$$

$$\log_{10}2^x=\log_{10}5$$

$$x\log_{10}2=\log_{10}5$$

$$x=\frac{\log_{10}5}{\log_{10}2}=\frac{0.6990}{0.3010}=2.32$$

Example 2

If $\log_{10}(x^3+3)-\log_{10}(x+7)+\log_{10}2=\log_{10}x$, find x.

Solution

Applying the laws of logarithms, we have

$\log_{10}(x^3+3)-\log_{10}(x+7)+\log_{10}2=\log_{10}x$

$$\log_{10}2\left(\frac{x^3+3}{x+7}\right)=\log_{10}x$$

Taking antilog of both sides

$$\frac{2(x^3+3)}{x+7}=x$$

$$2x^3+6=x^2+7x$$

$$2x^3-x^2-7x+6=0$$

Using method of factorization of polynomials (i.e. the factor theorem)

$$(x+2)(x-1)(2x-3)=0$$

$$x=-2,\ 1\text{ or }\tfrac{3}{2}$$

Example 3

Find the truth set of $\log_2(x-3)-\log_2 x=2$, where x is real.

Solution

$$\log_2(x-3)-\log_2 x=2$$

$$\log_2\left(\frac{x-3}{x}\right)=2$$

$$\frac{x-3)}{x}=2^2=4$$

(By the definition of logarithms)

$$x-3=4x$$

$$3x=-3\quad\Rightarrow\quad x=-1$$

$\therefore$ The truth set is $(x : x=-1)$

Example 4

a) If $\log_5 3=0.682$, find the value of

$\log_5(\tfrac{3}{8})+\log_5 15+2\log_5(\tfrac{4}{5})-\log_5(\tfrac{2}{5})$

b) Find the solution set of

$\log_3(2x-1)-\log_3(2x+1)=1$.

Solution

a) $\log_5(\tfrac{3}{8})+\log_5 15+2\log_5(\tfrac{4}{5})-\log_5(\tfrac{2}{5})$

$=\log_5(\tfrac{3}{8}\times15)+\log_5(\tfrac{4}{5})^2-\log_5(\tfrac{2}{5})$

$=\log_5(\tfrac{3}{8}\times15)+\log_5(\tfrac{16}{25})-\log_5(\tfrac{2}{5})$

$=\log_5(\tfrac{3}{8}\times15\times\tfrac{16}{25})-\log_5(\tfrac{2}{5})$

$=\log_5(\tfrac{18}{5})-\log_5(\tfrac{2}{5})$

$=\log_5(\tfrac{18}{5}\div\tfrac{2}{5})=\log_5 9=2\log_5 3$

$=2\times0.682=1.364$

Since $\log_5 3=0.682$

b) $\log_3(2x-1)-\log_3(2x+1)=1$

$\log_3\left(\frac{2x-1}{2x+1}\right)=1$ [2nd law of logarithm]

$\frac{2x-1}{2x+1}=3^1=3$ (By the definition of log)

$$2x-1=3(2x+1)$$

$$2x-1=6x+3$$

$$4x=-4$$

$$x=-1$$

$\therefore$ The truth set is $\{x : x=-1\}$.

Example 5

Find the truth set of the equation $\log_{10}(2x^2-1)=1+\log_{10}(\frac{1}{2}-\frac{x}{10})$.

Solution

$$\log_{10}(2x^2-1)=1+\log_{10}(\tfrac{1}{2}-\tfrac{x}{10})$$

$$\log_{10}(2x^2-1)-\log_{10}(\tfrac{1}{2}-\tfrac{x}{10})=1$$

$$\log_{10}\left(\frac{2x^2-1}{\frac{1}{2}-\frac{x}{10}}\right)=1$$

$$\left(\frac{2x^2-1}{\frac{1}{2}-\frac{x}{10}}\right)=10^1=10 \text{ (By the definition of log)}$$

$$(2x^2-1)=10(\tfrac{1}{2}-\tfrac{x}{10})$$

$$2x^2-1=5-x$$

$$2x^2+x-6=0$$

$$(2x-3)(x+2)=0$$

$$x=-2 \text{ or } \tfrac{3}{2}$$

$\therefore$ The truth set of the equation is $\{x : x=-2 \text{ or } \frac{3}{2}\}$

Example 6

If $\log_{10}(5x+2)-\log_{10}(x-1)=0.7782$, find the value of x.

Solution

$$\log_{10}(5x+2)-\log_{10}(x-1)=0.7782$$

$$\log_{10}\frac{(5x+2)}{(x-1)}=0.7782$$

$$\frac{5x+2}{x-1}=10^{0.7782} \Rightarrow \frac{5x+2}{x-1}=6$$

$$5x+2=6x-6 \Rightarrow x=6+2=8$$

Example 7

Find the possible values of x for which

a) $\log_5(x-5)+\log_5(x-6)=1$

b) $\log_{\sqrt{3}}9=x$

Solution

a) $\log_5(x-2)+\log_5(x-6)=1$

$$\log_5(x-2)(x-6)=1$$

$$(x-2)(x-6)=5$$

$$x^2-8x+12=5$$

$$x^2-8x+7=0$$

$$(x-1)(x-7)=0$$

$$x=1 \text{ and } x=7$$

b) $\log_{\sqrt{3}}9=x$

$$(\sqrt{3})^x=9$$

$$3^{\frac{x}{2}}=3^2$$

$$\frac{x}{2}=2 \Rightarrow x=4$$

Example 8

Solve the equations

i) $\log_{10}(2x^2+5x-2)=1$

ii) $\log_2(3x^2+8x-1)=1$

iii) $\log_3(x^2-x-2)=2\log_3(x+1)$

iv) $\log_2(x^2-5x-6)=3$

v) $\log_{10}(x^2+6x+28)=2$

vi) $\log_{10}(x^2-x-7)=0.1$

vii) $\log_{10}(x-8)=1-\log_{10}(x+1)$

viii) $\log_{10}(x+9)$
$=1+\log_{10}(x+1)-\log_{10}(x-2)$

ix) $\log_4(x^2+x+10)$
$=\frac{1}{2}+\log_4(x^2+x-5)$

x) $\log_{10}(x^2+13x)=1+\log_{10}(1+x)$

Solution

i) $x=\frac{3}{2}$ or -4 ii) $x=\frac{1}{3}$ or -3

iii) no solution iv) $x=7$ or -2

v) $x=6$ or -12 vi) no solution

vii) $x=9$ viii) $x=7$

ix) $x=4$ or -5 x) $x=2$

Note: In each case the final answer has been provided. Work out the missing steps.

2. Logarithmic equations leading to simultaneous equations

Illustrative examples

Example 1

If $3\log_2 a = b$ and $\log_2 4a = b + 4$, where a and b are real numbers, find the values of a and b

Solution

$3\log_2 a = b$ and $\log_2 4a = b + 4$ we can write

$\log_2 a^3 = b$ ------------(1)

$\log_2 4a = b + 4$ --------(2)

Eqn(2) – eqn(1)

$$\log_2 4a - \log_2 a^3 = 4$$

$$\log_2\left(\tfrac{4a}{a^3}\right) = 4 \quad \Rightarrow \quad \log_2\left(\tfrac{4}{a^2}\right) = 4$$

$$\tfrac{4}{a^2} = 2^4 = 16 \quad \Rightarrow 16a^2 = 4$$

$$a^2 = \tfrac{4}{16} = \tfrac{1}{4} \quad \Rightarrow a = \pm\tfrac{1}{2}.$$

We shall take only $\frac{1}{2}$ since $a = -\frac{1}{2}$ will give us logarithm of a negative number which does not exist.

$\therefore\ a = \frac{1}{2}$. Substituting this value into (1)

$b = 3\log_2 \frac{1}{2}$ but $\log_2 \frac{1}{2} = -1 \ \Rightarrow b = -3$

$\therefore\ a = \frac{1}{2}$ and $b = -3$

Example 2

Solve the simultaneous equations

$$xy = 80$$
$$\log x - 2\log y = 1$$

Solution

$xy = 80$ ……….(1)

$\log x - 2\log y = 1$………….(2)

Using the laws of logarithm, (2) becomes

$$\log\left(\frac{x}{y^2}\right) = 1 \ \Rightarrow \ \frac{x}{y^2} = 1 \ \text{ or } x = 10y^2$$

Substituting $x = 10y^2$ into (1) we get

$$10y^3 = 80 \quad \Rightarrow \ y^3 = 8 \quad \Rightarrow \ y = 2$$

Substituting $y = 2$ into (1), we get $x = 40$

$\therefore\ y = 2$ and $x = 40$

Example 3

Solve the simultaneous equations

i) $\log_x y = 2$

$xy = 8$

ii) $2\log y = \log 2 + 3\log x$

$2^y = 4^x$

iii) $\log(x + y) = 0$

$2\log x = \log(y + 1)$

iv) $\log_y x = 2$

$\log_2 x + \log_2 y = 3$

Solution

i) $x = 2$ and $y = 4$

ii) $x = \frac{1}{2}$ and $y = 1$

iii) $x = 1$ and $y = 0$

iv) $x = 4$ and $y = 2$

Note: In each case the final answer has been provided. Work out the missing steps.

3. Logarithmic equations requiring the use of change of base

Some logarithmic equations require the use of change of base as in the examples below.

Note 1: $\log_a x = \dfrac{\log_b x}{\log_b a}$

Note 2: $\log_a x = \dfrac{1}{\log_x a}$

Illustrative examples

Example 1

Solve the equation $\log_3 x - 3\log_x 3 = 2$

Solution

$\log_3 x - 3\log_x 3 = 2$

Changing $\log_x 3$ to base 3, we have

$$\log_3 x - \frac{3\log_3 3}{\log_3 x} = 2$$

$$\log_3 x - \frac{3}{\log_3 x} = 2$$

$$(\log_3 x)^2 - 3 = 2\log_3 x$$

(i.e. Multiplying through by $\log_3 x$)

$$(\log_3 x)^2 - 2\log_3 x - 3 = 0$$

Now let $\log_3 x = y$

$$y^2 - 2y - 3 = 0$$

$$(y-3)(y+1) = 0$$

$$y = 3 \text{ or } -1$$

But $y = \log_3 x$

$$\log_3 x = 3 \quad \Rightarrow \quad x = 3^3 = 27$$

and $\log_3 x = -1 \quad \Rightarrow \quad x = 3^{-1} = \frac{1}{3}$

$\therefore x = \frac{1}{3}$ or 27

Example 2

Find the truth set of the equation

$\log_3 x - 4\log_x 3 + 3 = 0$

Solution

Changing the base of $\log_x 3$ to base 3,

we have $\log_x 3 = \frac{1}{\log_3 x}$

(Refer to change of base identity)

So the given equation becomes

$$\log_3 x - \frac{4}{\log_3 x} + 3 = 0$$

Multiplying through by $\log_3 x$.

$$(\log_3 x)^2 - 4 + 3\log_3 x = 0$$

Let $y = \log_3 x$

$$y^2 + 3y - 4 = 0$$

$$(y-1)(y+4) = 0$$

$$y = -4 \text{ or } 1$$

But $y = \log_3 x$

i.e. either $\log_3 x = -4$ or $\log_3 x = 1$.

Therefore $x = 3^{-4} = \frac{1}{81}$ or $x = 3$

GRAPHS OF EXPONENTIAL RELATIONS AND THEIR APPLICATIONS

Introduction

Very often, it is necessary to find a relationship between two connected variables. A table of experimental data, showing corresponding values of the two given quantities and a suggestion as to the form of the expected relationship are given. The data are usually displayed and, if the expected relationship is confirmed, the graph is used to obtain the unknown constants in the relationship.

Linear relations

The simplest situation is when the expected relation is a linear one of the form $y = mx + c$.

In this case we simply plot y against x. If the set of points plotted lie approximately in a straight line then the given linear relation is approximately true. m is the gradient of the line and c is the intercept on the y – axis.

Exponential relations

If it is assumed that a certain exponential relationship exists between two variable quantities x and y for example, then a graph of these two quantities will lie on a curve. It is therefore advisable to reduce the exponential relationship to a linear form such as $y = mx + c$ which gives a straight-line graph. The reduction of the exponential relation to the linear form usually requires the **application of logarithms**.

The following examples illustrate some of the exponential relationships which can be reduced to the linear forms. The following exponential relations will be considered:

1. $y = ax^n$ 2. $y = ab^x$

1. The relationship $y = ax^n$

Given $y = ax^n$ where a and n are constants, we can reduce the equation to a linear form by taking logarithms of both sides to base 10

i.e. $\log_{10} y = \log_{10} ax^n = \log_{10} a + \log_{10} x^n$

$\log_{10} y = \log_{10} a + n\log_{10} x$ ------------(1)

Equation (1) is of the form $Y = mX + c$ where $Y = \log_{10} y$, $X = \log_{10} x$, $m = n$ and $c = \log_{10} a$

Hence a graph of $\log_{10} y$ against $\log_{10} x$ will give a straight line with gradient n.

Illustrative examples

Example 1

The table below gives the corresponding values of two related variables x and y.

x	1.0	1.5	2.0	2.5	3.0
y	2.00	8.27	22.26	49.39	93.53

The relation between x and y is of the form $y = ax^n$, where a and n are constants. Draw a suitable linear graph and use it to estimate, correct to two decimal places, the values of a and n.

Solution

Given $y = ax^n$ where a and n are constants.

Taking logarithms of both sides to base 10.

$\log_{10} y = \log_{10} ax^n = \log_{10} a + \log_{10} x^n$

$\log_{10} y = \log_{10} a + n\log_{10} x$ ------------(1)

Equation (1) is of the form $Y = mX + c$ where $Y = \log_{10} y$, $X = \log_{10} x$, $m = n$ and $c = \log_{10} a$

A graph of $\log_{10} y$ against $\log_{10} x$ will give a straight line with gradient n.

Therefore we have to form a new table of $\log_{10} y$ and $\log_{10} x$ using the x and y values given.

$X = \log_{10} x$	0	0.176	0.301	0.398	0.477
$Y = \log_{10} y$	0.301	0.917	1.355	1.694	1.971

Use these new values to draw a graph of $\log_{10} y$ against $\log_{10} x$ as shown on the next page. From the graph, the gradient = 3.45

$\therefore$ $n = 3.45$ to 2 decimal places.

$\log_{10} y = \log_{10} a + 3.45\log_{10} x$.......(2)

But from the graph when $\log_{10} x = 0$; $\log_{10} y = 0.301$

Therefore substitute these values into (2) to determine the value of a.

i.e. $0.0301 = \log_{10} a + 3.45(0)$

$0.301 = \log_{10} a$

Taking antilogarithm of both sides

$a = 2.00$ to two decimal places.

$\therefore$ $y = 2x^{3.45}$

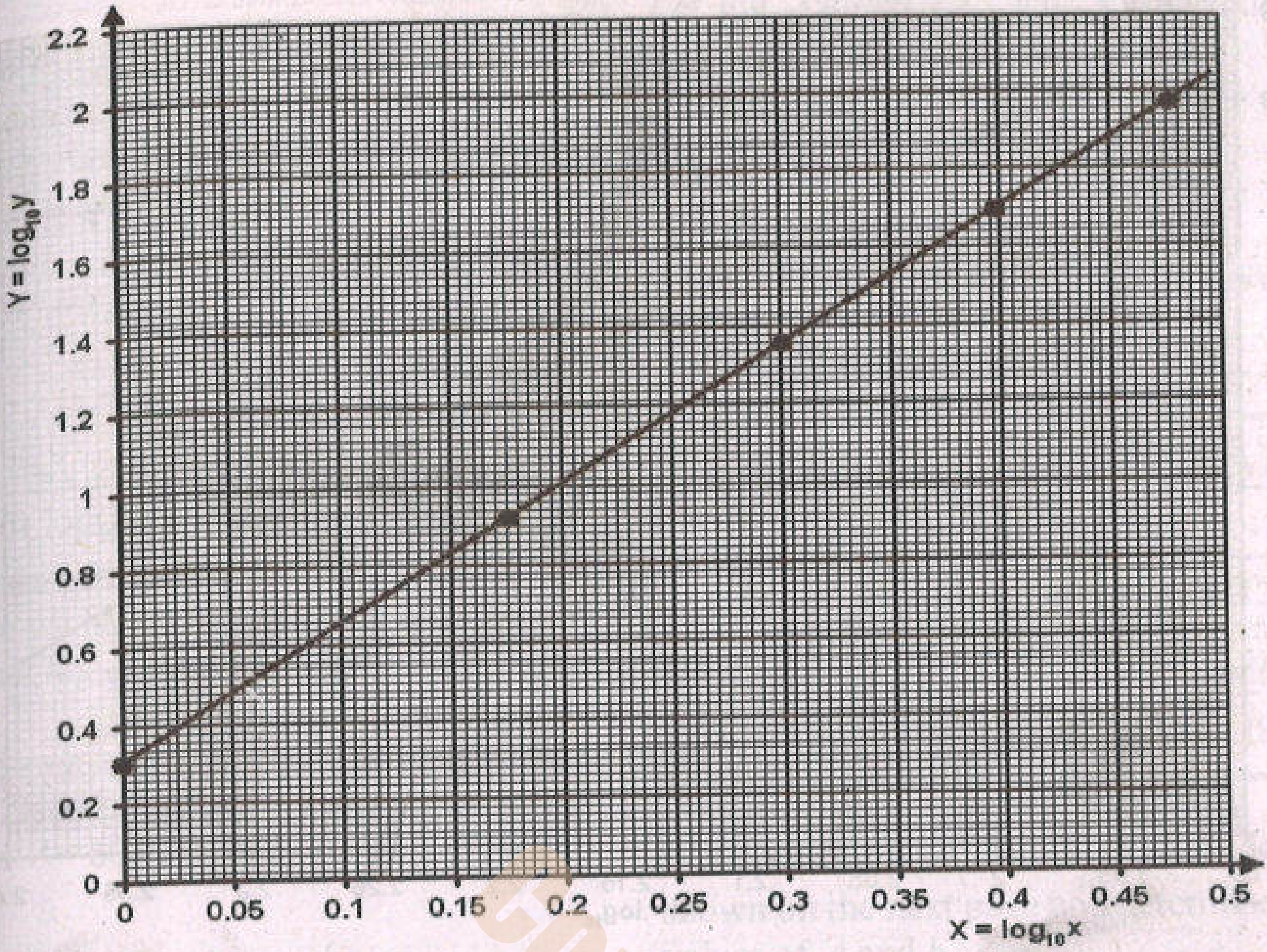
***Note 1*:** The points rarely lie exactly on a straight line, so the **line of best fit** is drawn 'by eye'.

The values of m and c can be found from the graph by either

a) finding the gradient (m) of the line and its intercept on the y – axis (c) or

b) substituting the coordinates of two points on the line into the linear equation formed and solving the resulting simultaneous equations.

***Note 2*:** Do not use given values to find m and c since they may not give points on the line of 'best fit'.



Example 2

A given mass of air expands adiabatically and the following measurements are taken of the pressure P and volume V.

V	100	125	150	175	200
P	58.6	42.4	32.8	27.0	22.3

Confirm that $P = kV^n$ and determine the values of the constants k and n.

Solution

Assuming that $P = kV^n$ and taking logarithms to base 10 of each side, we have

$$\log_{10} P = \log_{10} V^n + \log_{10} k$$

$$\log_{10} P = n\log_{10} V + \log_{10} k \text{ --------(1)}$$

Putting $\log_{10} P = Y$, $\log_{10} V = X$ and $\log_{10} k = c$ we have $Y = nX + c$

$\therefore$ A graph of $\log_{10} P$ against $\log_{10} V$ is a straight line with gradient n as shown on the next page. Therefore we have to form a new table of $\log_{10} V$ and $\log_{10} P$.

$X = \log_{10} V$	2.000	2.097	2.176	2.243	2.301
$Y = \log_{10} P$	1.768	1.627	1.516	1.431	1.348

Use these new values to draw a graph of $\log_{10} P$ against $\log_{10} V$ as shown on the next page.

The straight line graph obtained confirms that $P = kV^n$.

From the graph the gradient is -1.41

$\therefore\ n = -1.41$.

Substituting the value of n into equation (1)

$$\log_{10} P = -1.41\log_{10} V + \log_{10} k$$

But from the graph, when $\log_{10} V = 2$, $\log_{10} P = 1.768$

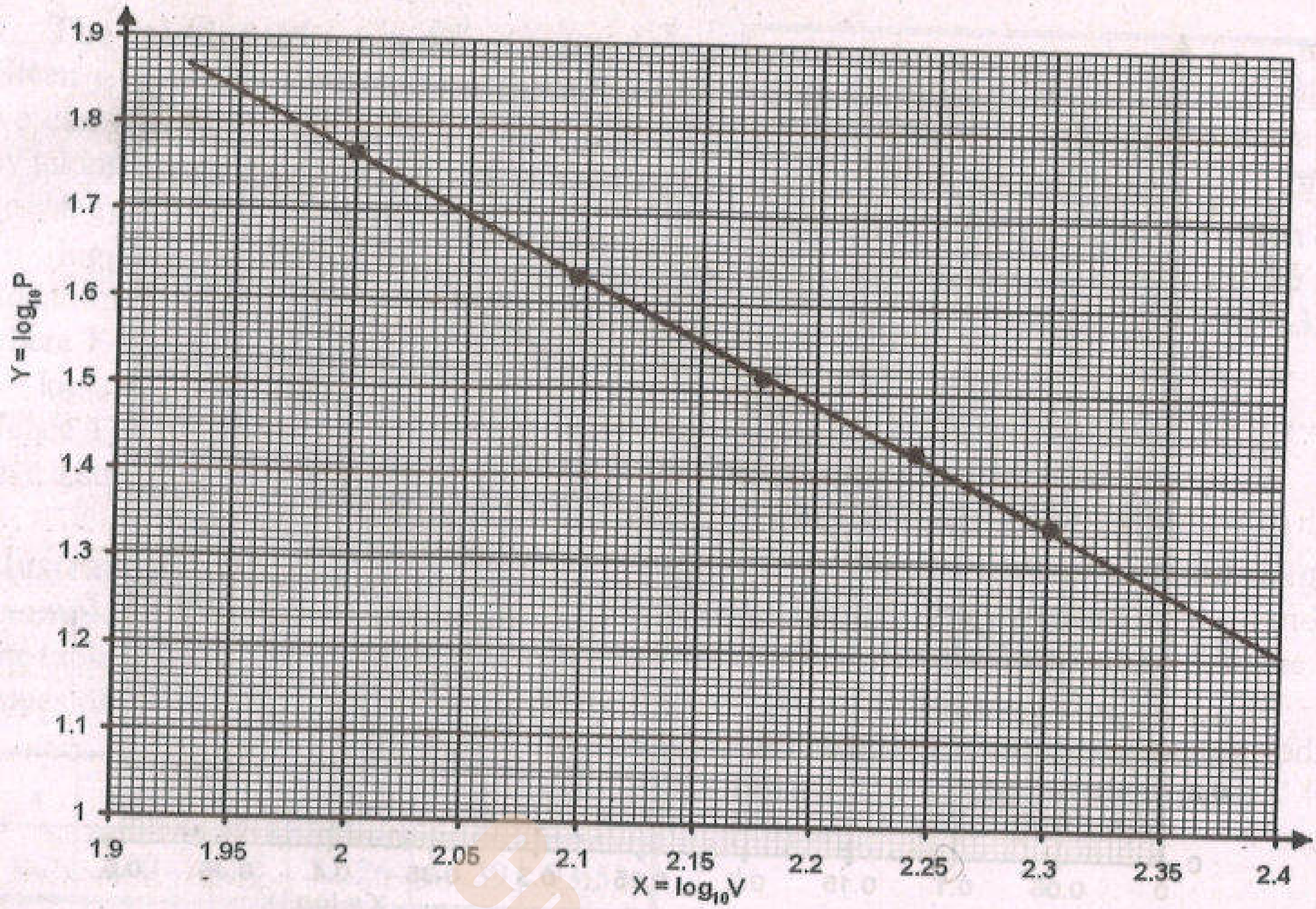
$$\therefore\ 1.768 = (-1.41)(2) + \log_{10} k$$

$$\log_{10} k = 1.768 + 2.820 = 4.588.$$

Taking antilogarithm of both sides

$$k = 38820$$

$$\therefore\ P = 38820V^{-1.41}$$



Example 3

The relation between the pressure p and the volume v of a mass of gas is given by the formula $pv^n = c$ where n and c are constants. If $p = 180$ when $v = 8$ and $p = 80$ when $v = 12.4$. Find the values of n and c.

Solution

Given that $c = pv^n$, taking logarithms of both sides to base 10

$$\log c = \log pv^n = \log p + \log v^n$$
$$= \log p + n\log v$$

From the data given, $p = 180$ when $v = 8$

$$\log c = \log 180 + n\log 8 \text{ ....(1)}$$

Also $p = 80$ when $v = 12.4$

$$\log c = \log 80 + n\log 12.4 \text{ ....(2)}$$

Eqn(1) – eqn(2)

$$0 = \log 180 - \log 80 + n(\log 8 - \log 12.4)$$
$$n(\log 12.4 - \log 8) = \log 180 - \log 80$$
$$n = \frac{\log 180 - \log 80}{\log 12.4 - \log 8} = 1.850$$

Substituting this value into eqn(1)

$$\log c = \log 180 + 1.85\log 8$$
$$c = 8.433$$

Hence $n = 1.850$ and $c = 8.433$

Example 4

The pressure, P and volume, V of a gas under certain conditions are related by the equation $P = kV^n$. The following experimental results were obtained.

V	200	150	100	80	50
P	76	115	200	280	530

a) Express the relation $P = kV^n$ in the form $y = mx + c$. By drawing a suitable graph, find the values of k and n, correct to one decimal place.

b) Use your graph in (a) to find, correct to the nearest whole number, the value of P when $V = 60$.

Solution

Given that $P = kV^n$ and taking logarithms of both sides to base 10

$$\log P = \log kV^n = \log k + \log V^n$$
$$= \log k + n\log V$$

Let $\log_{10} P = Y$, $\log_{10} V = X$ and $\log_{10} k = c$

$\therefore\ Y = c + nX = mx + c$

A graph of $\log_{10} P$ against $\log_{10} V$ is a straight line with gradient n.
Therefore we have to form a new table of $\log_{10} V$ and $\log_{10} P$ and use it to draw a graph.

From the graph;

a) $k = 79, 432.9$ and $n = -1.4$

b) when $V = 60$, $P = 355$

Example 5

The following corresponding values of x and y are believed to be related by the equation $y = x + ax^b$

x	2	3	4	5	6	10
y	7.54	9.33	11.00	12.59	14.12	719.90

Draw a suitable graph to show that this may be so and use your graph to find the probable values of a and b.

Solution

$$y = x + ax^b$$
$$y - x = ax^b$$

Taking logarithms gives:

$$\log_{10}(y - x) = \log_{10} a + b\log_{10} x$$

Let $\log_{10}(y - x) = Y$, $\log_{10} x = X$ and $\log_{10} a = c$.

$\therefore\ Y = c + bX$

A graph of $\log_{10}(y - x)$ against $\log_{10} x$ is a straight line with gradient b.
Therefore we have to form a new table of $\log_{10}(y - x)$ and $\log_{10} x$.

x	2	3	4	5	6	10
y	7.54	9.33	11.00	12.59	14.12	719.90
$y - x$	5.54	6.33	7.00	7.59	8.12	9.90
$\log_{10}(y-x)$	0.74	0.80	0.85	0.88	0.91	1.00
$\log_{10} x$	0.30	0.48	0.60	0.70	0.78	1.00

Plot a graph of $\log_{10}(y - x)$ against $\log_{10} x$ as shown on the next page and determine the values of a and b.

From the graph,

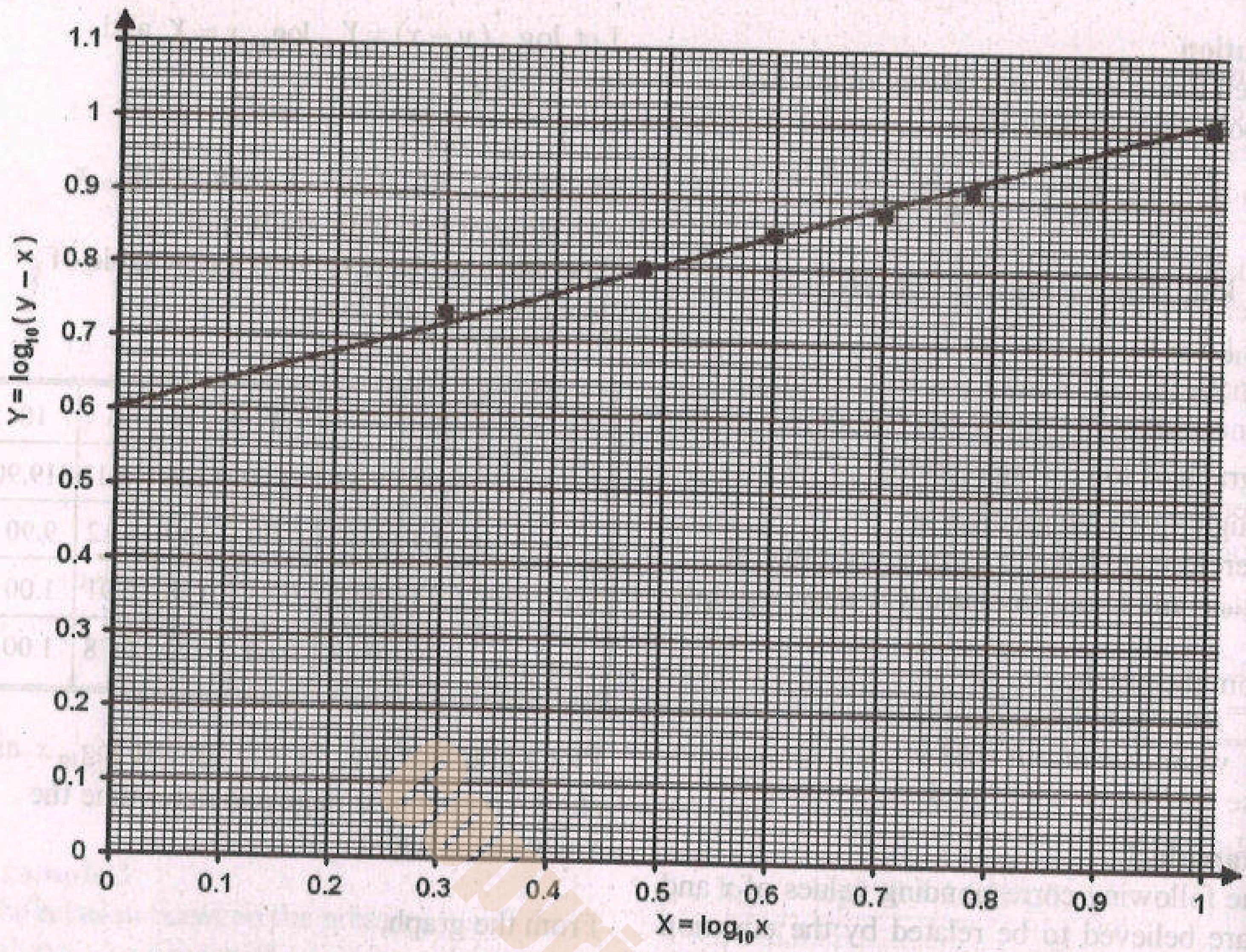
the gradient gives: $b = \dfrac{1.1 - 0.6}{1} = 0.50$

and the intercept on the Y-axis is:

$\log_{10} a = 0.6 \ \Rightarrow\ a = 3.98$

$\therefore\ \ y = x + 3.98x^{0.50}$

Note that this graph starts from (0, 0), therefore we can easily use the intercept on the y-axis.



2. The relationship $y = ab^x$

Given that $y = ab^x$, where a and b are constants we can reduce the equation to the linear form by taking logarithms of both sides to base 10.

i.e. $\log_{10} y = \log_{10} ab^x = \log_{10} b^x + \log_{10} a$

$\log_{10} y = x\log_{10} b + \log_{10} a$ ------------(1)

Equation (1) is of the form $Y = mx + c$ where $Y = \log_{10} y$, $m = \log_{10} b$ and $c = \log_{10} a$

$\therefore$ A graph of $\log_{10} y$ against x will give a straight line with gradient $\log_{10} b$.

Note that in this case the variable x is an exponent.

Illustrative examples

Example 1

The table below gives the values of two related variables x and y.

x	2	3	4	5	6
y	10.74	16.22	24.32	36.41	54.70

a) Assuming that the relation between x and y is of the form $y = mn^x$ where m and n are constants, draw a suitable linear graph for the relation.

b) Use your graph to estimate, correct to two significant figures the values of m and n.

Solution

Given $y = mn^x$ and taking logarithms of both sides to base 10 we have,

$\log_{10} y = \log_{10} mn^x = \log_{10} n^x + \log_{10} m$

$\log_{10} y = x\log_{10} n + \log_{10} m$ ----------(1)

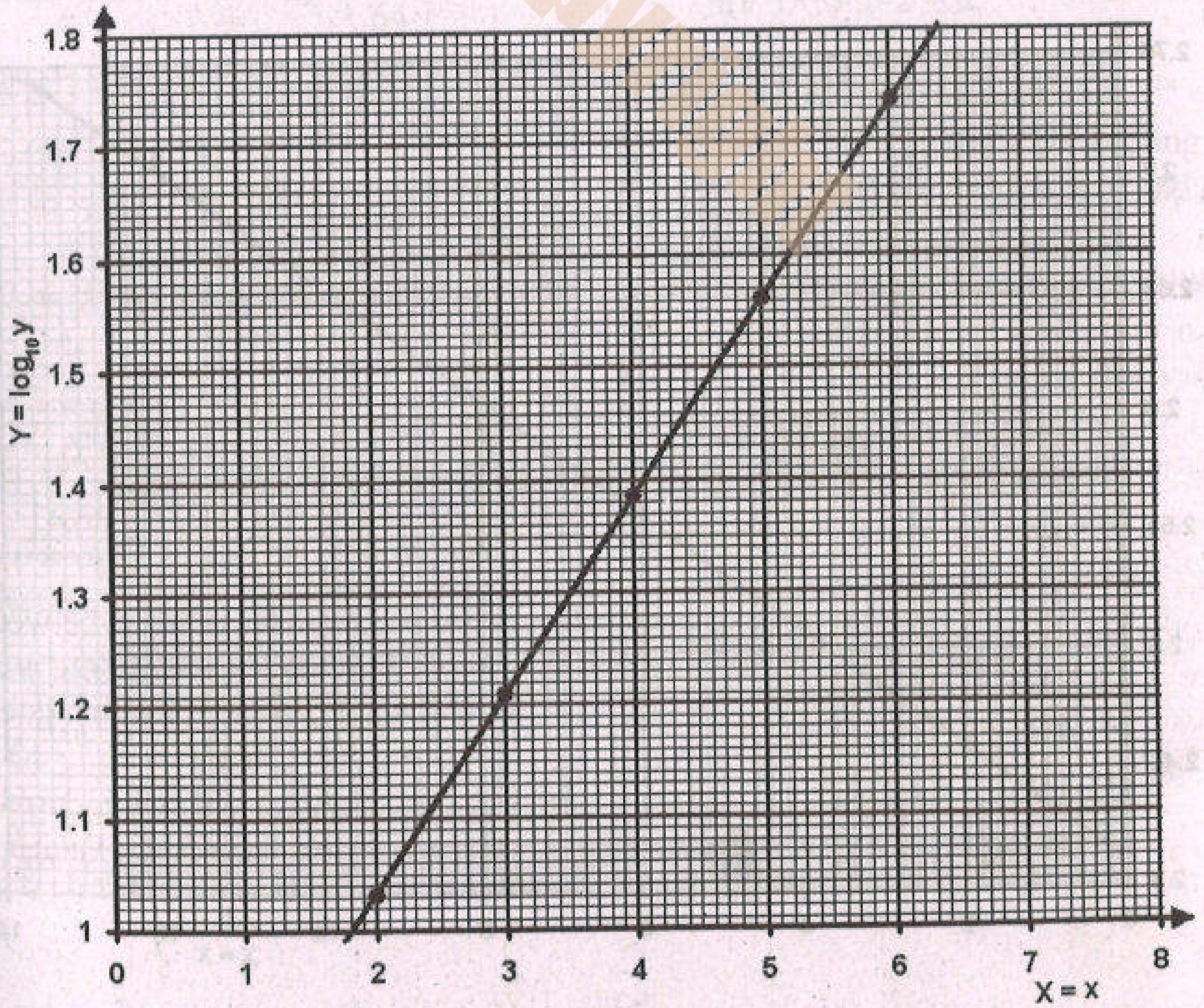
Equation (1) is of the form $Y = ax + c$ where $Y = \log_{10} y$, $X = x$, $a = \log_{10} n$ and $c = \log_{10} m$,

Hence a graph of $\log_{10} y$ against x will give a linear graph with gradient $\log_{10} n$.

Therefore we have to form a new table of $\log_{10} y$ and x.

$X = x$	2	3	4	5	6
$Y = \log_{10} y$	1.031	1.210	1.386	1.561	1.738

Use these new values to draw a graph of $\log_{10} y$ against x as shown below.



From the graph the gradient is 0.175

$\therefore \log_{10} n = 0.175$

(Since $\log_{10} n$ is the gradient).

Taking antilogs of both sides we have,

$n = 1.496 \Rightarrow n = 1.5$ to two significant figures

$\therefore$ From eqn(1),

$\log_{10} y = (0.175)x + \log_{10} m$ --------- (2)

When $x = 2$; $\log_{10} y = 1.031$.

Substituting into (2)

$1.031 = (0.175)2 + \log_{10} m$

$\log_{10} m = 0.681$

$\therefore$ Taking antilogarithm of both sides

$m = 4.797 = 4.8$ to two significant figures.

Example 2

The table below gives the values of two related variables x and y.

x	0	2	4	5	7	9	11	12
y	256	287	323	342	384	431	483	512

Assuming that x and y are related by a law of the form $y = ab^x$, determine the values of the constants a and b.

Solution

Assuming that $y = ab^x$ and taking logarithms of both sides to base 10, we have

$\log_{10} y = \log_{10} b^x + \log_{10} a$

$\log_{10} y = x\log_{10} b + \log_{10} a$ --------(1)

Equation (1) is of the form $Y = mx + c$ where $Y = \log_{10} y$, $X = x$, $m = \log_{10} b$ and $c = \log_{10} a$.

A graph of $\log_{10} y$ against x will give a straight line with gradient $\log_{10} b$.

Therefore we have to form a new table of $\log_{10} y$ and x.

$X = x$	0	2	4	5	7	9	11	12
$Y = \log y$	2.41	2.46	2.51	2.53	2.58	2.63	2.68	2.71

The gradient of the line is 0.0252

$\therefore \ \log_{10} b = 0.0252$

Taking antilogarithm of both sides

$b = 1.059$ (Use antilog tables)

$\therefore \ \log_{10} y = 0.0252x + \log_{10} a$

Now taking a point on the straight line, $x = 0$, $\log_{10} y = 2.408$

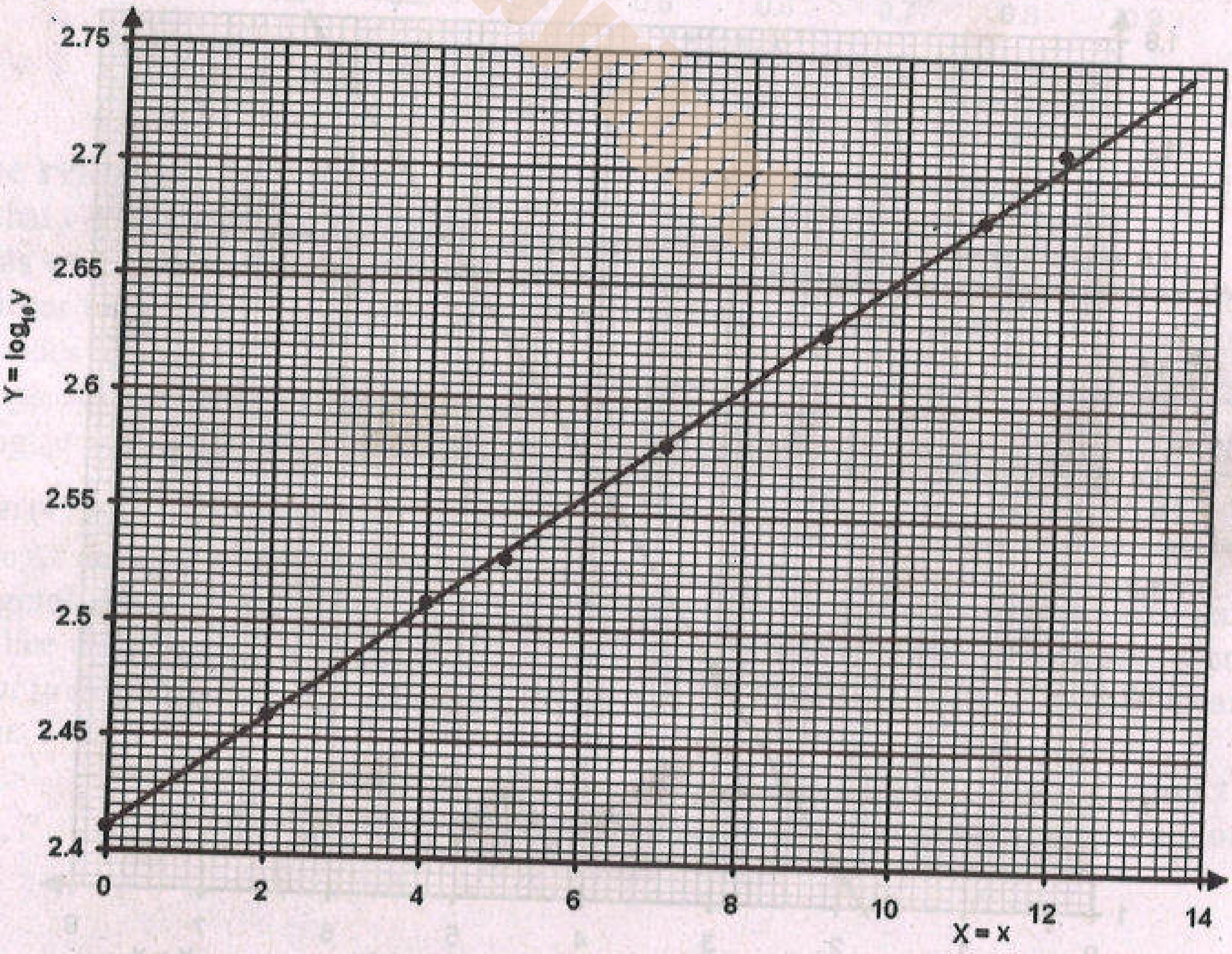
$2.408 = 0.0252(0) + \log_{10} a$

$\log_{10} a = 2.408$

Taking antilogarithm of both sides

$a = 256$

$\therefore \ y = 256\,(1.059)^x$



Review exercise 7.1

1. If $3^{5x+3} = 81^{x+3}$, find x.

2. If $2^{9x+4} = \frac{1}{32}$, find x.

3. Without using tables evaluate the following:

   i) $64^{-\frac{2}{3}}$ ii) $\left(\frac{9}{4}\right)^{-\frac{3}{2}}$ iii) $81^{\frac{3}{4}}$

   iv) $125^{1\frac{2}{3}}$ v) $\left(1\frac{7}{9}\right)^{\frac{1}{2}}$ vi) $(-27)^{-\frac{2}{3}}$

   vii) $(0.027)^{\frac{2}{3}}$

4. Given $125^{(x+2)} = 5^{2y}$, find x in terms of y.

5. If $2^{-n} = x$, find 2^{n+1}

6. Simplify the following:

   i) $288^{\frac{1}{2}} \div 64^{\frac{1}{3}}$ ii) $\left(\frac{27}{64}\right)^{-\frac{2}{3}}$

   iii) $8^n \times 2^{2n} \div 4^{3n}$ iv) $32^{-\frac{2}{5}}$

   v) $\left(\frac{256}{81}\right)^{-\frac{3}{4}}$

7. Without using tables evaluate

   i) $\left(2\frac{1}{4}\right)^{\frac{1}{2}}$ ii) $(0.04)^{-\frac{3}{2}}$

   iii) $\frac{27^{\frac{1}{2}} \times 243^{\frac{1}{2}}}{243^{\frac{4}{5}}}$ iv) $\frac{6^{\frac{1}{2}} \times 1296^{\frac{1}{4}}}{216^{\frac{1}{4}}}$

8. Find the value of x in the following equations:

   i) $32^x = 0.25$ ii) $8^{2x+1} = \frac{1}{512}$

   iii) $\left(\frac{1}{8}\right)^{4x-9} = 64^{3x+1}$ iv) $3^{3x} = 729$

   v) $10^x = 0.001$

9. Find, correct to two decimal places, the values of x for which $5^{2x} - 5^{(x+1)} + 4 = 0$.

10. Find the truth set of $2^{2x} - 3(2^{x+1}) + 5 = 0$.

11. Find the truth set of $(2^x)^{½} = 8^x$.

12. Solve the equation $2^{2x+8} - 32(2^x) + 1 = 0$.

13. Find the truth set of the equation $4^x + 2 = 3(2^x)$, where x is a real number. {Hint: $4^x = 2^{2x} = (2^x)^2$}

14. Find, correct to three significant figures, the truth set of $2^x - 27(2^{-x}) = 6$.

15. Find the truth set of the equations

    i) $2^{2+2x} + 3(2^x) - 1 = 0$

    ii) $4^x - 5(2^x) + 4 = 0$

16. Find the truth set of the equation $6(9^x) + 3^x - 2 = 0$.

17. If $3^m \times 3^n = 243$ and $3^m \div 3^{2n} = 9$ write down two equations connecting m and n. Hence find the values of m and n.

18. Find the value of x in the following equations:

    i) $3^{3x} - 3^{x+1} - 3^x + 3 = 0$

    ii) $2^{2x} + 4(2^x) - 32 = 0$

    iii) $3^{2(x-1)} - 8(3^{x-2}) = 1$

    iv) $4^x - 3.2^{x+1} = -8$

    v) $2^{2x-3} - 2^{x-2} - 1 = 0$

    vi) $2^x + 2^{-x} = 2$

19. Simplify the following:

    i) $\frac{27^{n+2} - 6.3^{3n+3}}{3^n 9^{n+2}}$

ii) $\dfrac{(x^{\frac{3}{2}}+x^{\frac{1}{2}})(x^{\frac{1}{2}}-x^{-\frac{1}{2}})}{(x^{\frac{3}{2}}-x^{\frac{1}{2}})^2}$

iii) $\dfrac{x}{y^{\frac{1}{2}}+x^{\frac{1}{2}}}+\dfrac{x}{y^{\frac{1}{2}}-x^{\frac{1}{2}}}$

iv) $\dfrac{\sqrt{(a^4b^5)}\times b^{\frac{1}{3}}}{a}$

v) $\left(\dfrac{8a^3}{27b^6}\right)^{-\frac{4}{3}}$

20. Solve for x in the following

i) $5x^{\frac{2}{3}}=80$

ii) $3^x\times 2^{x-1}=1$

iii) $8^{\frac{x}{2}}=2^{\frac{3}{8}}\times 4^{\frac{3}{4}}$

21. Solve the simultaneous equations

i) $\left(\dfrac{1}{3}\right)^{2x}=\left(\dfrac{1}{9}\right)^{y+2}$ and $3^{x-1}=27^{y-1}$

ii) $3^{x+y}=243$ and $3x+y=9$

22. Solve for x correct to two significant figures, the equation $4^x-2^{x+1}-3=0$

23. Without using tables evaluate

i) $\left(\dfrac{16}{49}\right)^{\frac{1}{2}}$ ii) $(121)^{\frac{3}{2}}$

iii) $\left(\dfrac{100}{9}\right)^{-\frac{3}{2}}$ iv) $\left(\dfrac{27}{8}\right)^{\frac{2}{3}}$

v) $\left(\dfrac{125}{27}\right)^{-\frac{2}{3}}$ vi) $(0.36)^{\frac{1}{2}}$

vii) $(0.64)^{-\frac{1}{2}}$ viii) $\left(\dfrac{25}{9}\right)^{-\frac{3}{2}}$

24. Solve for the equation
$2^{2x+1}-9(2^x)+4=0$

25. Solve the following simultaneous equations

i) $3^{x+1}=27^{y-1}$ and $\left(\frac{1}{3}\right)^{2x}=\left(\frac{1}{9}\right)^{y+2}$

ii) $2^{5x+1}=16^{y+1}$ and $2^{x-13}=32^{1-y}$

iii) $2^{2x+y}=32$ and $2^{2(x-y)}=4$

iv) $3^x-2^{y+2}=49$ and $2^y-3^{x-2}+1=0$

v) $2^{x-y}=8$ and $2^{3x-y}=128$

vi) $8^{4-x}=2^{1-y}$ and $9^{-y}=3^{2x+y}$

26. Solve the equation $3^x=4^{x-2}$ giving your answer to three significant figures.

27. Solve the equation
$3^{2y}-3^{y+1}-3^y+3=0$

28. Find the truth set of the equations

i) $2^{2x}-3(2^x)+2=0$

ii) $2^{2x}-5(2^x)+4=0$

iii) $2^{2x+3}+1=9(2^x)$

iv) $3^{2x-3}-4(3^{x-2})+1=0$

v) $3^{2x-1}-28(3^{x-2})+1=0$

vi) $3^x+3^{x-1}=4$

vii) $4(4^x-2^x)+1=2^x$

viii) $5^{x-1}=0.2$

ix) $3^{x^2+2}=27^x$

x) $3^{2x+1}-3^{x+2}-3^{x+1}+1=0$

xi) $3^{2x+1}+26(3^x)-9=0$

xii) $2^{2x+1}-33(2^x)+16=0$

xiii) $2^{2x+1}-15(2^x)=8$

xiv) $2^{2x+2}+8=33(2^x)$

xv) $2^x+2^{-x}=2$

xvi) $5^{2x}+1=26(5^{x-1})$

29. Solve the equation
$3^{2x}-30(3^x)+81=0$.

30. If $x=\frac{1}{81}$, $p=\frac{1}{4}$ and $q = 3$, find the value of x^{pq}

31. If $27^{x+3} \div 81^8 = \frac{1}{9}$, find the value of x.

32. Solve the following exponential equation.

i) $\frac{5^{x+1}}{5^{1-x}} = \frac{1}{25^x}$

ii) $\left(\frac{1}{3}\right)^x 27^{x+1} = 9^{3-2x}$

iii) $\left(\frac{1}{3}\right)^{x^2} = 27^{(2/3)x-1}$

iv) $32^{1-x} = 8^{x^2-x}$

v) $125^{2-3x} = 25^{3x-5}$

vi) $(0.25)^{2x+1} = 8^{2-x}$

vii) $(0.2)^{x-3} = 25^{1-x}$

viii) $4^x \cdot 16^{2-3x} = 8$

Review exercise 7.2

1. a, b, x and y are real positive numbers

i) If $\log_{10} a = x$ and $\log_{10} b = y$, express ab in terms of x and y. Deduce that $\log_{10} ab = \log_{10} a + \log_{10} b$.

ii) If $a = 2^{4.136}$ and $2 = 10^{0.3010}$, calculate $\log_{10} 2a$ to three significant figures.

2. Without using tables, simplify
$$\frac{(\log 27)(\log 36)}{(\log 216)(\log 3}$$

3. Given $\log_3 5 = 1.465$, evaluate, without using tables $\log_3 25 + \log_3 15$.

4. Find the truth set of

i) $\log_{10}(10 + 9x) - \log_{10}(11 - x) = 2$

ii) $\log(2x + 3) = \log 43$

5. Find the solution set of the following simultaneous equations.
$2\log_3 x - 3\log_3 y = 7$ and
$\log_3 x - 2\log_3 y = 4$

6. If $2\log y - \log 2x = \log 2(y - x)$, express y in terms of x.

7. Without using logarithms tables show that $\log 27 + \log 16 - \log 9 - \log 12 = 2\log 2$

8. Solve the equation

i) $\log_{10}(x^2 + 1) - 2\log_{10} x = 1$

ii) $\log_{81} x = -\frac{1}{2}$

9. Use logarithm to evaluate
$$\frac{86.19 \times (0.0462)^2}{\sqrt{0.846}}$$

10. Given that $\frac{\log(x-3)^{4x+1}}{\log(x-3)} = 9$, find x.

11. Without using tables, simplify
$$\frac{\log_2 8 + \log_2 16 - 4\log_2 2}{\log_4 16}$$

12. Simplify $\log_2 5 . \log_5 2$

13. Solve the equation

i) $\log_{\frac{1}{2}} 8 = x$

ii) $\log_{10} x - \frac{1}{2}\log_{10} 2 = \frac{1}{2}\log_{10}(x+4)$

iii) $3\log_3 9 = x$

14. Express the following in terms of x and y.

i) $2\log_y x = 3$

ii) $\log y + 2\log x \le \log x$

15. Evaluate the following

i) $(\log_5 625)^2 - \log_5 25$

ii) $\frac{1}{3}\log_2 64$

iii) $3 + 2\log_3 243$

16. Given that $\log_7 2 = 0.356$ and $\log_7 3 = 0.565$, find the values of

i) $\log_7 6$

ii) $\log_7 1.5$

iii) $\log_7 8$

iv) $\log_7 18$

v) $\log_7 12$

vi) $\log_7 0.5$

vii) $\log_7 \sqrt{3}$

viii) $\log_7 13.5$

ix) $\log_7 \left(\frac{1}{4}\right)$

x) $\log_7 \left(\frac{3}{8}\right) + 2\log_7 \left(\frac{4}{7}\right) - \log_7 \left(\frac{2}{7}\right)$

17. Simplify the following:

i) $\log_5 12 + \log_5 10$

ii) $2\log_7 9 - \log_7 81$

iii) $\log_3 24 + \log_3 15 - \log_3 10$

iv) $\log_7 98 - \log_7 30 + \log_7 15$

v) $\log_5 \sqrt{6} + \log_5 9$

vi) $\dfrac{\log \sqrt{27} - \log \sqrt{8}}{\log 3 - \log 2}$

18. Evaluate $\dfrac{\log_m 256}{\log_m 64}$.

19. Express $3\log_{10} x - \frac{1}{2}\log_{10} y + 1$ as a single logarithm

20. Solve the following equations for x.

i) $\log_3 \sqrt[3]{3x - 5} = 1$

ii) $1 + \log_2 \sqrt{x} = 5$

iii) $\frac{1}{2}\log_2 (x-1) = \log_2 (x-3)$

iv) $2 - \log_9 x^2 = 1$

v) $1 + \log_4 (x+1)^2 = 4$

vi) $\log_4 6x - \log_4 (x-5) = 2$

vii) $\log (3x - 1) = 1 + \log (5x - 2)$

viii) $\log x + \log (7 - x) = 1$

ix) $2 \log_3 x = \log_3 (5x - 4)$

x) $1 - \log (x-5) = \log \left(\dfrac{x}{5}\right)$

xi) $\log x = 2 - \log (x + 21)$

xii) $\log_5 (2x-3) - \log_5 x = 1$

xiii) $\log_3 (4x + 5) - \log_3 (x-7) = 2$

xiv) $\log_5 x(x + 2) = 0$

xv) $\log x + \log (4 + x) = \log 5$

xvi) $\log (x + 6) - 2 \log x = \log 12$

xvii) $2 \log_3 x + \log_3 (10 - x^2) = 2$

xviii) $\log_3 x^2 + \log_3 (10 - x^2) = 2$

xix) $\log_2 x + 4\log_x 2 = 5$

Review exercise 7.3

1. The following table gives the corresponding values of two variables h and t

h	2	3	4	5	6
t	6.4	32.4	102.4	250	518.4

If the relation between h and t is of the form $t = kh^n$, where k and n are constants.

i) Determine graphically the values of k and n

ii) Use your graph to estimate, correct to one decimal place, the value of t for which $h = 4.5$.

[Hint: Before you can use the graph, find the logarithm of 4.5 and then read the value of log t from the graph. To get t, find the antilog of the value you have read]

2. The following table gives the value of two related variables p and q as obtained in an experiment.

p	1	2	3	4	5
q	3	10	51	190	760

It is thought that p and q are connected by a scientific law of the form $q = kc^p$ where c and k are constants.

i) By plotting a suitable linear graph, verify that such a law is approximately valid.

ii) Use your graph to estimate
   a) the values of c and k, each correct to one decimal place;
   b) the value of q when $p = 3.5$.

3. The following table gives the values of two related variables p and q

q	0.10	0.32	2.14	3.14	10.0	15.9
p	0.0002	0.0064	1.78	5.6	180	710

The variable p and q are connected by a relation of the form $p^2 = Aq^{n-i}$, where A and n are constants.

By drawing a suitable linear graph, determine the values of A and n, each correct to one significant figure.

4. The following table gives the value of two related variables x and y as obtained in an experiment.

x	0.5	1	2	4
y	0.28	0.45	1.13	7.03

It is assumed that x and y are related by $y = ba^x$, where b and a are constants.

By drawing a suitable linear graph determine the values of b and a, each correct to one significant figure.

5. The population P thousands to one decimal place, of a new town T years after 1980 is summarised in the table below.

P	1	2	3	4	5
T	15.4	24.6	47.2	88.3	144.4

It is assumed that P and T are connected by the relationship of the form $P = kc^T$ where k and c are constants.

i) By plotting a suitable linear graph, verify that such a law is approximately valid.

ii) Use your graph to estimate the values of k and c, each correct to two decimal place.

6. The table below gives the corresponding values of two related variables x and y.

x	1.6	2.0	2.8	3.9	5.0
y	5.1	7.1	11.7	19.2	28.0

The relation between x and y is of the form $y = ax^k$, where a and k are constants.

Draw a suitable linear graph and use it to estimate, correct to one decimal place, the values of a and k.

Use your graph to estimate

a) the value of x when $y = 10$

b) the value of y when $x = \sqrt{10}$

7. It is believed two variables x and y are related by $y = ax^b$, where a and b are constants. For the following values, draw a suitable linear graph and use it to estimate, correct to one decimal place, the values of a and b.

x	0.6	1.2	1.8	2.4	3.0
y	0.35	1.04	1.95	3.06	4.33

8. It is believed two variables x and y are related by $y = ab^x$, where a and b are constants. For the following values, draw a suitable linear graph and use it to estimate, correct to one decimal place, the values of a and b.

x	1	2	3	4	5
y	14.1	15.8	17.8	19.9	22.4

9. It is known that the two variables x and y are related by $y = kx^n$, where a and b are constants. For the following values, draw a suitable linear graph and use it to estimate, correct to one decimal place, the values of k and n.

x	1.34	3.58	7.60	12.1	14.8
y	208.0	10.9	1.14	0.283	0.155

10. It is known that the two variables x and y are related by $y = ax^n$, where a and b are integers. For the following values, draw a suitable linear graph and obtain the integer values of a and n.

x	2	3	4	5	6	7
y	50	250	775	1875	3900	7200

11. It is believed that x and y satisfy a relationship of the form $y = ab^x$. A sample of data is given in the table below.

x	2.39	4.17	7.23	8.19	10.2
y	2.51	9.62	17.62	20.21	25.73

a) Draw up an appropriate table to draw a straight line graph. On the straight line graph the Y intercept is 0.69 and the gradient is 0.095.

b) Find
   i) the value of a
   ii) the value of b

Give your answers to two significant figures.

12. The following table gives the values of two related variables t and h as obtained in an experiment.

t	1	2	3	4	5	6
h	6.3	7.93	9.98	12.56	15.81	19.90

The relation between h and t is of the form $h = A(10^{kt})$. By means of a suitable graph, estimate, correct to one decimal place,

i) the values of A and k.

ii) the value of h when $t = 4.6$

CHAPTER 8

THE BINOMIAL THEOREM

Definition

A **binomial** is an expression which contains two terms connected by a plus or minus sign. For example, $(x - y)$, $(a + b^2)$, $(2p + q)$ are binomial expressions. It is often necessary to expand a power of a binomial such as $(x + y)^4$ but the multiplication is tedious when the power is 3 or more. We shall now describe a far quicker way of obtaining such expansions using Pascal's triangle and the Binomial theorem.

Pascal's triangle

The *coefficients* in the binomial expansion of $(a + x)^n$ form a pattern which is useful for small values of n. This pattern is known as **Pascal's triangle.**

Consider the following expansions for $(a + x)^n$ for integer values of n from 0 to 7:

Binomial	**Coefficient**	**Expansion**
$(a+x)^0$	1	1
$(a+x)^1$	1 1	$1a + 1x$
$(a+x)^2$	1 2 1	$1a^2 + 2ax + 1x^2$
$(a+x)^3$	1 3 3 1	$1a^3 + 3a^2x + 3ax^2 + 1x^3$
$(a+x)^4$	1 4 6 4 1	$1a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + 1x^4$
$(a+x)^5$	1 5 10 10 5 1	$1a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + 1x^5$
$(a+x)^6$	1 6 15 20 15 6 1	$1a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + 1x^6$
$(a+x)^7$	1 7 21 35 35 21 7 1	$1a^7 + 7a^6x + 21a^5x^2 + 35a^4x^3 + 35a^3x^4 + 21a^2x^5 + 7ax^6 + 1x^7$

It can be seen from the expansions above that the following patterns emerge:

1. The sum of the powers of a and x is equal to the power of the binomial in each term. For example, if the binomial is $(a + x)^6$, the sum of the powers of a and x is equal to 6.

2. 'a' *decreases* in power moving from left to right.

3. 'x' *increases* in power moving from left to right.

4. The coefficients form a pattern known as Pascal's triangle. For example, the coefficients of the expansion of $(a + x)^6$ are : 1, 6, 15, 20, 15, 6, 1.

5. The coefficient of a term may be obtained by adding the two adjacent coefficients immediately above in the previous row.

6. Pascal's triangle method is used for expansion of the form $(a + x)^n$ for *integer values* of n less than 8.

Illustrative example

Expand the following binomial expressions using the Pascal's triangle

i) $(2x-3y)^3$ ii) $(2x-3)^4$ iii) $(p-q)^4$ iv) $(2a-b)^6$ v) $(1+x)^7$

Solution

i) The initial coefficients will be 1, 3, 3, 1.

From Pascal's triangle: $(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$

Replacing a by $2x$ and x by $-3y$ gives

$$(2x-3y)^3 = (2x)^3 + 3(2x)^2(-3y) + 3(2x)(-3y)^2 + (-3y)^3$$
$$= 8x^3 - 36x^2y + 54xy^2 - 27y^3$$

Note: $(2x-3y)^3 = [2x+(-3y)]^3$

ii) The initial coefficients will be 1, 4, 6, 4, 1.

From Pascal's triangle $(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$

Replacing a by $2x$ and x by -3 gives

$$(2x-3)^4 = (2x)^4 + 4(2x)^3(-3) + 6(2x)^2(-3)^2 + 4(2x)(-3)^3 + (-3)^4$$
$$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$$

Note: $(2x-3)^4 = [2x+(-3)]^4$

Work out the solution of the following;

iii) $(p-q)^4 = p^4 - 4p^3q + 6p^2q^2 - 4pq^3 + q^4$

iv) $(2a-b)^6 = 64a^6 - 192a^5b + 240a^4b^2 - 160a^3b^3 + 60a^2b^4 - 612ab^5 + b^6$

v) $(1+x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$

The binomial theorem for positive integral index

Using the Pascal's triangle to expand $(1+x)^{20}$ will be tedious since the construction of the triangular array will be lengthy. A more general method known as the **Binomial Theorem** will be used for expanding powers of a binomial.

The *Binomial Theorem* is a formula for raising a binomial expression to a higher power without lengthy multiplication. The general binomial expansion of $(a+x)^n$ for all x and *positive integral values* of n, is given by:

$$\boxed{(a+x)^n = a^n + {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 + {}^nC_3a^{n-3}x^3 + \ldots + {}^nC_ra^{n-r}x^r + \ldots + x^n}$$

The notation ${}^nC_r$ reads 'n combination r. An alternative notation for ${}^nC_r$ is $\begin{pmatrix} n \\ r \end{pmatrix}$.

Thus ${}^5C_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$. Values of ${}^nC_r$ can be found in Mathematical tables or Scientific calculators.

We can also calculate them as follows:

$${}^nC_r = \frac{n!}{(n-r)!r!};\quad {}^nC_1 = n\ ;\quad {}^nC_1 = \frac{n(n-1)}{2!};\quad {}^nC_2 = \frac{n(n-1)}{2!};\quad {}^nC_3 = \frac{n(n-1)(n-2)}{3!}$$

and so on.

Therefore in its expanded form, the Binomial Theorem is given as:

$$\boxed{(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \ldots + x^n}$$

where 3! means $3 \times 2 \times 1$ and is termed '3 factorial' and 4! means $4 \times 3 \times 2 \times 1$
This expansion holds for *positive powers* of n.

The general term of the binomial expansion $(a+x)^n$
In the binomial expansion of $(a+x)^n$ above;

1. There are $(n+1)$ terms in this finite expression.
2. The sum of the powers of a and x in each term is n.
3. The 4th term is ${}^nC_3a^{n-3}x^3$ or $\dfrac{n(n-1)(n-2)}{3!}a^{n-3}x^3$. The number 3 is evident in this expression. The 5th term is ${}^nC_4a^{n-4}x^4$ or $\dfrac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}x^4$. The number 4 is evident. The third term is the term in x^2. The sixth term is the term in x^5 and so on.
4. The **general term** is the $(r+1)^{\text{th}}$ term i.e. the term containing x^r and is given by:

$$^nC_ra^{n-r}x^r \quad \text{or} \quad \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}x^r$$

Using the sigma notation ($\sum$) to write the binomial expansion

The binomial expansion of $(a+x)^n$ is given as a sum of terms:
$(a+x)^n = a^n + {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 + \ldots + {}^nC_ra^{n-r}x^r + \ldots + x^n$
Instead of writing down each term in the sum in this way, a shorthand notation can be used. We write down the general term and then use the sigma '$\sum$' to denote the sum. That is:

$$(a+x)^n = a^n + {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 + \ldots + {}^nC_ra^{n-r}x^r + \ldots + x^n$$

$$= \sum_0^n {}^nC_ra^{n-r}x^r = \sum_0^n \binom{n}{r}a^{n-r}x^r$$

The Greek letter '$\sum$' denotes the *sum of the terms*, where the general term is ${}^nC_ra^{n-r}x^r$ and the values of r ranges from integral values $r = 0$ to $r = n$.

Finding a term in a binomial expansion
Illustrative examples
Example 1 Write down and simplify the fifth term in the binomial expansion of $(3+x)^7$

Solution
To find the fifth term in the expansion of $(3+x)^7$, compare with $(a+x)^n$,
i.e. $a = 3, n = 7$ and $r + 1 = 5$ (i.e. fifth term) $\Rightarrow$ $r = 4$ and substitute into the general term.

Using the general term, $^nC_r a^{n-r} x^r$; the fifth term = $^7C_4 3^3 x^4 = 35(27)x^4 = 945x^4$ or

Using $\dfrac{n(n-1)(n-2)...(n-r+1)}{r!}a^{n-r}x^r$; fifth term = $\dfrac{7(6)(5)(4)}{4!}3^{7-4}x^4 = 35(3)^3x^4 = 945x^4$

i.e. the 5th term is $945x^4$. Note that the coefficient of the fifth term is 945.

Example 2 Without fully expanding $(2a-b)^8$, determine the sixth term.

Solution

The general term or the $(r+1)^{\text{th}}$ term for the expansion of $(a+x)^n$ is:

$$\frac{n(n-1)(n-2)...(n-r+1)}{r!}a^{n-r}x^r$$

Substitute $a = 2a,\ \ x = -b,\ \ n = 8$ and $r + 1 = 6 \ \Rightarrow\ r = 5$

i.e. 6th term = $\dfrac{8(7)(6)(5)(4)}{5!}(2a)^{8-5}(-b)^5 = -56(2a)^3b^5 = -448a^3b^5$

Note that the coefficient of the sixth term is -448.

Use $^nC_r a^{n-r} x^r$ to arrive at the same answer.

Example 3 Write down and simplify, the term indicated, in the binomial expansions of the following.

i) $(1+x)^9$, 4th term ii) $\left(2-\frac{1}{2}x\right)^{12}$, 4th term

iii) $(3+x)^7$, 5th term iv) $(1+x)^{20}$, 3rd term

v) $(2x-1)^5$, 5th term vi) $(3x-1)^6$, 3rd term

Solution

i) $84x^3$ ii) $-14080x^3$ iii) $945x^4$

iv) $190x^2$ v) $80x^4$ vi) $-540x^2$

Finding the coefficient of a term in a binomial expansion

Illustrative examples

Example 1 Find the coefficient of the term in x^3 of the binomial expansion of $(1+3x)^{12}$

Solution

Using the general term $^nC_r a^{n-r} x^r$, where $n = 12$, $a = 1$ and $x = 3x$

The term of the expansion containing x^3 (i.e. $r = 3$) = $^{12}C_3(1)^9(3x)^3 = \dfrac{12!}{9!\,3!} \times 27x^3 = 5940x^3$

Therefore the coefficient of the term in x^3 is 5940.

Example 2 Find the coefficient of the term in x^2 in the binomial expansion of $(2+x)^{10}$

Solution

Using the general term ${}^{n}C_r a^{n-r} x^r$, where $n = 10$, $a = 2$ and $x = x$

The term of the expansion containing x^2 (i.e. $r = 2$) $= {}^{10}C_2(2)^8(x)^2 = \dfrac{10!}{8!\,2!} \times 256x^2 = 11520x^2$

Therefore the coefficient of the term in x^2 is 11520.

Example 3 Write down and simplify, the coefficients of the term indicated, in the binomial expansions of the following.

i) $(1+2x)^5$, term containing x^4 ii) $(4+\frac{3}{4}x)^6$, term containing x^3

iii) $(2x-3)^7$, term containing x^5 iv) $(3+\frac{1}{3}x)^{11}$, term containing x^5

Solution

i) 80 ii) 540 iii) 6048 iv) 1386

Writing down and simplifying the terms in a binomial expansion

Illustrative examples

Example 1 Using the binomial theorem, write down and simplify the terms of the expansion of $(2+x)^7$ in ascending powers of x.

Solution

Using the binomial expansion of $(a+x)^n$, where $n = 7$, $a = 2$ and $x = x$

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \dots + x^n$$

$$(2+x)^7 = 2^7 + 7(2)^6x + \frac{(7)(6)}{2!}(2)^5x^2 + \frac{(7)(6)(5)}{3!}(2)^4x^3 + \frac{(7)(6)(5)(4)}{4!}(2)^3x^4 + \frac{(7)(6)(5)(4)(3)}{5!}(2)^2x^5 + \frac{(7)(6)(5)(4)(3)(2)}{6!}(2)^1x^6 + x^7$$

i.e. $(2+x)^7 = 128 + 448x + 672x^2 + 560x^3 + 280x^4 + 84x^5 + 14x^6 + x^7$

Example 2 Using the binomial theorem, write down and simplify the terms of the expansion of $(3-2x)^5$ in ascending powers of x.

Solution

Using the binomial expansion of $(a+x)^n$, where $n = 5$, $a = 3$ and $x = -2x$

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \dots + x^n$$

$$(3-2x)^5 = (3)^5 + 5(3)^4(-2x) + \frac{(5)(4)}{2!}(3)^3(-2x)^2 + \frac{(5)(4)(3)}{3!}(3)^2(-2x)^3 + \frac{(5)(4)(3)(2)}{4!}(3)(-2x)^4 + (-2x)^5$$

$$(3-2x)^5 = 243 - 810x + 360x^2 - 720x^3 + 240x^4 - 32x^5$$

The binomial theorem for any rational index

If n is any rational value, positive or negative, and $-1 \le x < 1$, then the general binomial is:

$$\boxed{(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots}$$

The first term of the binomial must be 1.

Thus to expand $(a+x)^n$ it must be first written as $a^n\left(1+\frac{x}{a}\right)^n$.

For example, $(2-x)^{\frac{1}{2}} = 2^{\frac{1}{2}}(1-\frac{x}{2})^{\frac{1}{2}}$. Note that n can be positive or negative or a fraction.

Illustrative examples

Example 1 Write down the first three terms in the expansion of

i) $(1-\frac{x}{2})^{10}$ ii) $(2x-1)^5$ iii) $(3-6x)^8$ iv) $\sqrt[3]{1+x}$

Solution

i) Using the binomial expansion of $(1+x)^n$, where $n = 10$ and $x = -\frac{x}{2}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$(1-\tfrac{x}{2})^{10} = 1 + 10\,(-\tfrac{x}{2}) + \frac{10(9)}{2!}(-\tfrac{x}{2})^2 + \ldots \qquad \textbf{Note:} \quad (1-\tfrac{x}{2})^{10} = (1+(-\tfrac{x}{2}))^{10}$$

$$= 1 - 5x + \tfrac{45}{4}x^2 + \ldots$$

ii) Writing $(2x-1)^5$ as $(-1)^5(1-2x)^5$, we can use the binomial expansion for $(1+x)^n$ by replacing x by $-2x$ and n by 5

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n$$

$$(2x-1)^5 = (-1)^5(1-2x)^5 = (-1)^5\left[1 + 5(-2x) + \frac{5(4)}{2!}(-2x)^2 + \ldots\right]$$

$$= -1 + 10x - 40x^2 + \ldots$$

Continuing the expansion, the other terms are : $160x^3 - 80x^4 + 32x^5$

iii) Writing $(3-6x)^8$ as $3^8(1-2x)^8$, we can use the binomial expansion for $(1+x)^n$ by replacing x by $-2x$ and n by 8

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$(3-6x)^8 = 3^8(1-2x)^8 = 3^8\left[1 + 8(-2x) + \frac{8(7)}{2!}(-2x)^2 + \ldots\right]$$

$$= 3^8 - (3^8)(16)x + (3^8)(112)x^2 + \ldots$$
$$= 6561 - 104976x + 734832x^2 + \ldots$$

iv) $\sqrt[3]{1+x} = (1+x)^{\frac{1}{3}}$ Using the Binomial expansion for $(1+x)^n$, where $n = \frac{1}{3}$ and $x = x$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + x^n \ldots$$

$$(1+x)^{\frac{1}{3}} = 1 + \tfrac{1}{3}x + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}x^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!}x^3 + \ldots$$

$$(1+x)^{\frac{1}{3}} = 1 + \frac{1}{3}x - \frac{\frac{2}{9}}{2}x^2 + \frac{\frac{10}{27}}{6}x^3 + \ldots$$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 + \ldots$$

Example 2 Write down the first three terms in the binomial expansion of $(1-2x)(1+\frac{x}{2})^{10}$.

Solution

Using the binomial expansion of $(1+x)^n$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \text{ (replace } n \text{ by 10 and } x \text{ by } \tfrac{x}{2})$$

$$(1+\tfrac{x}{2})^{10} = 1 + 10(\tfrac{x}{2}) + \frac{10(9)}{2!}(\tfrac{x}{2})^2 + \ldots$$

$$= 1 + 5x + \tfrac{45}{4}x^2 + \ldots$$

$$\therefore \quad (1-2x)(1+\tfrac{x}{2})^{10} = (1-2x)(1+5x+\tfrac{45}{4}x^2+\ldots)$$
$$= 1 + 5x + \tfrac{45}{4}x^2 + \ldots - 2x - 10x^2 + \ldots$$
$$= 1 + 3x + \tfrac{5}{4}x^2 + \ldots$$

Example 3 Write down the first three terms in the expansion of

i) $(1+x)^{-3}$ ii) $(1-2x)^{-2}$

iii) $(1-\frac{1}{3}x)^{-3}$ iv) $(1+x)^{\frac{1}{2}}$

Solution

i) $(1+x)^{-3} = 1 - 3x + 6x^2 + \ldots$

ii) $(1-2x)^{-2} = 1 + 4x + 12x^2 + \ldots$

iii) $(1-\frac{1}{3}x)^{-3} = 1 + x + \frac{2}{3}x^2 + \ldots$

iv) $(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \ldots$

Continue the expansions up to the fifth term in each case.

Approximations

The binomial theorem in the form $(1+x)^n$, may be used to obtain approximations and to evaluate roots of numbers to any degree of accuracy i.e. 3 decimal places, 4 significant figures etc. Note that when finding an approximate value of 3.006^5, we use $(3+0.006)^5 = 3^5(1+0.002)^5$. Similarly, when finding 0.99^6, we use $(1-0.01)^6$.

Illustrative examples

Example 1 Use the expansion of $(1-2x)^6$ to find the value of 0.98^6, correct to four significant figures.

Solution

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$(1-2x)^6 = 1 - 12x + 60x^2 - 160x^3 + \ldots \quad \text{to four terms}$$

$0.98^6 = (1-0.02)^6$. Substituting $2x = 0.02$ i.e. $x = 0.01$ in the above expansion

$$\therefore\ 0.98^6 = (1-0.02)^6 = 1 - 12(0.01) + 60(0.01)^2 - 160(0.01)^3 + \ldots$$
$$= 1 - 0.12 + 0.006 - 0.00016 + \ldots$$
$$= 0.8858 \text{ to four decimal places}$$

Example 2 Use the expansion of $(1-x)^7$ to find the value of 1.998^7 correct to five significant figures.

Solution

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \text{ (replace } n \text{ by 7 and } x \text{ by } -x)$$

$$(1-x)^7 = 1 + 7(-x) + \frac{7(6)}{2!}(-x)^2 + \frac{7(6)(5)}{3!}(-x)^3 + \ldots$$

$$(1-x)^7 = 1 - 7x + 21x^2 - 35x^3 + \ldots$$

$1.998^7 = (2-0.002)^7 = 2^7(1-0.001)^7$. Substituting $x = 0.001$

$$\therefore\ 1.998^7 = 2^7(1-0.001)^7 = 2^7(1 - 7(0.001) + 21(0.001)^2 + \ldots$$
$$= 128(1 - 0.007 + 0.0000210)$$
$$= 127.11 \text{ correct to five significant figures.}$$

Note that only three terms need to be considered since the answer is to be given in five significant figures.

Example 3 Use the expansion of $(1+x)^6$ to find the value of 1.98^6 correct to five decimal places.

Solution

$$(1+x)^6 = 1+6x+15x^2+20x^3+15x^4+6x^5+x^6$$

$1.98^6 = (2-0.02)^6 = 2^6(1-0.01)^6$. Substituting $x = -0.01$ in the above expansion

$$\therefore\ 1.98^6 = 2^6(1-0.01)^6 = 2^6\left(1+6(-0.01)+15(-0.01)^2+20(-0.01)^3+\dots\right)$$

$$= 64(1+-0.06+0.0015-0.00002+\dots)$$

$$= 64\times 0.9414801 = 60.25473 \text{ correct to five decimal places}$$

Example 4 Write down the binomial expansion of $(1+x)^n$, where n is a rational number, as far as the term in x^3. Use your expansion to evaluate $\sqrt{104}$ correct to three decimal places.

Solution

The binomial expansion of $(1+x)^n$, where n is a rational number is given by:

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots$$

$\sqrt{104} = 104^{\frac{1}{2}} = (100+4)^{\frac{1}{2}} = (100[1+0.04])^{\frac{1}{2}} = 10(1+0.04)^{\frac{1}{2}}$. Substituting $x = 0.04$ and $n = \frac{1}{2}$

$$\therefore\ \sqrt{104} = 10(1+0.04)^{\frac{1}{2}} = 10\left(1+\tfrac{1}{2}(0.04)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(0.04)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(0.04)^3+\dots\right)$$

$$= 10\left(1+\tfrac{1}{2}(0.04)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}(0.04)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}(0.04)^3+\dots\right)$$

$$= 10\left(1+0.02-\frac{0.0016}{8}+\frac{0.000064}{16}+\dots\right)$$

$$= 10(1+0.02-0.0002+0.000004+\dots)$$

$$= 10(1.019804) = 10.198 \text{ to three decimal places}$$

Example 5 Expand $(1-x)^{-\frac{1}{3}}$ in ascending powers of x up to the fourth term.

Use these terms to find an approximate value of $\left(\frac{100}{99}\right)^{\frac{1}{3}}$, correct to two decimal places.

Solution

The binomial expansion of $(1+x)^n$, where n is a rational number is given by:

$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots$ Replacing n by $-\frac{1}{3}$ and x by $-x$

$$(1-x)^{-\frac{1}{3}} = \left(1+\left(-\tfrac{1}{3}\right)(-x)+\frac{-\left(\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}(-x)^2+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}(-x)^3+\dots\right)$$

$$(1-x)^{-\frac{1}{3}} = \left(1+\left(-\tfrac{1}{3}\right)(-x)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2}(-x)^2+\frac{\left(-\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}(-x)^3+\dots\right)$$

$$(1-x)^{-\frac{1}{3}} = \left(1+\tfrac{1}{3}x+\tfrac{2}{9}x^2+\tfrac{5}{81}x^3+\dots\right)$$

Second part

$$\left(\frac{100}{99}\right)^{\frac{1}{3}} = \left(\frac{99}{100}\right)^{-\frac{1}{3}} = (0.99)^{-\frac{1}{3}} = (1-0.01)^{-\frac{1}{3}}$$

Substituting $x = 0.01$ in the expansion for $(1-x)^{-\frac{1}{3}}$, we have

$$\left(\frac{100}{99}\right)^{\frac{1}{3}} = 1 + \frac{0.01}{3} + \frac{2(0.01)^2}{9} + \frac{5(0.01)^3}{81} + \dots$$

$$= 1 + 0.003333 + 0.000022222 + 0.0000003125 + \dots$$

$= 1.003$ correct to three decimal paces

Example 6

i) Write down the first four terms of the binomial expansion of $\sqrt[3]{1+x}$.

ii) Use your expansion in (i) to evaluate $\sqrt[3]{9}$, correct to **three** decimal places.

Solution

$\sqrt[3]{1+x} = (1+x)^{\frac{1}{3}}$ Applying Binomial theorem :

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + x^n$ Replacing n by $\frac{1}{3}$

$$(1+x)^{\frac{1}{3}} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 + \dots$$

ii) $\sqrt[3]{9} = (9)^{\frac{1}{3}} = (8+1)^{\frac{1}{3}} = 8^{\frac{1}{3}}(1+\frac{1}{8})^{\frac{1}{3}}$

$= 2(1+\frac{1}{8})^{\frac{1}{3}} = 2(1+0.125)^{\frac{1}{3}}$

Putting $x = 0.125$ in the expansion in (i) above

$$\sqrt[3]{9} = 2(1+0.125)^{\frac{1}{3}} = 2\left[1 + \tfrac{1}{3}(0.125) - \tfrac{1}{9}(0.125)^2 + \tfrac{5}{81}(0.125)^3\right]$$

$= 2[1.0401] = 2.080$ (3 d.p.)

Example 7

i) Write down the first four terms of the binomial expansion of $(2+\frac{x}{4})^{10}$ in ascending powers of x.

ii) Using your expansion in (i), find the value of 2.025^{10}, correct to the nearest whole number.

Solution

i) $(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \dots + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \dots + x^n$

Replacing n by 10, a by 2 and x by $\frac{x}{4}$

$$(2+\tfrac{x}{4})^{10} = 2^{10} + 10\times 2^{9}(\tfrac{x}{4}) + \frac{10(10-1)}{2!}\times 2^{10-2}(\tfrac{x}{4})^{2} + \frac{10(10-1)(10-2)}{3!}\times 2^{10-3}(\tfrac{x}{4})^{3} + \ldots$$

$$= 1024 + 1280x + \frac{45\times 256x^{2}}{16} + \frac{120\times 128x^{3}}{64} + \ldots$$

$$= 1024 + 1280x + 720x^{2} + 122x^{3} + \ldots$$

ii) $2.025^{10} = (2+0.025)^{10} = (2+\frac{0.1}{4})^{10}$

Comparing $(2+\frac{0.1}{4})^{10}$ with $(2+\frac{x}{4})^{10} \Rightarrow x = 0.1$

Putting $x = 0.1$ in the expansion in (i), above

$(2+\frac{0.1}{4}) = 1024 + 1280(0.1) + 720(0.1)^{2} + 122(0.1)^{3} = 1024 + 128 + 7.2 + 0.122 = 1159.322$

$\therefore\ 2.025^{10} = 1159$ (to the nearest whole number)

Example 8

a) Find the first three terms of the binomial expansion of $(2-3x)^{5}$ in ascending powers of x.

b) Use your expansion in (a) to evaluate correct to two decimal places $(1.97)^{5}$.

Solution

Using the Binomial theorem;

a) $(2-3x)^{5} = 2^{5} - 5(2^{4})3x + 10(2^{3})(3x)^{2} = 32 - 240x + 720x^{2}$

b) $(1.97)^{5} = (2-3(0.01))^{5} \Rightarrow x = 0.01$

$(1.97) = 32 - 240(0.01) + 720(0.01)^{2}$

$= 32 - 2.4 + 0.072 = 29.672$

$= 29.67$ correct to two decimal places

Example 9 Use the binomial expansion of $(2+x)^{7}$ to find the value of 2.08^{7} correct to three decimal places.

Solution

$2.08^{7} = 168.439$ correct to three decimal places. [Work out the missing steps]

Example 10 Use the binomial expansion of $\left(1+\dfrac{x}{5}\right)^{6}$ to find the value of 1.01^{6} correct to four significant figures.

Solution

$1.01^{6} = 1.602$ correct to three decimal places. [Work out the missing steps]

Example 11 Use the binomial theorem to expand $(1-x)^{\frac{1}{2}}$. By substituting 0.02 for x, find the value of $\sqrt{2}$ correct to five decimal places.

Solution

$$(1-x)^{\frac{1}{2}} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} + \dots$$

Substituting 0.02 for *x* gives:

$$(0.98)^{\frac{1}{2}} = 1 - 0.01 - 0.00005 - 0.0000005 + \dots = 0.9899495$$

$$\sqrt{\frac{98}{100}} = 0.9899495 \Rightarrow \sqrt{\frac{49\times 2}{100}} = 0.9899495 \qquad [\textbf{Note: } 0.98^{\frac{1}{2}} = \sqrt{\tfrac{98}{100}}\,]$$

$$\tfrac{7}{10}\sqrt{2} = 0.9899495$$

$$\sqrt{2} = \frac{0.9899495\times 10}{7} = 1.41421 \text{ correct to five decimal places}$$

Example 12 Evaluate the following using the binomial theorem.

i) 1.002^9 correct to four decimal places
ii) 0.97^6 correct to four decimal places
iii) 3.039^6 correct to three decimal places
iv) $\sqrt{4.02}$ correct to three decimal place
v) $\sqrt{1.004}$ correct to three decimal place

Solution

i) $1.002^9 = 1.0181$ correct to four decimal places
ii) $0.97^6 = 0.8330$ correct to four decimal places
iii) $3.039^6 = 787.742$ correct to three decimal places
iv) $\sqrt{4.02} = 2.005$ correct to three decimal place
v) $\sqrt{1.004} = 1.002$ correct to three decimal place

Note: In each case the final answer has been provided. Work out the missing steps.

Solved examination questions

The following questions will help you to understand all what you have learnt so far.

Example 1 Write down the first four terms in the binomial expansion of

i) $(1+3x)^{12}$ ii) $(1-2x)^9$ iii) $(2+x)^{10}$

iv) $(1-\frac{x}{3})^{20}$ v) $(2-\frac{3}{2}x)^7$ vi) $(1-2x)^{\frac{1}{2}}$ vii) $(1+\frac{x}{2})^{-\frac{1}{2}}$

Solution

i) $(1+3x)^{12} = 1 + 36x + 594x^2 + 5940x^3$

ii) $(1-2x)^9 = 1 - 18x + 144x^2 - 672x^3$

iii) $(2+x)^{10} = 1024 + 5120x + 11520x^2 + 15360x^3$

iv) $\left(1-\frac{x}{3}\right)^{20}=1-\frac{20}{3}x+\frac{190}{9}x^2-\frac{380}{9}x^3$

v) $\left(2-\frac{3}{2}x\right)^{7}=128-672x+1512x^2-1890x^3$

vi) $(1-2x)^{\frac{1}{2}}=1-x-\dfrac{x^2}{2}-\dfrac{x^3}{2}$

vii) $\left(1+\frac{x}{2}\right)^{-\frac{1}{2}}=1-\dfrac{x}{4}+\dfrac{3x^2}{32}-\dfrac{5}{128}x^3$

Note: In each case the final answer has been provided. Work out the missing steps.

Example 2 Write down the binomial expansion of $(1+x)^5$, simplifying all the terms. Use your expansion to evaluate $(1.01)^5$ correct to four decimal places.

Solution
Using the binomial expansion of $(1+x)^n$, where $n=5$

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$$

$$(1+x)^5=1+5x+\frac{5(4)}{2!}x^2+\frac{5(4)(3)}{3!}x^3+\frac{5(4)(3)(2)}{4!}x^4+\frac{5(4)(3)(2)(1)}{5!}x^5$$
$$=1+5x+10x^2+10x^3+5x^4+x^5$$

$(1.01)^5=(1+0.01)^5$. Comparing this with the binomial $(1+x)^5 \Rightarrow x=0.01$

$\therefore$ Substitute $x=0.01$ into the expansion of $(1+x)^5$ above.

$$(1+0.01)^5=1+5(0.01)+10(0.01)^2+10(0.01)^3+5(0.01)^4+\ldots$$
$$=1+0.05+0.0010+0.000010+0.00000005=1.05101005$$
$$=1.0510 \text{ correct to four decimal places}$$

Example 3 Using the binomial theorem, write down and simplify the first seven terms of the expansion of $(1+2x)^{10}$ in ascending powers of x.
Use your expansion to show that $1.2^{10}>6.19$.

Solution
Using the binomial expansion of $(1+x)^n$, where $n=10$ and $x=2x$

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$$

Thus replace n by 10 and x by $2x$

$$(1+x)^{10}=1+10(2x)+\frac{10(9)}{2!}(2x)^2+\frac{10(9)(8)}{3!}(2x)^3+\frac{10(9)(8)(7)}{4!}(2x)^4+$$
$$\frac{10(9)(8)(7)(6)}{5!}(2x)^5+\frac{10(9)(8)(7)(6)(5)}{6!}(2x)^6+\ldots$$
$$=1+20x+180x^2+960x^3+3360x^4+8064x^5+13440x^6+\ldots$$

Now $(1.2)^{10} = (1 + 0.2)^{10}$. Comparing with $(1 + 2x)^{10} \Rightarrow 2x = 0.2 \Rightarrow x = 0.1$
Therefore replace x by 0.1 in the expansion of $(1 + 2x)^{10}$, we have

$$\begin{aligned}(1.2)^{10} &= 1 + 20(0.01) + 180(0.01)^2 + 960(0.01)^3 + 3360(0.01)^4 + 8064(0.01)^5 + 13440(0.01)^6 + ..\\ &= 1 + 2 + 1.8 + 0.960 + 0.3360 + 0.8064 + 0.013440 + \ldots\\ &= 6.19008 > 6.19 \text{ as required.}\end{aligned}$$

Example 4 Using the binomial theorem, write down and simplify all the terms of the expansion of $(1 + \frac{1}{2}a)^6$. Given that $3.15^6 = 3^6 \times b$, use your results to estimate the value of b, correct to four decimal places.

Solution

Using the binomial expansion of $(1 + x)^n$, where $n = 6$ and $x = \frac{1}{2}a$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

Thus replace n by 6 and x by $\frac{1}{2}a$

$$\begin{aligned}(1 + \tfrac{1}{2}a)^6 &= 1 + 6(\tfrac{1}{2}a) + \frac{6(5)}{2!}(\tfrac{1}{2}a)^2 + \frac{6(5)(4)}{3!}(\tfrac{1}{2}a)^3 + \frac{6(5)(4)(3)}{4!}(\tfrac{1}{2}a)^4 + \\ &\quad \frac{6(5)(4)(3)(2)}{5!}(\tfrac{1}{2}a)^5 + \frac{6(5)(4)(3)(2)(1)}{6!}(\tfrac{1}{2}a)^6\\ &= 1 + 3a + \tfrac{15}{4}a^2 + \tfrac{5}{2}a^3 + \tfrac{15}{16}a^4 + \tfrac{3}{16}a^5 + \tfrac{1}{64}a^6\end{aligned}$$

$3.15^6 = 3^6 \times b \quad \Rightarrow \quad b = (\frac{3.15}{3})^6 = 1.05^6 = (1 + 0.05)^6$

Therefore comparing $(1 + 0.05)^6$ with $(1 + \frac{1}{2}a)^6$

$\frac{1}{2}a = 0.05 \quad \Rightarrow \quad a = 2(0.05) = 0.1$

Hence substituting $a = 0.1$ into the binomial expansion of $(1 + \frac{1}{2}a)^6$ above, we shall obtain the value of b.

i.e. $b = 1 + 3(0.1) + \frac{15}{4}(0.1)^2 + \frac{5}{2}(0.1)^3 + \frac{15}{16}(0.1)^4 + \frac{3}{16}(0.1)^5 + \ldots$
$= 1 + 0.3 + 0.0375 + 0.0025 + 0.00009375 + 0.000001875 = 1.340096$

$\therefore \quad b = 1.3401$ correct to four decimal places.

Example 5 Using the binomial theorem, write down and simplify the first four terms in the expansion of $(1 - \frac{1}{2}x)^6$. Use your results to estimate the value of $(0.99)^6$, giving your answer to four decimal places.

Solution

Using the binomial expansion of $(1 + x)^n$, where $n = 6$ and $x = -\frac{1}{2}x$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

Therefore replace n by 6 and x by $-\frac{1}{2}x$

$$(1-\tfrac{1}{2}x)^6 = 1 + 6(-\tfrac{1}{2}x) + \frac{6(5)}{2!}(-\tfrac{1}{2}x)^2 + \frac{6(5)(4)}{3!}(-\tfrac{1}{2}x)^3 + \ldots \text{ to four terms}$$

$$= 1 - 3x + \tfrac{15}{4}x^2 - \tfrac{5}{2}x^3 + \ldots \text{ to four terms}$$

$(0.99)^6 = (1-0.01)^6$ Therefore comparing $(1-0.01)^6$ with $(1-\frac{1}{2}x)^6$

$$\tfrac{1}{2}x = 0.01 \quad \Rightarrow \quad x = 2(0.01) = 0.02$$

Hence substitute $x = 0.02$ into the binomial expansion of $(1-\frac{1}{2}x)^6$ above, we have

$$(0.99)^6 = (1-0.01)^6 = 1 - 3(0.02) + \tfrac{15}{4}(0.02)^2 - \tfrac{5}{2}(0.02)^3 + \ldots$$

$$= 1 - 0.06 + 0.0015 - 0.00002 +$$

$$= 0.9415 \text{ correct to four decimal places.}$$

Example 6

a) Using the binomial theorem, write down and simplify all the terms of the expansion of $(2+x)^6$ and $(2-x)^6$.
b) Use (a) above to evaluate $(2+x)^6 - (2-x)^6$.
c) Use your result in b) to calculate the value of $(2.01)^6 - (1.99)^6$ correct to four decimal places.

Solution

a) Using the binomial expansion of $(a+x)^n$, where $n = 6$, $a = 2$ and $x = x$

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \ldots + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \ldots + x^n$$

$$(2+x)^6 = 2^6 + 6(2^5)x + \frac{6(5)}{2!}2^4x^2 + \frac{6(5)(4)}{3!}2^3x^3 + \frac{6(5)(4)(3)}{4!}2^2x^4 +$$

$$\frac{6(5)(4)(3)(2)}{5!}2^1x^5 + \frac{6(5)(4)(3)(2)(1)}{6!}x^6$$

$$= 64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6$$

By inspection, the expansion of $(2-x)^6$, is obtained by replacing x by $-x$ in the expansion of $(2+x)^6$

$$(2-x)^6 = 64 + 192(-x) + 240(-x)^2 + 160(-x)^3 + 60(-x)^4 + 12(x)^5 + (-x)^6$$

$$= 64 - 192x + 240x^2 - 160x^3 + 60x^4 - 12x^5 + x^6$$

b) $(2+x)^6 - (2-x)^6 = (64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6)$

$$- (64 - 192x + 240x^2 - 160x^3 + 60x^4 - 12x^5 + x^6)$$

$$= 384x + 320x^3 + 24x^5 \text{ ---------------(1)}$$

c) To calculate the value of $(2.01)^6 - (1.99)^6$ using the result in (b), we have $(2.01)^6 = (2 + 0.01)^6$ and $(1.99)^6 = (2 - 0.01)^6$

Therefore $(2.01)^6 - (1.99)^6 = (2 + 0.01)^6 - (2 - 0.01)^6$

Comparing $(2 + 0.01)^6 - (2 - 0.01)^6$ with $(2 + x)^6 - (2 - x)^6 \quad \Rightarrow \quad x = 0.01$

Hence to evaluate $(2.01)^6 - (1.99)$, we shall substitute $x = 0.01$ in (1) above

i.e. $(2.01)^6 - (1.99)^6 = 384(0.01) + 320(0.01)^3 + \ldots$

$$= 3.84 + 0.000320$$

$$= 3.8403 \text{ correct to four decimal places}$$

Example 7 Expand $(x + 3y)^6$ by the binomial theorem and use your results to evaluate $(1.03)^6$ correct to five decimal places.

Solution

Using the binomial expansion of $(a + x)^n$, where $n = 6$, $a = x$ and $x = 3y$

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \ldots + x^n$$

$$(x+3y)^6 = x^6 + 6x^5(3y) + \frac{6(5)}{2!}x^4(3y)^2 + \frac{6(5)(4)}{3!}x^3(3y)^3 + \frac{6(5)(4)(3)}{4!}x^2(3y)^4 + \frac{6(5)(4)(3)(2)}{5!}x^1(3y)^5 + \frac{6(5)(4)(3)(2)(1)}{6!}(3y)^6$$

$$= x^6 + 18x^5y + 135x^4y^2 + 540x^3y^3 + 1215x^2y^4 + 1458xy^5 + 729y^6$$

$(1.03)^6 = (1 + 0.03)^6$. Comparing $(1 + 0.03)^6$ with $(x + 3y)^6$

$x = 1$ and $3y = 0.03 \Rightarrow \quad y = 0.01$

Hence to evaluate $(1.03)^6$, we shall substitute $x = 0.01$ in the expansion of $(x + 3y)^6$ above

$$(1.03)^6 = 1 + 18(0.01) + 135(0.01)^2 + 540(0.01)^3 + 1215(0.01)^4 + 1458(0.01)5 + 729(0.01)^6$$

$$= 1.19405 \text{ correct to five decimal places.}$$

Example 8 Find the exact value of $(1.02)^5$ using the binomial theorem.

Solution

$(1.02)^5 = (1 + 0.02)^5$

Using the binomial expansion of $(1 + x)^n$, where $n = 5$ and $x = 0.02$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$ Replace n by 5 and x by 0.02

$$(1+0.02)^5 = 1 + 5(0.02) + \frac{5(4)}{2!}(0.02)^2 + \frac{5(4)(3)}{3!}(0.02)^3 + \frac{5(4)(3)(2)}{4!}(0.02)^4 + \frac{5(4)(3)(2)(1)}{5!}(0.02)^5$$

$$= 1 + 5(0.02) + 10(0.0004) + 10(0.000008) + 5(0.00000016) + 0.0000000032$$

$$= 1.1040808032$$

Example 9 Expand $(1+x+x^2)^8$ in ascending powers of x, as far as the term in x^3.

Solution

We can treat $1+x+x^2$ as the binomial of the form $[1+(x+x^2)]$

$\therefore \quad (1+x+x^2)^8 = [1+(x+x^2)]^8$.

Using the binomial expansion of $(1+x)^n$, where $n=8$ and $x=(x+x^2)$

$(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots$ Replace n by 8 and x by $(x+x^2)$

$$\begin{aligned}[1+(x+x^2)]^8 &= 1 + 8(x+x^2) + \frac{6(5)}{2!}(x+x^2)^2 + \frac{6(5)(4)}{3!}(x+x^2)^3 + \ldots\\ &= 1 + 8(x+x^2) + 28(x+x^2)^2 + 56(x+x^2)^3 + \ldots\\ &= 1 + 8x + 8x^2 + 28(x^2+2x^3+x^4) + 56(x^3+\ldots) + \ldots\\ &= 1 + 8x + 8x^2 + 28x^2 + 56x^3 + 56x^3 + \ldots\\ &= 1 + 8x + 36x^2 + 112x^3 + \ldots\end{aligned}$$

Neglecting terms in x^4 and higher powers of x.

Example 10

i) Write down the binomial expansion of $(1-y)^6$, simplify all the terms.

ii) By putting $y=(x-x^2)$ in (i), deduce the expansion of $(1-x+x^2)^6$ in ascending powers of x as far as the term in x^5.

iii) By taking $x=0.1$ in (ii), evaluate 0.91^6, correct to two decimal places.

Solution

i) Using the binomial expansion of $(1+x)^n$, where $n=6$ and $x=-y$

$(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots$ Therefore replace n by 6 and x by $-y$

$$\begin{aligned}(1-y)^6 &= 1 + 6(-y) + \frac{6(5)}{2!}(-y)^2 + \frac{6(5)(4)}{3!}(-y)^3 + \frac{6(5)(4)(3)}{4!}(-y)^4 + \\ &\quad \frac{6(5)(4)(3)(2)}{5!}(-y)^5 + \frac{6(5)(4)(3)(2)(1)}{6!}(-y)^6\\ &= 1 - 6y + 15y^2 - 20y^3 + 15y^4 - 6y^5 + y^6\end{aligned}$$

ii) By putting $y=(x-x^2)$ in (i) we can deduce the expansion of $(1-x+x^2)^6$ in ascending powers of x as far as the term in x^5. Now

$$\begin{aligned}&(1-x+x^2)^6\\ &= [1-(x-x^2)]^6\\ &= 1 - 6(x-x^2) + 15(x-x^2)^2 - 20(x-x^2)^3 + 15(x-x^2)^4 - 6(x-x^2)^5 + (x-x^2)^6\end{aligned}$$

Using the coefficients of the binomial expansion, we obtain

$$\begin{aligned}(1-x+x^2)^6 &= 1 - 6x + 6x^2 + 15[x^2 - 2x^3 + x^4] - 20[x^3 + 3(x^2)(-x^2) + 3(x)(-x^2)^2 + x^6] + \\ &\quad 15[x^4 + 4x^3(x^2)^2 + 6x^2(-x^2)^3 + 4x(-x^2)^3 + (-x^2)^4] - 6(x^5 + \ldots)\\ &= 1 - 6x + 6x^2 + 15x^2 - 30x^3 + 15x^4 - 20x^3 + 60x^4 - 60x^5 - 20x^4 + 15x^4 - 60x^5 - 6x^5\\ &= 1 - 6x + 21x^2 - 50x^3 + 55x^4 - 126x^5 + ..\end{aligned}$$

i.e. if terms in x^6 and higher powers of x are neglected.

iii) By taking $x = 0.1$ in (ii) to evaluate 0.91^6 we have,

$$0.91^6 = 1 - 6(0.1) + 21(0.1)^2 - 50(0.1)^3 + 55(0.1)^4 - 126(0.1)^5$$
$$= 1 - 0.6 + 0.21 - 0.050 + 0.0055 - 0.00126$$
$$= 0.56424 = 0.56 \text{ (to two decimal places)}$$

Note : We can obtain the value of $x = 0.1$ by solving the quadratic equation $1 - x + x^2 = 0.91$ i.e. $1 - x + x^2 = 0.91 \Rightarrow x^2 - x + 0.09 = 0$ Using the general formula $x = 0.1$ or 0.9

Example 11

a) Write down the binomial expansion of $(1 + y)^8$, simplifying all the terms.

b) Use the substitution $y = x - x^2$ in (a) to deduce the expansion of $(1 + x - x^2)^8$, in ascending powers of x as far as the term in x^4.

c) Find by inspection, a value of x such that $1 + x - x^2 = 1.09$ and hence evaluate 1.09^8, correct to three decimal places.

Solution

a) Using the binomial expansion of $(1 + x)^n$, where $n = 8$ and $x = y$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

Therefore replace n by 8 and x by y

$$(1 + y)^8 = 1 + 8(y) + \frac{8(7)}{2!}(y)^2 + \frac{8(7)(6)}{3!}(y)^3 + \frac{8(7)(6)(5)}{4!}(y)^4 + \frac{8(7)(6)(5)(4)}{5!}(y)^5 + \frac{8(7)(6)(5)(4)(3)}{6!}(y)^6 + \frac{8(7)(6)(5)(4)(3)(2)}{7!}(y)^7 + \frac{8(7)(6)(5)(4)(3)(2)(1)}{8!}(y)^8$$
$$= 1 + 8y + 28y^2 + 56y^3 + 70y^4 + 56y^5 + 28y^6 + 8y^7 + y^8$$

b) By putting $y = (x - x^2)$ in (a) we can deduce the expansion of $(1 - x + x^2)^8$ in ascending powers of x as far as the term in x^4.

Now $(1 - x + x^2)^8 = \{1 + (x - x^2)\}^8 = 1 + 8(x - x^2) + 28(x - x^2)^2 + 56(x - x^2)^3 + 70(x - x^2)^4 + \ldots$

Using the coefficients of the binomial expansion, we obtain

$$(1 - x + x^2)^8 = 1 + 8x - 8x^2 + 28[x^2 - 2x^3 + x^4] + 56[x^3 + 3(x^2)(-x^2) + \ldots] + 70[x^4 + \ldots] + \ldots$$
$$= 1 + 8x - 8x^2 + 28x^2 - 56x^3 + 28x^4 + 56x^3 - 168x^4 + 70x^4$$
$$= 1 + 8x + 20x^2 - 70x^4 + \ldots$$

i.e. if terms in x^5 and higher powers of x are neglected.

c) By inspection the value of x such that $1 + x - x^2 = 1.09$ is $x = 0.1$

We can also obtain the value of $x = 0.1$ by solving the quadratic equation $1 - x + x^2 = 1.09$

i.e. $1 - x + x^2 = 1.09 \Rightarrow x^2 - x + 0.09 = 0 \Rightarrow x = 0.1$ or 0.9

Use the general formula for solving the quadratic equation $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } a = 1,\ b = -1 \text{ and } c = 0.09$$

Hence to evaluate 1.09^8 we shall put $x = 0.1$ in expansion for $(1 - x + x^2)^8$

i.e. $1.09^8 = 1 + 8(0.1) + 20(0.1)^2 - 70(0.1)^4$

$= 1 + 0.8 + 0.2 - 0.0070$

$= 1.993$ correct to three decimal places.

Example 12 If the first three terms of the expansion of $(1 + ax)^n$ in ascending powers of x are : $1 + 12x + 64x^2$, find n and a.

Solution

Using the binomial expansion of $(1 + x)^n$, where $n = n$ and $x = ax$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$(1+ax)^n = 1 + nax + \frac{n(n-1)}{2!}(ax)^2 + \ldots$$ Note that we need only the first three terms

$$= 1 + nax + \frac{n(n-1)}{2}a^2x^2 +$$

We are given the first three terms of the expansion of $(1 + ax)^n = 1 + 12x + 64x^2$.

$$\therefore \quad 1 + nax + \frac{n(n-1)}{2}a^2x^2 = 1 + 12x + 64x^2$$

Now comparing coefficients

$na = 12$ --------(1) and

$$\frac{n(n-1)}{2}a^2 = 64 \quad \Rightarrow \quad n^2a^2 - na^2 = 128 \text{ --------(2)}$$

From eqn(1), $n = \frac{12}{a}$. Putting $n = \frac{12}{a}$ in eqn(2)

$$\left(\tfrac{12}{a}\right)^2 a^2 - \left(\tfrac{12}{a}\right) a^2 = 128 \quad \Rightarrow \quad 144 - 12a = 128$$

$$12a = 16 \quad \Rightarrow \quad a = \tfrac{16}{12} = \tfrac{4}{3}$$

Putting $a = \frac{4}{3}$ in eqn(1) $\Rightarrow$ $4n = 36$ $\Rightarrow$ $n = 9$

$\therefore \quad a = \frac{4}{3}$ and $n = 9$

Example 13

Write down the first three terms of the binomial expansion of $(1 + ax)^n$ in ascending powers of x. If the coefficient of x and x^2 are 2 and $\frac{3}{2}$ respectively, find the values of a and n.

Solution

$$(1+ax)^n = 1 + nax + \frac{n(n-1)}{2!}a^2x^2 + \ldots$$

Coefficient of $x = 2 \quad \Rightarrow \quad na = 2 \;\Rightarrow\; a = \frac{2}{n} \;\Rightarrow\; a^2 = \frac{4}{n^2}$ …………(1)

Coefficient of $x^2 = \frac{3}{2} \;\Rightarrow\; \dfrac{n(n-1)}{2}a^2 = \dfrac{3}{2} \quad \Rightarrow \; n(n-1)a^2 = 3$ ….(2)

From $a^2 = \frac{4}{n^2}$, eqn(2) becomes $n(n-1)\dfrac{4}{n^2} = 3$

$$4n^2 - 4n = 3n^2$$
$$n^2 - 4n = 0$$
$$n(n-4) = 0$$
$$n = 0 \quad \text{or}\, n = 4$$
$$\therefore \quad a = \frac{2}{n} = \frac{2}{4} = \frac{1}{2}$$

Example 14

Use the binomial theorem to expand $(1-x)^{-3}$ in ascending powers of x up to the term in x^4.

Solution

Using the binomial expansion of $(1+x)^n$, where $n = -3$ and $x = -x$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$(1-x)^{-3} = 1 + (-3)(-x) + \frac{-3(-4)}{2!}(-x)^2 + \frac{-3(-4)(-5)}{3!}(-x)^3 + \frac{-3(-4)(-5)(-6)}{4!}(-x)^4 + \ldots$$
$$= 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \ldots$$

Example 15

a) Write down the binomial expansion of $(2+x)^4$

b) Use your expansion to evaluate $(1.97)^4$, correct to two decimal places.

Solution

Using the binomial expansion of $(a+x)^n$, where $n = 4, a = 2$ and $x = x$

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \ldots + x^n$$

Therefore replacing $n = 4, a = 2$ and $x = x$, we have

$$(2+x)^4 = 2^4 + 4(2)^3(x) + \frac{4(3)}{2!}2^2x^2 + \frac{4(3)(2)}{3!}(2)(x)^3 + \frac{4(3)(2)(1)}{4!}x^4$$
$$= 16 + 32x + 24x^2 + 8x^3 + x^4$$

b) $(1.97)^4 = (2-0.03)^4$. Comparing $(2-0.03)^4$ with $(2+x)^4 \Rightarrow x = -0.03$

Hence to evaluate $(1.97)^4$, we shall substitute $x = -0.03$ in the expansion for $(2+x)^4$ above.

$\therefore \quad (1.97)^4 = (2-0.03)^4 = 16 + 32(-0.03) + 24(-0.03)^2 + \ldots$

$= 15.04$ correct to two decimal places.

Example 16

Using the binomial theorem, write down and simplify all the terms of the expansion $(1+2x)^6$. Use your expansion to evaluate $(1.02)^6$, correct to four decimal places.

Solution

Using the binomial expansion of $(1+x)^n$ where $n = 6$ and $x = 2x$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

Therefore replacing n by 6 and x by $2x$, we have

$$(1+2x)^6 = 1 + 6(2x) + \frac{6(5)}{2!}(2x)^2 + \frac{6(5)(4)}{3!}(2x)^3 + \frac{6(5)(4)(3)}{4!}(2x)^4 +$$
$$\frac{6(5)(4)(3)(2)}{5!}(2x)^5 + \frac{6(5)(4)(3)(2)(1)}{6!}(2x)^6$$
$$= 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$$

$(1.02)^6 = (1+0.02)^6$ Comparing $(1+0.02)^6$ with $(1+2x)^6 \Rightarrow 2x = 0.02 \Rightarrow x = 0.01$.

Therefore substituting $x = 0.01$ in the expansion of $(1+2x)^6$ we have,

$$(1.02)^6 = (1+0.02)^6 = 1 + 12(0.01) + 60(0.01)^2 + 160(0.01)^3 + \ldots$$
$$= 1 + 0.12 + 0.006 + 0.00016 + \ldots$$
$$= 1.12616 = 1.1262 \text{ correct to four decimal places.}$$

Example 17

Without fully expanding $(1+2x)^5$, determine the coefficient of the fifth term.

Solution

The general term of the expansion of $(a+x)^n$ is the $(r+1)$th term

i.e. the term containing x^r and it is given by: $\dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}x^r$

For the fifth term in the expansion of $(1+2x)^5$,

substitute $a = 1$, $n = 5$ and $r+1 = 5 \Rightarrow r = 4$ into $\dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}x^r$

i.e. 5th term is $\dfrac{5(4)(3)(2)}{4!}(2x)^4 = 5 \times 16x^4 = 80x^4$

The coefficient of the fifth term is 80.

Example 18

Expand $(1-x)^{\frac{1}{3}}$ in ascending powers of x, up to the third term.

Hence evaluate $\sqrt[3]{0.997}$, correct to three decimal places.

Solution

The binomial theorem for any rational index

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \quad \text{where } -1 \le x < 1$$

Therefore replacing n by $\frac{1}{3}$ and x by $-x$, we have

$$(1-x)^{\frac{1}{3}} = 1 + \tfrac{1}{3}(-x) + \frac{(\frac{1}{3})(\frac{1}{3}-1)}{2!}(-x)^2 + \ldots = 1 - \tfrac{1}{3}x - \frac{\frac{1}{3}(-\frac{2}{3})}{2}x^2 + \ldots$$

$$= 1 - \tfrac{1}{3}x - \tfrac{1}{9}x^2 + \ldots$$

$$\sqrt[3]{0.997} = (0.997)^{\frac{1}{3}} = (1-0.003)^{\frac{1}{3}}$$

Hence to evaluate $\sqrt[3]{0.997}$, we shall put $x = 0.003$ in expansion for $(1-x)^{\frac{1}{3}}$

$$\sqrt[3]{0.997} = 1 - \tfrac{1}{3}(0.003) - \tfrac{1}{9}(0.003) + \ldots$$

$$= 1 - 0.001 - 0.000001$$

$$= 0.99899 = 0.999 \text{ correct to three decimal places.}$$

Example 19

Expand $\sqrt{\left(\dfrac{1+x}{1-2x}\right)}$ as a series of ascending powers of x up to and including the term in x^2.

Solution

$$\sqrt{\left(\frac{1+x}{1-2x}\right)} = (1+x)^{\frac{1}{2}}(1-2x)^{-\frac{1}{2}}$$

Now $(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x + \dfrac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \ldots$ and

$$(1-2x)^{-\frac{1}{2}} = 1 + (-\tfrac{1}{2})(-2x)\frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}(-2x)^2 + \ldots$$

Hence $\sqrt{\left(\dfrac{1+x}{1-2x}\right)} = (1+x)^{\frac{1}{2}}(1-2x)^{-\frac{1}{2}}$

$$= \left(1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \ldots\right)\left(1 + x + \tfrac{3}{2}x^2 + \ldots\right)$$

$$= 1 + (\tfrac{1}{2}x + x) + (\tfrac{1}{2}x^2 - \tfrac{1}{8}x^2 + \tfrac{3}{2}x^2) + \ldots$$

$$= 1 + \tfrac{3}{2}x + \tfrac{15}{8}x^2 + \ldots$$

Example 20

i) Expand $(1+x)^{12}$ in ascending powers of x, up to the fifth term.

ii) A sum of P cedis invested at $r\%$ compound interest amounts to $P(1+\frac{r}{100})^n$ after n years. Using the binomial expansion in (i), estimate, correct to the nearest thousand cedis, how much ¢50,000.00 will amount to at 7% compound interest after 12 years.

Solution

i) $(1+x)^{12} = 1 + 12x + 66x^2 + 220x^3 + 495x^4 + \ldots$ [Work out the missing steps]

ii) Replacing x by $\frac{r}{100}$ and $n = 12$, the amount after 12 years is given by:

Amount, $A = P(1+\frac{r}{100})^{12} = P(1 + \frac{12r}{100} + \frac{66r^2}{10000} + \frac{220r^3}{1000000} + \frac{495r^4}{100000000} + \ldots)$

Putting $r = 7$ and $P = ¢50,000$

$$A = ¢50000\left(1 + \tfrac{84}{100} + \tfrac{3234}{10000} + \tfrac{75460}{1000000} + \tfrac{1188495}{100000000} + \ldots\right)$$

$$= 50000 + 42000 + 16170 + 3773 + 594.25 + \ldots$$

$= ¢112537.25 = ¢113,000.00$ to the nearest thousand cedis.

Example 21

Find the first three terms of $\sqrt{\dfrac{1+x}{1-x}}$. By substituting $x = \frac{1}{28}$, find the approximate value of $\sqrt[3]{29}$ correct to three decimal places.

Solution

Work out the solution.

Linear approximations

The binomial expansion for $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots$

If x is sufficiently small, $x^2, x^3, x^4, \ldots$ will be smaller and the expansion for $(1+x)^n$ can be approximated to $1 + nx$. Thus $(1+x)^n \approx 1 + nx$ by ignoring the terms in $x^2, x^3, x^4, \ldots$ The smaller the value of x, the better the approximation.

Similarly, $(1-x)^n \approx 1 - nx$

Illustrative example

Find the linear approximation for

i) $(1.005)^{10}$ ii) $\sqrt{1.004}$ iii) $\sqrt[3]{1.006}$ iv) $(4.01)^3$

Solution

i) $(1.005)^{10} = (1 + 0.005)^{10} = 1 + 10(0.005) = 1.05$

ii) $\sqrt{1.004} = (1.004)^{\frac{1}{2}} = (1 + 0.004)^{\frac{1}{2}} = 1 + \frac{1}{2}(0.004) = 1.002$

iii) $\sqrt[3]{1.006} = (1.006)^{\frac{1}{3}} = (1 + 0.006)^{\frac{1}{3}} = 1 + \frac{1}{3}(0.006) = 1.002$

iv) $(4.01)^3 = (4 + 0.01)^3 = 4^3(1 + \frac{0.01}{4})^3 = 64(1 + 0.0025)^3$

But $(1 + 0.0025)^3 = 1 + 3(0.0025) = 1.0075$

$\therefore \quad (4.01)^3 = 64(1.0075) = 64.48$

Other applications

Binomial expansions may be used for calculations involving small variations.

Illustrative examples

Example 1

If the radius of a sphere is increased by 2 %, what is the approximate percentage increase in its volume.

Solution

Volume of sphere = $\frac{4}{3}\pi r^3$

Let r be the original value of the radius. The new value is $1.02r$ or $(1 + 0.02)r$

The new volume = $\frac{4}{3}\pi[(1+0.02)r]^3 = \frac{4}{3}\pi r^3(1+0.02)^3$

Now $(1+0.02)^3 = 1 + 3(0.02) + 3(0.02)^2 + \ldots$neglecting powers of small terms

$= 1 + 0.06$ neglecting products of small terms

Hence the new volume $\approx \frac{4}{3}\pi r^3\,(1 + 0.06)$ or $1.06\frac{4}{3}\pi r^3$ i.e. 106 % Of the original volume.

Hence the volume is increased by approximately 6 %.

Example 2

The radius of a cylinder is reduced by 4 % and its height is increased by 2 %. Determine the approximate percentage change in its volume.

Solution

Volume of cylinder = $\pi r^2 h$

Let r and h be the original values of the radius and height.

The new values are $0.96r$ or $(1 - 0.04)r$ and $1.02h$ or $(1 + 0.02)h$

The new volume = $\pi[(1-0.04)r]^2[(1+0.02)h]$

$= \pi r^2 h(1-0.04)^2(1+0.02)$

Now $(1-0.04)^2 = 1 - 2(0.04) + (0.04)^2 = 1 - 0.08$ … neglecting products of small terms

Hence the new volume $\approx \pi r^2 h(1-0.08)(1+0.02)$

$\approx \pi r^2 h(1-0.08+0.02)$ neglecting products of small terms

$\approx \pi r^2 h(1-0.06)$ or $0.94\pi r^2 h$ i.e. 94 % of the original volume

Hence the volume is decreased by approximately 6 %.

Example 3

An error of +1.5 % is made when measuring the radius of a sphere. Find the error in measuring the volume.

Solution

Volume of sphere = $\frac{4}{3}\pi r^3$

Let r be the original value of the radius. The new value is $1.015r$ or $(1 + 0.015)r$

The new volume = $\frac{4}{3}\pi[(1+0.015)r]^3 = \frac{4}{3}\pi r^3(1+0.015)^3$

Now $(1+0.015)^3 = 1 + 3(0.015) + 3(0.015)^2 + \ldots$neglecting powers of small terms.

$= 1 + 0.045$ neglecting products of small terms.

Hence the new volume $\approx \frac{4}{3}\pi r^3\,(1 + 0.045)$ or $1.045\frac{4}{3}\pi r^3$ i.e. 104.5 % of the original volume. Hence the volume is increased by approximately 4.5 %.

Review exercise 8

1. Write down the binomial expansion of $(1 + 2x)^7$ in ascending powers of x, simplifying all coefficients. Use your expansion to calculate the value of $(0.98)^7$ correct to four decimal places.

2. a) Without using tables, find the value of $(2 + \sqrt{3})^6 + (2 - \sqrt{3})^6$.
   b) Use the binomial theorem to evaluate 1.004^8, correct to five decimal places.

3. Write down the first four terms of the binomial expansion of $(2 + \frac{1}{2}x)^7$ simplifying all its coefficients. Use your result to evaluate 1.998^7 correct to four decimal places.

4. i) Use the binomial theorem to expand $[1 + (\frac{1}{2}x + x^2)]^4$ in ascending powers of x, up to the term in x^3.
   ii) By putting $x = 0.02$ in the expansion in (i), evaluate $(1.0104)^4$, correct to five decimal places.

5. If the first three terms of the expansion of $(1 + px)^n$ in ascending powers of x are $1 + 15x + 90x^2$, find the values of p and n. (Hint: expand and compare coefficients)

6. a) Write down and simplify the first five terms of the binomial expansion $(1 - \frac{1}{2}x)^8$
   b) Use your expansion to evaluate $(0.995)^8$, correct to five decimal places.

7. Expand $(2 - \frac{1}{2}x)^5$ in ascending powers of x. Use the first four terms of the expansion to find an approximate value of $(1.99)^5$

8. a) Write down the first four terms of the binomial expansion $(1 - 4x)^{\frac{1}{2}}$ in ascending powers of x.
   b) Use your expansion to evaluate $\sqrt{15.36}$, correct to five decimal places.

9. a) Write down the first four terms of the binomial expansion $(2 - 3x)^8$ in ascending powers of x.
   b) Use your expansion to evaluate $(1.997)^8$, correct to the nearest whole number.

10. Expand $(2 - x)^5$ in ascending powers of x up to and including the term in x^2
Use these terms to find an approximate value of $(2.01)^5$ correct to three decimal places.

11. Expand $(1 - x)^{-\frac{1}{2}}$ in ascending powers of x up to the third term.
Use these terms to find an approximate value of $\left(\frac{100}{99}\right)^{\frac{1}{2}}$, correct to two decimal places.

12. Find the coefficient of y in the binomial expansion of $(x - 2y)^4$.

13. Evaluate the following using the binomial theorem correct to four significant figures
    i) $(0.97)^6$ ii) $(3.039)^4$
    iii) $(1.002)^5$ iv) $(1.003)^8$
    v) $(0.98)^7$ vi) $(2.01)^9$

14. Find the linear approximations of the following

i) $(0.97)^6$ ii) $(3.039)^4$

iii) $(1.002)^5$ iv) $(1.003)^8$

v) $(0.98)^7$ vi) $(2.01)^9$

15. Evaluate the following using the binomial theorem

i) $(2+x)^7$ ii) $(x-y)^7$

iii) $2x-3y)^4$ iv) $(1-\frac{1}{2}x)^5$

v) $(1+2x)^{-\frac{1}{2}}$ vi) $(2x+3y)^5$

16. The side of a cube increases by 5%. Find the corresponding increase in the volume.

17. The radius of a cylinder is reduced by 4% and its height is increased by 2%. Determine the approximate percentage change in its curved surface area. [Hint: Curved surface area of a cylinder $=2\pi rh$]

18. If the radius of a sphere is increased by 4%, what is the approximate percentage increase in its volume?

19. The radius of a cone is increased by 2.7% and its height is reduced by 0.9%. Determine the approximate percentage change in its volume. [Hint : Volume of a cone $=\frac{1}{3}\pi r^2 h$]

20. Expand $\dfrac{1}{(1+x)^2}$ in ascending powers of x up to the fourth term.

21. Write down the term indicated in the binomial expansions of the following.

i) $(1-4x)^7$, 3rd term

ii) $(1-\frac{1}{2}x)^{20}$, 6th term

iii) $(2-x)^{15}$, 12th term

iv) $(2-3x)^{30}$, 9th term

v) $(p-2q)^{10}$, 5th term

vi) $(3a+2b)^8$, 2nd term

vii) $(1+2x)^{12}$, term containing x^4

viii) $(2+\frac{1}{2}x)^9$, term containing x^5

22. Write down the binomial expansions of the following as far as and including the term in x^2.

i) $(1+x)(1-x)^9$

ii) $(1-x)(1+2x)^{10}$

iii) $(2+x)(1-\frac{x}{2})^{20}$

iv) $(1+x)^2(1-5x)^{14}$

23. By substituting 0.01 for x in the binomial expansion of $(1-2x)^{10}$, find the value of $(0.98)^{10}$ correct to four significant figures.

24. By substituting 0.05 for x in the binomial expansion of $(1-\frac{x}{5})^6$, find the value of $(1.01)^6$ correct to four significant figures.

25. By using the the binomial expansion of $(2+x)^7$, show that $(2.08)^7=168.439$ correct to three decimal places.

26. Write down the binomial expansions of the following as far as and including the term in x^3.

i) $(1-2x)^{\frac{1}{2}}$ ii) $(3+x)^{-1}$

iii) $\sqrt{\dfrac{1}{1+x}}$ iv) $(1+x)\sqrt{(1-x}$

27. By substituting 0.08 for x in the binomial expansion of $(1+x)^{\frac{1}{2}}$, find the value of $\sqrt{3}$ correct to four significant figures.

28. By substituting $\frac{1}{10}$ for x in the binomial expansion of $(1-x)^{-\frac{1}{2}}$, find the value of $\sqrt{10}$ correct to four significant figures.

29. By using a suitable binomial expansion, find the value of $\sqrt{1.01}$ correct to five decimal places.

30. Expand $\dfrac{1-2x}{\sqrt{1+2x}}$ as far as and including the term in x^2

31. Expand $\sqrt{\dfrac{1+2x}{1-2x}}$ as far as and including the term in x^2. By substituting $x=\frac{1}{1000}$, find an approximation for $\sqrt{51}$ correct to four significant figures.

32. By substituting $\frac{1}{1000}$ for x in the binomial expansion of $(1-x)^{\frac{1}{3}}$, find the value of $\sqrt[3]{37}$ correct to four significant figures.

33. Expand $\dfrac{(1-x)^2}{(1+x)^{\frac{1}{2}}}$ as far as and including the term in x^3.

34. Expand $(1-2x)^{-\frac{1}{2}}$ as far as and including the term in x^3. By substituting $x=\frac{1}{8}$, find an approximation for $\sqrt{5}$ correct to four significant figures.

35. An error of +2 % is made when measuring the radius of a sphere. Find the error in measuring the volume.

36. An error of +1.5 % is made when measuring the radius of a sphere. Find the error in measuring the surface area.

37. An error of 3 % is made when measuring the radius of a sphere. Find the error in measuring the volume.

38. For small values of x the values of y on the curve $y=(2x+3)(1-x)^{10}$ can be found approximately by using $y \approx ax^2+bx+c$. Find the values of a, b and c.

39. The first three terms in the expansion $\left(a-\frac{1}{3}x\right)^6$ in ascending powers of x are $64-64x+bx^2$, where a and b are constants. Find the values of a and b.

40. Write down the binomial expansions of $(2+x)^4$ and $(1-3x)^4$, simplifying the coefficients as far as possible. Find the coefficient x^2 of in the expansion $(2-5x-3x^2)^4$

41. i) Expand $(1+x)^{10}$ in ascending powers of x, up to the fifth term.

ii) A sum of P cedis invested at $r\%$ compound interest amounts to $P(1+\frac{r}{100})^n$ after n years.

Using the binomial expansion in (i), estimate, correct to the nearest thousand cedis, how much ¢100,000.00 will amount to at 5% compound interest after 10 years.

42. The coefficient of x of the binomial expansion of $(4+x)^n$ is equal to the coefficient of x^3 in the binomial expansion of $(4+x)^{n+1}$, where n is a positive integer. Find the value of n.

43. a) i) Using the binomial theorem expand $(1+2x)^5$, simplifying all the terms.

ii) Use your expansion to calculate the value of 1.02^5 correct to six significant figures.

b) If the first three terms of the expansion of $(1+px)^n$ in ascending powers of x are $1+20x+160x^2$, find the values of n and p.

44. a) Find the first four terms of $\dfrac{1}{\sqrt{1+x}}$.

b) By using the linear approximation of the expansion in (a), find the approximate percentage decrease in $y=\dfrac{1}{\sqrt{x}}$ if x is increased by 0.4%.

45. Expand each of the binomials $(1+x)^4$ and $(2-x)^5$ as far as the terms in x^3. Hence find the coefficient of x^3 in the expansion of $(1+x)^4(2-x)^5$

46. a) Use the binomial theorem to expand $(2+3x)^5$

b) Use your expansion to evaluate $(2.15)^5$ correct to two decimal places.

47. Expand the following in ascending powers of x, as far as the terms in x^3

a) $(2+x)^{-3}$ b) $(8+x)^{\frac{1}{3}}$ c) $(1+3x)^{\frac{1}{2}}$

48. Find the ratio of the term in x^5 to the term in x^3 in the binomial expansion of $(2x+3)^6$.

49. i) Use the binomial theorem to expand $\left(1+\frac{1}{4}x\right)^6$

ii) Use your expansion to evaluate $(1.0025)^6$ correct to four decimal places.

50. Expand in terms of ascending powers of x up to and including the term in x^3.

i) $(1-x)^{\frac{1}{2}}$ ii) $\sqrt[3]{(1+3x)}$

iii) $\dfrac{1}{1-x}$ iv) $\dfrac{1}{(1+2x)^3}$

v) $\sqrt[3]{27+x}$ vi) $\dfrac{1}{\sqrt{9+x}}$

51. The first three terms of the binomial expansion of $(1+3x)^6$ is given by $1+18x+135x^2$. From this information, what is the value of $(0.97)^6$, correct to **three** decimal places?

52. i) Use the binomial theorem to expand $\dfrac{1}{\sqrt[3]{8+x}}$, up to the fourth term.

ii) Use your expansion to evaluate $\dfrac{1}{\sqrt[3]{8.08}}$ correct to four decimal places.

CHAPTER 9

INEQUALITIES IN TWO VARIABLES AND LINEAR PROGRAMMING

Linear inequalities in two variables

Linear inequality in two variables are of the form $ax + by > c$ or $ax + by < c$ or $ax + by \leq c$ or $ax + by \geq c$ where a, b and c are constants. Typical examples of inequalities in two variables are:

$x + 2y \geq 4$; $2x - 3y \leq 6$

$x + y > 2$; $3x + 4y > 12$

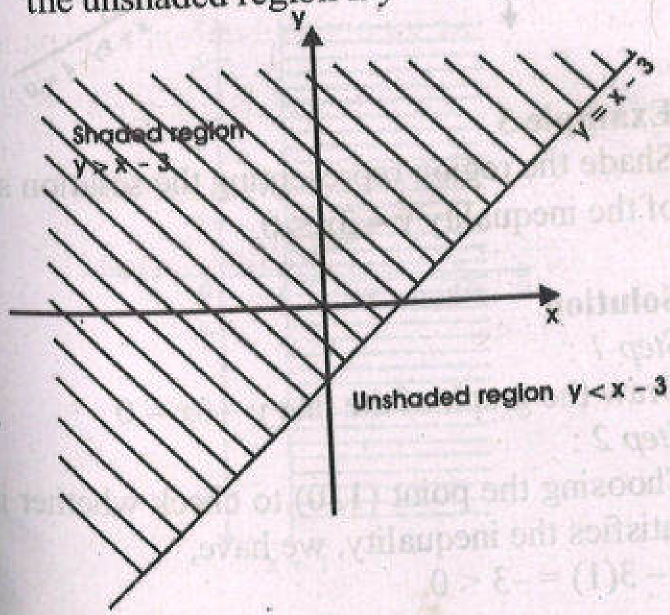
Graphical solution of a single inequality in two variables

A straight line divides the number plane into *three sets of points*. They are:

1. on the line
2. in the region on one side of the line
3. in the region on the other side of the line

Inequalities can be used to describe the regions on each side of the line.
For example, on the following diagram:

the line is $y = x - 3$,
the shaded region is $y > x - 3$,
the unshaded region is $y < x - 3$.



How to draw graphs of inequalities

Step 1:
Draw the *boundary line* of the inequality. The corresponding equation gives this line. If the inequality is ≤ or ≥, then the line is included in the region. Show this by drawing a continuous line: ———
If the inequality is < or >, then the line is *not* included in the region.
Show this by drawing a dotted line: – – – –

Step 2:
Work out which side of the line you want for the inequality. To do this, choose an easy point on one side of the line. Find out whether its coordinates satisfy the inequality or not.

Step 3:
Shade the region you want, i.e. the region where the inequality is satisfied.
Note that some people shade the unwanted region. **Always** make it clear in your answer which region you are shading.

Illustrative examples

Example 1
Solve the inequality $2x - 3y - 6 \leq 0$ graphically.

Solution
Step 1:
Plot the graph with equation $2x - 3y - 6 = 0$ using any two points.
When $x = 0, \Rightarrow -3y = 6 \Rightarrow y = -2$ i.e. $(0, -2)$
When $y = 0 \Rightarrow 2x = 6 \Rightarrow x = 3$ i.e. $(3, 0)$

Plot these two points and draw a straight line through them. Note that the line should be continuous.

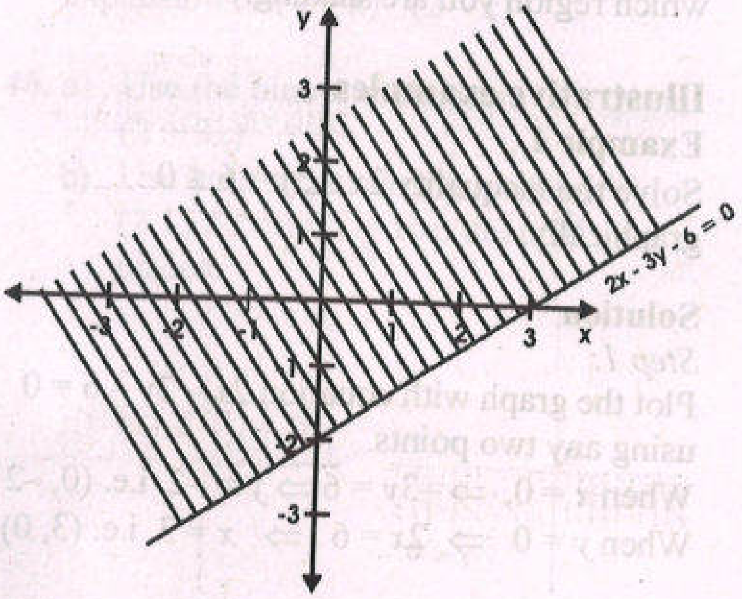
Step 2 :
Choose any point above or below the line drawn (i.e. on either side of the line) and check whether it satisfies the inequality given. Thus choosing the point (0, 0), which is above the line, we have
$2(0) - 3(0) - 6 = -6 < 0$ (i.e. substitute $x = 0$ and $y = 0$ into the inequality).

Therefore since (0, 0) satisfies the inequality, the solution set is the region above the line as shaded in the graph below. Note that a point below the line can be chosen i.e. choosing (2, –2) which is below the line,

$2(2) - 3(-2) - 6 = 4$ which is not less than 0.

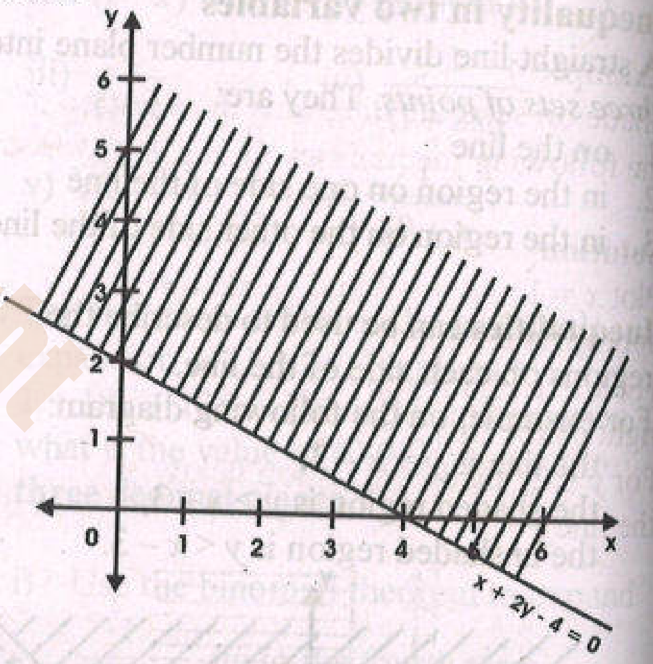
Therefore the solution set is not this region but rather the region above the line.
Thus if a point chosen from a region does not satisfy the inequality, then the other region is the solution set.

Note that if the line does not pass through the origin it is advisable to choose the origin (0, 0) for the test.



Example 2
Shade the region representing the solution set of the inequality $x + 2y - 4 \geq 0$.

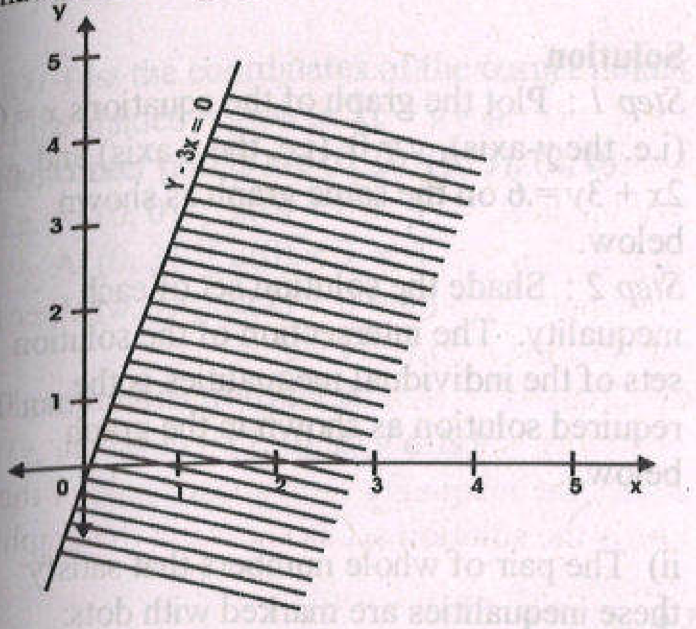
Solution
Step 1:
Plot the graph with equation $x + 2y - 4 = 0$
i.e. $x = 0 \Rightarrow y = 2$ and $y = 0 \Rightarrow x = 4$.
Therefore using the points (0, 2) and (4, 0), plot the line.
Step 2 :
Choosing the point (0, 0) to check whether it satisfies the inequality we have,
$0 + 2(0) - 4 = -4$, which is not greater than 0.
Therefore since the point (0, 0) does not satisfy the inequality the opposite side of the line is the solution set as shown in the graph below.



Example 3
Shade the region representing the solution set of the inequality $y - 3x \leq 0$.

Solution
Step 1 :
Draw the graph of the line $y - 3x = 0$
Step 2 :
Choosing the point (1, 0) to check whether it satisfies the inequality, we have,
$0 - 3(1) = -3 < 0$.

Note that the line passes through the origin, so we did not use (0, 0). Therefore that side of the line is the solution set since the point satisfies the inequality as shaded on the graph below.



Example 4

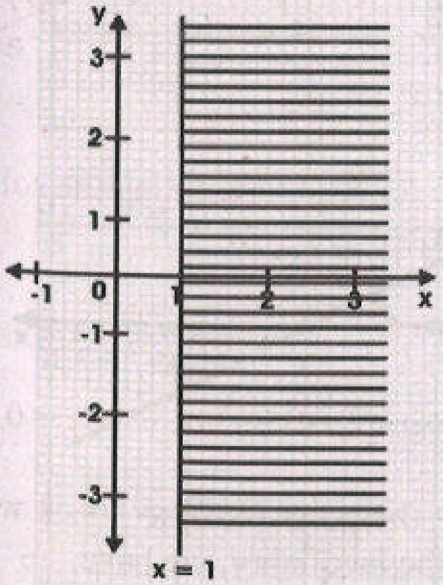
Shade on separate diagrams the solution of the following inequalities i) $x \geq 1$ ii) $y \leq 3$

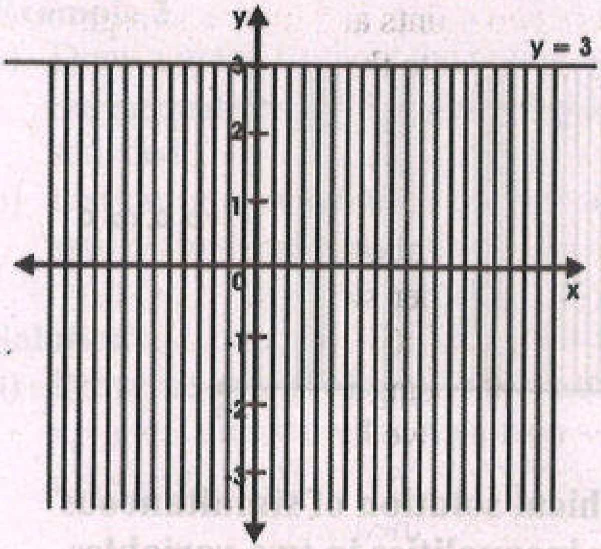
Solution

Plot $x = 1$ and $y = 3$ and shade the appropriate regions.

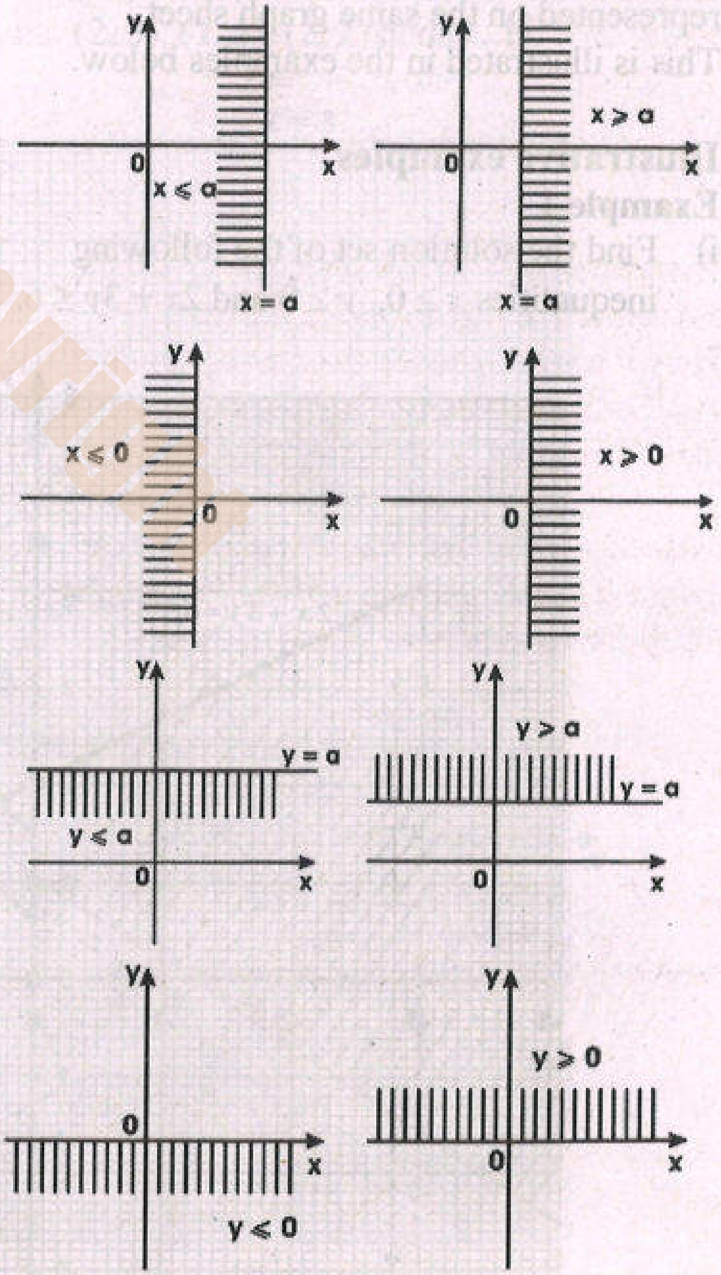
For $x \geq 1$, shade the region to the right of the line $x = 1$.

For $y \leq 3$, shade the region below the line $y = 3$.





Note the following graphs and the shaded regions carefully.


y
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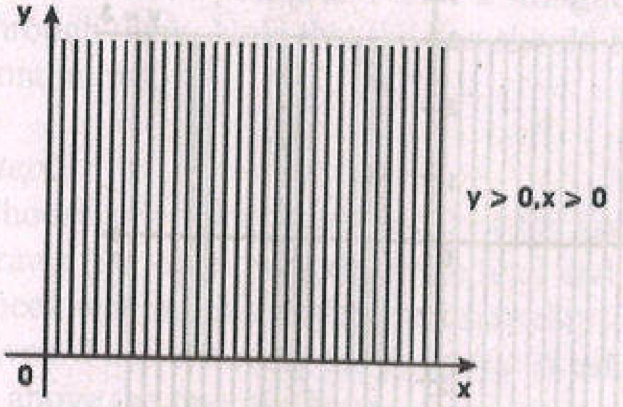


Graphical solution of simultaneous linear inequalities in two variables

The solution set of a system of inequalities in x and y is the *region of intersection* of the solution of all the individual inequalities represented on the same graph sheet.
This is illustrated in the examples below.

Illustrative examples

Example 1

i) Find the solution set of the following inequalities $x \geq 0$, $y \geq 0$ and $2x + 3y \leq 6$.

ii) Find the pair of whole numbers that satisfy these inequalities.

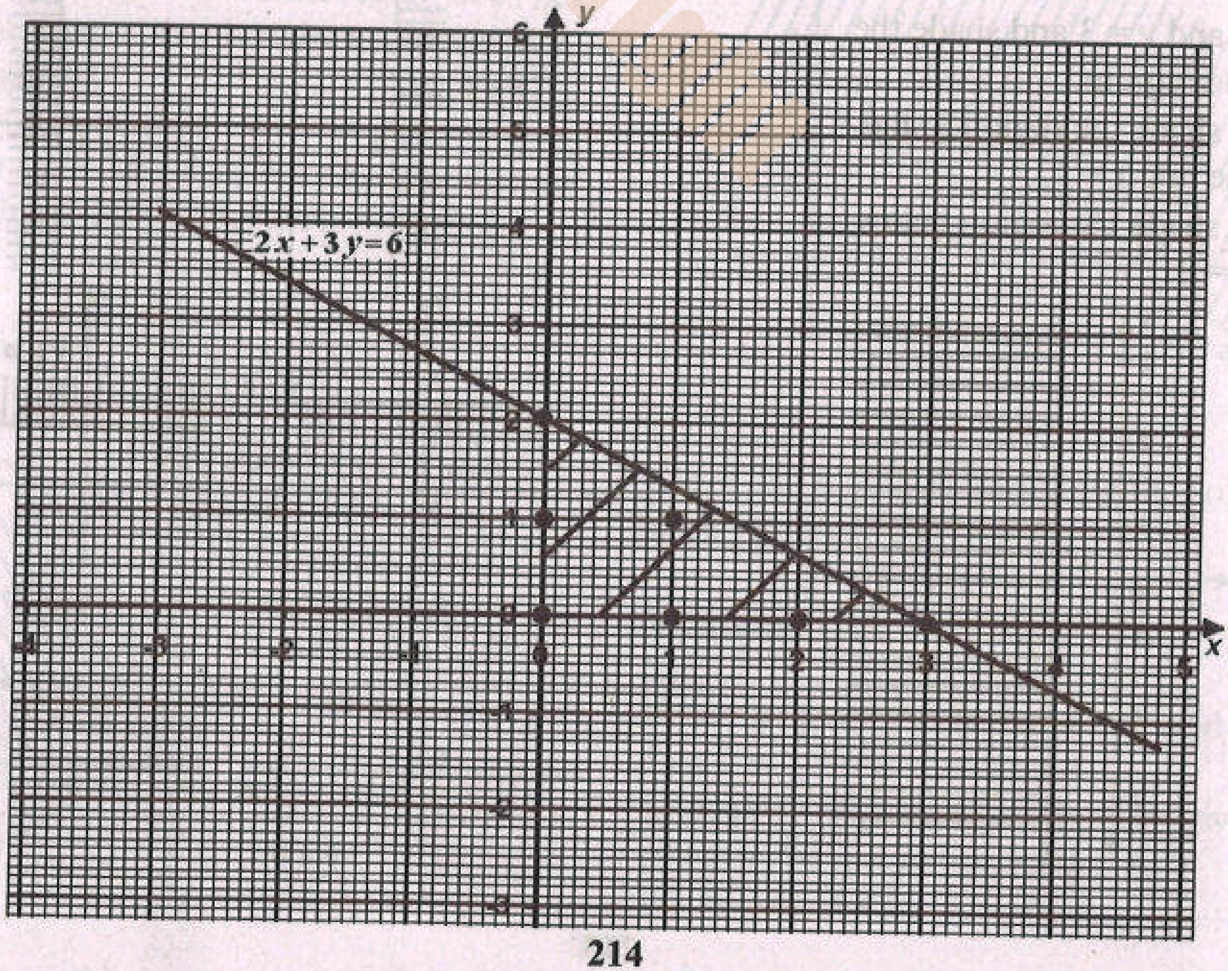
iii) Find the coordinates of the vertices of the region which satisfy the equation $2x + y = 4$

Solution

Step 1 : Plot the graph of the equations $x = 0$ (i.e. the y-axis), $y = 0$ (i.e. the x-axis) and $2x + 3y = 6$ on the same graph as shown below.

Step 2 : Shade the solution set of each inequality. The intersection of the solution sets of the individual inequalities is the required solution as shown in the graph below.

ii) The pair of whole numbers that satisfy these inequalities are marked with dots: (0, 0), (0, 1), (0, 2), (1, 0), (2, 0), (3, 0) and (1, 1).



Note:
If the sign of the inequality $2x + 3y \leq 6$ were to be < instead of ≤, the line will be a dotted one and the points (3, 0) and (0, 2) will not be part of the solution.

iii) Test the coordinates of the corner points of the shaded region in $2x + y = 4$.
The corner points are (0, 0), (0, 3), (2, 0)
i.e. At (0, 0) ; $2(0) + 0 \neq 4$
At (0, 3) ; $2(0) + 3 \neq 4$
At (2, 0) ; $2(2) + 0 = 4$

Therefore the truth set is
$\{(x, y) : (x, y) = (2, 0); x, y \in \mathbb{N}\}$

Example 2

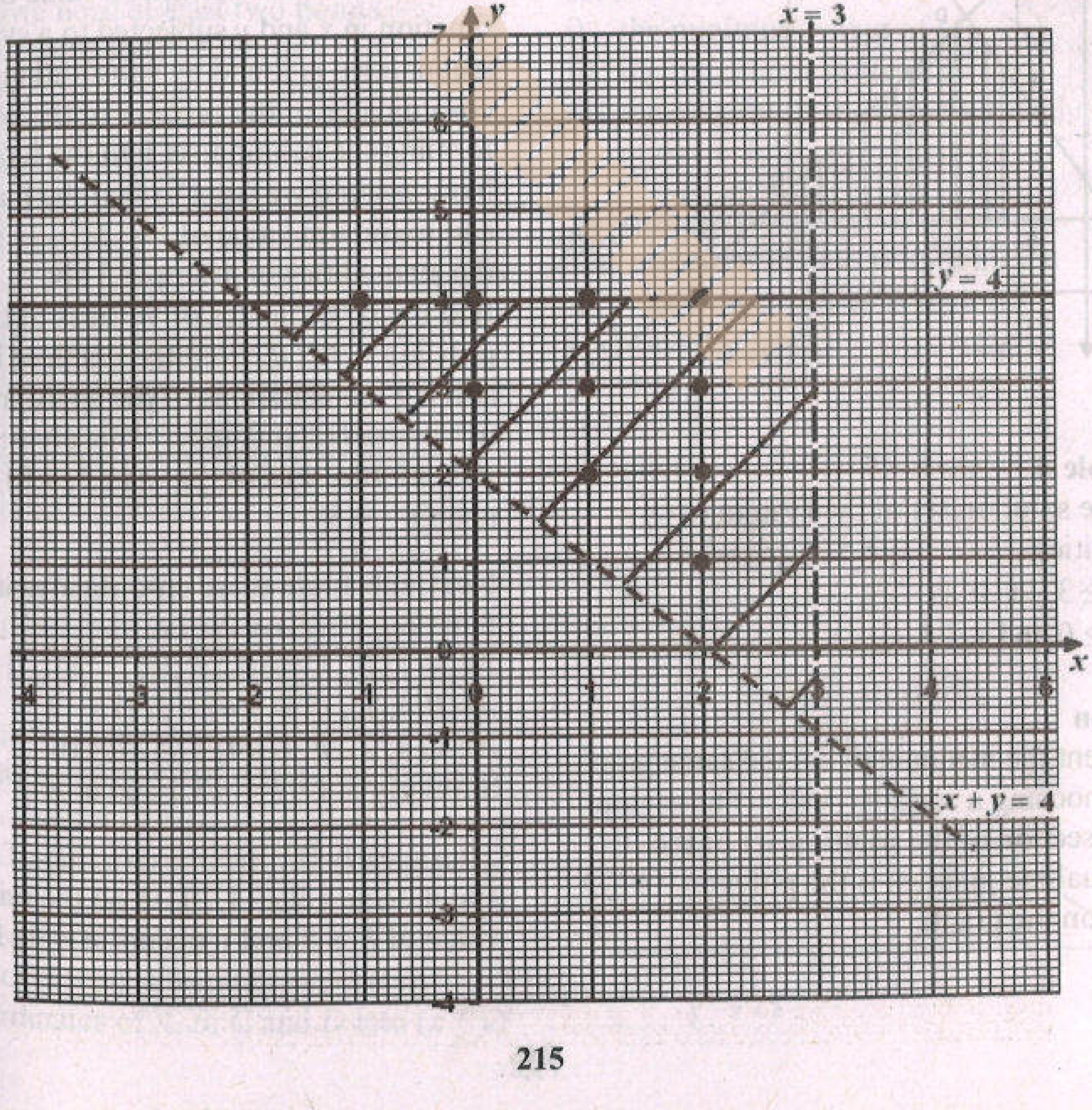
a) Draw a graph to show the region where the inequalities $x < 3$, $y \leq 4$, $x + y > 2$ are satisfied.

b) List the integer solutions (x, y) which satisfy these inequalities.

Solution

a) Draw the graphs of $x = 3$, $y = 4$ and $x + y = 2$ and show the regions $x < 3$, $y \leq 4$, $x + y > 2$.

b) The pair of integers that satisfy these inequalities are marked with dots.
These are:
(–1, 4), (0, 4), (1, 4), (2, 4), (0, 3), (1, 3), (2, 3), (1, 2), (2, 2) and (2, 1).

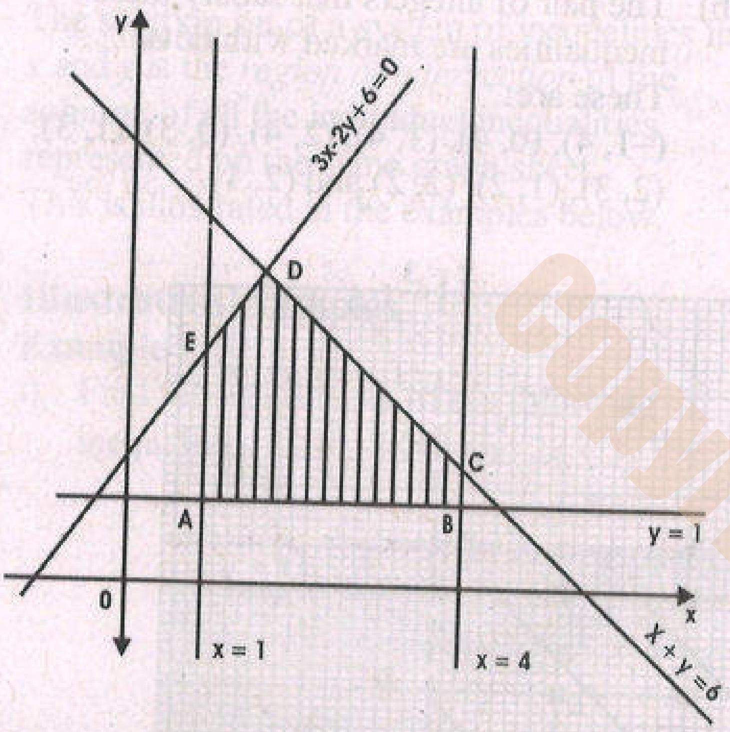


Example 3

Find the solution set of the system of inequalities: $3x - 2y + 6 \ge 0$, $x + y \le 6$, $1 \le x \le 4$ and $y \ge 1$

Solution

Using the procedure in the earlier examples, represent the inequalities on the same graph sheet choosing a suitable scale. The region of intersection of the solution set of the individual inequalities is the polygon *ABCDE* shown on the graph.



Example 4

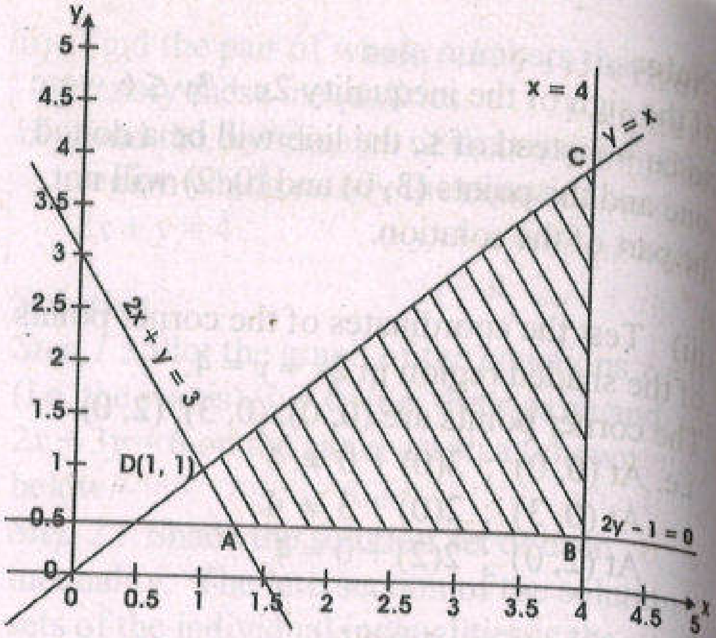
Find the solution set of the system of inequalities

$2x + y \ge 3$, $x \ge y$,

$2y - 1 \ge 0$ and $x \le 4$.

Solution

Represent the inequalities on the same graph sheet choosing a suitable scale. The region of intersection of the solution set of the individual inequalities is the polygon *ABCD* shown on the graph.



To find the maximum or minimum value of a function

The maximum or minimum value of a function in x and y subjected to a system of inequalities is found by substituting *all* the co-ordinates of the *vertices* of the polygon forming the solution set into the function. Now assume we want to find the maximum and minimum value of $F = 3x + 2y$ subject to the inequalities in example 3 above. We first find the co-ordinates of vertices of the polygon *ABCDE* from the graph or by solving the equations of the appropriate pair of lines simultaneously. This gives: $A(1, 1)$, $B(4, 1)$ $C(4, 2)$, $D(1.2, 4.8)$ and $E(1, 4.5)$.

Next, substitute these co-ordinates into the function, $F = 3x + 2y$, i.e.

At $A(1, 1)$, $F = 3(1) + 2(1) = 5$

At $B(4, 1)$, $F = 3(4) + 2(1) = 14$

At $C(4, 2)$, $F = 3(4) + 2(2) = 16$

Similarly at D, $F = 13.2$ and at E, $F = 12$.

Thus clearly the maximum value of F is 16 and it occurs at $C(4, 2)$. Also the minimum value of F is 5 and it occurs at $A(1, 1)$.

Illustrative examples

Example 1

Illustrate graphically the set P of all points (x, y) which satisfy simultaneously the following inequalities.

i) $8x + 5y \geq 80$

ii) $\frac{1}{2}x + \frac{1}{5}y \leq 12$

iii) $x - y \geq 0$, and $y \geq 0$.

Using your diagram, calculate on the set P.

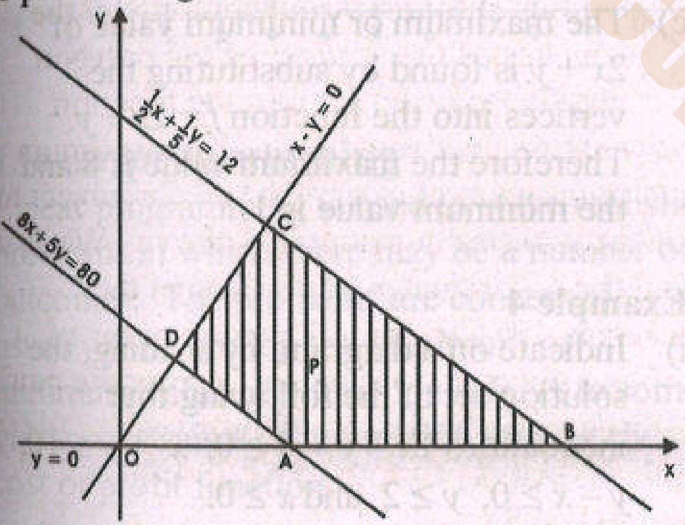
a) the maximum and minimum values of
   i) x ii) y iii) $x + y$

b) the maximum value of $8x + 5y$

c) the minimum value of $3x + y$

Solution

The region of intersection is region P.
i.e. polygon $ABCD$.
Note that we need at least two points to plot a straight line.



From the graph, the co-ordinates of the vertices of polygon $ABCD$ are $A(10, 0)$, $B(24, 0)$, $C(17.1, 17.1)$ and $D(6.2, 6.2)$.

a) i) From the co-ordinates, the maximum and minimum values of x are 24 and 6.2 respectively.
   ii) Also the maximum and minimum values of y are 17.1 and 0 respectively.
   iii) For $(x + y)$, substitute the co-ordinates of A, B, C and D into $(x + y)$ and choose the maximum and minimum values. Therefore the maximum value of $(x + y)$ is 34.2 and minimum value of $(x + y)$ is 10.

b) In the same way as a(iii) above, substitute the co-ordinates the vertices A, B, C and D into $(8x + 5y)$. Therefore the maximum value of $(8x + 5y)$ is 222.3 i.e. at $C(17.1, 17.1)$.

c) Similarly, the minimum value of $(3x + y)$ is 24.8 i.e. it occurs at D.

Example 2

Illustrate graphically the set P of all points (x, y) in the plane which satisfy simultaneously $x \geq 2$, $y \geq 3$, $x + y \leq 9$ and $2x + y \leq 12$. Use your graph to find on the set P.

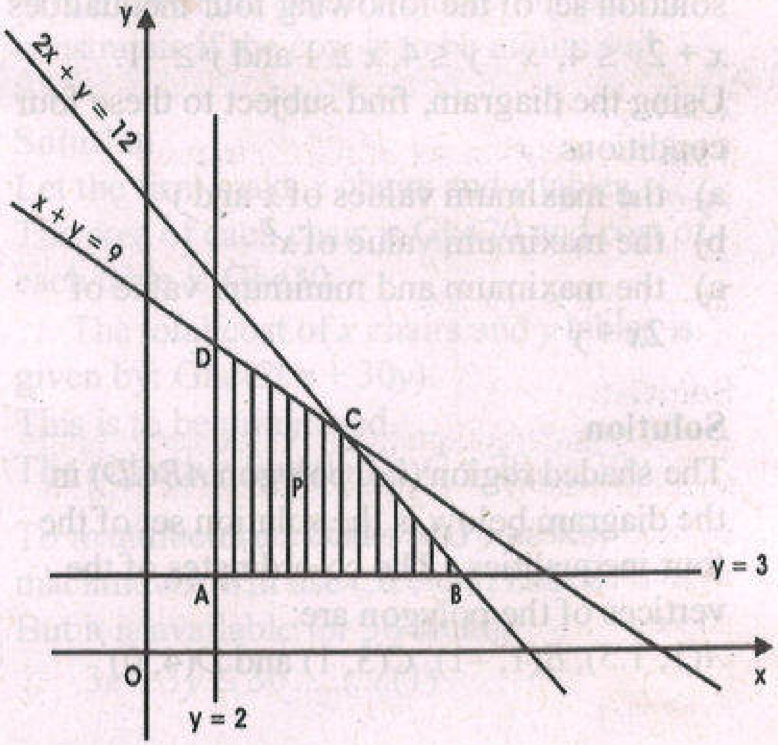
i) the minimum value of $(4 - 2x)$

ii) the maximum value of $(y + 3)$

iii) the maximum and minimum values of $(4x + 3y)$.

Solution

The region of intersection of all the individual inequalities is region P.
i.e. polygon $ABCD$.



From the graph, the co-ordinates of the vertices of polygon *ABCD* are *A*(2, 3) *B*(4.5, 3), *C*(3, 6) and *D*(2, 7).

i) For the minimum values of $(4 - 2x)$ substitute only the x-co-ordinates of all the vertices into $(4 - 2x)$ since it is a function of x only and pick the minimum value. Thus the minimum value of $(4 - 2x)$ is -5 and it occurs at *B*(4.5, 3)

ii) Similarly, since $(y + 3)$ is a function of only y, substitute only the y-co-ordinates of all the vertices into it and pick the maximum value; maximum value of $(y + 3)$ is 10 i.e. at *D*(2, 7)

iii) To find the maximum and minimum values of $(4x + 3y)$, substitute both the x and y co-ordinates of all the vertices since it is a function of x and y. Therefore the maximum and minimum values of $(4x + 3y)$ are 30 and 17 respectively.

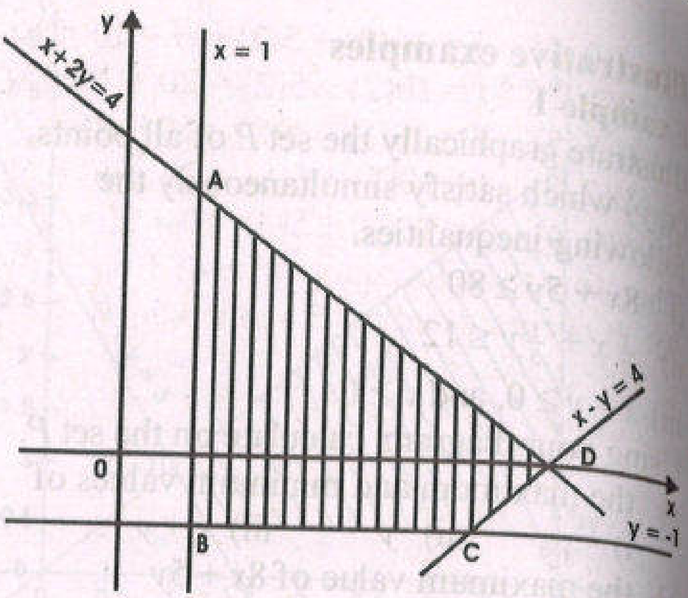
Example 3

Indicate on a diagram, by shading, the solution set of the following four inequalities $x + 2y \leq 4,\ x - y \leq 4, x \geq 1$ and $y \geq -1$.
Using the diagram, find subject to these four conditions

a) the maximum values of x and y
b) the maximum value of x^2
c) the maximum and minimum value of $2x + y$

Solution

The shaded region (i.e. polygon *ABCD*) in the diagram below is the solution set of the four inequalities. The co-ordinates of the vertices of the polygon are:
A(1, 1.5), *B*(1, –1), *C*(3, 1) and *D*(4, 0)



a) From the co-ordinates of the vertices the maximum value of x is 4 and the maximum value of y is 1.5.

b) The maximum value of x^2 is $4^2 = 16$ [Substitute only the x coordinates of the vertices into x^2].

c) The maximum or minimum value of $2x + y$ is found by substituting the vertices into the function $f = 2x + y$. Therefore the maximum value is 8 and the minimum value is 1.

Example 4

i) Indicate on a diagram, by shading, the solution set of the following four inequalities $2x - y + 4 \geq 0,\ x + y \leq 10,\ y - x \geq 0,\ y \geq 2$ and $x \geq 0$.

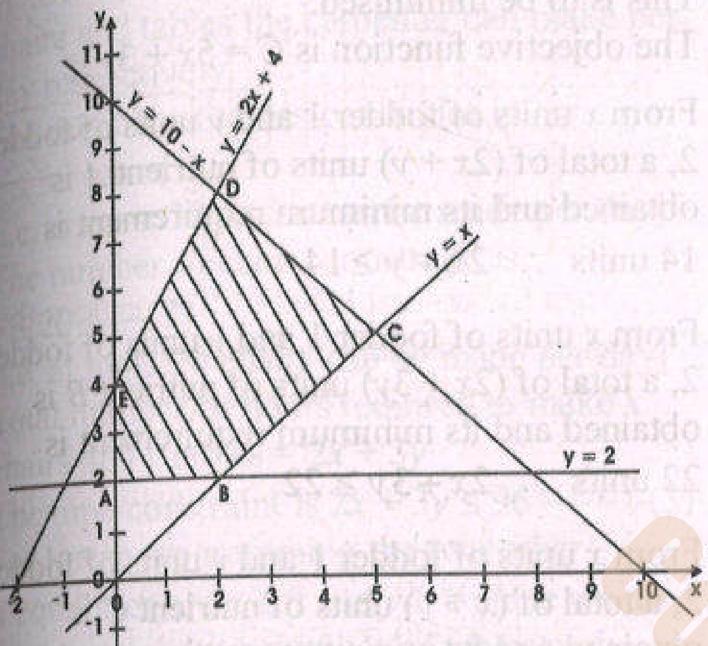
ii) Using the diagram, find the values of x and y for which $10x + 5y$ is maximum.

Solution

The shaded region (i.e. polygon *ABCDE*) in the diagram below is the solution set of the four inequalities. The co-ordinates of the vertices of the polygon are *A*(0, 2), *B*(2, 2), *C*(5, 5), *D*(2, 8) and *E*(0, 4)
The maximum or minimum of $10x + 5y$ is found by substituting the vertices into the function: $f = 10x + 5y$.

For $A(0, 2)$, $f = 10(0) + 5(2) = 0 + 10 = 10$
$B(2, 2)$, $f = 10(2) + 5(2) = 20 + 10 = 30$
$C(5, 5)$, $f = 10(5) + 5(5) = 50 + 25 = 75$
$D(2, 8)$ $f = 10(2) + 5(8) = 20 + 40 = 60$
$E(0, 4)$ $f = 10(0) + 5(4) = 0 + 20 = 20$

Therefore the maximum value of $10x + 5y$ is 75. It occurs when $x = 5$ and $y = 5$



Linear programming

Linear programming is used to solve realistic problems in which there may be a number of outcomes. The problems are concerned usually with finding the maximum or minimum value of a function subject to some given constraints. The function may be the cost or profit function.

The graphical method of solving linear programming problems

Step 1 : Write down the **objective function**. The objective function is the linear function which is to be maximised or minimised.

Step 2 : Write down the **constraints** according to the construction of the problem. Include the non-zero constraints $x \geq 0$, $y \geq 0$ if the problem requires that.

Step 3 : (
region of in the inequalities to obtain the required solution, which gives the

Step 4 : Use the co
of the region of interseates of the vertices minimize the objective fu to maximize or n.

Illustrative examples

Example 1

A firm makes two types of furnitu hairs and tables. The cost of production fo product as calculated is Gh¢20 per chair a Gh¢30 per table. Both products are processed by three machines M_1, M_2 and M_3. The time required in hours by each product and the total time available in hours per week on each machine is as follows:

	Chair	***Tables***	***Available time***
M_1	3	3	36
M_2	5	2	50
M_3	2	6	60

Write down the objective function and the constraints if the cost is to be minimised.

Solution

Let the firm make x chairs and y tables.
The cost of each chair is Gh¢20 and cost of each table is Gh¢30.
$\therefore$ The total cost of x chairs and y tables is given by: Gh¢$(20x + 30y)$.
This is to be minimised.
The objective function is $C = 20x + 30y$

To manufacture x chairs and y tables, machine M_1 will use $(3x + 3y)$ hours.
But it is available for 36 hours.

$\therefore \quad 3x + 3y \leq 36 \text{ ....(1)}$

To manufacture x chairs and y tables, machine M_2 will use $(5x + 2y)$ hours. But it is available for 50 hours.

$\therefore \quad 5x + 2y \leq 50$ ....(2)

To manufacture x chairs and y tables, machine M_3 will use $(2x + 6y)$ hours. But it is available for 60 hours.

$\therefore \quad 2x + 6y \leq 60$ ....(3)

Since the number of chairs and tables cannot be negative, $x \geq 0,\ y \geq 0$.

Therefore the mathematical formulation of the given problem is:

To minimise, $C = 20x + 30y$

Subject to the following constraints:

$3x + 3y \leq 36$

$5x + 2y \leq 50$

$2x + 6y \leq 60$

$x \geq 0,\ y \geq 0$

Example 2

At a cattle breeding firm, it is prescribed that the food constituents for one animal must contain at least 14, 22 and 1 units of nutrients A, B and C respectively. Two different kinds of fodder are available. Each unit weight of these two contains the following amount of the three nutrients,

	Fodder 1	***Fodder 2***
Nutrient A	2	1
Nutrient B	2	3
Nutrient C	1	1

It is given that the cost of unit quantity of fodder 1 and 2 are Gh¢5 and Gh¢3 respectively. Write down the objective function and the constraints if the cost is to be minimised.

Solution

Let the firm purchase x units of fodder 1 and y units of fodder 2.

The cost of one unit of fodder 1 is Gh¢5 and cost of one unit of fodder 2 is Gh¢3

$\therefore$ The total cost of x units of fodder 1 and y units of fodder 2 is given by: Gh¢$(5x + 3y)$.

This is to be minimised.

The objective function is $C = 5x + 3y$

From x units of fodder 1 and y units of fodder 2, a total of $(2x + y)$ units of nutrient A is obtained and its minimum requirement is 14 units $\therefore \quad 2x + y \geq 14$

From x units of fodder 1 and y units of fodder 2, a total of $(2x + 3y)$ units of nutrient B is obtained and its minimum requirement is 22 units $\therefore \quad 2x + 3y \geq 22$

From x units of fodder 1 and y units of fodder 2, a total of $(x + y)$ units of nutrient C is obtained and its minimum requirement is 1 units $\therefore \quad x + y \geq 1$

Since the number of units of fodder 1 and fodder 2 cannot be negative, $x \geq 0,\ y \geq 0$

Therefore the mathematical formulation of the given problem is:

To minimise, $C = 5x + 3y$

Subject to the following constraints:

$2x + y \geq 14$

$2x + 3y \geq 22$

$x + y \geq 1$

Example 3

A carpenter can make at most 20 tables and at most 30 chairs per day. Each table requires 3 hours of labour and each chair 2 hours of labour. The maximum total hours of labour that the carpenter has at his disposal is 96.

a) Give three inequalities that express the condition above.

b) Graph and shade the common region that satisfies these inequalities.

c) Find the maximum number of chairs and tables that he can make within the time at his disposal.

Solution

a) Let x and y represent the number of chairs and tables the carpenter can make per day respectively.

The number of tables constraint is

$x \leq 30$ -------(1)

[i.e. at most 30 tables can be made per day]

The number of chairs constraint is

$y \leq 20$------(2)

[i.e. at most 20 chairs can be made per day]

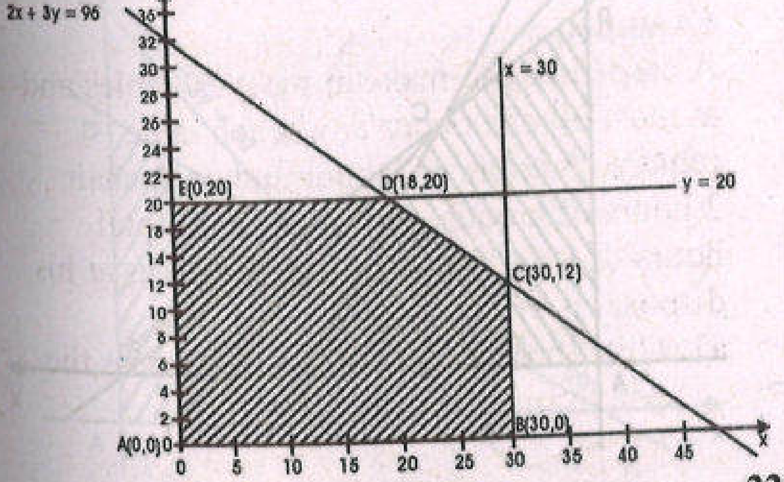
Total number of hours required to make x chairs and y tables $= 2x + 3y$

The time constraint is $2x + 3y \leq 96$ ---------(3)

For linear programming the two other inequalities are $x \geq 0$ and $y \geq 0$ since we do not have negative number of tables and chairs.

b) The diagram below shows the solution set of the above inequalities i.e. polygon $ABCDE$ where $A(0, 0)$, $B(30, 0)$, $C(30, 12)$, $D(18, 20)$ and $E(0, 20)$.

c) The sum of the coordinates at C gives a maximum of 42 chairs and tables that he can make within the time at his disposal. Thus the carpenter can make a maximum of 30 chairs and 12 tables within the time at his disposal.



Example 4

Out of a stock of 200 resistors, 120 transistors and 150 capacitors, it is required to produce two types of circuits. Circuit type I requires 20 resistors, 10 transistors and 10 capacitors, while printed circuit type II requires 10 resistors, 20 transistors and 30 capacitors. If a type I circuit sells at Gh¢5 each and type II circuit at Gh¢12 each, how many of each type should be made to get the maximum financial returns.

Solution

The information is set out as follows:

	Resistors	***Transistors***	***Capacitors***
Type I	20	10	10
Type II	10	20	30
Total No. Available	200	120	150

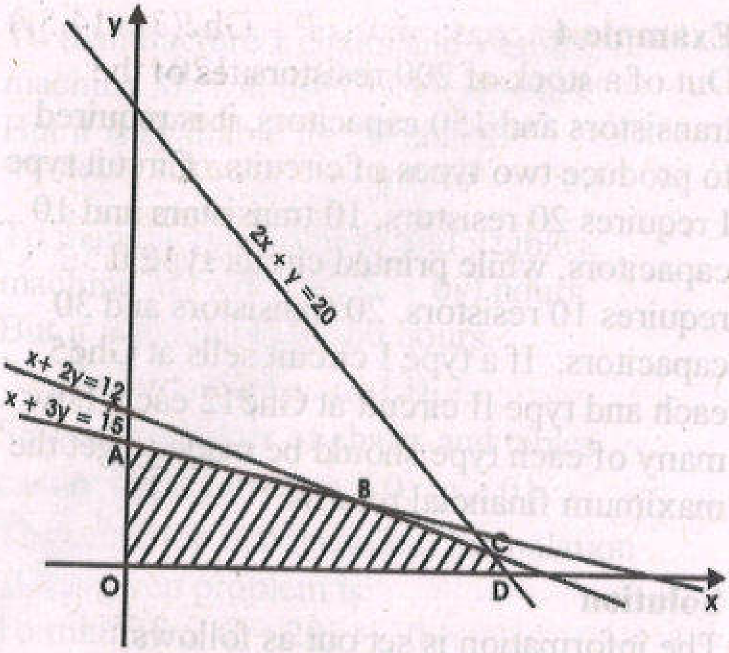
1. Let x and y represent the number of type I and type II circuits made respectively.

2. We do not have negative number of circuits $\therefore \quad x \geq 0, \ y \geq 0$.

3. The inequalities using the number of components are :
   $20x + 10y \leq 200$, $10x + 20y \leq 120$ and $10x + 30y \leq 150$

   These can be simplified to obtain
   $2x + y \leq 20$, $x + 2y \leq 12$ and $x + 3y \leq 15$

4. Graph these inequalities together with $x \geq 0$ and $y \geq 0$ to get the solution set. i.e. polygon $OABCD$.



From the graph, the co-ordinates of the vertices of the polygon are: $O(0, 0)$, $A(0, 5)$, $B(6, 3)$, $C(9.3\ 1.3)$ and $D(10\ ,0)$.

Now from the question, the financial returns is $F = \text{Gh¢}(5x + 12y)$

$\therefore$ Substitute these co-ordinates into F and choose the maximum value of F

i.e. at O, $F = 0$, at A, $F = 60$, at B, $F = 66$, at C, $F = 62.66$ and at D, $F = 50$

Hence the maximum value 66 occurs at $B(6, 3)$.

Therefore 6 circuits of type I and 3 circuits of type II should made for maximum financial returns of Gh¢66.

Example 5

Red Biro pens and black Biro pens are produced by a machine operator using any one of three machines *A*, *B* and *C*. The machine operator produces *x* packets of red and *y* packets of black Biro pens each hour. The maximum number of minutes required on each machine to produce one packet of pens is as follows.

	A	*B*	*C*
Red	2.5	4	3.5
Black	3.5	2	2.5

Write down the three inequalities representing the above information. If the profit on a packet of the red pens is GH¢2.00 and that of the black pens is GH¢2.50, find the maximum profit which the machine operators can make in 1 hour from the production of the two types of Biro pens?

Solution

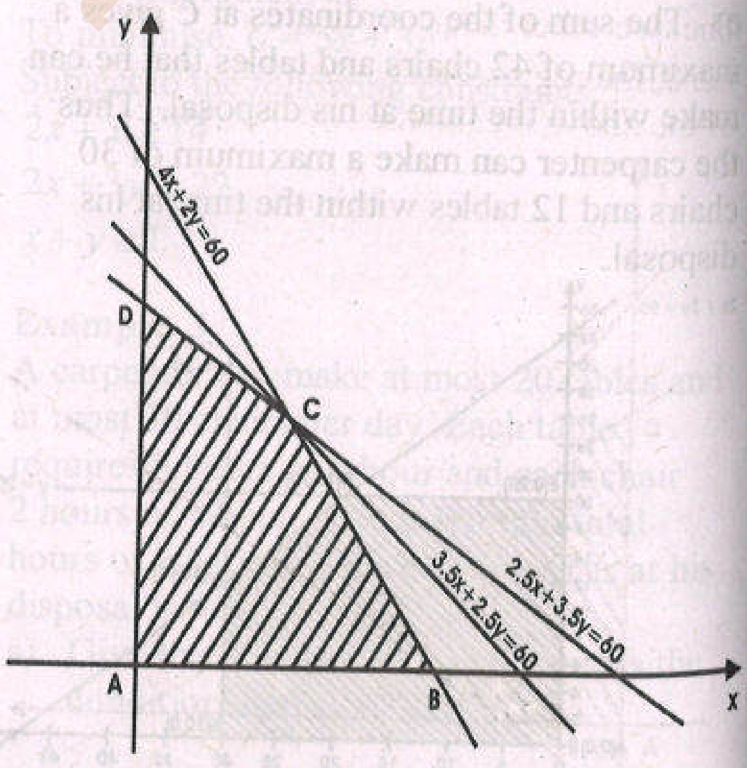
The number of minutes in 1 hr = 60

$\therefore$ If x and y represent the number of packets of red and black pens produced in 1 hr respectively, then the 3 inequalities are $2.5x + 3.5y \leq 60$, $4x + 2y \leq 60$ and $3.5x + 2.5y \leq 60$. Since we do not have negative packets we can also write the two inequalities $x \geq 0$ and $y \geq 0$.

The solution set of these inequalities is shown below i.e. polygon *ABCD*.

The profit on the production of x packets of red pens and y packets of black pens P is given by:

$P = \text{Gh¢}(2x + 2.5y)$. Therefore to find the maximum profit, substitute the co-ordinates of the vertices of the polygon into P. Hence, the maximum profit is Gh¢44.70 and it occurs at $C(10.5, 9.5)$.



Example 6

A seamstress buys 90 m of material to make dresses for both adults and children. She uses 1 m of material for a child's dress and 3 m for an adult's dress. It takes her 75 minutes to complete a child's dress and 2 hours to complete an adult's dress. She uses at most 80 hours to make x dresses for children and y dresses for adults.

a) Write down two inequalities involving x and y and illustrate them on a graph.

b) If she makes a profit of Gh¢7.00 on each child's dress and GH¢14.50 on each adult's dress, find how many of each kind of dress she should sew in order to make the greatest profit.

c) Find the maximum profit in (b) above.

Solution

a) If x and y represent the number of child dress and adult dress respectively.

The material constraint is

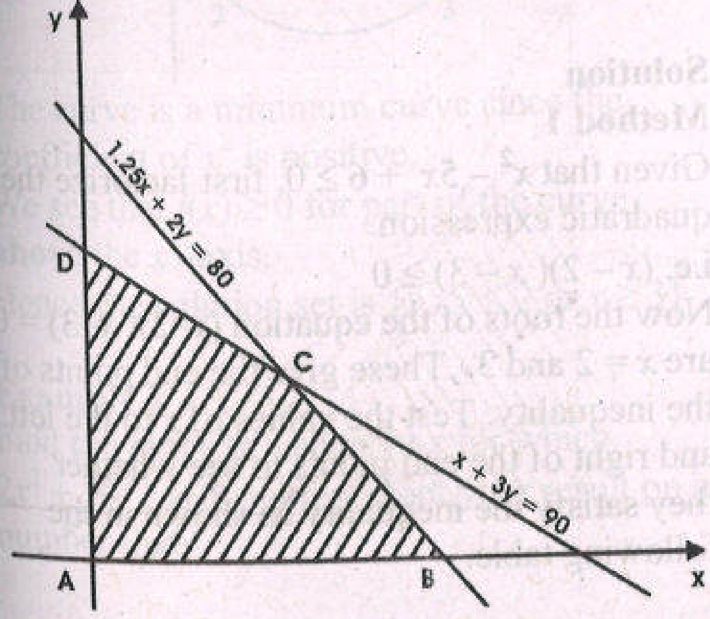
$x + 3y \leq 90$ ------------(1)

The time constraint is

$1.25x + 2y \leq 80$ --------(2)

Note: 75 min = 1.25 hrs and
'at most' means $\leq$

The two other inequalities are $x \geq 0$ and $y \geq 0$, the region below shows the solution set of the above inequalities i.e. polygon *ABCD*.



b) The profit constraint is $P = \text{Gh¢}(7x+14.5y)$ Now substituting the co-ordinates of the vertices of the polygon into P, the vertex $C(34, 19)$ gives the maximum profit. Hence to make the greatest profit, the seamstress should sew 54 child's dresses and 12 adult's dresses.

c) The maximum profit as given by $C(34, 19)$ is:
$P = 7(34) + 14.5(9) = \text{Gh¢}513.50$.

Example 7

A seamstress makes two types of dress *X* and *Y*. Each type of dress requires the use of three materials *A*, *B* and *C*. The table gives the number of square metres of each material required to make each type of dress, the total number available and the estimated profit for each type of dress.

a) If the seamstress makes x units of type *X* and y units of type *Y*, write down all the inequalities on x and y.

b) Represent, by shading on a graph, the solution set of the inequalities in a), identifying all the vertices on the graph.

c) Find how many of each dress the seamstress should make to get the maximum profit. What is the maximum profit?

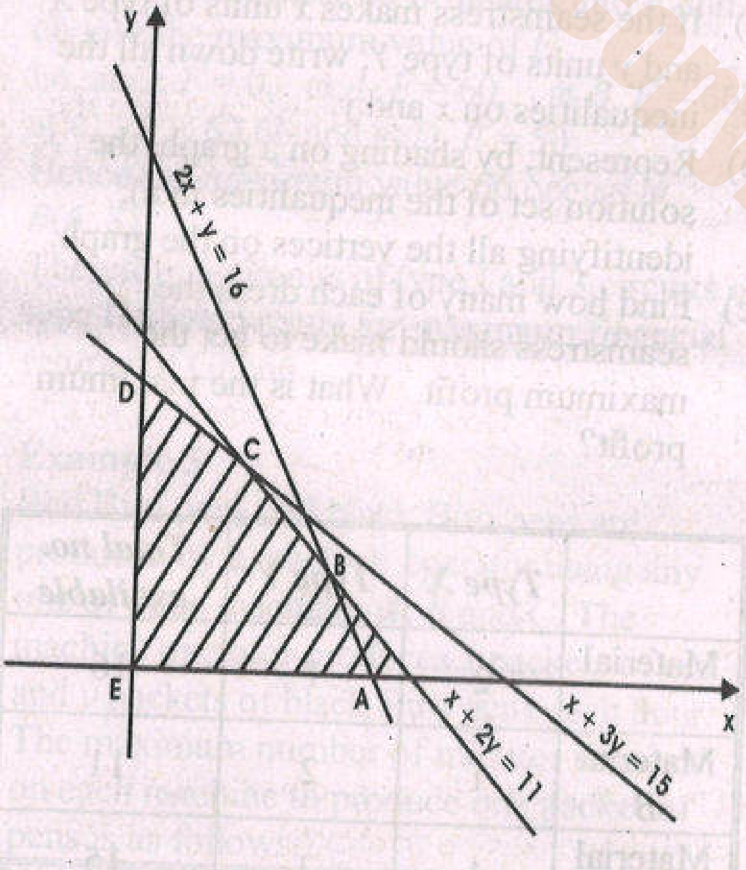
	Type X	***Type Y***	***Total no. available***
Material *A*	2	1	16
Material *B*	1	2	11
Material *C*	1	3	15
Profit	Gh¢30	Gh¢50	

Solution

a) The inequalities are:
$2x + y \le 16, x + 2y \le 11$ and $x + 3y \le 15$
We have also $x \ge 0$ and $y \ge 0$.
Note that the profit function is $P = 30x + 50y$

b) The diagram below shows the solution set of the above inequalities i.e. polygon *ABCDE* where $A(8, 0)$, $B(7, 2)$, $C(3, 4)$, $D(0, 5)$ and $E(0, 0)$.

c) The profit function $P =$ Gh¢$(30x + 50y)$. Therefore substitute the vertices into the function.
The maximum value of P is Gh¢310 and it occurs at the vertex $B(7, 2)$. Hence the seamstress should make 7 of type X and 2 of *type Y* to make a maximum profit of Gh¢310.



Note: To test your understanding, you are advised to draw the graph all the examples under this chapter and work out the missing steps.

Quadratic inequalities

An inequality of the form $ax^2 + bx + c \ge 0$ is called a **quadratic inequality**. To solve such an inequality factorize the quadratic expression first.
The range of values can be found by :

1. Testing a convenient point to identify the range which satisfies the inequality.

2. Sketching the graph of the quadratic function and identifying parts of the curve that lie below or above the x–axis. If the inequality sign is $\ge$ or $>$, the solution is the range of values of x corresponding to parts of the curve **above** the x–axis
   If the inequality sign is $\le$ or $<$, the solution is the range of values of x corresponding to parts of the curve **below** the x–axis

The two methods are illustrated in the examples below.

Illustrative examples

Example 1

Find the range of values of x for which $x^2 - 5x + 6 \ge 0$ and represent your result on a number line.

Solution

Method 1

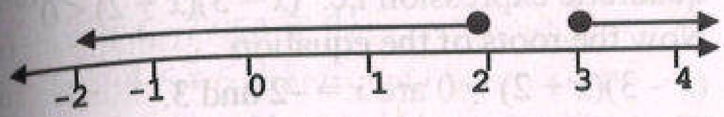
Given that $x^2 - 5x + 6 \ge 0$, first factorize the quadratic expression.
i.e. $(x - 2)(x - 3) \ge 0$
Now the roots of the equation $(x–2)(x–3) = 0$ are $x = 2$ and 3. These give the end points of the inequality. Test the values of x to the left and right of the end points to see whether they satisfy the inequality as shown in the following table.

x	$x < 2$	$2 < x < 3$	$x > 3$
$x - 2$	$-$	$+$	$+$
$x - 3$	$-$	$-$	$+$
$(x - 2)(x - 3)$	$+$	$-$	$+$

We can see that $x \leq 2$ and $x \geq 3$ satisfy the inequality.

Hence the solution set is $\{x : x \leq 2 \text{ or } x \geq 3\}$.

The graph of this set is shown below.



Method 2

Given that $x^2 - 5x + 6 \geq 0$, first factorize the quadratic expression

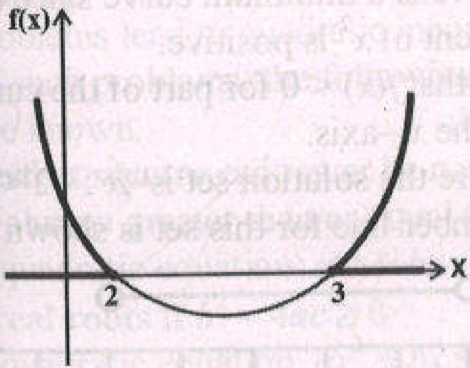
i.e. $x^2 - 5x + 6 \geq 0$

Now the roots of the equation $(x - 2)(x - 3) = 0$ are $x = 2$ and 3.

These give the end points of the inequality.

Let $f(x) = (x - 2)(x - 3)$.

The graph of this set is shown below.



The curve is a minimum curve since the coefficient of x^2 is positive.

We see that $f(x) \geq 0$ for part of the curve **above** the x –axis.

Hence the solution set is $\{x : x \leq 2 \text{ or } x \geq 3\}$

Example 2

Find the range of values of x for which $2x^2 + x - 1 \geq 0$ and represent your result on a number line.

Solution

Method 1

Given that $2x^2 + x - 1 \geq 0$, first factorize the quadratic expression i.e. $(2x - 1)(x + 1) \geq 0$

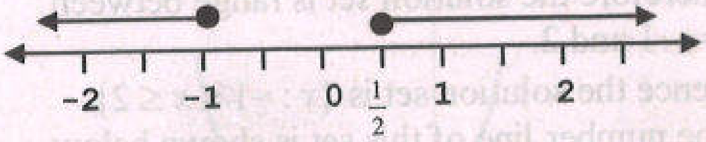
Now the roots of the equation $(2x-1)(x+1)=0$ are $x = -1$ and $\frac{1}{2}$. These give the end points of the inequality. Test $x = 0$ to see whether it satisfies the inequality.

i.e. $(-1)(1) = -1 < 0$ i.e. a value of x within the range $x = -1$ and $\frac{1}{2}$ **does not** satisfy the inequality. Therefore the solution set lies outside the range $x = -1$ and $\frac{1}{2}$.

Hence the solution set is

$\{x : x \leq -1 \text{ or } x \geq \frac{1}{2}\}$.

The number line is shown below.



Method 2

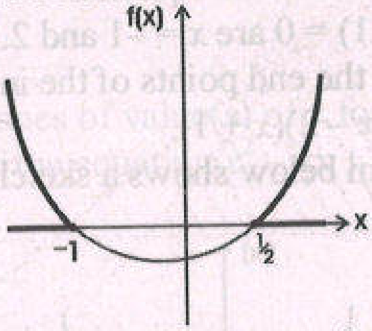
Given that $2x^2 + x - 1 \geq 0$, first factorize the quadratic expression i.e. $(2x - 1)(x + 1) \geq 0$

Now the roots of the equation $(2x - 1)(x + 1) = 0$ are $x = -1$ and $\frac{1}{2}$.

These give the end points of the inequality.

Let $f(x) = (2x - 1)(x + 1)$.

The diagram below shows a sketch of $f(x)$.



The curve is a minimum curve since the coefficient of x^2 is positive.

We see that $f(x) \geq 0$ for part of the curve **above** the x –axis.

Therefore the solution set is

$\{x : x \leq -1 \text{ or } x \geq \frac{1}{2}\}$

Example 3

Solve the inequality $x^2 - x - 2 \leq 0$ and represent your result on a number line.

Solution

Method 1

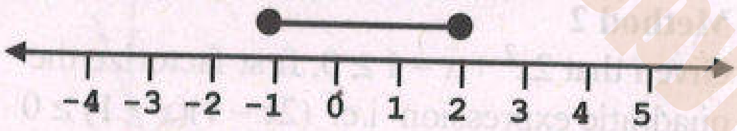
Given that $x^2 - x - 2 \leq 0$, first factorize the quadratic expression i.e. $(x-2)(x+1) \leq 0$

Now the roots of the equation $(x-2)(x+1) = 0$ are $x = -1$ and 2. These give the end points of the inequality. We then choose a value of x within the end points for the test i.e. test $x = 0$ to see whether it satisfies the inequality. i.e. $(-2)(1) = -2 \leq 0$ (true) i.e. a value of x within the range $x = -1$ and 2 **does** satisfy the inequality.

Therefore the solution set is range between $x = -1$ and 2.

Hence the solution set is $\{x : -1 \leq x \leq 2\}$.

The number line of this set is shown below.



Method 2

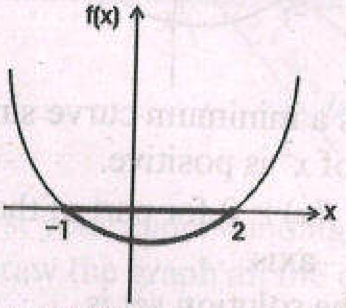
Given that $x^2 - x - 2 \leq 0$, first factorize the quadratic expression i.e. $(x-2)(x+1) \leq 0$

Now the roots of the equation $(x-2)(x+1) = 0$ are $x = -1$ and 2.

These give the end points of the inequality.

Let $f(x) = (x-2)(x+1)$.

The diagram below shows a sketch of $f(x)$.



The curve is a minimum curve since the coefficient of x^2 is positive.

We see that $f(x) \leq 0$ for part of the curve **below** the x-axis.

Therefore the solution set is $\{x : -1 \leq x \leq 2\}$.

Example 4

Solve the inequality $x^2 - x - 6 < 0$ and represent your result on a number line.

Solution

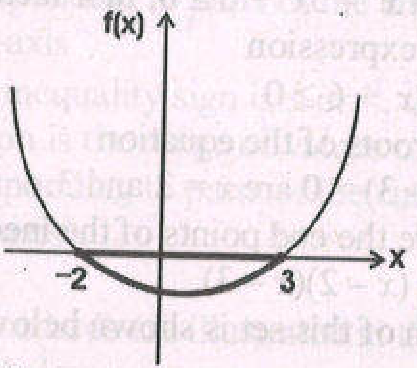
Given that $x^2 - x - 6 < 0$, first factorize the quadratic expression i.e. $(x-3)(x+2) < 0$

Now the roots of the equation $(x-3)(x+2) = 0$ are $x = -2$ and 3.

These give the end points of the inequality.

Let $f(x) = (x-3)(x+2)$.

The diagram below shows a sketch of $f(x)$.

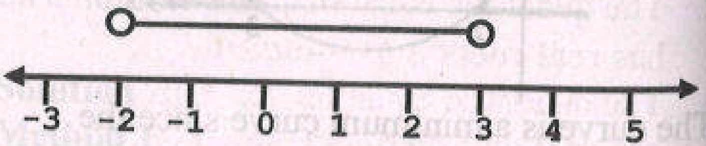


The curve is a minimum curve since the coefficient of x^2 is positive.

We see that $f(x) < 0$ for part of the curve **below** the x-axis.

Therefore the solution set is $\{x : -2 < x < 3\}$.

The number line for this set is shown below.



Example 5

Find the ranges of values of x for $2x^2 + 5x - 12$ is positive.

Solution

$x^2 + 5x - 12$ is positive

$x^2 + 5x - 12 > 0$

$\{x : x < -4 \text{ or } x > \frac{3}{2}\}$

Work out the solution.

Example 6

Find the set of values of x that satisfy the following inequalities.

i) $x^2 + 3x - 28 \ge 0$ ii) $2x^2 + 5x - 12 < 0$

iii) $x^2 - 2x - 3 < 0$ iv) $(2x - 1)(x + 1) < 0$

v) $(x - 3)(x - 5) < 0$ vi) $5x^2 > 3x + 2$

vii) $3x > x^2 + 2$ viii) $(x - 1)^2 > 4x^2$

ix) $x^2 - 4x > 5$

Solution

i) $\{x : x \le -7 \text{ or } x \ge 4\}$

ii) $\{x : -4 < x < \frac{3}{2}\}$

iii) $\{x : -1 < x < 3\}$

iv) $\{x : -1 < x < \frac{1}{2}\}$

v) $\{x : 3 < x < 5\}$

vi) $\{x : x < -\frac{2}{5} \text{ or } x > 1\}$

vii) $\{x : 1 < x < 2\}$

viii) $\{x : -1 < x < \frac{1}{3}\}$

ix) $\{x : x < -1 \text{ or } x > 5\}$

[Work out the solution in each case]

Problems leading to quadratic inequalities

Some problems lead to quadratic inequalities. To solve such problems, the following facts should be known.

1. A **perfect square** can never be negative or is always greater than or equal to zero
2. The quadratic equation $ax^2 + bx + c = 0$ has **real roots** if $b^2 - 4ac \ge 0$
3. The quadratic equation $ax^2 + bx + c = 0$ has **real and distinct roots** if $b^2 - 4ac > 0$
4. The quadratic equation $ax^2 + bx + c = 0$ has **complex roots** if $b^2 - 4ac < 0$

Illustrative examples

Example 1

Find the range of values of k for which the quadratic equation $x^2 - 2x - k = 0$ has real roots.

Solution

If $x^2 - 2x - k = 0$ has real roots, then $b^2 - 4ac \ge 0$, where $a = 1$, $b = -2$ and $c = -k$

i.e. $(-2)^2 - 4(1)(-k) \ge 0$

$4 + 4k \ge 0 \quad \Rightarrow \quad k \ge -1$

Example 2

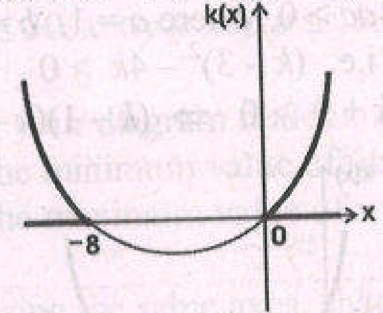
Find the ranges of value(s) of x for which the roots of the equation $kx^2 + kx - 2 = 0$ are real.

Solution

For $kx^2 + kx - 2 = 0$ to have real roots $b^2 - 4ac \ge 0$, where $a = k$, $b = k$ and $c = -2$

i.e. $k^2 - 4(k)(-2) \ge 0$

$k^2 + 8k \ge 0 \quad \Rightarrow \quad k(k + 8) \ge 0$



From the sketch we see that $k(k + 8) \ge 0$ for $k \le -8$ or $k \ge 0$.

Therefore the equation $kx^2 + kx - 2 = 0$ has real roots if the value of k lies in either of the ranges $k \le -8$ or $k \ge 0$.

Example 3

Find the ranges of value(s) of x for which the roots of the equation $x^2 - kx + (k + 3) = 0$ are real.

Solution

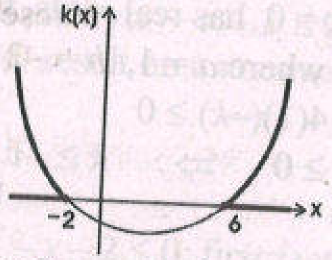
For $kx^2 - kx + (k + 3) = 0$ to have real roots $b^2 - 4ac \ge 0$, where $a = k$, $b = -k$ and $c = (k + 3)$

i.e. $(-k)^2 - 4(k + 3) \ge 0$

$k^2 - 4k - 12 \ge 0$

$(k - 6)(k + 2) \ge 0$

The diagram below shows a sketch of $f(kx)$.



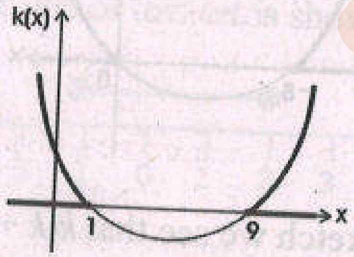
From the sketch we see that
$(k-6)(k+2) \geq 0$ for $k \leq -2$ or $k \geq 6$

Example 4

Find the ranges of value(s) of x for which the roots of the equation $x^2 + (k-3)x + k = 0$ are real distinct roots.

Solution

For $x^2 + (k-3)x + k = 0$ to have real distinct roots $b^2 - 4ac > 0$, where $a = 1$, $b = (k-3)$ and $c = k$ i.e. $(k-3)^2 - 4k > 0$
$\Rightarrow k^2 - 10k + 9 > 0 \Rightarrow (k-1)(k-9) > 0$



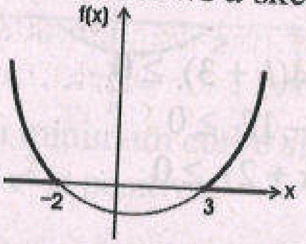
From the sketch we see that
$(k-1)(k-9) > 0$ for $k < 1$ or $k > 9$

Example 5

Find the values of x for which $x^2 - x - 6 > 0$

Solution

Given that $x^2 - x - 6 > 0$, first factorize the quadratic expression i.e. $(x-3)(x+2) > 0$
Now the roots of the equation $(x-3)(x+2) = 0$ are $x = -2$ and 3.
These give the end points of the inequality.
Let $f(x) = (x-3)(x+2)$.
The diagram below shows a sketch of $f(x)$.



The curve is a minimum curve since the coefficient of x^2 is positive. We see that $f(x) > 0$ for part of the curve **above** the x–axis. Hence the solution set is $\{x : x < -2 \text{ or } x > 3\}$

Example 6

Find the ranges of value(s) of p for which the roots of the equation $px^2 + 4(p-1)x + 9 = 0$ are real roots.

Solution

$p \leq \frac{1}{4}$ or $p \geq 4$

Work out the solution.

Review exercise 9.1

1. Show, by shading, the region of all the points (x, y) which satisfy simultaneously the following set of inequalities

i) $4x + 3y \geq 18$
$x - y \leq 8$
$2x + 3y \geq 12$
$x \geq 0$
$y \geq 0$

ii) $x - y \geq 5$
$x + y \leq 0$
$x \geq 0$
$y \geq 0$

iii) $-x + y \geq 3$
$x + y \leq -2$
$x \geq 0$
$y \geq 0$

iv) $2x + 3y \leq 18$
$x \geq 0$
$y \geq 0$

v) $-2x + y \leq 8$
$x \leq 5$
$x \geq 0$
$y \geq 0$

vi) $y \geq x$
$y \leq 8$
$x \geq 0$
$y \geq 0$

vii) $x \leq 6$
$y \leq 4$
$x \geq 0$
$y \geq 0$

viii) $2x - y \leq 4$
$y \leq 6$
$x \geq 0$
$y \geq 0$

ix) $2x - y \le 6$ x) $-x + y \le 1$

$x + y \ge 0$ $3x + 4y \le 12$

$x \ge 0$ $x \ge 0$

$y \ge 0$ $y \ge 0$

2. a) Show, by shading, the region S of all the points (x, y) which satisfy simultaneously the following four inequalities $x + 2y \le 6$, $x - y \le 6$, $y \le 2$, $x \ge -1$
   b) Use your diagram to find
      i) the maximum values of x and y
      ii) the minimum value of x^2
      iii) the maximum and minimum values of $(3x + y)$.

3. The region P of all points (x, y) in a plane satisfy simultaneously the following inequalities $5x - 2y \le 5$, $2x + y \ge 1$, $-x + y \le 2$, $4x + 2y \le 12$
   a) Illustrate graphically (by shading) the region representing the truth set of the system of inequalities.
   b) Use your graph to find, for the region P
      i) The maximum values of x and y
      ii) The minimum value of $3 - 2x$

4. A certain man lives on types X and Y of food. The number of proteins, carbohydrates and fats contained in 1 kg of each type of food is as follows:

Food	Protein	Carbohydrates	Fats
X	2	6	1
Y	1	1	3

In order to keep his health, he needs at least 8 units of proteins, 12 units of carbohydrates and 9 units of fats per week. If he buys x kg of X and y kg of Y per week, write down three inequalitie involving x and y and represent the solution set of these inequalities on a graph.

   a) If X costs 85p per kg and Y costs 40p per kg, how many of each should he buy to minimize his cost and still satisfy his minimum requirements?
   b) If the price of x drops to 75p per kg while Y still sells at 40p per kg, how many kilogramme of each should he then buy.

5. Indicate on a diagram, by shading, the solution set of the following six inequalities
   $x + y \le 30, x + y \ge 20,\ y \le 15,\ x \le 20,$
   $y \ge 0,\ x \ge 0$
   Using your diagram find
   i) The minimum value of $(4x + 2y + 10)$
   ii) The maximum value of x

6. a) Using the same axes, shade the region P of all points (x, y) which satisfy the following inequalities simultaneously;
      $x + y \le 10,\quad 10x + 3y \ge 30,$
      $4x - 3y \le 12$
   b) From your diagram find the maximum value of $5x + 3y$ in the region P.

7. i) Indicate, by shading, the set of all points $P(x, y)$ in the Oxy plane that satisfy simultaneously the inequalities
      $2x - y + 4 \ge 0,\quad x + y \le 10,$
      $y - x \ge 0,\quad y \ge 2$ and $x \ge 0$
   ii) Using your diagram, find the values of x and y for which $10x + 5y$ is maximum.

8. a) Shade the region P in the $x-y$ plane which satisfies simultaneously the inequalities;

$x+y \geq 5,\ 2y-x \geq 0,\ x+5y \leq 21$

b) Use your diagram in (a) to find on the set P

i) the minimum value of x

ii) the maximum value of $(3x+2y)$

9. Indicate, by shading, the set of all points $P(x, y)$ in the Oxy plane that satisfy simultaneously the inequalities $2x+y \geq 3,\ 2y-1 \geq 0,\ x \geq y$ and $x \leq 4$

10. i) Indicate, by shading, the set of all points $P(x, y)$ in the Oxy plane that satisfy simultaneously the inequalities $x+2y \leq 6, y \geq x-3$ and $x \geq 1$

ii) Using your diagram, find the maximum value of $x+3y$ in this region.

11. i) Indicate, by shading, the set of all points $P(x, y)$ in the Oxy plane that satisfy simultaneously the inequalities

$x+y+2 \geq 0,\ y \geq 2x-3$ and $y \leq x+4$

ii) Using your diagram, find the maximum value of $2x+3y$ in this region.

12. a) Indicate, by shading, the set of all points $P(x, y)$ in the Oxy plane that satisfy the following simultaneously the inequalities

i) $x \geq 0,\ y \geq 0$ and $x+y-2 \leq 0$

ii) $x \leq 2,\ y+2 \geq 0$ and $x-2y \geq 0$

iii) $y \geq 1,\ x+y \leq 3$ and $x \geq y-3$

b) Write down the coordinates of all points $P(x, y)$ which satisfy all the given inequalities in (a) and whose coordinates are integers.

13. a) Indicate, by shading, the set of all points $P(x, y)$ in the Oxy plane that satisfy the following simultaneously the inequalities. Find the coordinates of the vertices of this region.

i) $x \geq 1,\ y \leq 0$ and $y \geq x-2$

ii) $x \geq 0,\ y \leq 3$ and $x-y \geq 1$

iii) $y+1 \geq 0,\ x-1 \geq 0$ and $x-y \geq 0$

14. Two liquids X and Y contain the following quantities of Vitamin A & C.

	X	Y
Vitamin A	4 Units per kg	4 Units per kg
Vitamin C	2 Units per kg	6 Units per kg

A Druggist is to make a Vitamin mixture by mixing X and Y so that the final mixture contains at least 32 units of Vitamin A and at least 24 units of Vitamin C. If he takes x kg of X and y kg of Y, write down the two inequalities expressing the relation above. Draw on the same axis the graphs of these inequalities. If X cost Gh¢2.00 per kg and Y cost Gh¢3.00 per kg, find the values of x and y so that the Druggist could make the mixture as cheap as possible. Find also the corresponding least cost of the mixture.

15. Indicate, by shading, the set of all points $P(x, y)$ in the Oxy plane that satisfy simultaneously the inequalities $-x+5y \leq 10, 3x-4y \leq 8$ and $x > -1$.

16. Indicate, by shading, the set of all points $P(x, y)$ in the Oxy plane that satisfy simultaneously the inequalities $3x+y<9$ and $y>-8$.

17. Mr Mensah and Mr Amakye run a small business assembling two types of product. The cost of components and the labour needed for each product is shown in the table below.

	Cost of components	Labour man-hours
Type *A*	36	16
Type *B*	24	24

The business has \$156 available to buy components each week. The total labour available each week is 96 man-hours. If they make x of type *A* product and y of type *B* product each week:

a) Write down all the inequalities on x and y.

b) Represent, by shading on a graph, the solution set of the inequalities.

18. Illustrate graphically the set P of all points (x, y) in the plane which satisfy simultaneously

$x \geq 2, y \geq 3, \; x + y \leq 9$ and $2x + y \leq 12$.

Use your graph to find on the set P.

i) the minimum value of $(4 - 2x)$

ii) the maximum value of $(y + 3)$

iii) the maximum and minimum values of $(4x + 3y)$.

Review exercise 9.2

1. Find the values of x for which

$2x^2 > 11x - 12$

2. Find the truth set of the following inequalities and represent them on a number line

i) $x^2 + 2x \leq 15$

ii) $2x^2 - 5x + 3 > 0$

iii) $4x^2 - 4x - 3 > 0$

iv) $2x^2 - 5x + 2 < 0$

v) $x^2 - 2x - 3 > 0$

vi) $x^2 - 3x < -2$

vii) $x^2 - 2x - 8 \leq 0$

viii) $x^2 - 5x + 6 \geq 0$

3. Find the range of values of q for which the roots of the equation $x^2 + 6x + q^2 - 7 = 0$ are real.

4. Find the set of values of k for which the equation $x^2 + kx + 2k - 3 = 0$ has no real roots.

5. Find the range of values of p for which the quadratic equation has real roots.

i) $x^2 + (p + 3)x + 4p = 0$

ii) $x^2 + 3x + 1 = px$

6. Find the range of values of a for which the quadratic equation $x^2 + ax + (a + 3) = 0$ has no real roots.

7. Given that $px^2 + (p - 3)x + 1 = 0$, find the range of value(s) of p for which

a) the equation has real different roots.

b) the roots of the equation are not real

8. Find the set of values of x that satisfy the following inequalities.

i) $x^2 - 4x > 3$

ii) $4x^2 < 1$

iii) $5x^2 - 3x - 2 > 0$

iv) $(x - 1)^2 > 9$

v) $(x + 1)(x + 2) \leq 4$

vi) $(1 - x)(4 - x) > x + 11$

9. Find the range of values of k for which the quadratic equation has real different roots.

i) $kx^2 - 2x + k = 0$

ii) $x^2 - 3kx + k = 0$

10. Find the set of real values of x for which $x^2 - 9x + 20$ is negative.

11. Find the set of values of x such that

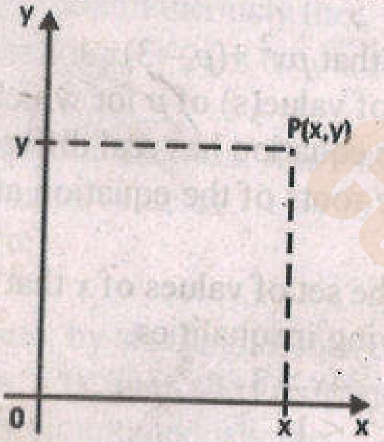
$3x - 2 > x^2 + 2x - 14$

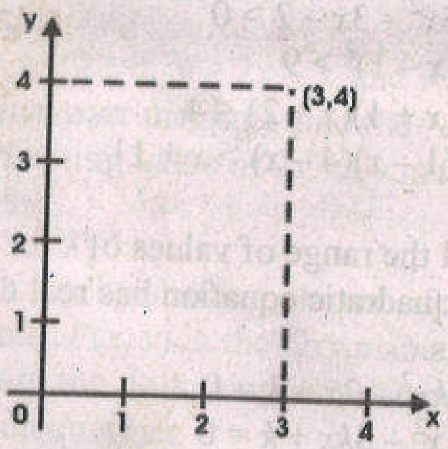
CHAPTER 10

COORDINATE GEOMETRY 1 (THE STRAIGHT LINE)

Introduction

Any point P in the x–y plane can be described by an ordered pair of numbers (x, y) where x is the **x-co-ordinate** (or **abscissa**) and y is the **y-co-ordinate** (or **ordinate**). For example, the point $Q(3, 4)$ has 3 as the x-co-ordinate and 4 as the y-co-ordinate as shown in the figure below. The **origin** O has co-ordinates $(0, 0)$.



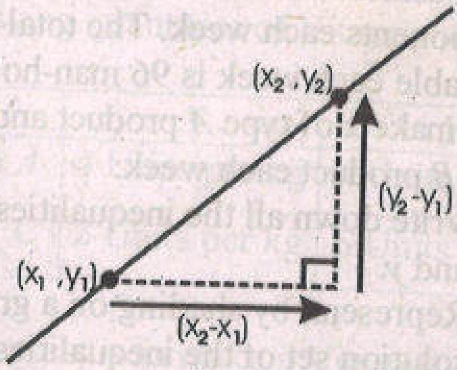


Length or distance between two points

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points, then the length of the straight line joining A and B is given by:

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Note: This is obtained by applying Pythagoras theorem to the right-angled triangle in the figure below.



Illustrative examples

Example 1

Find the length of the line joining the points $A(1, 3)$ and $B(4, 7)$.

Solution

$$|AB| = \sqrt{(4-1)^2 + (7-3)^2}$$
$$= \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \text{ units}$$

Example 2

Find the length of the line joining the points $P(-1, -3)$ and $Q(4, 9)$.

Solution

$$|PQ| = \sqrt{[4-(-1)]^2 + [9-(-3)]^2}$$
$$= \sqrt{5^2 + 12^2} = \sqrt{169} = 13 \text{ units}$$

Example 3

What is the distance between the points $P(5, -6)$ and $Q(-2, 5)$

Solution

$$|PQ| = \sqrt{[(5-(-2)]^2 + [(-6-(5)]^2}$$

$$= \sqrt{(7)^2 + (-11)^2} = \sqrt{(49 + 121)}$$

$$= \sqrt{170} \text{ units}$$

Example 4

$S(4, 6)$ and $R(12, 6)$ are points in the Cartesian plane. A point $P(x, y)$ moves in the plane such that $|PS| = |PR|$. Find the equation of the locus of P.

Solution

$|PS| = |PR|$

$$\sqrt{(x-4)^2 + (y-6)^2}$$

$$= \sqrt{(x-12)^2 + (y-6)^2}$$

Squaring both sides

$$(x-4)^2 + (y-6)^2 = (x-12)^2 + (y-6)^2$$

$$x^2 - 8x + 16 + y^2 - 12y + 36$$

$$= x^2 - 24x + 144 + y^2 - 12y + 36$$

$$16x = 128 \quad \Rightarrow \quad x = 8$$

The equation of the locus of P is $x = 8$

Example 5

Find the length of the lines passing through the following pairs of points;

i) (1, 2) and (4, 6)
ii) (3, 1) and (2, 0)
iii) (4, 2) and (2, 5)
iv) (–1, 4) and (2, 6)
v) (0, 0) and (–1, –2)
vi) (–1,–4) and (–3, –2)

Solution

i) 5 units　　ii) $\sqrt{2}$ units
iii) $\sqrt{13}$ units　　iv) $\sqrt{13}$ units
v) $\sqrt{5}$ units　　vi) $\sqrt{8}$ units

[Work out the solution]

Example 6

$A(0, 2)$, $B(1, 5)$ and $C(-1, 4)$ are the vertices of a triangle. Find the perimeter of the triangle leaving your answer in the surd form.

Solution

$|AB| = \sqrt{10} = \sqrt{5}\sqrt{2}$

$|AC| = \sqrt{5}$ and $|BC| = \sqrt{5}$

Perimeter of triangle

$= |AB| + |AC| + |BC|$

$= \sqrt{5}\sqrt{2} + \sqrt{5} + \sqrt{5}$

$= \sqrt{5}\sqrt{2} + 2\sqrt{5} = \sqrt{5}(\sqrt{2} + 2)$

Example 7

$P(6, 4)$, $Q(-3, 1)$ and $R(9, -5)$ are the vertices of a triangle. Find the perimeter of the triangle leaving your answer in the surd form.

Solution

$|PQ| = \sqrt{90} = 3\sqrt{10}$

$|PR| = \sqrt{90} = 3\sqrt{10}$

$|QR| = \sqrt{180} = 3\sqrt{20} = 3\sqrt{10}\sqrt{2}$

Perimeter of triangle

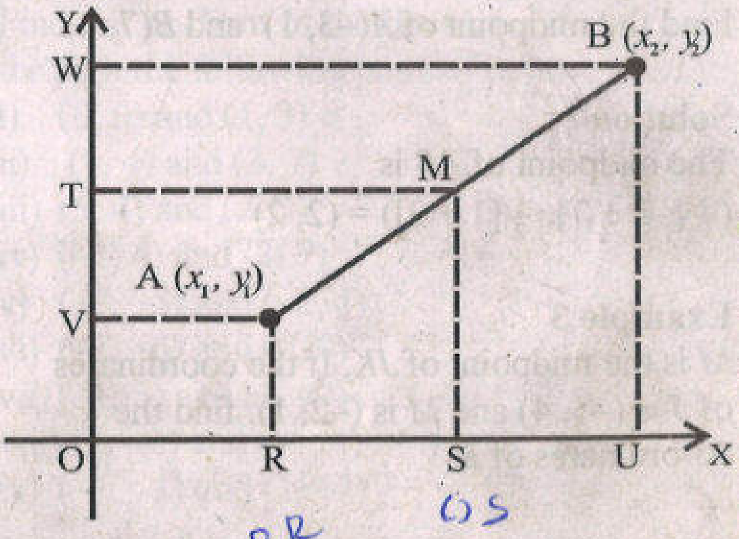
$= |PQ| + |PR| + |QR|$

$= 3\sqrt{10} + 3\sqrt{10} + 3\sqrt{10}\sqrt{2}$

$= 6\sqrt{10} + 3\sqrt{10}\sqrt{2} = 3\sqrt{10}(2 + \sqrt{2})$

The midpoint of a line segment

Suppose a point M is the midpoint of the straight line joining the two points $A(x_1, y_1)$ and $B(x_2, y_2)$.



OR　　OS

OT

We see from the diagram that the x-coordinate of M is OS, where

$$\begin{aligned} OS &= OR + \tfrac{1}{2}RU \\ &= x_1 + \tfrac{1}{2}(x_2 - x_1) \\ &= \tfrac{1}{2}(x_1 + x_2) \end{aligned}$$

Similarly, the y-coordinate of M is OT, where

$$\begin{aligned} OT &= OV + \tfrac{1}{2}VW \\ &= y_1 + \tfrac{1}{2}(y_2 - y_1) \\ &= \tfrac{1}{2}(y_1 + y_2) \end{aligned}$$

Therefore, if M is the midpoint of the straight line joining the two points $A(x_1, y_1)$ and $B(x_2, y_2)$, then the co-ordinates of M are given by:

$$[\tfrac{1}{2}(x_1 + x_2),\ \tfrac{1}{2}(y_1 + y_2)]$$

Note that the midpoint divides the distance between A and B into two equal parts.

Illustrative examples

Example 1

If $C(1, 3)$ and $D(5, 7)$ are two points. Find the co-ordinates of the point B which is the midpoint of CD.

Solution

The co-ordinates of $B = (\tfrac{1}{2}[1 + 5], \tfrac{1}{2}[3 + 7])$
$= (3, 5)$

Example 2

Find the midpoint of $A(\overset{x^1}{-3}, \overset{y^1}{1})$ and $B(\overset{x^2}{7}, \overset{y^2}{3})$.

Solution

The midpoint of AB is
$(\tfrac{1}{2}[-3 + 7], \tfrac{1}{2}[1 + 3]) = (2, 2)$

Example 3

M is the midpoint of JK. If the coordinates of J is $(-5, 4)$ and M is $(-2, 1)$, find the co-ordinates of K.

Solution

Let the coordinates of K be (x, y)
The co-ordinates of M
$= (\tfrac{1}{2}[-5 + x], \tfrac{1}{2}[4 + y]) = (-2, 1)$

Note: M is the midpoint of JK

Comparing, $\tfrac{1}{2}(-5 + x) = -2$

$-5 + x = -4 \quad \Rightarrow \quad x = 1$ and

$\tfrac{1}{2}(4 + y) = 1 \quad \Rightarrow \quad 4 + y = 2 \quad \Rightarrow \quad y = -2$

Therefore the coordinates of K is $(1, -2)$.

Example 4

M is the midpoint of the line AB. If the coordinates of A is $(5, 7)$ and M is $(0, 2)$, find the co-ordinates of B.

Solution

The co-ordinates of B is $(-5, -3)$
[Work out the solution]

Example 5

Find the coordinates of the midpoint of the lines joining;

i) $(-1, -4)$ and $(2, 6)$
ii) $(0, 0)$ and $(-1, -2)$
iii) $(-1, -4)$ and $(-3, -2)$

Solution

i) $(\tfrac{1}{2}, 1)$ ii) $(-\tfrac{1}{2}, -1)$ iii) $(-2, -3)$
[Work out the solution]

Example 6

A, B and C are points $(7, 3)$, $(-4, 1)$ and $(-3, -2)$ respectively.

a) Show that triangle ABC is isosceles
b) Find the midpoint of BC

Solution

a) $|AB| = \sqrt{125} = 5\sqrt{5}$
$|AC| = 5\sqrt{5}$
$|BC| = \sqrt{10}$

Since the length of two sides are the same (i.e. $|AB| = |AC|$), triangle ABC is isosceles.

b) The midpoint of BC is $(-3\frac{1}{2}, -\frac{1}{2})$
[Work out the solution]

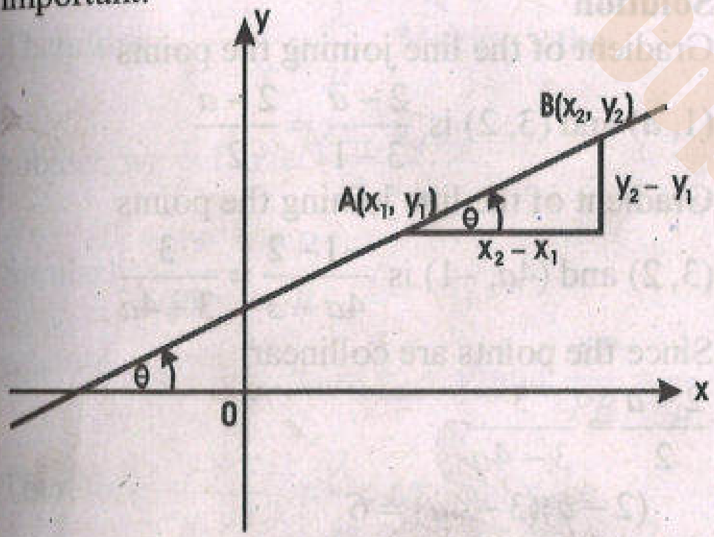
Gradient of a line joining two points

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are any two points, then the **gradient or slope** of the line joining A and B is given by:

$$m = \frac{\text{Difference in } y\text{ - values}}{\text{Difference in } x\text{ - values}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} = \tan\theta$$

Note: The *order* of subtraction is very important.



Note: The gradient of a line is also given by the tangent of the angle the line makes with the positive x–axis. From the diagram above, the gradient of line AB is $\tan\theta$. For example, if a line makes an angle of 45° to the positive x–axis, then the gradient of the line is $\tan 45° = 1$. Also if a line makes an angle of 60° to the positive x–axis, then the gradient of the line is $\tan 60° = \sqrt{3}$.

Illustrative examples

Example 1

Find the gradient of the line which passes through the points;

i) . $P(1, 1)$ and $Q(7, 2)$
ii) $A(2, 1)$ and $B(5, 5)$
iii) $C(3, -2)$ and $D(-3, 4)$

Solution

i) The gradient of line PQ

$$= \frac{2-1}{7-1} = \frac{1-2}{1-7} = \frac{1}{6}.$$

But not $\frac{2-1}{1-7}$ or $\frac{1-2}{7-1}$

Note: The order of subtraction is very important.

ii) The gradient of line AB

$$= \frac{5-1}{5-2} = \frac{1-5}{2-5} = \frac{4}{3}$$

iii) The gradient of line CD

$$= \frac{4-(-2)}{-3-3} = \frac{6}{-6} = -1$$

Example 2

Find the gradient of the line joining $(3, -2)$ and $(-5, -8)$

Solution

$$\text{The gradient} = \frac{-8-(-2)}{-5-3} = \frac{-6}{-8} = \frac{3}{4}$$

Example 3

Find the gradient of the lines passing through the following pairs of points;

i) (0, 0) and (1, 3)
ii) (1, 4) and (3, 7)
iii) (5, 4) and (2, 3)
iv) (–1, 4) and (3, 7)
v) (–1, –3) and (–2, 1)
vi) (–1, –6) and (0, 0)
vii) (–1, 5) and (1, –2)
viii) (3, –2) and (–1, 4)
ix) (–1, –1) and (1, –7)

Solution

i) 3 ii) $\frac{3}{2}$ iii) $\frac{1}{3}$ iv) $\frac{3}{4}$

v) -4 vi) 6 vii) $-\frac{7}{3}$ viii) $-\frac{3}{2}$

ix) -3

Example 4

Find the gradient of the lines passing through the following pairs of points;

i) (2, 5) and (5, 9)

ii) (–1, 2) and (2, –3)

iii) (1, 3) and (5, 3)

iv) (2, –3) and (2, 5)

v) (–4, –1) and (–6, –6)

vi) (–1, –4) and (–6, –16)

Solution

i) $\frac{4}{3}$ ii) $-\frac{5}{3}$ iii) 0

iv) no gradient v) $\frac{5}{2}$ vi) $\frac{12}{5}$

[Work out the solution in each case]

Example 5

Find the gradient of the line which makes an angle of 135° with the positive *x*-axis.

Solution

Gradient of the line $= \tan 135° = -1$

Collinear points

If three or more points are **collinear**, then they lie on the same straight line which has a constant gradient. Therefore if the gradients between adjacent pairs of points are *equal*, the points are collinear. For example, the set of points (7, –3), (–2, 1) and $(-6\frac{1}{2}, 3)$ are collinear since the gradient of the line joining the points (7, –3) and (–2, 1) is $-\frac{4}{9}$ and the gradient of the line joining (–2, 1) and $(-6\frac{1}{2}, 3)$ is $-\frac{4}{9}$.

Illustrative examples

Example 1

Which of the following sets of lines are collinear?

i) (2, 5), (–1, –1) and (–4, –7)

ii) (1, –1), (–2, 4) and (0, 1)

iii) (0, –1), (1, 1) and (2, 3)

iv) (0, 2), (2, 5) and (3, 7)

Solution

i) collinear ii) not collinear

iii) collinear iv) not collinear

Example 2

The points (1, *a*), (3, 2) and (4*a*, –1) are collinear points. Find the possible values of *a*.

Solution

Gradient of the line joining the points (1, *a*) and (3, 2) is $\frac{2-a}{3-1} = \frac{2-a}{2}$

Gradient of the line joining the points (3, 2) and (4*a*, –1) is $\frac{-1-2}{4a-3} = \frac{3}{3-4a}$

Since the points are collinear:

$$\frac{2-a}{2} = \frac{3}{3-4a}$$

$$(2-a)(3-4a) = 6$$

$$6-8a-3a+4a^2 = 6$$

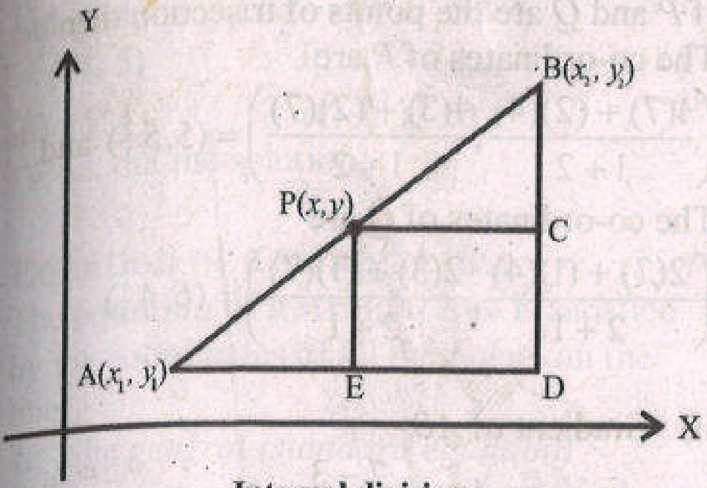
$$4a^2 - 11a = 0 \quad \Rightarrow \quad a(4a-11) = 0$$

$$a = 0 \text{ or } a = \tfrac{11}{4}$$

Division of a line segment in a given ratio

Suppose a point $P(x, y)$ divides the line joining the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ **internally** in the ratio $\lambda : \mu$.

i.e. $\frac{AP}{PB} = \frac{\lambda}{\mu}$



Internal division

In the diagram, triangles APE and PBC are similar. Therefore $\frac{AP}{PB}=\frac{AE}{PC}$

But $AE = x - x_1$, $PC = x_2 - x$ and $\frac{AP}{PB}=\frac{\lambda}{\mu}$

Therefore $\frac{x-x_1}{x_2-x}=\frac{\lambda}{\mu}$. Making x the subject, we get $x=\frac{\lambda x_2+\mu x_1}{\lambda+\mu}$

Similarly, $\frac{AP}{PB}=\frac{EP}{CB}$

But $EP = y - y_1$, $PC = y_2 - y$ and $\frac{AP}{PB}=\frac{\lambda}{\mu}$

Therefore $\frac{y-y_1}{y_2-y}=\frac{\lambda}{\mu}$. Making y the subject, we get $y=\frac{\lambda y_2+\mu y_1}{\lambda+\mu}$

Hence the co-ordinates of P are given by:

$$x=\frac{\lambda x_2+\mu x_1}{\lambda+\mu} \text{ and } y=\frac{\lambda y_2+\mu y_1}{\lambda+\mu}$$

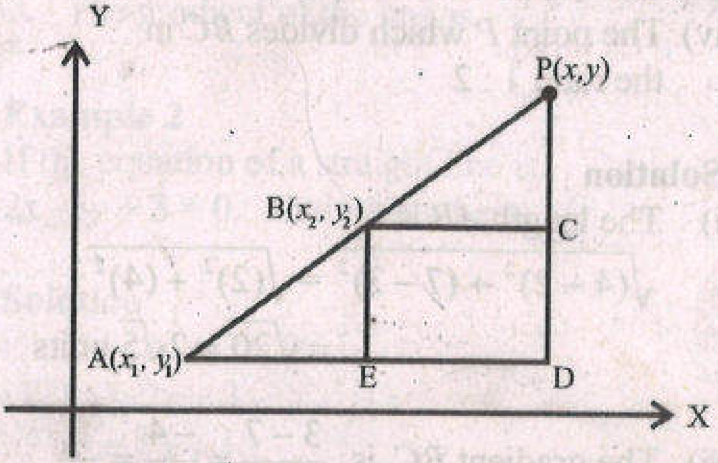
The same formulae apply for both external and internal division.

However, for **external division**, we negate either λ or μ (i.e. the smaller of them)

Thus if a point $P(x, y)$ divides the line joining the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ **externally** in the ratio λ : μ, then the co-ordinates of P are given by:

$$x=\frac{\lambda x_2-\mu x_1}{\lambda-\mu} \text{ and } y=\frac{\lambda y_2-\mu y_1}{\lambda-\mu}$$

(i.e. negate the smaller number in the ratio)



External division

Illustrative examples

Example 1

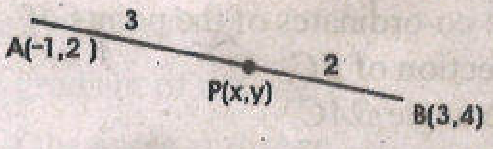
P and Q divide the line AB, $A(-1, 2)$ and $B(3, 4)$ internally and externally in the ratio 3 : 2. Find their co-ordinates.

Solution

For internal division, λ = 3 and μ = 2

The co-ordinates of P are:

$$\left(\frac{3(3)+2(-1)}{3+2}, \frac{3(4)+2(2)}{3+2}\right)=\left(\tfrac{7}{5}, \tfrac{16}{5}\right)$$

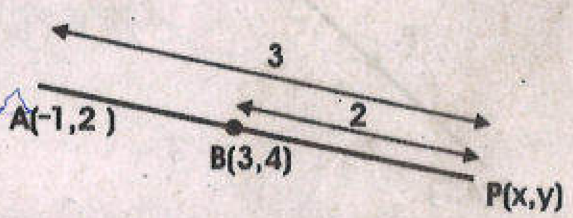


For external division, λ = 3 and μ = –2

(i.e. negate the smaller number in the ratio)

∴ The co-ordinates of Q are

$$\left(\frac{3(3)+(-2)(-1)}{3+(-2)}, \frac{3(4)+(-2)(2)}{3+(-2)}\right)=(11,8)$$



Example 2

A, *B*, *C* are the points (2, 3), (4, 7), (7, 3) respectively. Find

i) The length *AB*
ii) The gradient *BC*
iii) The midpoint of *AC*
iv) The point *P* which divides *BC* in the ratio 1 : 2

Solution

i) The length *AB* is

$$\sqrt{(4-2)^2+(7-3)^2}=\sqrt{(2)^2+(4)^2}$$
$$=\sqrt{20}=2\sqrt{5}\text{ units}$$

ii) The gradient *BC* is $\dfrac{3-7}{7-4}=\dfrac{-4}{3}=-\frac{4}{3}$

iii) The midpoint of *AC* is

$$(\tfrac{1}{2}[2+7],\ \tfrac{1}{2}[3+3])=(4\tfrac{1}{2},\ 3)$$

iv) The point *P* which divides *BC* in the ratio 1 : 2 is

$$\left(\frac{1(7)+(2)(4)}{1+2},\frac{1(3)+(2)(7)}{1+2}\right)=(5,5\tfrac{2}{3})$$

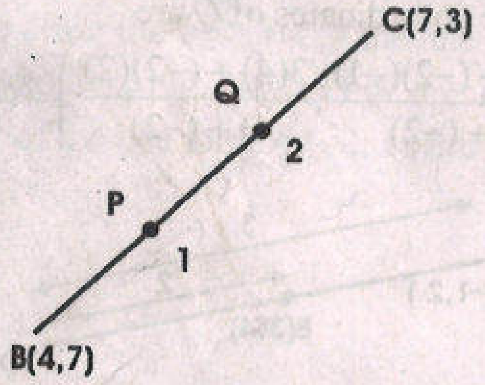
Example 3

A, *B*, *C* are the points (2, 3), (4, 7), (7, 3) respectively. Find

i) The co-ordinates of the points of trisection of *BC*
ii) The angle *BAC*

Solution

i)



If *P* and *Q* are the points of trisection then The co-ordinates of *P* are

$$\left(\frac{1(7)+(2)(4)}{1+2},\frac{1(3)+(2)(7)}{1+2}\right)=(5,5\tfrac{2}{3})\text{ and}$$

The co-ordinates of *Q* are

$$\left(\frac{2(7)+(1)(4)}{2+1},\frac{2(3)+(1)(7)}{2+1}\right)=(6,4\tfrac{1}{3})$$

ii) Gradient of *AB*

$$=\tan\angle BAC=\frac{7-3}{4-2}=2$$

$\therefore\ \angle BAC=63°$

Example 4

P and *Q* divide the line *AB*, *A*(–2, 5) and *B*(4, 2) internally and externally respectively in the ratio 2 : 1. Find their co-ordinates.

Solution

P divides the line *AB* internally in the ratio 2 : 1, so $\lambda=2$ and $\mu=1$

$$x=\frac{2(4)+1(-2)}{2+1}=2\text{ and}$$
$$y=\frac{2(2)+1(5)}{2+1}=3\quad\Rightarrow\quad P(2,3)$$

Q divides the line *AB* externally in the ratio 2 : 1, so $\lambda=2$ and $\mu=-1$

$$x=\frac{2(4)-1(-2)}{2-1}=10\text{ and}$$
$$y=\frac{2(2)-1(5)}{2-1}=-1\quad\Rightarrow\quad Q(10,-1)$$

Example 5

Find the coordinates of the point that divided *AB* in the given ratio in each of the following cases.

i) *A*(2, 4) and *B*(–3, 9) 1 : 4 internally
ii) *A*(–3, –4) and *B*(3, 5) 3 : 1 externally
iii) *A*(1, 5) and *B*(8, –2) 4 : 3
iv) *A*(–1, –6) and *B*(3, –2) 3 : –2

Solution

i) (1, 5) ii) $(6, \frac{19}{2})$

iii) (5, 1) iv) (11,–6)

[Work out the solution]

Equation of a straight line

The **equation of a straight line** is satisfied by the coordinates of all the points on the line.

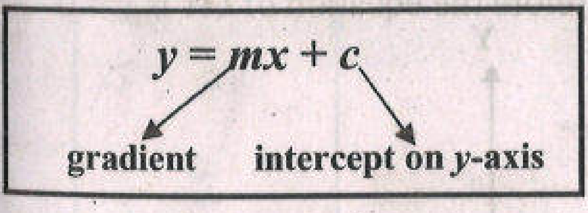
1. The *general (standard equation) equation* of a straight line is of the form: $ax + by + c = 0$, where a, b and c are constants

2. The *slope equation* is of the form $y = mx + c$, where m is the gradient of the line and c is the intercept on the y-axis.

3. The *intercept equation* is of the form $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are the intercepts on the x and y axes respectively. Thus any equation of this form cuts the x-axis at $(a, 0)$ and the y-axis at $(0, b)$.

i) *To find the gradient of a line when the equation is given*

To find the gradient, express the given equation in the form $y = mx + c$ and take the coefficient of x (i.e. m) as the gradient.



Illustrative examples

Example 1

Find the gradient of the line with equation $2x + 4y = 3$

Solution

Making y the subject of the equation, we have $2x + 4y = 3$

$$4y = -2x + 3 \quad \Rightarrow \quad y = -\tfrac{1}{2}x + \tfrac{3}{4}$$

The coefficient of x is $-\frac{1}{2}$

$\therefore$ The gradient of the line is $-\frac{1}{2}$.

Example 2

If the equation of a straight line is $2x - 8y + 3 = 0$. Find its gradient.

Solution

$$2x - 8y + 3 = 0$$
$$8y = 2x + 3$$
$$y = \tfrac{1}{4}x + \tfrac{3}{8}$$

The coefficient of x is $\frac{1}{4}$

$\therefore$ The gradient of the line is $\frac{1}{4}$.

Example 3

Find the angle of the straight line whose equation is $3x + 2y - 1 = 0$ makes with the positive x-axis.

Solution

$$3x + 2y - 1 = 0$$
$$2y = -3x + 1$$
$$y = -\tfrac{3}{2}x + \tfrac{1}{2}$$

The coefficient of x is $-\frac{3}{2}$

$\therefore$ The gradient of the line is $-\frac{3}{2}$.

Hence if the angle is θ, then

$\tan\theta = -\frac{3}{2} = -1.5$

$$\theta = \tan^{-1}(-1.5) = 123.7°$$

Note that the gradient of a line is equal to the tangent of the angle the line makes with the positive x–axis.

Example 4

Find the angles the straight lines whose equations are given make with the positive x-axis.

i) $y = 2x - 1$

ii) $y = 5x - 3$

iii) $x - 2x + 1 = 0$

Solution

i) 63.4° ii) 78.7° iii) 26.6°

[Work out the solution in each case]

Example 5

State the gradient and hence find the angles the straight lines whose equations are given make with the positive x-axis.

i) $x + y = 2$ ii) $y = x - 1$

iii) $y + \sqrt{3}x = 3$

Solution

i) -1, 135° ii) 1, 45°

iii) $-\sqrt{3}$, 120°

[Work out the solution in each case.]

ii) *To find the intercept on the x - axis*

To find the intercept of a straight line on the x-axis, put $y = 0$ in the equation and solve for x.

iii) *To find the intercept on the y - axis*

To find the intercept of a straight line on the y-axis, put $x = 0$ in the equation and solve for y.

Example 6

Find the gradient of the straight line with the equation $7x + 4y + 2 = 0$ and find the intercepts on the x and y - axis.

Solution

To find the gradient of the line, make y the subject of the equation of the line.

$7x + 4y + 2 = 0$

$4y = -7x - 2 \quad \Rightarrow \quad y = -\frac{7}{4}x - \frac{1}{2}$

$\therefore$ The gradient of the line is $-\frac{7}{4}$

To find the intercept on the y-axis put $x = 0$ in the equation and solve for y.

$7(0) + 4y + 2 = 0$

$4y = -2 \quad \Rightarrow \quad y = -\frac{1}{2}$

$\therefore$ The intercept on the y-axis is $(0, -\frac{1}{2})$

To find the intercept on the x-axis put $y = 0$ in the given equation and solve for x.

$7x + 4(0) + 2 = 0$

$7x = -2 \quad \Rightarrow x = -\frac{2}{7}$

$\therefore$ The intercept on the x-axis is $(-\frac{2}{7}, 0)$

Example 7

State the gradient and the intercept on the y – axis of the following lines.

i) $x + y = 4$ ii) $2x + y = 3$

iii) $2x - y = 3$ iv) $x + 3y = 6$

v) $3x - 4y - 12 = 0$ vi) $2x + 3y + 1 = 0$

Solution

i) -1, 4 ii) -2, 3 iii) 2, -3

iv) $-\frac{1}{3}$, 2 v) $\frac{3}{4}$, -3 vi) $-\frac{2}{3}$, $-\frac{1}{3}$

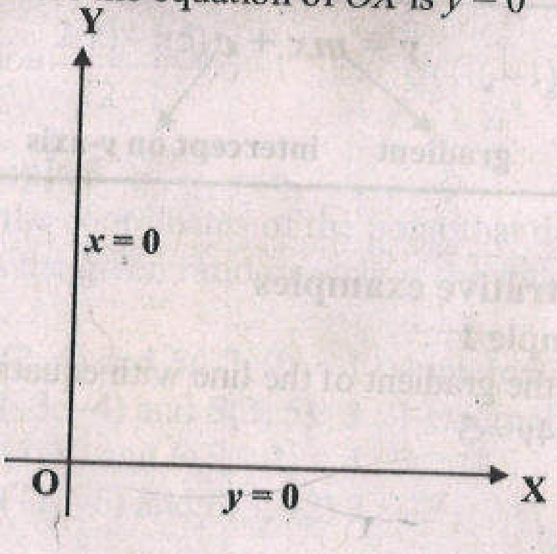
[Work out the solution in each case]

Finding the equation of a straight line

1. Equation of axes of coordinates

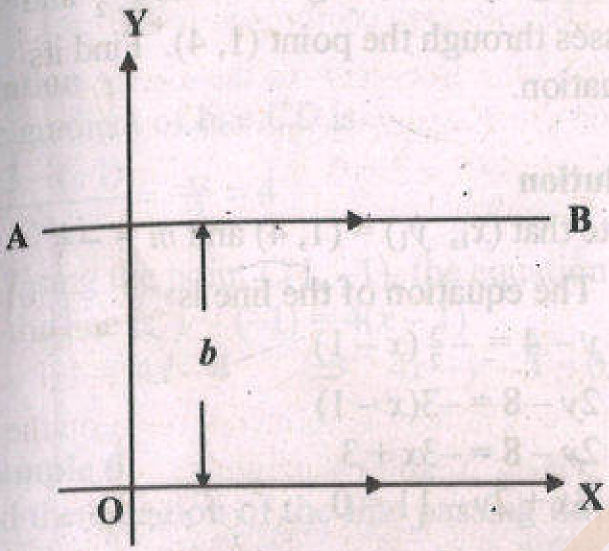
The x-coordinates of every point on OY is 0. Therefore the equation of OY is $x = 0$.

The y coordinates of every point on OX is 0. Therefore the equation of OX is $y = 0$



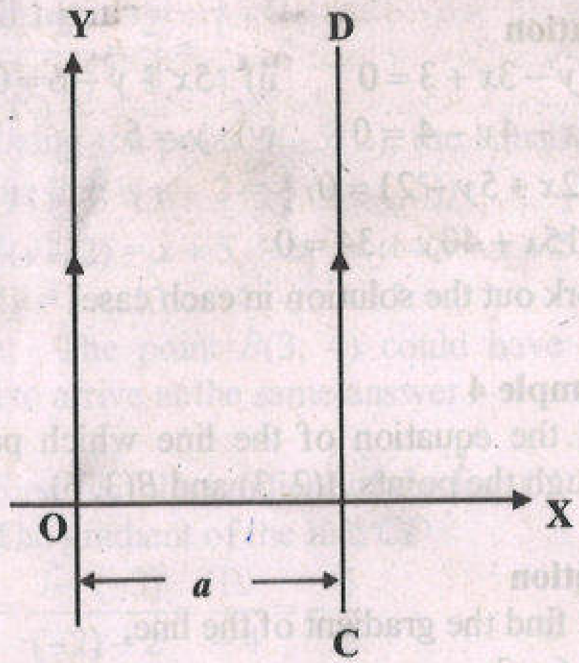
2. Equation of a straight line parallel to the *x*-axis

Let *AB* be a straight line parallel to the *x*-axis and at distance *b* from *x*-axis.
Then *y* coordinate of every point on *AB* is *b*.
$\therefore$ The equation of *AB* is $y = b$.

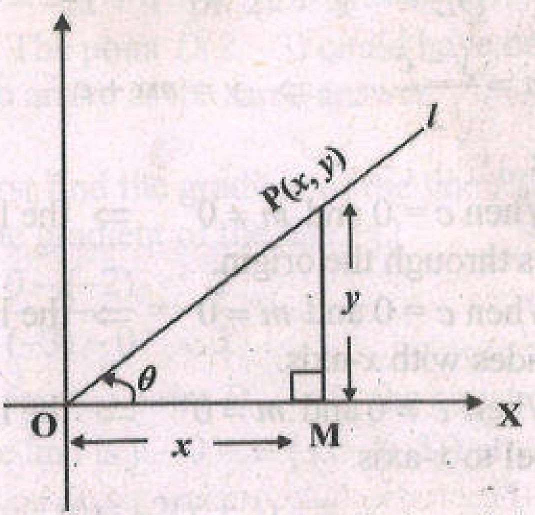


3. Equation of a straight line parallel to the *y*-axis

Let *CD* be a straight line parallel to *y*-axis and at a distance *a* from *y*-axis. Then the *x*-coordinate of every point on *CD* is *a*.
$\therefore$ The equation of *CD* is $y = a$



4. Equation of a straight line passing through the origin and whose slope is *m*

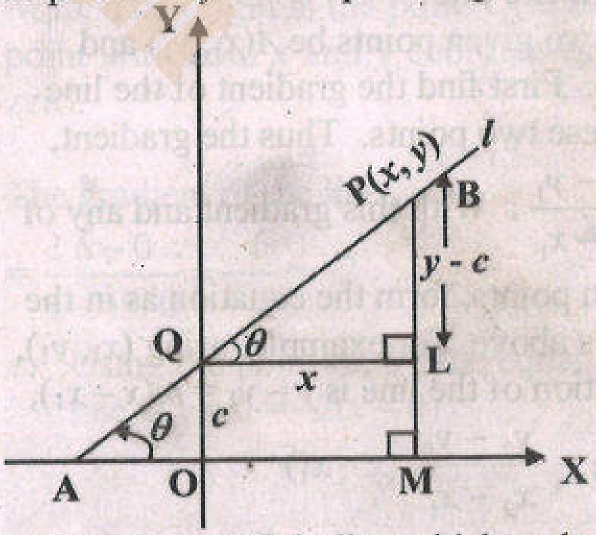


Let *l* be a straight line through the origin such that it makes an angle θ with the positive direction of the *x*-axis. Take any point $P(x, y)$ on the line *l*. Draw *PM* perpendicular to the *x*-axis. Then $OM = x$ and $PM = y$. Then

$$\tan\theta = \frac{y}{x} \quad \Rightarrow \quad m = \frac{y}{x} \quad \Rightarrow \quad y = mx$$

5. Slope-intercept form of the equation of a straight line

To find the equation of a straight line when slope *m* and *y*-intercept *c* are given.



Let *AB* be a straight line which makes an angle θ with *OX* and cuts *OY* at *Q*.
Let $OQ = c$. Let $P(x, y)$ be any point on *AB*, through *P*. Draw *PM* perpendicular to *x*-axis and *QL* perpendicular to *MP*. Then
$\angle PQL = \theta$ (Corresponding angles)

From right angled triangle PQL,

$$\tan\theta = \frac{PL}{QL} = \frac{y-c}{x}$$

$$m = \frac{y-c}{x} \quad \Rightarrow \quad y = mx + c$$

Note:

1. When $c = 0$ and $m \neq 0$ $\Rightarrow$ the line passes through the origin.
2. When $c = 0$ and $m = 0$ $\Rightarrow$ the line coincides with x-axis.
3. When $c \neq 0$ and $m = 0$ $\Rightarrow$ the line is parallel to x-axis.

6. To find the equation of a line given the gradient and a point on the line

Let m be gradient of the line and (x_1, y_1) be the given point, then the equation of the line is:

$$\boxed{y - y_1 = m(x - x_1)}$$

7. To find the equation of a line given two points on the line

Let the two given points be $A(x_1, y_1)$ and $B(x_2, y_2)$. First find the gradient of the line using these two points. Thus the gradient, $m = \frac{y_2 - y_1}{x_2 - x_1}$. With this gradient and any of the given points, form the equation as in the examples above. For example, using (x_1, y_1), the equation of the line is $y - y_1 = m(x - x_1)$,

or $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

Illustrative examples

Example 1

Find the equation of the line whose gradient is 3, passing through the point (0, 1).

Solution

Here $m = 3$ and $(x_1, y_1) = (0, 1)$

$\therefore$ The equation of the line is $y - 1 = 3(x - 0)$

$y - 1 = 3x$ or $3x - y + 1 = 0$.

Example 2

A straight line has a gradient of $-\frac{3}{2}$ and passes through the point (1, 4). Find its equation.

Solution

Note that $(x_1, y_1) = (1, 4)$ and $m = -\frac{3}{2}$

$\therefore$ The equation of the line is

$$y - 4 = -\tfrac{3}{2}(x - 1)$$
$$2y - 8 = -3(x - 1)$$
$$2y - 8 = -3x + 3$$
$$3x + 2y - 11 = 0$$

Example 3

Find the equation of the line with the given gradient and passing through the given point.

i) 3, (4, 9) ii) −5, (2, −4)

iii) $\frac{1}{4}$, (4, 0) iv) 0, (−1, 5)

v) $-\frac{2}{5}$, $(\frac{1}{2}, 4)$ vi) $-\frac{3}{8}$, $(\frac{22}{5}, -\frac{5}{2})$

Solution

i) $y - 3x + 3 = 0$ ii) $5x + y - 6 = 0$

iii) $x - 4y - 4 = 0$ iv) $y = 5$

v) $2x + 5y - 21 = 0$

vi) $15x + 40y + 34 = 0$

[Work out the solution in each case]

Example 4

Find the equation of the line which passes through the points $A(2, 3)$ and $B(3, 6)$.

Solution

First find the gradient of the line,

$$m = \frac{6-3}{3-2} = 3$$

∴ Using the point $A(2, 3)$, the equation of the line is $y - 3 = 3(x - 2)$

$$y - 3 = 3x - 6 \quad \Rightarrow \quad 3x - y - 3 = 0$$

Example 5

Find the equation of a line CD which passes through the points $C(1, -1)$ and $D(-2, -13)$.

Solution

The gradient of line CD is

$$\frac{-13-(-1)}{-2-1} = \frac{-12}{-3} = 4$$

∴ Using the point $C(1, -1)$, the equation of the line is $y - (-1) = 4(x - 1)$

$$y + 1 = 4x - 4 \quad \Rightarrow \quad 4x - y - 5 = 0.$$

Example 6

Find the equation of the line passing through the points

i) $A(-5, 2)$ and $B(3, 4)$
ii) $C(-2, 7)$ and $D(2, -3)$
iii) $P(-3, 0)$ and $Q(0, -2)$

Solution

i) First find the gradient of the line AB.
The gradient of the line AB

$$= \frac{4-2}{3-(-5)} = \frac{2}{8} = \frac{1}{4}$$

∴ Using the point $A(-5, 2)$, the equation of the line is $y - 2 = \frac{1}{4}(x - (-5))$

$$4(y - 2) = x + 5 \quad \Rightarrow \quad 4y - 8 = x + 5$$

$$4y = x + 13 \text{ or } 4y - x - 13 = 0$$

Note: The point $B(3, 4)$ could have been used to arrive at the same answer

ii) First find the gradient of the line CD.
The gradient of the line CD

$$= \frac{7-(-3)}{(-2)-2} = \frac{10}{-4} = -\frac{5}{2}$$

∴ Using the point $C(-2,7)$, the equation of the line is $y - 7 = -\frac{5}{2}(x - (-2))$

$$2(y - 7) = -5(x + 2)$$

$$2y - 14 = -5x - 10$$

$$2y = -5x + 4 \text{ or } 2y + 5x - 4 = 0$$

Note: The point $D(2, -3)$ could have been used to arrive at the same answer.

iii) First find the gradient of the line PQ.
The gradient of the line PQ

$$= \frac{0-(-2)}{(-3)-0} = -\frac{2}{3}$$

∴ Using the point $P(-3, 0)$, the equation of the line is $y - 0 = -\frac{2}{3}(x - (-3))$

$$3(y - 0) = -2(x + 3)$$

$$3y - 0 = -2x - 6$$

$$3y = -2x - 6 \text{ or } 2x + 3y + 6 = 0$$

Note: The point $Q(0, -2)$ could have been used to arrive at the same answer.

Example 7

A straight line passes through the origin and the point $(-3, 6)$. Find the equation of the line.

Solution

Note: The origin is the point $O(0, 0)$ i.e. the point with both x and y coordinates equal to zero.

The gradient of the line

$$= \frac{6-0}{(-3)-0} = \frac{6}{-3} = -2$$

∴ Using the point $(0, 0)$, the equation of the line is $y - 0 = -2(x - 0)$

$$y = -2x \text{ or } 2x + y = 0$$

Example 8

The equation of the line through $(4, 2)$ and $(-8, -2)$ is $3y = px + q$, where p and q are constants. Find the value of p and q.

Solution

To find p and q, first find the equation of the line and compare it to the one given. To find the equation of the line through the two points, first find the gradient.

The gradient of the line

$$= \frac{(-2)-2}{(-8)-4} = \frac{-4}{-12} = \frac{1}{3}$$

$\therefore$ Using the point $P(4, 2)$, the equation of the line is $y - 2 = \frac{1}{3}(x - 4)$

$$3(y-2) = 1(x-4)$$
$$3y - 6 = x - 4$$
$$3y = x + 2$$

Now comparing $3y = x + 2$ with $3y = px + q$, we have $p = 1$ and $q = 2$.

Example 9

Find the equation of the line passing through the points

i) $(0, 1)$ and $(2, 4)$
ii) $(-1, 2)$ and $(1, 5)$
iii) $(3, -1)$ and $(3, 2)$
iv) $(2, 4)$ and $(5, 6)$
v) $(-1, -2)$ and $(3, 0)$
vi) $(-4, 3)$ and $(2, -5)$
vii) $(1, 2)$ and $(1, -4)$
viii) $(3, -2)$ and $(-3, -2)$
ix) $(-2, -6)$ and $(5, -4)$

Solution

i) $3x - 2y + 2 = 0$ ii) $3x - 2y + 7 = 0$
iii) $x = 3$ iv) $2x - 3y + 8 = 0$
v) $x - 2y - 3 = 0$ vi) $4x + 3y + 7 = 0$
vii) $x = 1$ viii) $y = -2$
ix) $2x - 7y - 38 = 0$

[Work out the solution in each case.]

Intersection of lines

If two lines **intersect**, the co-ordinates of the point of intersection must satisfy both equations. Hence the co-ordinates of the point of intersection can be obtained by solving the two equations *simultaneously*.

Illustrative examples

Example 1

Find the point of intersection of the two lines $2x - 3y = 8$ and $3x + 4y + 5 = 0$.

Solution

$2x - 3y = 8$ ------------(1)
$3x + 4y = -5$ --------(2)

Solving eqn(1) and eqn(2) simultaneously, we have

$4 \times$ eqn(1) $+ 3 \times$ eqn(2)

$$17x = 17 \Rightarrow x = 1$$

Substituting $x = 1$ into eqn(1) we have $y = -2$

$\therefore$ The point of intersection is $(1, -2)$

Example 2

Two lines with equations $x + y - 2 = 0$ and $2x - y + 5 = 0$ meet at a point A, find the co-ordinates of A.

Solution

$x + y = 2$ ------(1)
$2x - y = -5$ ----(2)

Solving eqns (1) and (2) simultaneously, we have Eqn (1) + eqn(2)

$$3x = -3 \Rightarrow x = -1$$

Substituting $x = -1$ into eqn(1) $\Rightarrow$ $y = 3$

$\therefore$ The co-ordinates of A are $(-1, 3)$

Example 3

Find the coordinates of the point of intersection of each pair of lines.

i) $y = 3x - 5$ and $y = 4 - 5x$
ii) $2x + y - 3 = 8$ and $x - 3y + 4 = 0$
iii) $y = 2x - 9$ and $4x - 7y + 5 = 0$

Solution

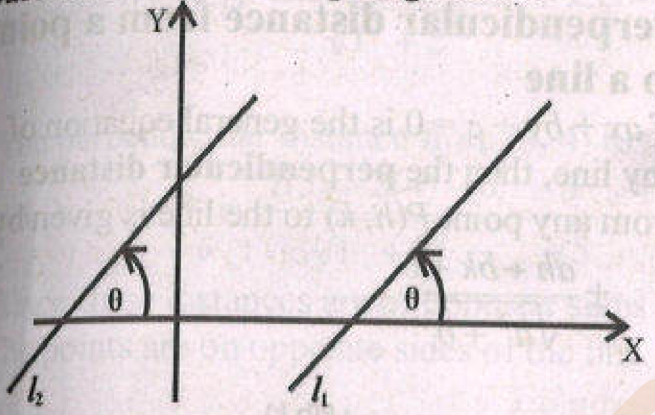
i) $(\frac{9}{8}, -\frac{11}{8})$ ii) $(\frac{5}{7}, \frac{11}{7})$ iii) $(\frac{34}{5}, \frac{23}{5})$

[Work out the solution in each case.]

Conditions for parallel and perpendicular lines

Parallel lines

If two lines l_1 and l_2 are parallel, they are equally inclined to the positive direction of the x-axis (i.e. corresponding angles). Therefore $\tan\theta$ is the gradient of both l_1 and l_2. Hence two lines are said to be **parallel** if they have **equal** gradient.

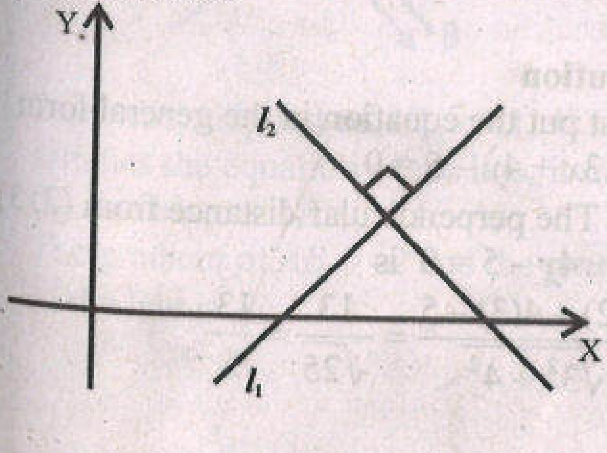


Perpendicular lines

If the gradient of one line is equal to the negative reciprocal of the other, then the two lines are said to meet at **right angles** or are said to be **perpendicular**. Thus if the gradient of one line is m the other is $-\dfrac{1}{m}$ then the lines are perpendicular. We can also say that the product of the gradients of two perpendicular lines is -1.

i.e. $m_1 = -\dfrac{1}{m_2}$ or $m_1 m_2 = -1$

Note that a line and its bisector or mediator are perpendicular.



Illustrative examples

Example 1

Find the equation of the line which passes through the point (1, 3) and perpendicular to the line $2y - 3x + 1 = 0$.

Solution

The gradient of the line $2y - 3x + 1 = 0$ is $\frac{3}{2}$. {i.e. making y the subject, $y = \frac{3}{2}x - \frac{1}{2}$ }

Therefore the gradient of the line which passes through (1, 3) is $-\frac{2}{3}$, since the two lines are perpendicular. The equation of this line is $y - 3 = -\frac{2}{3}(x - 1)$

$$3y - 9 = -2x + 2 \;\Rightarrow\; 2x + 3y - 11 = 0$$

Example 2

Find the equation of the straight line which passes through the mid-point of the line joining the points $R(1, 2)$ and $S(2, -3)$ and perpendicular to the line $3x + 2y - 5 = 0$.

Solution

The mid-point of the line joining the points R and S is $(\frac{1}{2}[1 + 2], \frac{1}{2}[2 + (-3)]) = (\frac{3}{2}, -\frac{1}{2})$

The gradient of the line $3x + 2y - 5 = 0$ is $-\frac{3}{2}$. {i.e. making y the subject, $y = -\frac{3}{2}x + \frac{5}{2}$ }

Therefore the gradient of the line which passes through the mid-point $(\frac{3}{2}, -\frac{1}{2})$ is $\frac{2}{3}$, since the two lines are perpendicular.

The equation of this line is $y + \frac{1}{2} = \frac{2}{3}(x - \frac{3}{2})$

$$6y + 3 = 4x - 6 \;\Rightarrow\; 4x - 6y - 9 = 0$$

Example 3

The lines $x + 2y - 3 = 0$ and $4x + 5y - 9 = 0$ intersect at M. Find

a) the co-ordinates of M

b) the equation of the line through M, parallel to the line $3x - 4y + 7 = 0$

Solution

a) To find the point of intersection M, we solve the two equations simultaneously,

$x + 2y = 3$ ....(1)

$4x + 5y = 9$ ....(2)

$4 \times \text{eqn}(1) - \text{eqn}(2) \;\Rightarrow\; 3y = 3 \;\Rightarrow\; y = 1$

Substituting $y = 1$ into eqn(1)

$x + 2 = 3 \quad \Rightarrow x = 1$

∴ The co-ordinates of M is (1, 1)

b) The gradient of the line through M is equal to the gradient of the line $3x - 4y + 7 = 0$ since they are parallel.

From $3x - 4y + 7 = 0 \;\Rightarrow\; y = \frac{3}{4}x + \frac{7}{4}$

Therefore the gradient of the line through $M(1, 1)$ is $\frac{3}{4}$

Hence the equation of this line is

$y - 1 = \frac{3}{4}(x - 1)$

$4y - 4 = 3x - 3 \;\Rightarrow\; 3x - 4y + 1 = 0$

Example 4

Find the equation of the line which passes through the point (2, 1) and perpendicular to:

i) $3x + y - 2 = 0$ ii) $2x - 4y - 1 = 0$

Solution

i) $x - 3y + 1 = 0$ ii) $2x + y - 5 = 0$

[Work out the solution]

Example 5

Find the equation of the line which passes through the point (3, –2) and parallel to:

i) $5x - y + 3 = 0$ ii) $x + 7y - 5 = 0$

Solution

i) $5x - y - 17 = 0$ ii) $x + 7y + 11 = 0$

[Work out the solution]

Example 6

Find the equation of the line which passes through the origin and perpendicular to:

i) $2x - y + 3 = 0$

ii) $3x + 2y - 4 = 0$

iii) $x - 2y + 3 = 0$

Solution

i) $x + 2y = 0$

ii) $2x - 3y = 0$

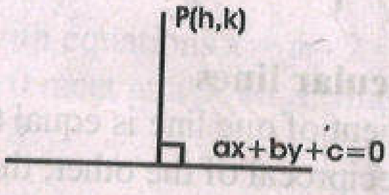
iii) $2x + y = 0$

[Work out the solution]

Perpendicular distance from a point to a line

If $ax + by + c = 0$ is the general equation of any line, then the **perpendicular distance** from any point $P(h, k)$ to the line is given by:

$$\pm\frac{ah + bk + c}{\sqrt{a^2 + b^2}}$$



The perpendicular distance from the origin is taken as positive.

Points on the same side of a line give the same sign.

Points on opposite sides of a line give different signs.

Illustrative examples

Example 1

Find the perpendicular distance from a point (2, 3) to the line $3x + 4y = 5$.

Solution

First put the equation in the general form i.e. $3x + 4y - 5 = 0$

∴ The perpendicular distance from (2, 3) to $3x + 4y - 5 = 0$ is

$$\frac{3(2) + 4(3) - 5}{\sqrt{3^2 + 4^2}} = \frac{13}{\sqrt{25}} = \frac{13}{5}$$

Example 2
Show that the points (–1, 2) and (3, 4) are on opposite sides of the line $x + 2y = 6$

Solution
First put the equation in the general form i.e. $x + 2y - 6 = 0$

The perpendicular distance from (–1, 2) to $x + 2y - 6 = 0$ is $\dfrac{1(-1)+2(2)-6}{\sqrt{1^2+2^2}} = \dfrac{-3}{\sqrt{5}}$

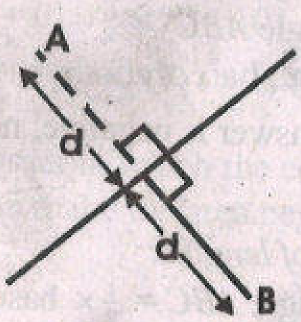
The perpendicular distance from (3, 4) to $x + 2y - 6 = 0$ is $\dfrac{1(3)+2(4)-6}{\sqrt{1^2+2^2}} = \dfrac{5}{\sqrt{5}}$

Since these distances are of opposite signs the points are on opposite sides of the line.

Example 3
Show that the point $A(-1, 5)$ is a reflection of the point $B(3, -3)$ in the line $x - 2y + 1 = 0$.

Solution
A point $A(-1, 5)$ is a reflection of the point $B(3, -3)$ in the given line if and only if :

1. The given line bisects AB
2. AB is perpendicular to the given line



1. Since the midpoint of AB i.e. (1, 1) satisfies the equation of the line, it means the line bisects AB.
2. The gradient of $AB = -2$ and the gradient of the line is $\frac{1}{2}$.

Since the product of the two gradients equals –1, it means AB is perpendicular to the given line.
Therefore $A(-1, 5)$ is a reflection of the point $B(3, -3)$ in the line $x - 2y + 1 = 0$

Bisectors of angles between two lines

The **equations of the bisectors** of the angles between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given by:

$$\frac{a_1x+b_1y+c_1}{\sqrt{(a_1{}^2+b_1{}^2)}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{(a_2{}^2+b_2{}^2)}}$$

Illustrative example
Find the equation of the bisectors of the angles between $5x + 12y + 4 = 0$ and $3x - 4y + 1 = 0$.

Solution
Note: The equations should be rearranged in the form $ax + by + c = 0$ if they are not.
The equations are:

$$\frac{5x+12y+4}{\sqrt{(5^2+12^2)}} = \pm\frac{3x-4y+1}{\sqrt{(3^2+(-4^2))}}$$

$$\frac{5x+12y+4}{13} = \pm\frac{3x-4y+1}{5}$$

$$5(5x+12y+4) = \pm 13(3x-4y+1)$$

Taking each sign in turn, we have $2x - 16y - 1 = 0$ and $64x + 8y + 33 = 0$ as the equations of the angle bisectors.

The angle between two intersecting straight lines

In general, the **angle** θ between two straight lines with gradients m_1 and m_2 is given by:

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}, \text{ where } m_1 > m_2.$$

If the two lines are **parallel**, then $m_1 = m_2$.
If the two lines are **perpendicular**, then

$$m_1 m_2 = -1 \text{ or } m_1 = -\frac{1}{m_2}$$

The **acute angle** θ between two straight lines with gradients m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

In this case, it is irrelevant which gradient is taken as m_1 or m_2.
Note that $|-4| = 4$; $|-2| = 2$; $|4| = 4$ etc.

Illustrative examples

Example 1

Find the angle between the lines $2x - y = 4$ and $x - y = 3$

Solution

$2x - y = 4 \quad \Rightarrow y = 2x - 4$ and
$x - y = 3 \quad \Rightarrow y = x - 3$

Therefore the gradients of the lines are 2 and 1

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{2 - 1}{1 + (2)(1)} = \tfrac{1}{3}$$

$$\tan\theta = \tfrac{1}{3} = 0.333 \Rightarrow \theta = 18.4°$$

Example 2

Find the acute angle between the lines $2x + y = 4$ and $y - 3x + 7 = 0$.

Solution

$2x + y = 4 \quad \Rightarrow y = -2x + 4$ and
$y - 3x + 7 = 0 \quad \Rightarrow y = 3x - 7$

Therefore the gradients of the lines are -2 and 3

$$\text{Hence } \tan\theta = \left|\frac{-2 - 3}{1 + (-2)(3)}\right|$$

$$= \left|\frac{-5}{1 - 6}\right| = \left|\frac{-5}{-5}\right| = 1$$

$$\tan\theta = 1 \Rightarrow \theta = 45°$$

Example 3

Find the acute angle between the lines $y = x + 2$ and $3y - x - 2 = 0$

Solution

$y = x + 2$ and $3y - x - 2 = 0$

$$y = \tfrac{1}{3}x + \tfrac{2}{3}$$

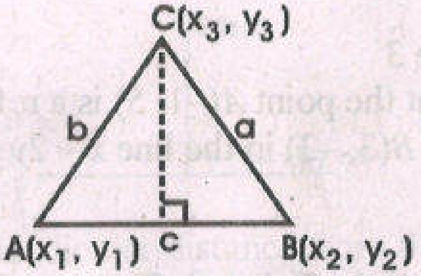
Therefore the gradients of the lines are 1 and $\frac{1}{3}$

$$\text{Hence } \tan\theta = \left|\frac{1 - \frac{1}{3}}{1 + (1)(\frac{1}{3})}\right| = \left|\frac{\frac{2}{3}}{\frac{4}{3}}\right| = \left|\frac{2}{4}\right| = 0.5$$

$$\tan\theta = 0.5 \Rightarrow \theta = 26.57°$$

The area of a triangle or parallelogram

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC.



The area of triangle ABC can be found as follows:

1. *In terms of coordinates:*

Area of triangle ABC

$$= \tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

If the final answer is negative, neglect the sign.

2. *In terms of lengths:*

Area of triangle $ABC = \frac{1}{2} \times$ base $\times$ altitude

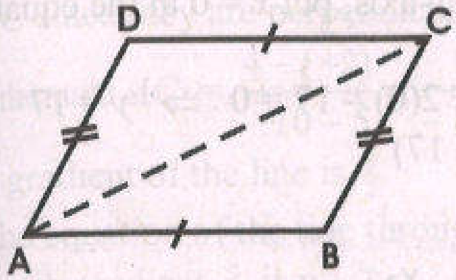
Choose one side (say AB) to be the base and find the equation of that side.

Next find the perpendicular distance from the other vertex (say C) to the base. This gives the altitude.

Also area of triangle ABC

$$= \sqrt{[s(s-a)(s-b)(s-c)]}$$

where $s = \frac{a+b+c}{2}$ and a, b and c are the length of the sides of the triangle.



The area of a parallelogram is *two times* the area of the triangle formed by any three of its vertices. The following are parallelograms: rhombus, square and rectangle .

Illustrative examples

Example 1

The vertices of a triangle ABC are $A(2, 3)$, $B(4, 7)$ and $C(7, 3)$. Find the area of the triangle.

Solution

Area of triangle ABC

$$=\tfrac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$$

$$=\tfrac{1}{2}[2(3-7)+7(7-3)+4(3-3)]$$

$$=\tfrac{1}{2}(-8+28+0]=\tfrac{1}{2}\times 20=10 \text{ units}^2$$

Use the other methods to find the area.

Note: The order in which the coordinates are chosen can make the area negative.

Summary

1. The slope of a line parallel to x-axis is 0, so slope is $\tan 0^\circ = 0$.
2. A line perpendicular to x-axis, its inclination is 90°, so slope is $\tan 90^\circ = \infty$.
3. Two straight lines are parallel if their slopes are equal.
4. Two straight lines are perpendicular, if product of their slopes is -1.
5. If angle of inclination is acute, slope is positive.
6. If angle of inclination is obtuse, slope is negative.
7. In terms of coordinates,

   slope, $m = \dfrac{y_1 - y_2}{x_1 - x_2}$

   m = angle of inclination or $\tan\theta$

   $m = \dfrac{\text{Difference of ordinates}}{\text{Difference of abscissa}}$
8. The slope of the x-axis and of any line parallel to it is zero.
9. The slope of the y-axis and of any line parallel to it is not defined.
10. Any point which satisfies the equation of a line lies on that line.

Solved examination questions

The following questions will help you understand what you have studied so far in this chapter. The questions require you to use all the concepts you have learnt in this chapter.

Example 1

a) The vertices of a quadrilateral $ABCD$ are $A(4, 0)$, $B(14, 11)$, $C(0, 6)$ and $D(-10,-5)$.
   i) Show that AC and BD bisect each other at right angles.
   ii) Calculate $|BD| : |AC|$
b) The points $A(7, 9)$ $B(3, 5)$ and $C(5, 1)$ are the vertices of a triangle ABC. If H and K are the midpoints of AB and AC respectively, find
   i) The equation of the line HK.
   ii) P and Q, the points of intersection of the line HK and the x and y axes respectively.
   iii) The area of ΔOPQ, where O is the origin.

Solution

a) i) If AC and BD meet at right angles, then the gradient of AC must be equal to the negative reciprocal of the gradient of BD.

$$\text{The gradient of } AC = \frac{6-0}{0-4} = \frac{6}{-4} = -\tfrac{3}{2}$$

$$\text{The gradient of } BD = \frac{11-(-5)}{14-(-10)} = \frac{16}{24} = \tfrac{2}{3}$$

∴ From the two gradients, AC and BC bisect each other at right angles.

ii) $$|BD| = \sqrt{(14+10)^2+(11+5)^2}$$
$$= \sqrt{24^2+16^2} = \sqrt{832}$$

$$|AC| = \sqrt{(4-0)^2+(0-6)^2}$$
$$= \sqrt{16+36} = \sqrt{52}$$

$$|BD| : |AC| = \frac{|BD|}{|AC|} = \frac{\sqrt{832}}{\sqrt{52}}$$
$$= \sqrt{\frac{832}{52}} = \sqrt{\frac{16}{1}} = \frac{4}{1}$$

$\therefore\ |BD| : |AC| = 4 : 1$

b) Given the points $A(7, 9)$, $B(3, 5)$ and $C(5, 1)$ as the vertices of a triangle ABC. If H is the midpoint of AB, then the co-ordinates of $H = (\tfrac{1}{2}\{7+3\}, \tfrac{1}{2}\{9+5\})$
$= (5, 7)$

If K is the midpoint of AC, then the co-ordinates of $K = (\tfrac{1}{2}\{7+5\}, \tfrac{1}{2}\{9+1\})$
$= (6, 5)$

i) To find the equation of line HK, first find the gradient of HK

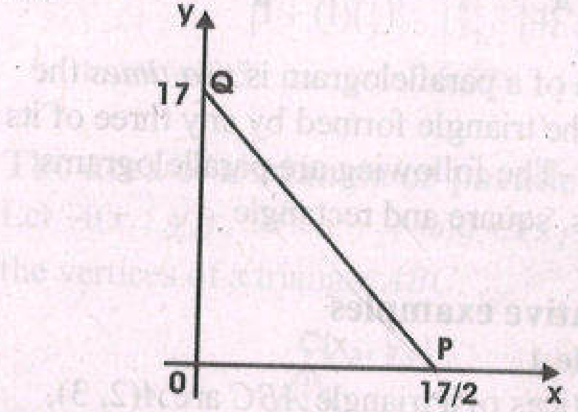
$$\text{The gradient of } HK = \frac{7-5}{5-6} = \frac{2}{-1} = -2$$

∴ The equation of HK, using the point $H(5, 7)$ is $y-7=-2(x-5)$
$2x+y-17=0$

ii) For the point of intersection, P, of line HK with the x-axis, put $y=0$ in the equation of line HK i.e. $0+2x-17=0$
$x=\frac{17}{2}$ $\therefore$ $P(\frac{17}{2}, 0)$

For the point of intersection, Q, of line HK with the y-axis, put $x=0$ in the equation of line HK
i.e. $y+2(0)-17=0 \Rightarrow y=17$
$\therefore$ $Q(0, 17)$

iii)



From the figure,

The area of $\Delta OPQ = \tfrac{1}{2}\,|OQ|\,|OP|$
$= \tfrac{1}{2}(17)(\tfrac{17}{2})$
$= 72.25$ sq. units

Example 2

The points $A(5,-2)$, $B(2,7)$ and $C(-10,4)$ are in the x-y plane. Find

i) the equation of the line passing through A and parallel to line BC

ii) the equation of the line passing through B, and perpendicular to line AC

iii) the point of intersection of the lines in (i) and (ii) above.

Solution

$A(5,-2)$ $B(2, 7)$ and $C(-10, 4)$

i) The gradient of the line passing through A = the gradient of BC since they are parallel
The gradient of BC is

$$\frac{7-4}{2-(-10)} = \frac{3}{12} = \tfrac{1}{4}$$

∴ The equation of the line through $A(5, 2)$ with gradient $\frac{1}{4}$ is $y + 2 = \frac{1}{4}(x - 5)$

$4y + 8 = x - 5 \Rightarrow x - 4y - 13 = 0$

ii) The gradient of the line passing through B = the negative reciprocal of the gradient of AC since they are perpendicular.

The gradient of $AC = \dfrac{4-(-2)}{-10-5} = \dfrac{6}{-15} = -\frac{2}{5}$

∴ The gradient of the line is $\frac{5}{2}$

Hence the equation of the line through $B(2, 7)$ with gradient $\frac{5}{2}$ is $y - 7 = \frac{5}{2}(x - 2)$

$2y - 14 = 5x - 10 \Rightarrow 5x - 2y + 4 = 0.$

iii) The point of intersection of the lines $4y - x = -13$ and $2y - 5x = 4$ is found by solving the two equations simultaneously.

$x - 4y = 13$---------(1)

$5x - 2y = -4$ --------(2)

$5 \times$ eqn(1) − eqn(2)

$-18y = 69 \Rightarrow y = -\frac{69}{18} = -\frac{23}{6}$

Eqn(1) − 2 × eqn(2)

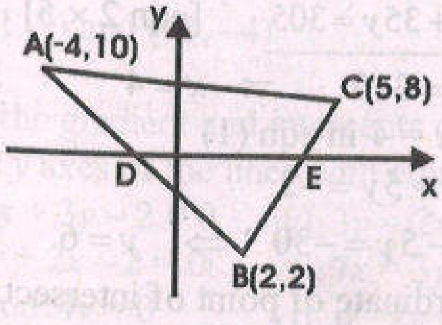
$-9x = 21 \Rightarrow x = -\frac{21}{9} = -\frac{7}{3}$

∴ The point of intersection is $(-\frac{7}{3}, -\frac{23}{6})$

Example 3

A triangle has vertices $A(-4, 10)$, $B(2, 2)$ and $C(5, 8)$. Calculate the co-ordinates of D, the midpoint of AB. The line through D parallel to AC meets BC at E. Calculate the equation of the line DE and the co-ordinates of the point E.

Solution



The co-ordinates of D are

$(\frac{1}{2}[-4 + 2], \frac{1}{2}[10 + 2]) = (-1, 6)$ since D is the midpoint of AB

The gradient of AC is $\dfrac{10-8}{-4-5} = -\frac{2}{9}$

Therefore the gradient of the line DE which is parallel to AC is $-\frac{2}{9}$.

Hence the equation of the line DE passing through $D(-1, 6)$ with gradient $-\frac{2}{9}$ is

$y - 6 = -\frac{2}{9}(x + 1)$

$9y - 54 = -2x - 2$

$2x + 9y - 52 = 0$…………(1)

The gradient of BC is $\dfrac{8-2}{5-2} = \frac{6}{3} = 2$

The equation of the line BC is

$y - 2 = 2(x - 2)$

$y - 2 = 2x - 4$

$2x - y - 2 = 0$……………(2)

Since E is the point of intersection of DE and BC, solve the equations of these two lines simultaneously to get E.

Solving equations (1) and (2) simultaneously, $x = \frac{7}{2}$ and $y = 5$.

Therefore the co-ordinates of E are $(\frac{7}{2}, 5)$

Example 4

The points $A(-4, 6)$ and $C(-1, 1)$ are opposite vertices of a parallelogram $ABCD$. The sides BC and CD of the parallelogram lie along the lines, $x + 3y - 2 = 0$ and $x - y + 2 = 0$ respectively. Calculate

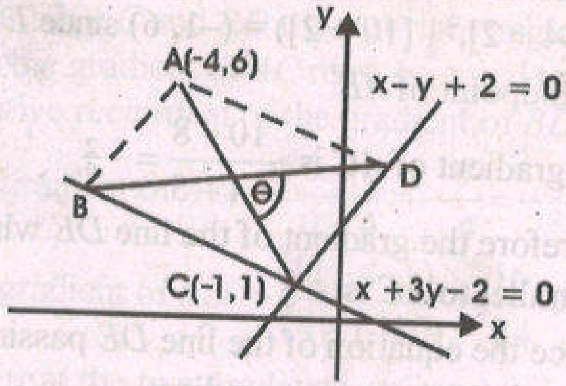
i) the co-ordinates of B and D

ii) the tangent of the acute angle between the diagonals of the parallelogram

iii) the length of the perpendicular from A to the side CD

iv) the area of the parallelogram.

Solution



For the parallelogram $ABCD$,
Gradient of BC = gradient of $AD = -\frac{1}{3}$
Gradient of AB = gradient of $CD = 1$
Equation of BC is $x + 3y - 14 = 0$
Equation of AB is $x - y + 10 = 0$

i) Since B is the point of intersection of AB and BC, solve the equations of these two lines simultaneously to get B. Therefore the co-ordinates of B are $(-7, 3)$
Similarly, since D is the point of intersection of AD and CD, solve the equations of these two lines simultaneously to get D.
Therefore the co-ordinates of D are $(2, 4)$

ii) Gradient of $AC = \dfrac{6-1}{-4+1} = -\frac{5}{3}$;

Gradient of $BD = \dfrac{4-3}{2+7} = \frac{1}{9}$

The tangent of the acute angle θ is given by $\tan\theta = \left|\dfrac{-\frac{5}{3} - \frac{1}{9}}{1 + (-\frac{5}{3})(\frac{1}{9})}\right|$

$$= \left|-\tfrac{16}{9} \times \tfrac{27}{22}\right| = \left|-\tfrac{24}{11}\right| = \tfrac{24}{11}$$

iii) Let p be the length of the perpendicular from $A(-4, 6)$ to the side CD with equation
$x - y + 2 = 0$, then

$$p = \frac{-4-6+2}{\sqrt{1^2 + (-1)^2}} = \frac{-8}{\sqrt{2}} = 4\sqrt{2} \text{ units}$$

Note: Discard the negative sign

iv) Length of $CD = \sqrt{(2+1)^2 + (4-1)^2}$

$$= \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

∴ The area of parallelogram

$$= p \times CD = 4\sqrt{2} \times 3\sqrt{2} = 24 \text{ units}$$

Example 5

$P(1, 8)$ and $Q(8, 3)$ are points in the $x - y$ plane. Find

i) the equation of a line normal to $\overline{PQ}$ and passing through the point R (6, 8);

ii) correct to the nearest whole number, the coordinates of the point of intersection of the normal in (i) and the line $\overline{PQ}$.

Solution

i) Gradient of $PQ = \dfrac{8-3}{1-8} = -\dfrac{5}{7}$

∴ Gradient of normal $= \dfrac{7}{5}$

Required equation of normal is

$$y - 8 = \frac{7}{5}(x-6)$$
$$5y - 40 = 7x - 42$$
$$7x - 5y = 2 \ldots\ldots\ldots\ldots(1)$$

ii) Equation of PQ

$$y - 8 = -\frac{5}{7}(x-1)$$
$$7y - 56 = -5x + 5$$
$$5x + 7y = 61 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Solving eqns (1) and (2) simultaneously

$$49x - 35y = 14 \qquad [\text{eqn } 1 \times 7]$$
$$25x + 35y = 305 \qquad [\text{eqn } 2 \times 5]$$
$$74x = 319 \quad \Rightarrow \quad x = 4$$

Putting $x = 4$ in eqn (1)

$$7(4) - 5y = -2$$
$$-5y = -30 \quad \Rightarrow \quad y = 6$$

∴ Coordinate of point of intersection (4, 6)

Review exercise 10

1. Find the lengths of the straight lines joining the following pairs of points
   - i) $(-1, 3)$ and $(5, 2)$
   - ii) $(1, 2)$ and $(5, 2)$
   - iii) $(-2, 3)$ and $(4, 3)$
   - iv) $(3, 4)$ and $(7, 1)$
   - v) $(-6, 1)$ and $(6, 6)$
   - vi) $(-2, -4)$ and $(-10, -10)$

2. Find
   - a) the gradient
   - b) the midpoint of the lines joining the following pairs of points
     - i) $(-1, 3)$ and $(-8, 11)$
     - ii) $(4, 6)$ and $(9, 15)$
     - iii) $(5, -11)$ and $(-1, 3)$
     - iv) $(7, 0)$ and $(-3, -2)$
     - v) $(-2\frac{1}{2}, -\frac{1}{2})$ and $(4\frac{1}{2}, -1)$
     - vi) $(-1, -4)$ and $(-6, -6)$

3. Find the equation of the lines with
   - i) gradient -2 and passing through the point $(-1, -3)$
   - ii) gradient 2 and passing through the point $(2, 3)$
   - iii) gradient 1 and passing through the point $(7, -1)$

4. Find the equation of the line passing through the points
   - i) $(3, 4)$ and $(1, -2)$
   - ii) $(5, 7)$ and $(-1, 4)$
   - iii) $(1, 1)$ and $(3, 2)$
   - iv) $(1, 4)$ and $(3, 4)$
   - v) $(4, 2)$ and $(2, 4)$
   - vi) $(1, 2)$ and $(-1, -4)$

5. Find the gradient and intercepts on the x and y axes of the lines with equations.
   - i) $2x + 3y - 2 = 0$
   - ii) $y + 2x - 2 = 0$
   - iii) $5x + 4y = 0$
   - iv) $3y = 2x + 6$
   - v) $7x = 3y + 5$
   - vi) $x - 2y + 2 = 0$
   - vii) $3x + y + 6 = 0$
   - viii) $2y = 3x - 1 = 0$
   - ix) $2x - 5y - 8 = 0$
   - x) $2y + 4x - 1 = 0$
   - xi) $2y + 7x + 6 = 0$
   - xii) $4x - 3y - 24 = 0$

6. Two points P and Q have co-ordinates $(11, 1)$ and $(2, 7)$ respectively. Find the co-ordinates of the points which divide PQ internally and externally in the ratio
   - i) $2 : 1$.
   - ii) $3 : 5$

7. The midpoint of the line joining the points $A(3, 0)$ and $B(5, 6)$ is M. A point $C(t, 4t)$ is such that CM is perpendicular to AB. Calculate
   - i) the value of t
   - ii) the area of triangle ABC

8. $A(5, 0)$ and $B(0, 8)$ are two vertices of triangle OAB.
   - a) What is the equation of the bisector of angle AOB.
   - b) If E is the point of intersection of this bisector and the line through A and B, find the coordinates of E. Hence show that $OA : OB = AE : EB$.

9. Find the image of the point $(5, 6)$ in the line
   - i) $3x - y + 1 = 0$
   - ii) $y = 4x + 20$
   - iii) $2x + 5y + 18 = 0$

10. Find the equations of the perpendicular bisectors of the lines joining:
    - i) $(0, 0)$ and $(2, 4)$
    - ii) $(3, -1)$ and $(-5, 2)$
    - iii) $(5, -1)$ and $(0, 7)$

11. Given the three points $A(4, 0)$, $B(0, 2)$ and $C(-2, -2)$. Show that
    - i) $AB = BC$
    - ii) AB is perpendicular to BC

    If a square $ABCD$ is formed, calculate the co-ordinates of D

12. Three points have co-ordinates $A(1, 7)$, $B(7, 5)$ and $C(0, -2)$. Find
   i) the equation of the perpendicular bisector of AB
   ii) the point of intersection of this perpendicular bisector and BC

13. The midpoint of the line joining the points $A(7, 3)$ and $B(-1, -5)$ is C. Find both values of k if the straight line joining C to the point $P(k^2, k)$ is perpendicular to AB.

14. The perpendicular bisector of the line joining the points (1, 2) and (5, 4) meets the y – axis at the point $(0, k)$. Calculate k.

15. Find the equation of the perpendicular bisector of the line joining the points $A(0, 8)$ and $B(4, 2)$.

16. Find the equation of the line through the point (3, –5) which is parallel to the line with equation $3x + 2y - 5 = 0$

17. The coordinates of P and Q are (5, 3) and (6, 0) respectively. Find the equation of the mediator of PQ

18. Find the equation which makes an angle of 135° with the positive x-axis and passes through the point (0, 4).

19. Find the equation of the line which is parallel to the line $x + 4y - 1 = 0$ and passes through the point (1, 2).

20. Find the equation of the line which passes through the point (4, –3) and is perpendicular to the line $2x + 5y + 1 = 0$.

21. Find the equation of the line which is parallel to the line with equation $x + 2y - 3 = 0$ and passes through the midpoint of (–2, 3) and (4, 5)

22. Find the perpendicular distance from the point (2, 3) to the line $12x + 5y = 13$.

23. Find the perpendicular distance from the point (5, 3) to the line $2x - y + 4 = 0$.

24. Calculate the perpendicular distance of the point $P(-4, 3)$ from the line $3x+4y = 5$

25. Find the acute angle between the lines
   i) $3x - 2y = 8$ and $x - 5y = -7$
   ii) $x - y = 2$ and $x - 3y = -2$
   iii) $2x + 3y = 7$ and $x - 6y = 5$
   iv) $x + 4y - 1 = 0$ and $3x + 7y = -2$

26. Three points have co-ordinates $P(-2, -3)$, $Q(2, 0)$ and $R(8, -8)$
   i) Prove that angle $PQR = 90°$
   ii) Calculate the area of triangle PQR
   iii) Calculate the length of PR and hence or otherwise, find the perpendicular distance of Q from PR

27. One side of a rhombus lies along the line $5x + 7y = 1$ and one of the vertices is (3, –2). One diagonal of the rhombus is the line $3y = x + 1$. Find the coordinates of other vertices and the equations of the three remaining sides.

28. A rhombus $ABCD$ is such that the coordinates of A and C are (–3, –4) and (5, 4), respectively. Find the equation of the diagonal BD of the rhombus. If the side BC has gradient 2, obtain the coordinates of B and D. Prove that the rhombus has area $21\frac{1}{3}$.

29. Three points have coordinates $A(-2, -1)$, $B(6, 9)$ and $C(2, -3)$. The line through the midpoint of AB parallel to AC meets the x-axis at X and the y-axis at Y. Calculate the coordinates of X and Y. Hence deduce the area of triangle XOY, where O is the origin.

30. The points A and B have coordinates $A(-3, -1)$ and $B(7, 4)$ respectively. Find the coordinates of the point C which divides AB internally in the ratio 2 : 3. Find also the equation of the line through B perpendicular to AB.

31. The coordinates of two points are $A(2, 3)$ and $B(7, 6)$. Find
- i) the equation of AB
- ii) the coordinates of the point C, which is the point of intersection of the x-axis and the bisector of AB.

32. The points $C(6, 6)$, $O(0, 0)$ and $A(4, 3)$ are three of the vertices of a parallelogram $COAB$:
- i) Calculate the coordinates of the point B
- ii) Show that the tangent of the acute angle AOC is $\frac{1}{7}$ and hence write down the exact value of the sine of the angle AOC.
- iii) Calculate the area of the parallelogram $COAB$

33. A, B, C and D are the points $(-1, 3)$, $(2, 9)$, $(-2, 4)$ and $(9, 9)$ respectively.
- i) P and Q divide AB internally and externally in the ratio 2 : 1. Find the coordinates of P and Q.
- ii) Find the ratio $AP : AQ$
- iii) M is the midpoint of AD. Find the coordinates of M
- iv) Prove that CP is perpendicular to MQ.

34. The lines $y = 2x$, $2x + y - 12 = 0$ and $y = 2$ enclose a triangular region of the xy-plane. Find;
- a) the coordinates of the vertices of this region,
- b) the area of this region.

35. The line $3x + 4y = 15$ intersects the curve $2xy = 9$ at A and B. Find
- i) coordinates of A and B
- ii) the distance AB.

36. The vertices of triangle ABC are the points $A(-1, 5)$, $B(-5, 2)$ and $C(8, -7)$.
- i) Find in the form $px + qy + r = 0$, where p, q and r are integers, an equation of the line passing through B and C.
- ii) Show that AB and AC are perpendicular.
  The point D lies on the line BA produced and is such that $3BA = BD$
- iii) Determine the coordinates of D

37. A trapezium $ABCD$ in which AB is parallel to DC. The point A lies on the y-axis. Points B and D are $(6, 13)$ and $(1, -2)$ respectively. Angles ABC and BCD are 90°. Given that the equation of DC is $3y = 4x - 10$, find
- i) the equation of AB
- ii) the equation of BC
- iii) the coordinates of A and C
- iv) the area of the trapezium

38. Find the equation of the line through the point $(5, 2)$ and perpendicular to the line $x - y + 2 = 0$.

39. $E(5, 3)$and $F(2, -1)$ are the two vertices of a square $EFGH$, and H is on the x-axis. Find the coordinates of H and G.

40. Find the equation of the line through the origin which is parallel to the line $4x + 2y - 5 = 0$.

41. The line $4x - 5y + 20 = 0$ cuts the x-axis at A and the y-axis at B. Find the equation of the median through O of ΔOAB. Find the equation of the altitude through O of triangle OAB.

42. Find the equation of the perpendicular from (5, 3) to the line $2x - y + 4 = 0$.

43. The points $A(1, 4)$ and $B(5, 7)$ are two adjacent vertices of a parallelogram $ABCD$. The point $C(7, 10)$ is another vertex of the parallelogram. Find the equation of the side CD.

44. Find the equation of the line with given inclination to Ox passing through the given point. Use $\tan 60° = \sqrt{3}$ and $\tan 30° = \frac{1}{3}\sqrt{3}$

a) 45°, (1, –2) b) 30°, (0, 0)
c) 135°, (–1, 2) d) 120°, (2, –1)
e) 150°, (3, 1)
f) θ, where $\tan \theta = \frac{1}{3}$, (1, 5)

45. Find the inclination to Ox of the line whose equation is given. Give answers in degrees correct to 1 decimal place.

a) $y = 2x - 1$
b) $y = 5x - 3$
c) $x - 2y + 1 = 0$
d) $3x + 2y - 1 = 0$

46. Show that the triangle whose vertices are (1, 1), (3, 2) and (2, –1) is isosceles.

47. Find the area of the triangular region enclosed by the x and y axes and the line $2x - y - 1 = 0$.

48. Find the equation of the line passing through (–1, 2) and parallel to $2x - 3y = 5$.

49. Find the coordinates of the triangular region enclosed by the lines $y = 0$, $y = x + 5$ and $x + 2y - 6 = 0$.

50. Write down the equation of the perpendicular bisector of the line joining the points (2, –3) and $(-\frac{1}{2}, 3\frac{1}{2})$.

51. Find the equation of the line through $A(5, 2)$ which is perpendicular to the line $y = 3x - 5$. Hence find the coordinates of the foot of the perpendicular from A to the line.

52. Write down the equation of the line which goes through (7, 3) and which is inclined at 45° to the positive direction of the x-axis.

53. The equations of two sides of a square are $y = 3x - 1$ and $x + 3y - 6 = 0$. If (0, –1) is one vertex of the square, find the coordinates of the other vertices.

CHAPTER 11

STATISTICS

Introduction

Statistics is concerned with the collection, ordering and analysis of data. Data consist of sets of recorded observations or values. A statistical exercise normally consists of four stages:

(a) Collection of data by counting or measuring;

(b) Ordering and presentation of the data in a suitable way form;

(c) Analysis of the collected data;

(d) Interpretation of the results and drawing of conclusions.

Definition of terms

Discrete data

Values of discrete data are restricted to only certain, exact distinct numbers (but not measurements). For example, a number of pupils can be 0, 1, 2, 3… but not $-3, 2\frac{1}{2}, 4.7, \ldots$

It is usually found by counting.

Continuous data

Values of continuous data can be any real number. Continuous data is usually found by measuring (not counting) and rounded to a suitable degree of accuracy. For example, weight and height can be measured to the nearest kg or cm respectively.

Frequency

It is the *number* of occurrence of individual items that fall under a particular class. It can also be defined as the *number* of times an item or event occurs.

Frequency distribution tables

This is a tabular arrangement of data by grouping them into classes together with corresponding class frequencies. When repeated observations are made on a variable (which can be marks, heights, ages etc.), the result is a **frequency distribution**.

A frequency distribution is recorded in a **frequency table**. The values of the variable are often grouped in classes. The frequency table gives the possible values or class intervals of the variable and the corresponding frequencies (*f*). An example of a frequency distribution table is shown in the table below.

Class Marks	*No. of Candidates (i.e. frequency)*
35 – 39	3
40 – 44	2
45 – 49	4

A **tally table** is a simple frequency distribution table. When making a tally table, you must work carefully through the data item by item, putting a *tally mark*, /, for each item at correct place in the table. For example, 𝍸 shows five tallies.

Add the tallies up to get each *frequency*.

Check the total frequencies. If it is not the same as the total number of items, then you need to tally again. See the table below.

Marks	*Tally*	*Frequency*
0	𝍸 ////	9
1	𝍸 𝍸 𝍸	15
2	𝍸 𝍸 ///	13
3	𝍸	5
4	///	3

Note: A frequency table is usually given without the tally marks. The word 'frequency' may not appear in the table, so make sure you identify the values and frequencies. For example, another version of the frequency table above is given below.

Values ↓

Number of fillings	0	1	2	3	4
Number of boys	9	15	13	5	3

↑ *Frequencies*

Grouped frequency distributions

Grouping data is useful when there are many small frequencies in a distribution and/or a large range of values. A **grouped frequency distribution table** can be made to show the number of values (frequency) for each group as shown in the table below. The range of possible values for a group or a class is called its **class interval**.

Class marks	*No. of candidates (i.e. frequency)*
35 – 39	3
40 – 44	2
45 – 49	4

For example, the number of overtime hours per week worked by employees at a factory are as follows:

45 31 46 25 57 39 42 55 20 37
40 59 11 38 34 22 62 33 48 43
57 37 43 51 29 41 35 66 45 32
44 47 42 46 54 65 17 35 53 27
38 22 33 39 45 32 43 41 57 45

Lowest value of the data = 11
Highest value = 66
Therefore arrange 6 classes of interval 10.

To determine the frequency distribution, we set up the table as follows.

No. of runs	*Tally*	*Frequency*
10 – 19	//	2
20 – 29	𝍸 /	6
30 – 39	𝍸 𝍸 ////	14
40 – 49	𝍸 𝍸 𝍸 //	17
50 – 59	𝍸 ///	8
60 - 69	///	3

Class interval

A symbol defining a class such as 35 – 39 in the table above is called a **class interval.** The end numbers 35 and 39 are called **class limits**. The smaller number is the **lower class limit** and the larger number is **the upper class limit**. Hence for the class 35 – 39, 35 is the lower class limit and 39 is the upper class limit.

Although data are not always grouped into equal class intervals, they are in many situations grouped into equal class intervals. In an examination the class intervals to use may be given.

Illustrative examples

The number of runs scored by 36 batmen in a cricket competition is as follows.

31	40	20	35	21	12
11	39	16	0	28	49
29	32	17	2	12	24
14	1	10	23	19	13
26	30	10	21	18	0
15	11	29	8	12	1

Tally these data in a grouped frequency table. Use equal class intervals starting 0–9, 10–19 20–29,….

Solution

No. of runs	Tally	Frequency
0 – 9	𝍸 /	6
10 – 19	𝍸 𝍸 ////	14
20 – 29	𝍸 ////	9
30 – 39	𝍸	5
40 – 49	//	2

Ungrouped frequency distributions

This is data organised in such a way that the marks are not grouped as shown in the table below.

Marks	Frequency
4	5
5	2
6	3

Class boundaries

Class boundaries separate one class from another. For each class they are the *smallest* and *largest* values an item in that class can have. The smallest value is the lower class boundary and the largest value is called the upper class boundary. For a grouped frequency distribution, each class boundary is usually *half-way* between the upper limit of one class and the lower limit of the next class. i.e. $\frac{1}{2}$ (upper class limit of one class + lower class limit of the next class)

Alternatively, to find the class boundaries of a grouped data, follow the steps below:

1. First find the difference between the upper class limit of a particular class interval and the lower class limit of the next class and divide it by 2.

From the table below, the difference between the upper class limit of a particular class interval and the lower class limit of the next class i.e. (40 – 39 = 1) and half of the difference is 0.5

2. *Subtract* the 0.5 obtained from the *lower class limits* of each class and *add* the 0.5 to the *upper class limit* of each class to obtain the class boundaries. i.e. for the class 35 – 39, 34.5 is the lower class boundary and 39.5 is the upper class boundary as shown in the table below.

Marks	Frequency	Class Boundaries
35 – 39	3	34.5 – 39.5
40 – 44	2	39.5 – 44.5
45 – 49	4	44.5 – 49.5

Note that it is not always that you subtract 0.5 from the lower class limits and add 0.5 to the upper class limits. For the frequency table below, we subtract 40 (i.e. half of 80) from the lower class limits and add 40 to the upper class limits if we follow the steps above.

Marks	Frequency	Class Boundaries
980 – 1000	10	940 – 1040
1080 – 1100	15	1040 – 1140
1180 – 1200	24	1140 – 1240

Class size or width

The **class size** or **width** is the difference between the lower and upper class boundaries of a class interval.

i.e. class size or width = upper class boundary – lower class boundary.

From the table below, the class size of the class interval 35 – 39 is (39.5 – 34.5) = 5.

For the class interval 45 – 49, the class size is (49.5 – 44.5) = 5.

Marks %	Frequency	Class Boundaries	Class Size
35 – 39	3	34.5 – 39.5	5
40 – 44	2	39.5 – 44.5	5
45 – 49	4	44.5 – 49.5	5

Class mid-value or midpoint

The **class mid-value** or **midpoint** is the middle value of a class. It is obtained by adding the lower and the upper class limits and dividing the result by 2 or it is the average of the lower and the upper class boundaries. i.e.

Class-mid value

$= \frac{1}{2}$ (lower class limit + upper class limit)

$= \frac{1}{2}$ (lower class boundary + upper class boundary)

From the table below, the class mid-value of the class interval 35 – 39 is $\frac{1}{2}(35 + 39) = 37$.

The class mid-value (denoted by x) is often used in calculating the mean and standard deviation of a grouped frequency distribution.

Marks	*Frequency*	*Class mid-value or midpoint*
5 – 9	3	7
10 – 14	5	12
15 - 19	2	17

Graphical representation of data

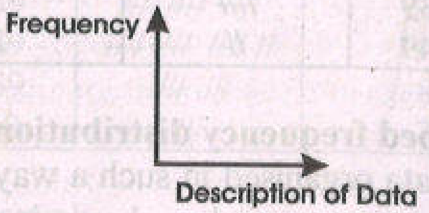
In statistics, data can be represented graphically in many forms depending on the data and the purpose for which it is intended. We shall consider the following graphs:

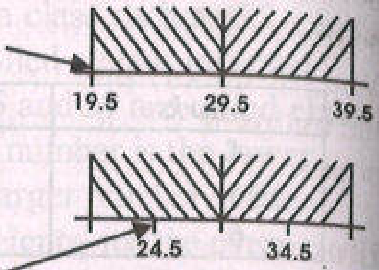
1. Histogram
2. Cumulative frequency curve.

Histogram

It is a graphical representation of frequency distributions. A **histogram** consists of a set of rectangles having a base on a horizontal axis (x-axis) with its centre at the *class mid-value* or *class mid-point* and lengths equal to the class sizes. The vertical axis corresponds to the frequency of the class.

Class boundaries can also be used as the horizontal axis instead of the class midpoints as shown in the diagrams below. The vertical axis is labelled 'frequency' and the horizontal axis is labelled with *a description of the data.*





- With the class boundaries the edge of the bars must be on these values.
- With the class mid points the values must be at the middle of the bars.

Note 1: The main difference between a histogram and a bar chart is that for a histogram there is no gap between the rectangles.

Note 2: Before any histogram is drawn, it is important to determine whether or not the class intervals are of equal size. *At this level we shall include histograms with unequal class intervals.* If the class intervals are of **equal size**, the heights of the rectangles are proportional to the class frequencies. It is therefore convenient to take the heights to be numerically equal to the class frequencies. Thus the vertical axis corresponds to the frequencies and the horizontal axis corresponds to the class boundaries of class marks.

Note 3. If the class intervals are of **unequal** sizes, the rectangles must have their heights adjusted so that their areas are proportional to the class frequencies. To do this, we divide each frequency by the class size to get what we call the **frequency density**. Thus, for each class, the frequency density is defined as:

$$\text{Frequency density} = \frac{\text{class frequency}}{\text{class size}}$$

In such cases, we plot the frequency density along the vertical axis and the class boundaries or class midpoints along the horizontal axis. Frequency polygon can also be drawn *without* drawing the bars first. You can simply plot the frequency against the midpoint of each group and join the points, in order, with straight lines.

Illustrative examples

Example 1

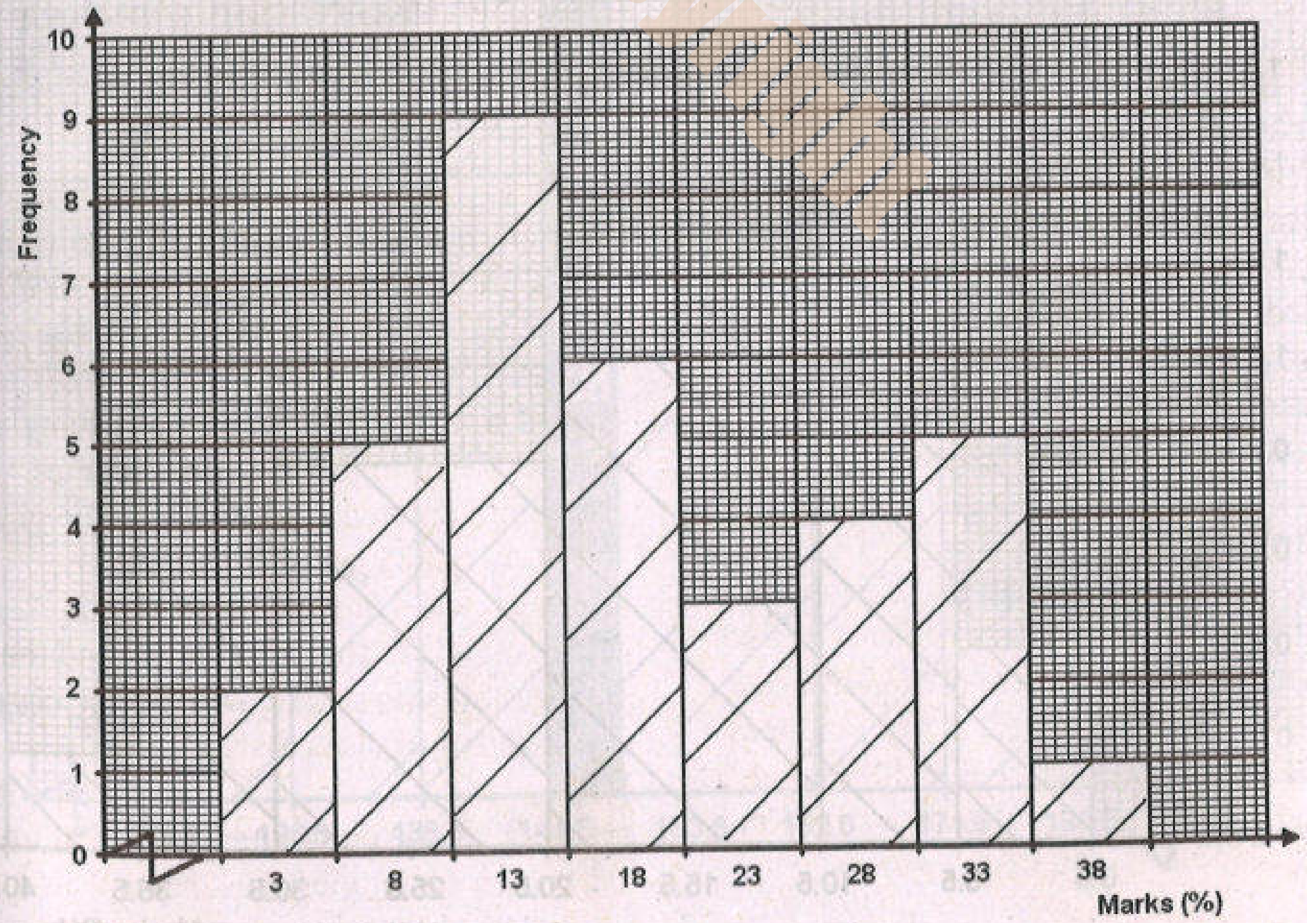
In an objective test marked out of 40, the marks scored by 35 students are given in the table. Draw a histogram for the distribution.

Marks scored	1–5	6–10	11–15	16–20	21–25	26–30	31–35	36–40
No. of students	2	5	9	6	3	4	5	1

Solution

Marks	*Frequency (f)*	*Class mid points*
1 – 5	2	3
6 – 10	5	8
11 – 15	9	13
16 – 20	6	18
21 – 25	3	23
26 – 30	4	28
31 – 35	5	33
36 – 40	1	38

Since the class sizes for the individual classes are **equal**, we have to plot the frequency along the vertical axis and the class midpoints along the horizontal axis, as shown in the graph below.



Example 2

In an objective test marked out of 40, the marks scored by 35 students are given in the table below.

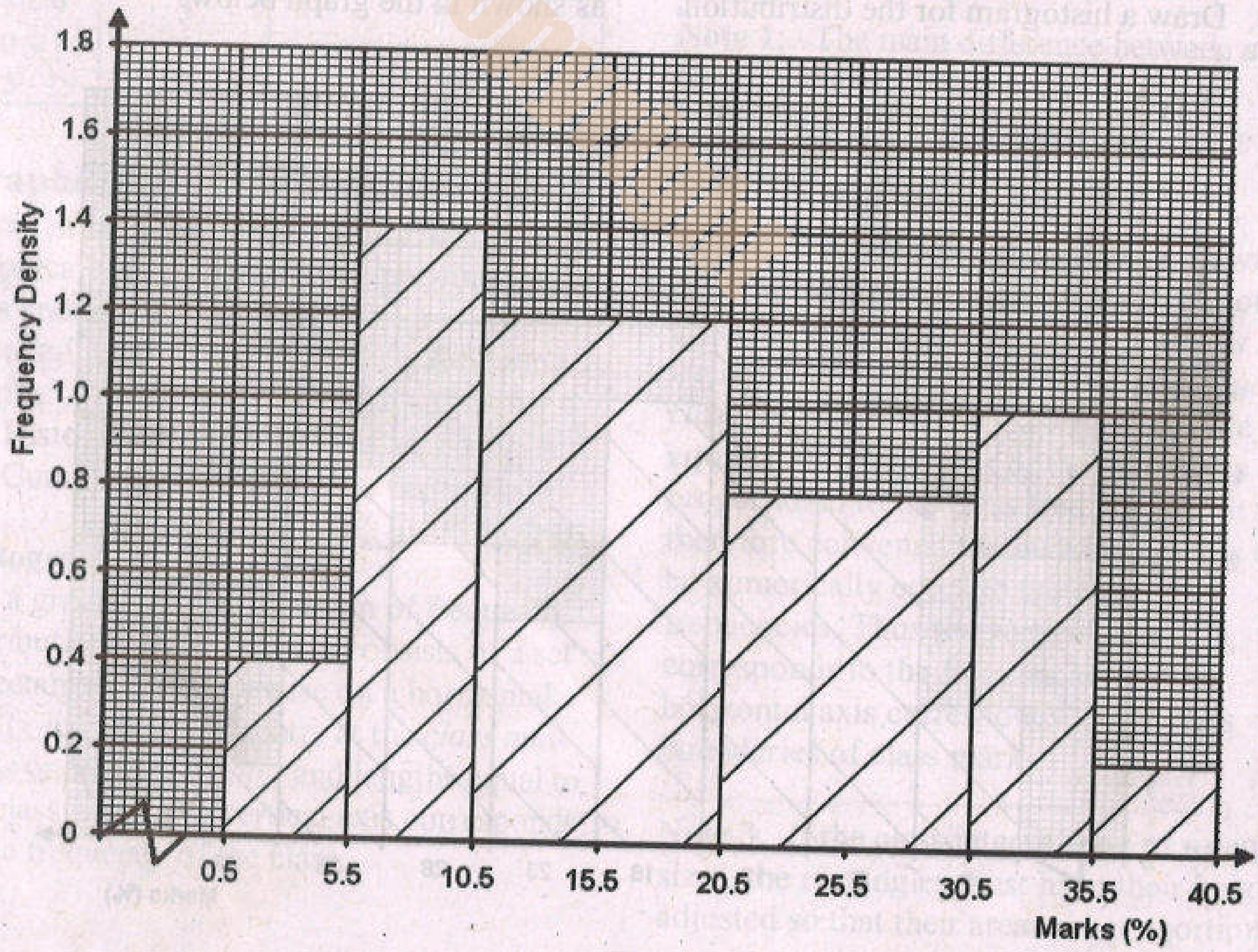
Marks scored	1 – 5	6 – 10	11 – 20	21 – 30	31 – 35	36 – 40
No. of Students	2	7	12	8	5	1

Draw a histogram for the distribution.

Solution

Marks	*Frequency (f)*	*Class boundaries*	*Class size (c)*	*Frequency density* $= \frac{\text{frequency}}{\text{class size}} = \frac{f}{c}$
1 – 5	2	0.5 – 5.5	5	0.4
6 – 10	7	5.5 – 10.5	5	1.4
11 – 20	12	10.5 – 20.5	10	1.2
21 – 30	8	20.5 – 30.5	10	0.8
31 – 35	5	30.5 – 35.5	5	1
36 – 40	1	35.5 – 40.5	5	0.2

Since the class sizes for the individual classes are **unequal**, we have to plot the frequency density along the vertical axis and the class boundaries along the horizontal axis, as shown in the graph below.



Finding the median from a histogram

From a histogram, the median is the value on the horizontal axis corresponding to the vertical line which divides the area of the histogram into two equal parts.

Illustrative examples

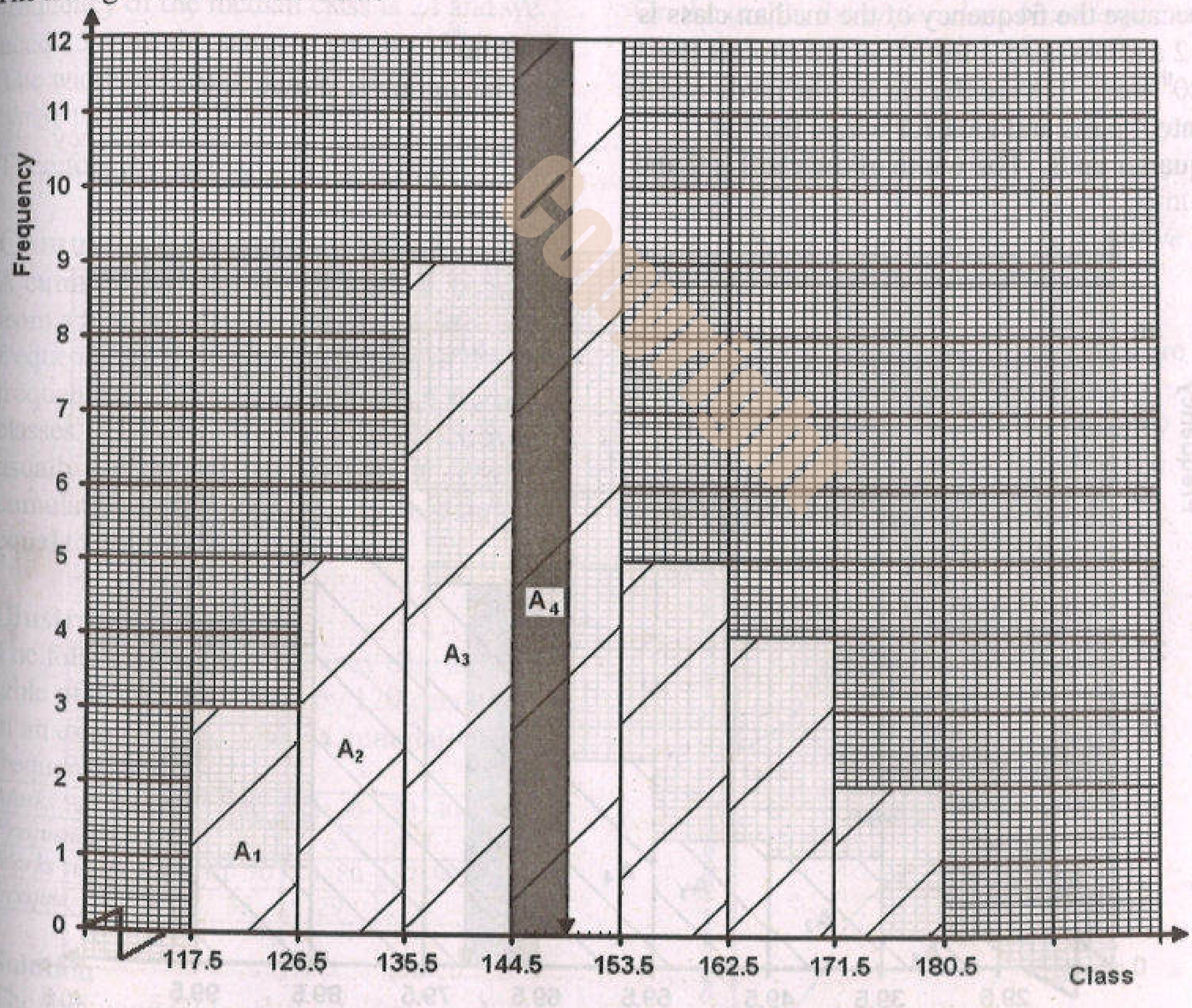
Example 1

Consider the data below. Draw a histogram and use it to estimate the median

Class	118–126	127–135	136–144	145–153	154–162	163–171	172–180
Frequency	3	5	9	12	5	4	2

Solution

The histogram is shown below.



The total frequency = 40. Half of it is 20
Therefore the median is the 20th item
Locate the median class. For this data, the median class is 145–153. This is found by adding the frequencies from the lower class up to half the total frequency (i.e. 20).
Since the class sizes are the same, we assume the area of the rectangles of the histogram is proportional to their frequencies.
Now considering the median class, the sum of the frequencies before the median class is 17 (i.e. $A_1 + A_2 + A_3$) but we need the 20th item. So we have to move 3/12th (i.e. one-fourth) inside the median class. This is because the frequency of the median class is 12 and we need 3 from this class to get the 20th item. Divide the bar for the median class into 4 parts and look for where the first quarter ends. The width of the class is 9 and $\frac{1}{4}$ of $9 = \frac{1}{4} \times 9 = 2.25 = A_4$. Therefore the median = 144.5 + 2.25 = 146.75

Example 2

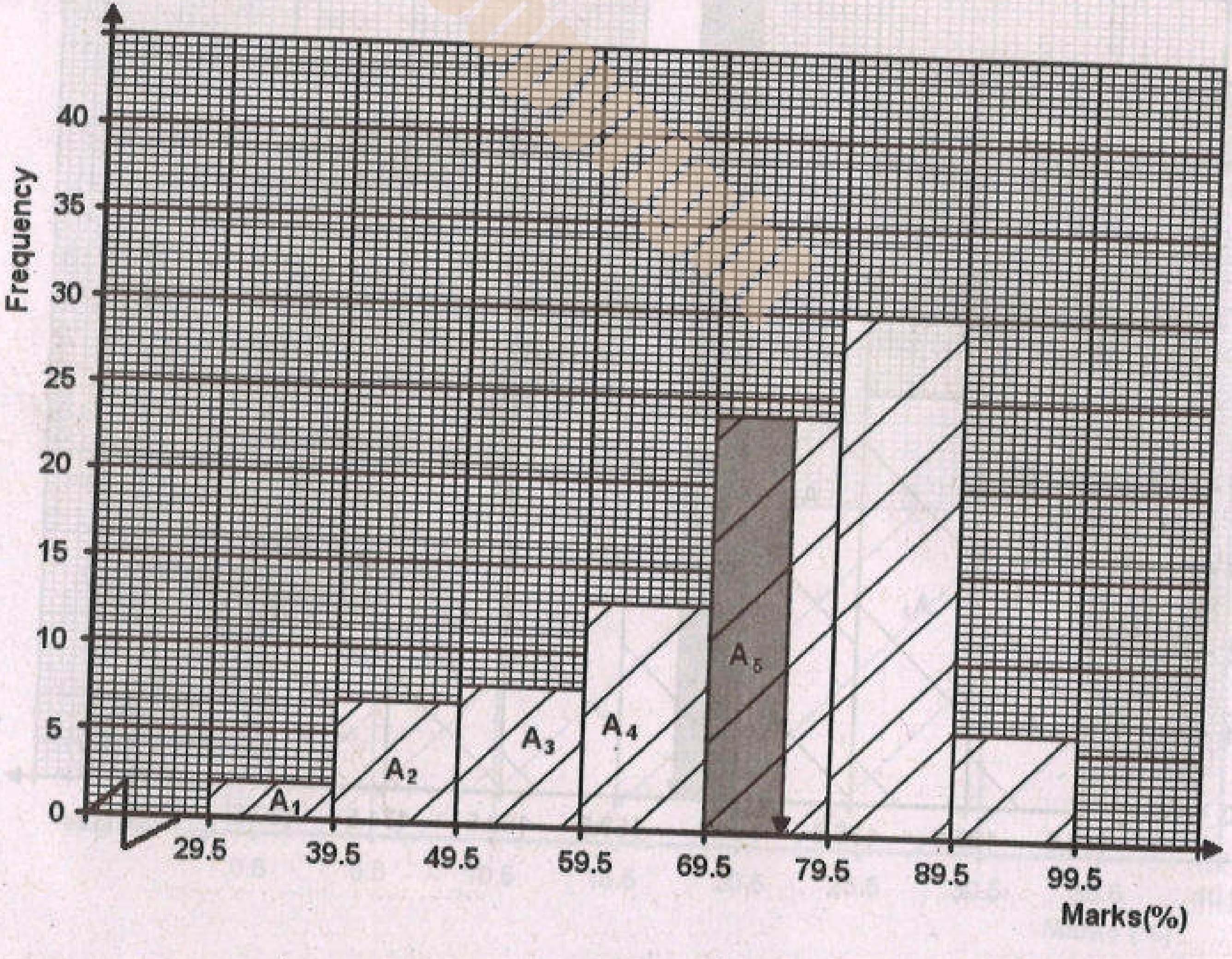
The grades of 90 students in an examination with the frequencies are shown in the table below:

Marks %	30–39	40–49	50–59	60–69	70–79	80–89	90–99
Freq	2	7	8	13	24	30	6

Draw a histogram and use it to estimate the median

Solution

The histogram is shown below.



The total frequency = 90. Half of it is 45
Therefore the median is the 45th item
Locate the median class. For this data, the median class is 70–79. This is found by adding the frequencies from the lower class up to half the total frequency (i.e. 45).
Since the class sizes are the same, we assume the area of the rectangles of the histogram is proportional to their frequencies.
Now considering the median class, the sum of the frequencies before the median class is 30 (i.e. $A_1 + A_2 + A_3 + A_4$). But we need the 45th item. So we have to move 15/24th inside the median class. This is because the frequency of the median class is 24 and we need 15 from this class to get the 45th item.
The width of the class is 10, and
$\frac{15}{24}$ of 10 = $\frac{15}{24} \times 10 = 6.25 = A_5$
Therefore the median = 69.5 + 6.25 = 75.75

Cumulative frequency tables

A **cumulative frequency table** can be made from a frequency table. The cumulative frequency for any class is the sum of the frequencies of that class and all the lower classes. Cumulative frequency tables are usually written in vertical column. The last cumulative frequency in the table should be equal to the total frequency.

Illustrative example

The following is the frequency distribution table of the marks scored by 120 candidates in an examination. Make a cumulative frequency table.

Marks %	1–10	11-20	21–30	31–40	41–50
Frequency	0	2	6	7	14
Marks %	51–60	61–70	71–80	81–90	91–100
Frequency	20	35	29	6	1

Solution

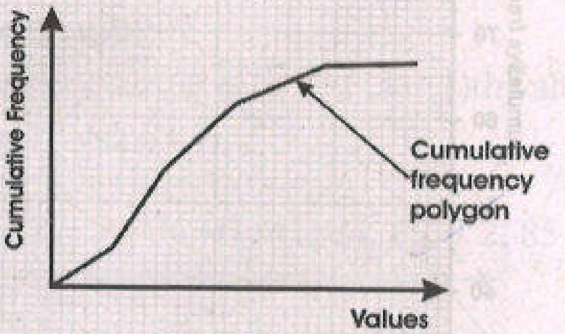
The following table shows the cumulative frequency.

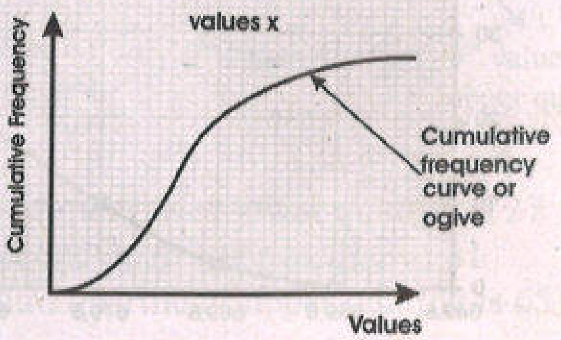
Marks less than	*Cumulative frequency*
10.5	0
20.5	2 + 0 = **2**
30.5	6 + 2 = **8**
40.5	15
50.5	29
60.5	49
70.5	84
80.5	113
90.5	119
100.5	120

Cumulative frequency diagram

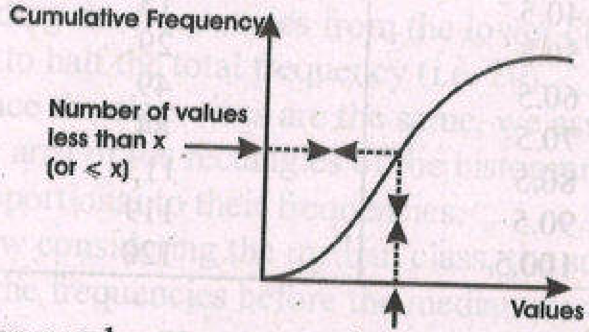
A **cumulative frequency diagram** can be drawn from a cumulative frequency table. On the graph, the *upper class boundaries* are always along the *horizontal axis* and the cumulative *frequencies* along the *vertical axis*. Joining all the consecutive points by a *straight line* gives a ***cumulative frequency polygon*** or joining all the consecutive points by the best shape *curve* gives a ***cumulative frequency curve*** (or ***ogive***).

See the figures below. Accurate values are obtained from the cumulative frequency curve than from the cumulative frequency polygon. But in an examination draw the diagram stated in the question.





A cumulative frequency diagram can be used to estimate values from the data. Always draw lines on your graph to show how you obtained your answers.



Given a value, you can estimate the number of values *less than* (or less than or equal to) that value and vice versa. You can also estimate the number of values *greater than* (or greater than or equal to) that value and vice versa. To make this estimate, you must relate the given *greater than* statement to a *less than* statement.

Illustrative example

The lives of 80 circuit components were recorded in hours to the nearest hour and grouped as follows:

Life (hours)	660–669	670–679	680–689	690–699	700–709	710–719	720–729	730–739
Freq	3	7	10	15	24	12	5	4

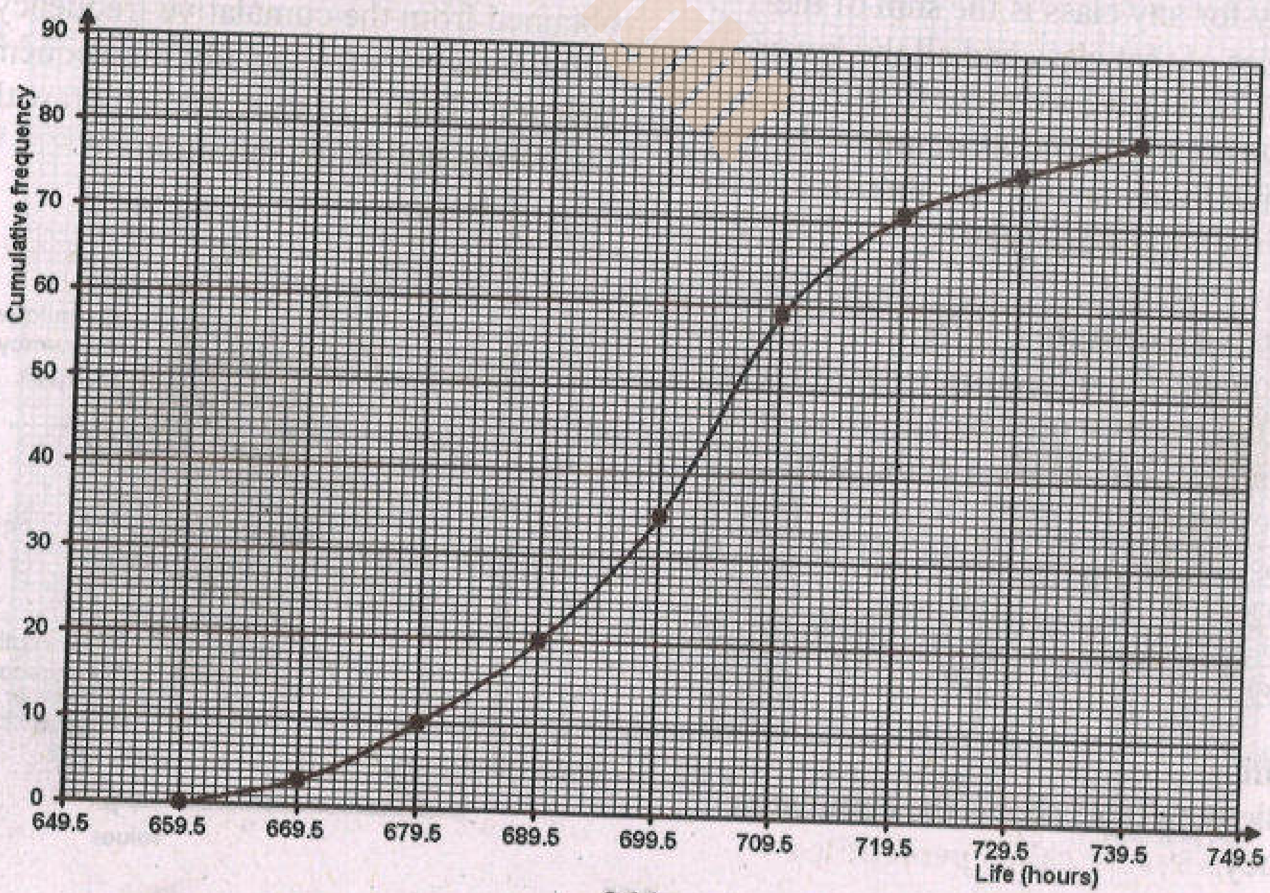
Make a cumulative frequency table and use it to draw the cumulative frequency curve.

Solution

Cumulative frequency table

Life less than	*Cumulative frequency*
659.5	**0**
669.5	$3 + 0 = \mathbf{3}$
679.5	$7 + 3 = \mathbf{10}$
689.5	$10 + 10 = \mathbf{20}$
699.5	$15 + 20 = \mathbf{35}$
709.5	$24 + 35 = \mathbf{59}$
719.5	$12 + 59 = \mathbf{71}$
729.5	$5 + 71 = \mathbf{76}$
739.5	$4 + 76 = \mathbf{80}$

The cumulative frequency curve is shown below.



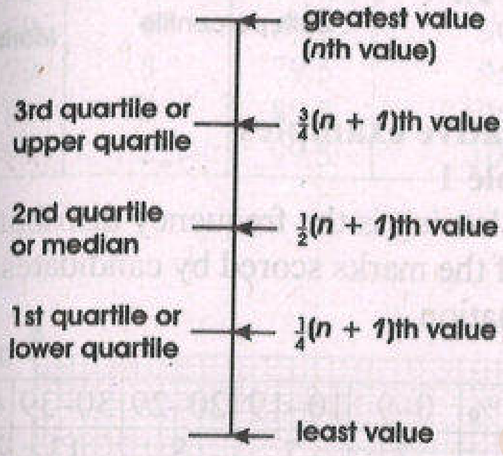
Quartiles, percentiles and deciles

Quartiles

The **quartiles** of a grouped data divide the set of data into *four equal parts*.

The *first or lower quartile* is denoted by $Q_1 = \frac{1}{4}\Sigma f$ on the cumulative frequency curve and the *third or upper quartile* by $Q_3 = \frac{3}{4}\Sigma f$.

The *second or middle quartile* is the median and is denoted by Q_2.


greatest value
(nth value)
3rd quartile or upper quartile
$\frac{3}{4}(n+1)$th value
2nd quartile or median
$\frac{1}{2}(n+1)$th value
1st quartile or lower quartile
$\frac{1}{4}(n+1)$th value
least value

The quartiles of a set of numbers are found in the same way as the median. Also the same method is applied when finding the quartiles from a frequency distribution table.

For example, to find the quartiles of a set of '*n*' numbers: $x_1, x_2, x_3, \ldots, x_n$

1. Arrange the numbers in order of magnitude.
2. The 1st quartile, Q_1 is the $\frac{1}{4}(n+1)$th item or observation.
3. The 3rd quartile, Q_3 is the $\frac{3}{4}(n+1)$th item or observation.

Note that when n is large,

Q_1 is the $\frac{1}{4}(n)$th item or observation.

Q_3 is the $\frac{3}{4}(n)$th item or observation.

The quartiles are usually found from the cumulative frequency curve.

Percentiles

The values which divide a set of data into *100 equal parts* are called **percentiles**.

Percentiles are denoted by $P_1, P_2, P_3, \ldots P_{99}$.

The 50th percentile, P_{50}, is the same as the median.

The *n*th percentile is given by $\frac{n}{100} \times \sum f$ on the cumulative frequency curve.

Therefore the 25th percentile

$= \frac{25}{100} \times \sum f = \frac{1}{4}\sum f$ = lower quartile.

Deciles

The values which divide a set of data into *10 equal parts* are called **deciles**.

Deciles are denoted by $D_1, D_2, D_3, \ldots, D_9$.

The 5th decile, D_5, is the same as the median.

The *n*th decile is given by $\frac{n}{10} \times \sum f$ on the cumulative frequency curve.

Illustrative example

Find the quartiles of the following set of numbers,

41, 26, 27, 64, 72, 65, 85, 20, 41, 30, 34

Solution

Arrange the data in order of size first. There are 11 values $\Rightarrow n = 11$

The lower quartile is the $\frac{1}{4}(11+1)$th value.

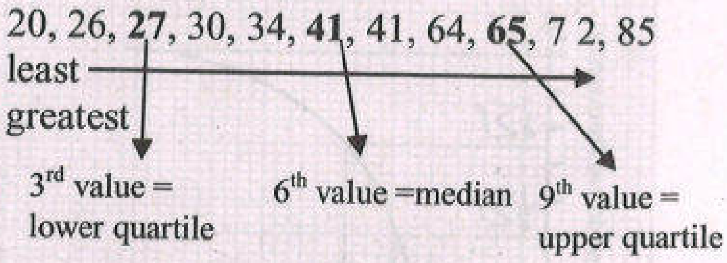
i.e. the 3rd value.

The median is the $\frac{1}{2}(11+1)$th value.

i.e. the 6th value.

The upper quartile is the $\frac{3}{4}(11+1)$th value.

i.e. the 9th value.


20, 26, 27, 30, 34, 41, 41, 64, 65, 7 2, 85
least
greatest
3rd value = lower quartile
6th value =median
9th value = upper quartile

The first quartile or lower quartile is 27

The second quartile or median is 41

The third quartile or upper quartile is 65

Inter-quartile range and semi -interquartile range

The **inter-quartile range** and *semi-interquartile range* are slightly better measures of dispersion than the range. They are not affected by extreme values because they are based on the 'middle-half' of the data, i.e. between the upper and lower quartiles.

The *inter-quartile range*
= upper quartile (Q_3) – lower quartile (Q_1)
i.e. $Q_3 - Q_1$.

The *semi-interquartile range* is half the inter-quartile range. i.e. $\frac{1}{2}(Q_3 - Q_1)$.

Finding quartiles and percentiles from a cumulative frequency curve

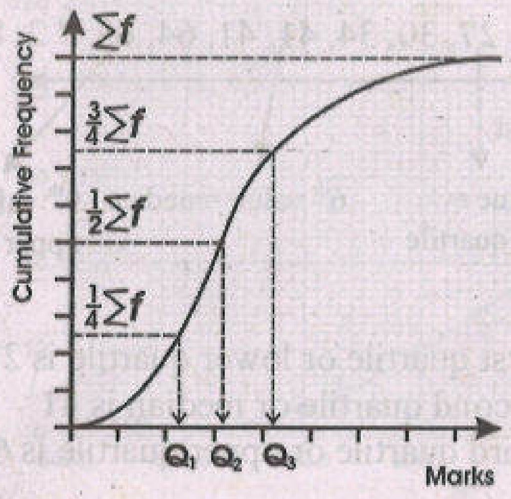
The cumulative frequency curve can also be used to estimate the **quartiles** and **percentiles.**

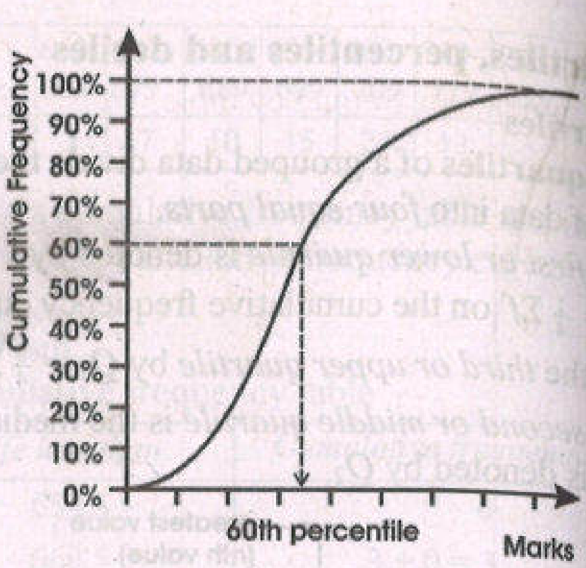
The *lower quartile* (Q_1) is the mark corresponding to $\frac{1}{4}\Sigma f$ on the cumulative frequency axis.

The *upper quartile* (Q_3) is the mark corresponding to $\frac{3}{4}\Sigma f$ on the cumulative frequency axis.

The n^{th} percentile is the mark corresponding to $\frac{n}{100}\times\Sigma f$ on the cumulative frequency axis.

For example, the 50th percentile is the mark corresponding to $\frac{50}{100}\times\Sigma f$ on the cumulative frequency axis.





Illustrative examples

Example 1

The following is the frequency distribution table of the marks scored by candidates in an examination.

Marks %	0–9	10–19	20–29	30–39	40–49
Freq	2	7	8	13	24
Marks%	50–59	60–69	70–79	80–89	90–99
Freq	30	6	5	3	2

a) Make a cumulative frequency table and use it to draw the cumulative frequency curve for the distribution.

b) Use your graph to estimate:
- i) the median mark
- ii) the lower quartile
- iii) the upper quartile
- iv) the inter-quartile range
- v) the pass mark if 60% of the students passed.
- vi) the 40th percentile
- vii) the 20th percentile

Solution

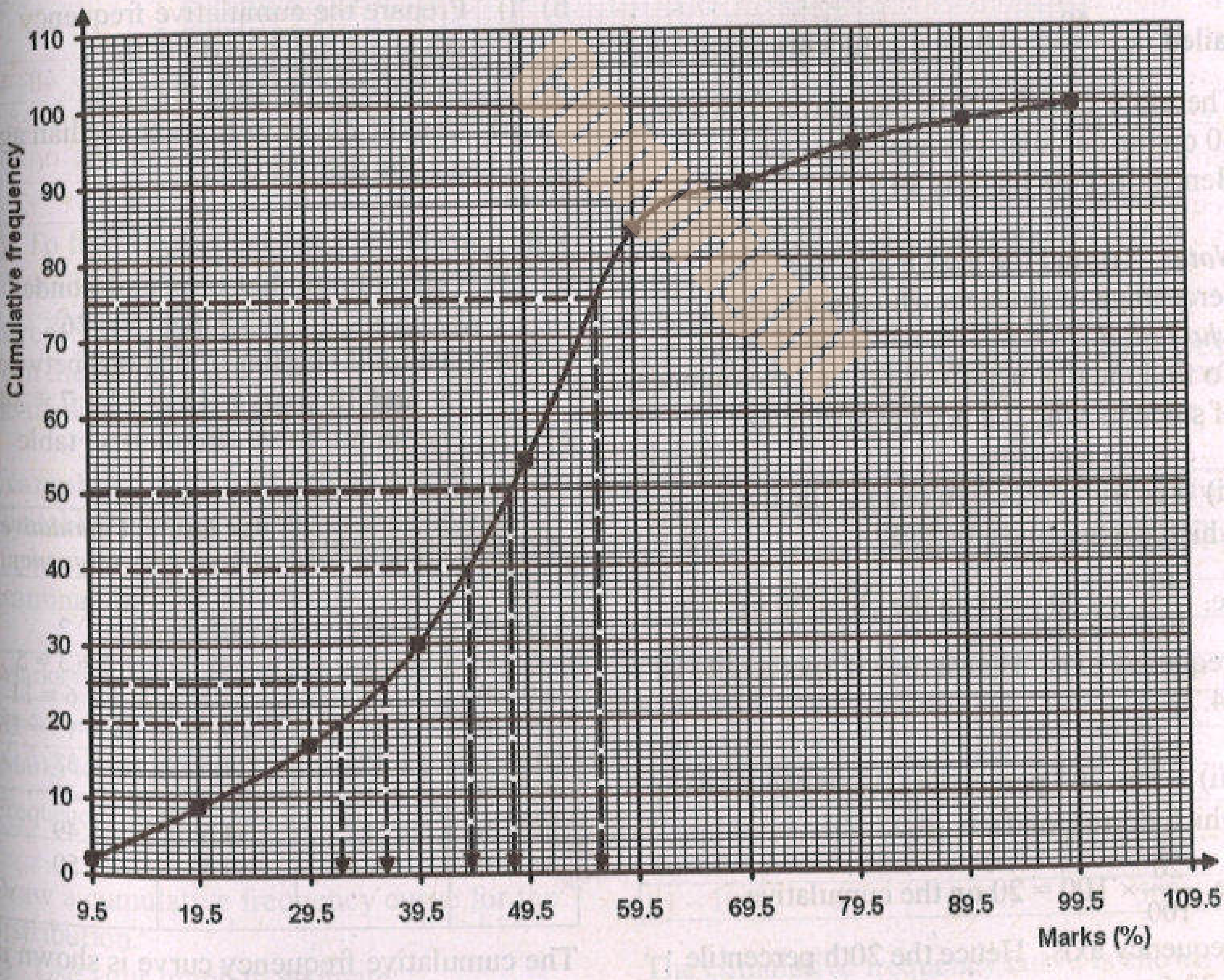
a) Cumulative frequency table.

Marks (%)	Frequency (f)	Marks less than	Cumulative frequency
0 – 9	2	9.5	2
10 – 19	7	19.5	2 + 7 = **9**
20 – 29	8	29.5	9 + 8 = **17**
30 – 39	13	39.5	30
40 – 49	24	49.5	54
50 – 59	30	59.5	84
60 – 69	6	69.5	90
70 – 79	5	79.5	95
80 – 89	3	89.5	98
90 – 99	2	99.5	100

Note that the third column gives the upper class boundaries.

a) Cumulative frequency curve is shown below

b) i) From the graph, the median mark is the mark corresponding to $\frac{1}{2}\Sigma f$ i.e. $\frac{1}{2}\times100 = 50^{th}$ item on the cumulative frequency axis. To find the median locate 50 on the cumulative frequency axis, trace it along the 50th line to curve and then trace it down to the marks axis for the median mark. Hence the median mark is 48.



b)(ii) The lower quartile is the mark that corresponds to $\frac{1}{4}\Sigma f$ i.e. $\frac{1}{4} \times 100 =$ 25th observation on the cumulative frequency axis. This is found in the same way as the median.
Hence the lower quartile is 36.5.

iii) The upper quartile is the mark that corresponds to $\frac{3}{4}\Sigma f$ i.e. $\frac{3}{4} \times 100 =$ 75th observation on the cumulative frequency axis. Hence the upper quartile is 56.5.

iv) The inter-quartile range
= The upper quartile – lower quartile
= 56.5 – 36.5 = 20.0

v) If 60 % of the students passed then 40% failed i.e. $\frac{40}{100} \times 100 = 40$ students failed.
Therefore find the mark that corresponds to 40 on the cumulative frequency axis.
Hence the pass mark is 44.7.

Note: To find the pass mark, use the percentage of students who *did not pass* or *who failed.*
To find the distinction mark, find the number of students who *did not get distinction.*

vi) The 40th percentile (P_{40}) is the mark which corresponds to $\frac{40}{100}\Sigma f$

i.e. $\frac{40}{100} \times 100 = 40$ on the cumulative frequency axis. Hence the 40th percentile is 44.7.

vii) The 20th percentile (P_{20}) is the mark which corresponds to $\frac{20}{100}\Sigma f$

i.e. $\frac{20}{100} \times 100 = 20$ on the cumulative frequency axis. Hence the 20th percentile is 33.5.

Example 2

The table gives the distribution of the ages (in years) of all persons (to the nearest thousand) in a town who were under the age of 40 years on 30th June 1990.

Age(yrs)	0 – 4	5–9	10–14	15–19
Nos. in thousands	2	3	6	15
Age(yrs)	20–24	25–29	30–34	35–39
Nos. in thousands	12	7	4	1

a) Using the table, calculate:
   i) the total number of persons under 20 years;
   ii) the total number of persons between 15 and 30 years.

b) i) Prepare the cumulative frequency table and use it to draw the cumulative frequency curve.
   ii) Use the curve to find the median age of the distribution.

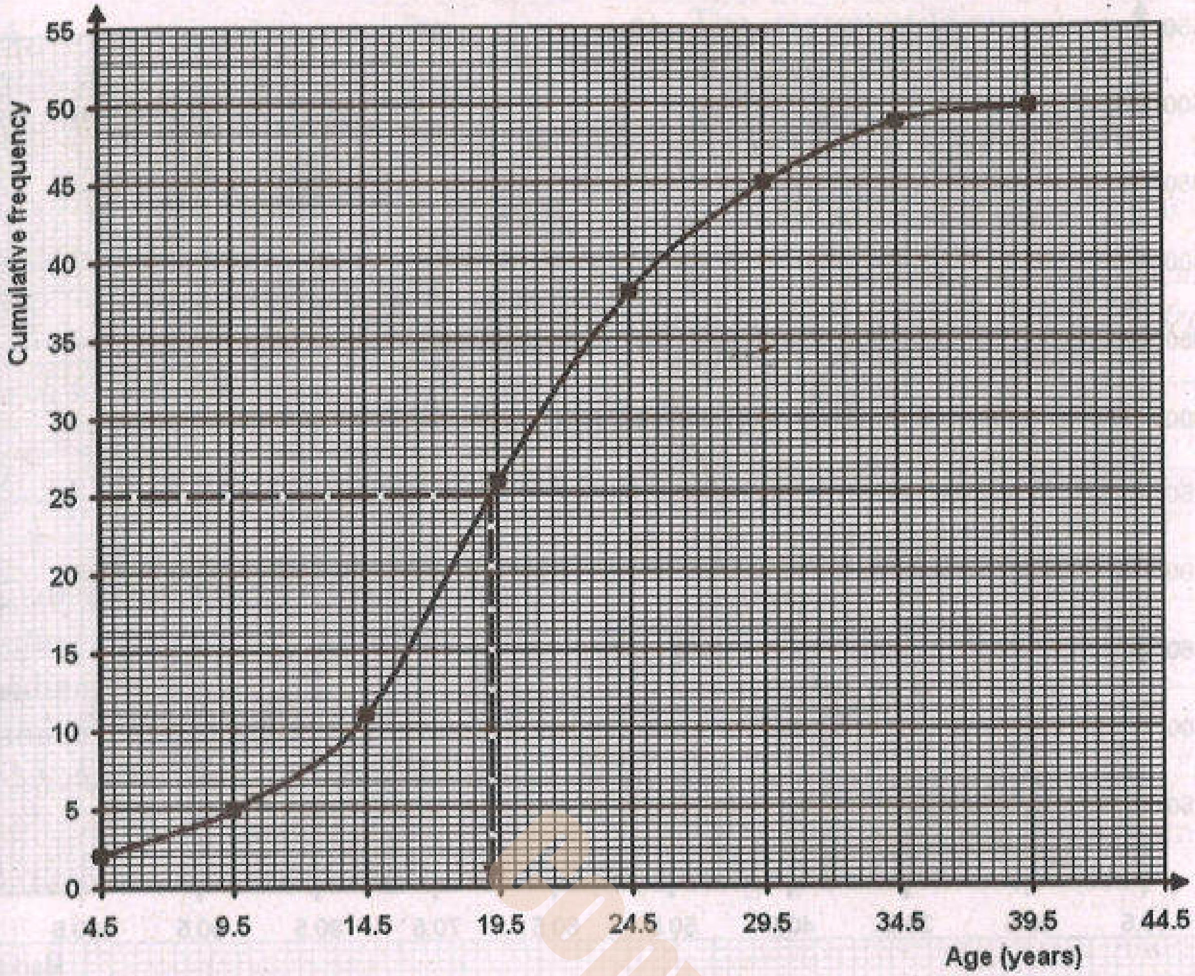
Solution

a) i) The total number of persons under 20 years = 2 + 3 + 6 +15 = 26
   ii) The total number of persons between 15 and 30 years = 15 + 12 + 7 = 34

b) i) The cumulative frequency table is shown below.

Age(yrs)	*Frequency*	*Age less than*	*Cumulative frequency*
0 – 4	2	4.5	2
5 – 9	3	9.5	2 + 3 = 5
11 – 14	6	14.5	5 + 6 = 11
15 – 19	15	19.5	11 + 15 = 26
20 – 24	12	24.5	38
25 – 29	7	29.5	45
30 – 34	4	34.5	49
35 – 39	1	39.5	50

The cumulative frequency curve is shown in the next page.



ii) To find the median age, first find $\frac{1}{2}\Sigma f = \frac{1}{2} \times 50 = 25$ and then find its corresponding age on the horizontal axis from the graph. Hence from the graph, the median age is 19 years.

Example 3
The table below gives the distribution of marks obtained by 500 candidates in an examination.

Marks %	1-10	11-20	21-30	31-40	41-50
Frequency	5	15	40	105	150
Marks %	51-60	61-70	71-80	81-90	91-100
Frequency	120	50	10	5	0

Draw a cumulative frequency curve for the distribution.

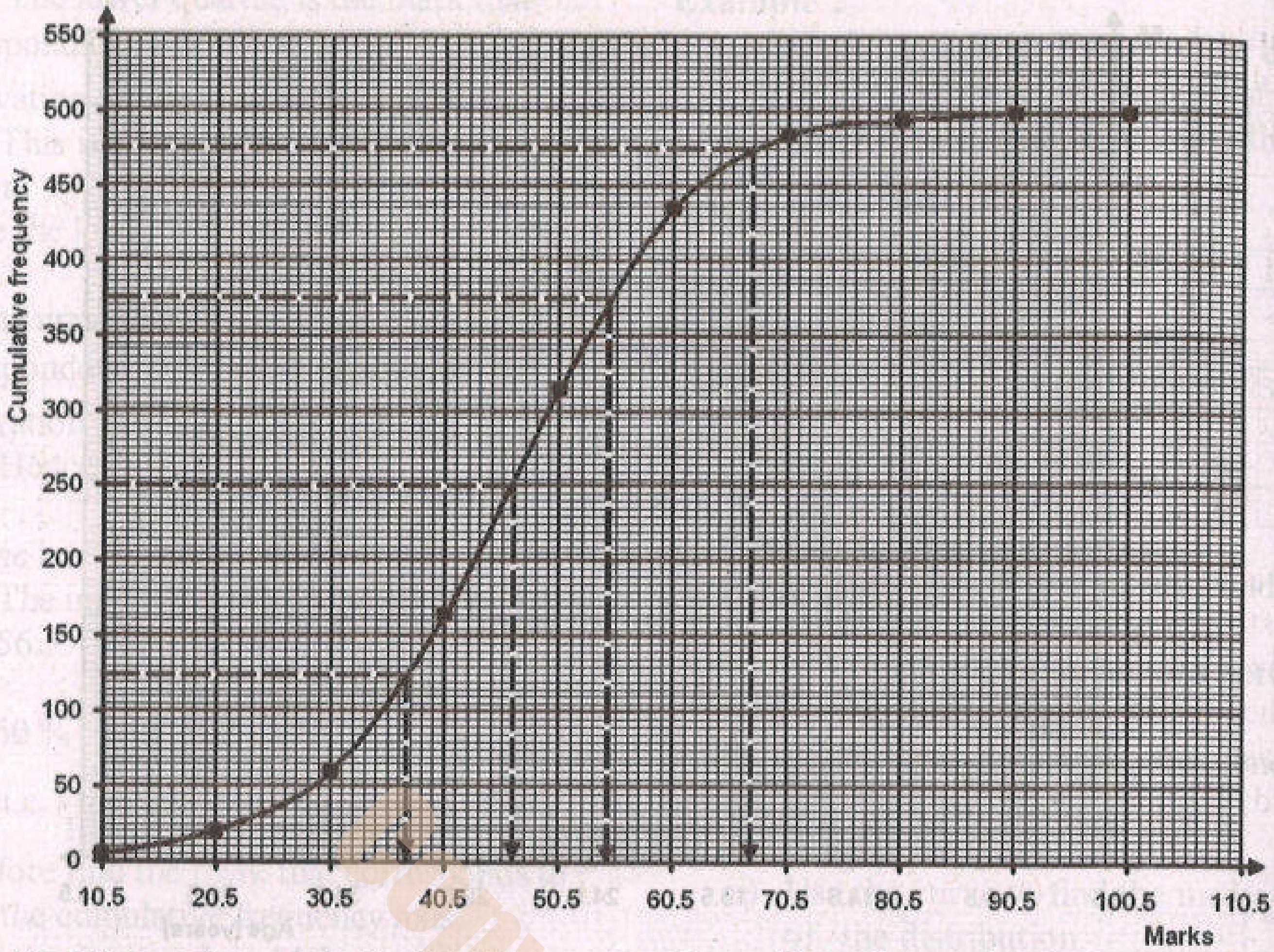
a) Use your graph to estimate:
   i) the median mark;
   ii) the inter quartile range and the semi-inter quartile range;

b) If only 5 % of the candidates attained the distinction level, estimate the lowest mark for this level.

Solution

Mark	*Frequency*	*Marks less than*	*Cumulative frequency*
1 – 10	5	10.5	5
11 – 20	15	20.5	20
21 – 30	40	30.5	60
31 – 40	105	40.5	165
41 – 50	150	50.5	315
51 – 60	120	60.5	435
61 – 70	50	70.5	485
71 – 80	10	80.5	495
81 – 90	5	90.5	500
91 – 100	0	100.5	500

The cumulative frequency curve is shown in the next page.



i) To find the median mark, first find $\frac{1}{2}\Sigma f = \frac{1}{2} \times 500 = 250$, and then find its corresponding mark on the horizontal axis from the graph. Hence from the graph, the mark corresponding to a cumulative frequency of 250 is 46.5. The median mark is 46.5.

ii) The inter-quartile range is the difference between the upper and lower quartiles. Therefore, we have to find the upper and lower quartiles and find the difference between them.
The upper quartile is the mark on the horizontal axis which corresponds to $\frac{3}{4}\Sigma f$ i.e. $\frac{3}{4} \times 500 = 375$. Hence from the graph, the upper quartile = 55.5.
The lower quartile is the mark corresponding to $\frac{1}{4}\Sigma f$ i.e. $\frac{1}{4} \times 500 = 125$. Therefore the mark corresponding to 125 on the horizontal axis is 36.5

Hence from the graph, the lower quartile is 36.5.

$\therefore$ The inter-quartile range
= upper quartile – lower quartile
= 55.5 – 36.5 = 19.

b) If 5 % of the students attained distinction level. Then, $\frac{5}{100} \times 500 = 25$ students attained distinction level. Those who did not attain distinction level is $500 - 25 = 475$.
The mark corresponding to 475 is 68.5. Hence the lowest mark for distinction is 68.5.

Note: To find the distinction or pass mark, always find the number of students who *did not attain distinction* or the number of students who *did not pass* (i.e. the number of students who failed) and find the corresponding mark.

Example 4

The following table gives the distribution by age of 4570 inhabitants of a village.

Age (years)	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50
No. of people	900	700	600	550	620

Age (years)	51 – 60	61 – 70	71 – 80	81 – 90	91–100
No. of people	450	360	250	130	10

Draw a cumulative frequency diagram for the distribution and use it to estimate:

a) The median;

b) The number of inhabitants who are at least 35 years old and at most 65 years old;

c) The interquartile range;

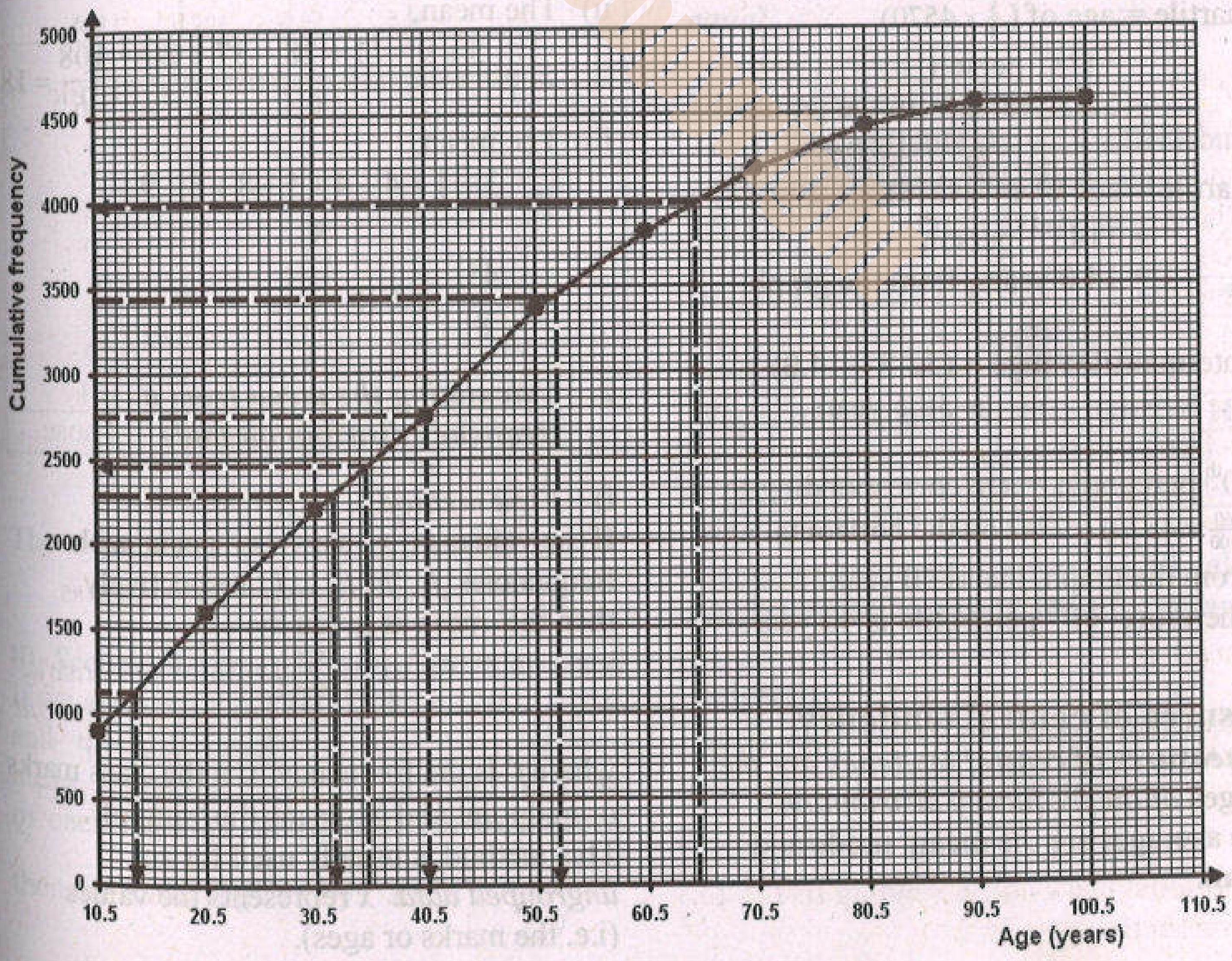
d) 60th percentile;

Solution

First construct a cumulative frequency table.

Age (years)	*Frequency*	*Age less than*	*Cumulative frequency*
1 – 10	900	10. 5	900
11 – 20	700	20 5	1600
21 – 30	600	30.5	2200
31 – 40	550	40.5	2750
41 – 50	620	50.5	3370
51 – 60	450	60.5	3820
61 – 70	360	70.5	4180
71 – 80	250	80.5	4430
81 – 90	130	90.5	4560
91 – 100	10	100.5	4570

The cumulative frequency diagram is shown on the graph below.



a) Median age

= age corresponding to $\frac{1}{2}\ \Sigma f$ or the 50th percentile

= age of ($\frac{1}{2} \times 4570$)th person

= 2285th person

= 32 years, from the graph.

b) From the graph, the number of people with age up to 35 years old = 2450 and the number of people with age up to 65 years = 3975.

∴ Number of inhabitants who are at least 35 years old and at most 65 years is:

$3975 - 2450 = 1525$.

c) Interquartile range

= upper quartile – lower quartile

= 3rd quartile – 1st quartile

3rd quartile = age of ($\frac{3}{4} \times 4570$)

= 3428th person

= 52.5 years, from the graph.

1st quartile = age of ($\frac{1}{4} \times 4570$)

= 1143rd person

= 14.0 years, from the graph.

∴ Interquartile range = (52.5 – 14.0)

= 38.5 years.

d) 60th percentile = age corresponding to ($\frac{60}{100} \times 4570$) = 2742 on the horizontal axis.

From the graph, this is 40 years.

Therefore 60th percentile is 40 years.

Measures of central tendency

The measures of central tendency *are the averages of the frequency distribution.* These averages are the **mean, mode** and **median.**

Mean

1. *The mean of a set of numbers or values*

The *mean* of a set of numbers $x_1, x_2, x_3, \ldots x_n$ is the average of the numbers. The mean is denoted by: $\bar{x}$. i.e.

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} = \frac{\sum x}{n}$$

Illustrative example

Find the mean of the following set of numbers

i) 0, 2, 3, 4, 5 and 4.

ii) 3, 9, 11, 28, 37, 20

iii) 3, 4, 4, 4, 5, 8, 8, 9

Solution

i) The mean,

$$\bar{x} = \frac{0+2+3+4+5+4}{6} = \frac{18}{6} = 3$$

ii) The mean,

$$\bar{x} = \frac{3+9+11+28+37+20}{6} = \frac{108}{6} = 18$$

iii) The mean,

$$\bar{x} = \frac{3+4+4+4+5+8+8+9}{8}$$

$$= \frac{45}{8} = 5.625$$

2. *Calculating the mean from a frequency distribution table*

i) *First method*

If the values $x_1, x_2, x_3, \ldots x_n$ occur with frequencies $f_1, f_2, f_3, \ldots f_n$ respectively, then the mean is given by:

$$\bar{x} = \frac{\sum fx}{\sum f}$$

where f is the frequency, x is the class marks or midpoint and $\sum$ means the 'sum of'. This method is usually used for an *ungrouped data.* x represents the values (i.e. the marks or ages).

However for a *grouped data, x* represents the class midpoints or mid-values of each class.

Illustrative example

In a workshop, the times taken by 120 trainees to identify a fault in a machine are shown in the table below.

Number of minutes	1	2	3	4	5	6	7	8	9	10
Number of trainees	2	3	5	10	15	30	25	15	10	5

Find the mean.

Solution

Form the table below. Note that a third column has been added for the product, *fx*.

No. of minutes (x)	No. of trainees *(f)*	*fx*
1	2	2
2	3	6
3	5	15
4	10	40
5	15	75
6	30	180
7	25	175
8	15	120
9	10	90
10	5	50
	$\sum f = 120$	$\sum fx = 753$

$$\text{The mean, } \bar{x} = \frac{\sum fx}{\sum f} = \frac{753}{120} = 6.275$$

ii) *Second method (Assumed mean method)*

If we assume or guess that the mean is say A, and the difference between the assumed mean, A and the class midpoint or mid-value of each class x_i is d_i(called the **deviation**), then the mean is given by: $\bar{x} = A + \frac{\sum fd}{\sum f}$

where $d_i = x_i - A$ and x_i = class midpoints

This method is usually used when we have a **grouped data** where the class intervals are *equal* or *unequal*. Note that for a *grouped data, x* represents the class midpoints.

iii) *Third method (Coding method)*

If the class intervals are equal, the working can be simplified further by dividing the deviations (d) by the class width (c). This is called the ***coding method.***

The mean is given by: $\bar{x} = A + c\frac{\sum fu}{\sum f}$

where $u = \frac{d}{c}$

This method can be used when we have a **grouped data** where the class intervals are ***equal.***

Illustrative example

The following table shows the distribution of weekly wages earned by some construction workers in Ghana.

Wages ¢	40-49	50-59	60-69	70-79
No. of Workers	4	12	18	11
Wages ¢	80-89	90-99	100-109	110-119
No. of Workers	7	5	2	1

Using an assumed mean of 74.50 calculate, correct to the nearest pesewa the mean.

Solution

Using an assumed mean, A = 74.50, we form the following table.

Wages ¢	No. of employees (f)	Class midpoint (x)	Deviation $d = (x - A)$	fd
40 – 49	4	44.50	−30	−120
50 – 59	12	54.50	−20	−240
60 – 69	18	64.50	−10	−180
70 – 79	11	74.50	0	0
80 – 89	7	84.50	10	70
90 – 99	5	94.50	20	100
100–109	2	104.50	30	60
110–119	1	114.50	40	40
	$\Sigma f = 60$			$\Sigma fd = -270$

Note that fd^2 column = the fd column x the d column

The mean, $\bar{x} = A + \dfrac{\sum fd}{\sum f}$, where $d = x - A$ and the symbols have their usual meanings.

From the table above,

$\bar{x} = 74.5 + \dfrac{(-270)}{60} = 74.5 - 4.5 =$ ¢70.00 to the nearest pesewa.

Alternative method

The above calculation can be simplified, since the class intervals are equal. The working can simplified by dividing the deviations by the class size, $c = 10$

Wages ¢	No. of employees (f)	Class midpoint (x)	$d=(x-A)$	$u = \dfrac{d}{c}$	fu
40 – 49	4	44.50	−30	−3	−12
50 – 59	12	54.50	−20	−2	−24
60 – 69	18	64.50	−10	−1	−18
70 – 79	11	74.50	0	0	0
80 – 89	7	84.50	10	1	7
90 – 99	5	94.50	20	2	10
100–109	2	104.50	30	3	6
110–119	1	114.50	40	4	4
	$\Sigma f = 60$				$\Sigma fu = -27$

$$\bar{x} = A + c\frac{\sum fu}{\sum f}$$

$= 74.5 + 10 \times \dfrac{(-27)}{60} = 74.5 - 4.5$

= ¢70.00 to the nearest pesewa.

Mode

1. *Mode of a set of numbers or values (raw data)*

The **mode** (or the modal value) of a set of numbers is the number which occurs most frequently.

Illustrative example

Find the mode of the following numbers

i) 2, 3, 2, 3, 4, 3, 5

ii) 3, 2, 2, 4, 3, 5, 4, 2, 4, 2, 3

Solution

i) The mode is 3, since it is the most frequent occurring number.

ii) The mode is 2, since it is the most frequent occurring number.

2. *Mode from an ungrouped frequency table*

The **mode** is the value with the largest frequency. The mode can be read directly from the frequency table. Remember that the mode is the *value*, **not the** *frequency*.

Illustrative example

This frequency table below shows the number of children in each of 30 families. What is the mode of the data?

Number of children	0	1	2	3	4	5
Number of families	3	9	11	4	2	1

Solution

The largest frequency is 11.

11 families each had 2 children.

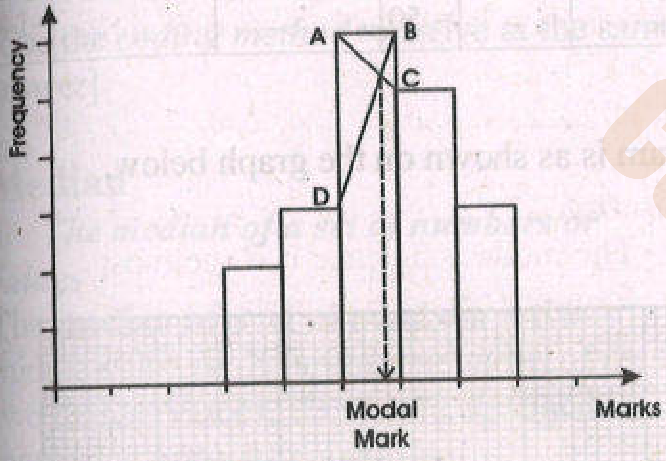
Therefore, the mode of the data is 2 children.

3. *The modal class of a grouped data*

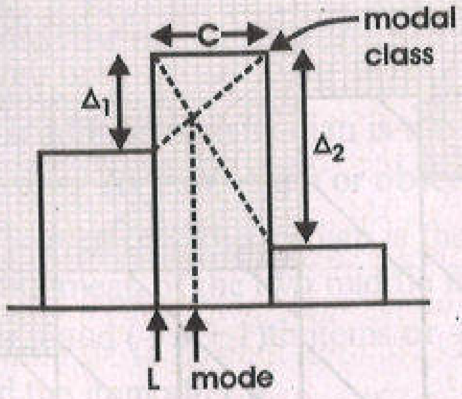
For a grouped data, the *modal class* is the class with the *largest frequency*.

4. *The mode from a histogram*

The value of the mode within the modal class of a grouped data can be found from a histogram of the distribution as shown in the diagram below. Using the highest rectangle (modal class), find the intersection of the lines AC and BD.



5. *Calculating the mode*



The value of the mode within the modal class of a grouped data can be calculated using the formula:

$$\text{Mode} = L + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)c$$

Where,

L_1 = lower *class boundary* of modal class

Δ_1 = frequency of modal class – frequency of next lower class

Δ_2 = frequency of modal class – frequency of next higher class

c = width of modal class

Illustrative examples

Example 1

Calculate the mode of the distribution below.

Class	10–14	15–19	20–24	25–29	30–34	35–40
Frequency	2	12	27	41	30	7

Solution

The modal class is '25–29' so

L = lower class boundary = 24.5,

$\Delta_1 = 41 - 27 = 14,\ \Delta_2 = 41 - 30 = 11$

and $c = 5$

$$\text{Therefore mode} = L + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)c$$

$$= 24.5 + \left(\frac{14}{14 + 11}\right)5 = 27.3$$

Example 2

The following is a record of marks obtained by a group of 50 candidates in an examination.

48	63	62	59	12
57	37	38	81	43
23	9	70	58	70
44	28	18	23	2
25	42	40	90	63
62	83	8	27	58
54	81	91	45	31
32	43	19	93	72
25	63	82	71	38
35	63	46	17	7

a) Tabulate the marks as a frequency distribution grouping in intervals of 10 marks, namely 0–9, 10–19, 20–29 etc.

b) Draw a histogram for this distribution and estimate the mode from the histogram.

c) Find i) the modal class
ii) the mean of the distribution.

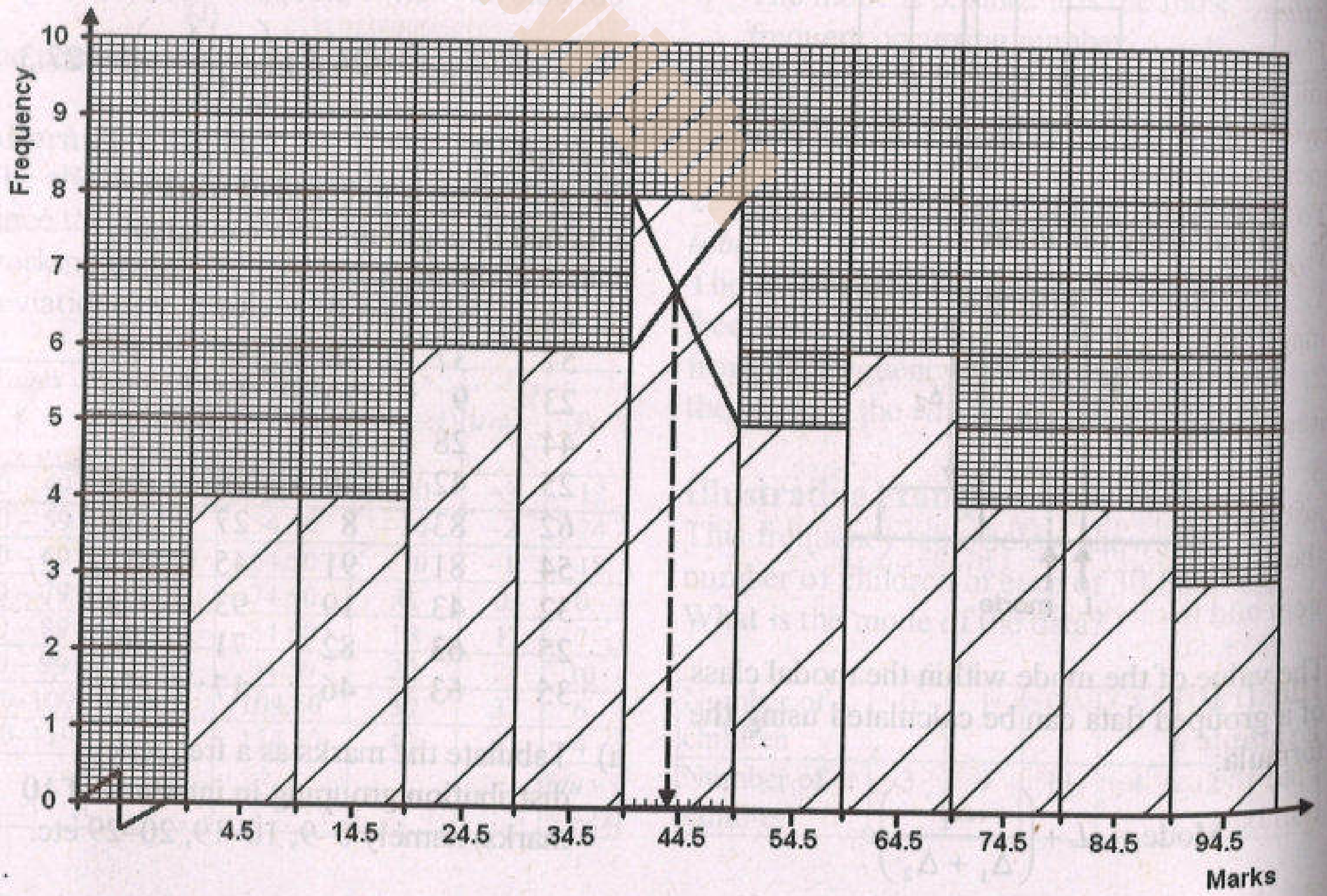
Solution

a) The table below is the frequency distribution table.

Note: When making a **tally table**, you must work carefully through the data item by item, putting a tally mark / for each item at correct place in the table. 𝍸 shows five tallies. Add the tallies up to get the frequency. Check the total frequencies. If it is not the same as the total number of items, then you need to tally again.

Marks	*Tally*	*Freq (f)*	*Class midpoint (x)*	*Deviation $d = x - A$*	*fd*
0 – 9	////	4	4.5	–40	–160
10 – 19	////	4	14.5	–30	–120
20 – 29	𝍸 /	6	24.5	–20	–120
30 – 39	𝍸 /	6	34.5	–10	–60
40 – 49	𝍸 ///	8	44.5	0	0
50 – 59	𝍸	5	54.5	10	50
60 – 69	𝍸 /	6	64.5	20	120
70 – 79	////	4	74.5	30	120
80 – 89	////	4	84.5	40	160
90 – 99	///	3	94.5	50	150
		$\sum f = 50$			$\sum fd = 140$

a) The method for finding the mode on the histogram is as shown on the graph below. The mode is 43.5.



c) i) The modal class is 40 – 49 (since it is the class with the highest frequency of 8).

ii) To find the mean for a grouped data, it is always advisable to use the assumed mean method. Let the assumed mean $A = 44.5$.

Note: The assumed mean, *A*, is conveniently chosen from the class midpoints. Usually the middle number or one of the two middle numbers (i.e. either 44.5 or 55.5).

Then the mean, $\bar{x} = A + \frac{\sum fd}{\sum f}$

$$= 44.5 + \frac{140}{50} = 44.5 + 2.8$$

$$= 47.3$$

[Use the *coding method* to arrive at the same answer]

Median

1. *The median of a set of numbers or values*

The **median** of a set of numbers is the middle value or the arithmetic mean of the two middle values, if they are arranged in increasing order of magnitude.

To find the median of a set of '*n*' numbers: $x_1, x_2, x_3, \ldots, x_n$

1. Arrange the numbers in order of magnitude
2. If *n* is odd then the median is the middle item i.e. the $\frac{1}{2}(n + 1)$th item or observation
3. If *n* is even then the median is the arithmetic mean of the two middle items i.e. the $\frac{1}{2}(n)$th and $(\frac{1}{2}n + 1)$th items or $\frac{1}{2}(n)$th item and the item after it.

Illustrative examples

Example 1

Find the median of the following set of numbers.

i) 2, 1, 2, 3, 5
ii) 3, 7, 2, 9, 8, 11, 12
iii) 3, 7, 4, 1, 2, 3, 6, 9, 8

Solution

i) Re-arrange the numbers in ascending order (starting from the smallest) i.e. 1, 2, 2, 3, 5.
$n = 5$ which is odd, therefore the median is the $\frac{1}{2}(5 + 1)$th i.e. 3rd item. Hence the median is 2.

ii) Re-arrange the numbers in ascending order i.e. 2, 3, 7, 8, 9, 11, 12
$n = 7$ which is odd, therefore the median is the $\frac{1}{2}(7 + 1)$th i.e. 4th item. Hence the median is 8.

iii) Re-arrange the numbers in ascending order i.e. 1, 2, 3, 3, 4, 6, 7, 8, 9
$n = 9$ which is odd, therefore the median is the $\frac{1}{2}(9 + 1)$th i.e. 5th item. Hence the median is 4.

Example 2

Find the median of the following numbers:

i) 2, 1, 3, 5, 4, 4
ii) 9, 3, 1, 5, 8, 5, 7, 4, 8, 4, 7, 6

Solution

i) Re-arranging the numbers in ascending order, we have 1, 2, 3, 4, 4, 5.
$n = 6$ which is even, therefore the median is the mean of $\frac{1}{2}(6)$th and $[\frac{1}{2}(6) + 1)]$th items i.e. mean of the 3rd and 4th items or observations. The 3rd and 4th items are 3 and 4 respectively.
Hence the median is $\frac{1}{2}(3 + 4) = 3\frac{1}{2}$

ii) Re-arrange the numbers in ascending order i.e. 1, 3, 4, 4, 5, 5, 6, 7, 7, 8, 8, 9

$n = 12$ which is even, therefore the median is the mean of $\frac{1}{2}(12)$th and $[\frac{1}{2}(12) + 1]$th items i.e. mean of the 6th and 7th items or observations. The 6th and 7th items are 5 and 6 respectively.

Hence the median is $\frac{1}{2}(5 + 6) = 5\frac{1}{2}$

2. *Median from an ungrouped frequency distribution table*

Step 1: Find the total frequency i.e. $\sum f$ or N.

Step 2: If N is odd, then the median is the $\frac{1}{2}(\sum f + 1)$th or the $\frac{1}{2}(N + 1)$th item or observation.

For example, if $\sum f = 29$, then the median is the $\frac{1}{2}(29 + 1)$th item or observation. i.e. the 15th item.

If n is even, then the median is the arithmetic mean of the two middle items i.e. the $\frac{1}{2}(\sum f)$th and $(\frac{1}{2}\sum f + 1)$th items or the mean of the $\frac{1}{2}(\sum f)$th item and the item after it.

For example, if $\sum f = 24$ then the median is the mean of 12th i.e. $\frac{1}{2}(24)$th item and the item after it i.e. the 13th item.

Remember that the median is the ***value*** not the ***frequency***.

Note that for a *grouped frequency distribution table*, we can find the *median class*.

Illustrative examples

Example 1

The table gives the number of accidents had by 60 drivers last year. What is the median of these data?

Number of accidents	0	1	2	3	4	5	6	7
Number of drivers	15	9	6	19	5	3	2	1

Solution

To find the median, first find the total frequency = 60, which is even.

Half the total frequency is $\frac{1}{2}(60) = 30$.

The median is the mean of the 30th and 31st observations.

Now add the frequencies, starting from 15, until you get the age corresponding to the 30th and 31st observations. From the table, the 30th observation is 2 and the 31st observation is 3.

Therefore the median age is $\frac{1}{2}(2 + 3) = 2.5$ accidents.

Example 2

The frequency table below gives the age distribution of students who offered A-Level Mathematics in a certain school.

Age(yrs)	17	18	19	20	21
No. of students	3	10	8	5	2

i) How many students offered A-Level Mathematics?

ii) What is the median age?

Solution

i) The number of students

$= 3 + 10 + 8 + 5 + 2$

$= 28$

ii) To find the median, first find the total frequency = 28, which is even.

Half the total frequency is $\frac{1}{2}(28) = 14$.

The median is the mean of the 14th and 15th observations. Now add the frequencies, starting from 3, until you get the age corresponding to the 14th and 15th observations. From the table, the 14th observation is 19 and the 15th observation is 19. Therefore the median age is

$\frac{1}{2}(19 + 19) = 19$ years.

Remember that the median is the *value* not the frequency.

Example 3
Find the median of x for the data below.

x	0	1	2	3	4
f	1	2	4	5	2

Solution
To find the median, first find the total frequency = 14 which is even.
Half the total frequency is $\frac{1}{2}(14) = 7$. The median is the mean of the 7th and 8th observations
Now add the frequencies, starting from 1, until you get the 7th and 8th observations.
From the table, the 7th observation is 2 and the 8th observation is 3.
Therefore the median age is $\frac{1}{2}(2+3) = 2.5$

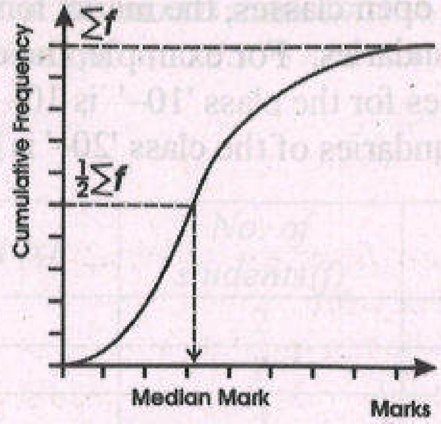
Example 4
Find the median of x for the data below

x	0	1	2	3	4
f	1	2	4	5	3

Solution
To find the median, first find the total frequency = 15 which is odd.
Therefore the median is $\frac{1}{2}(15+1)$th = 8th observation. Now add the frequencies, starting from 1, until you get the 8th observation. The 8th observation is 3.
Therefore the median is 3.

3. *Finding the median from a cumulative frequency curve*

This is used to estimate the median from a *grouped data.* From a cumulative frequency curve, the median is the value on the horizontal scale corresponding to $\frac{1}{2}\Sigma f$ (i.e. half of the total frequency). Note that it is difficult to obtain the actual value of the median of a grouped frequency distribution table since we can only find the median class. The cumulative frequency curve can therefore be used to estimate the actual value.



4. *Finding the median by calculation*

A more accurate value of the median may be found using the formula:

$$\text{median} = L + \left(\frac{\frac{1}{2}N - (\Sigma f)_L}{f_{\text{median}}}\right)c$$

where
L = lower boundary of median class
N = total frequency
$(\Sigma f)_L$ = sum of frequencies below median class
f_{median} = frequency of median class
c = width of median class

Illustrative examples

Example 1
The table below is the frequency distribution of marks obtained in a test by 200 students.

Marks %	10–	20–	30–	40–	50–	60–	70–	80–90
Frequency	18	34	58	42	24	10	6	8

a) Make a cumulative frequency table and use it to draw a cumulative frequency polygon.
b) Estimate the median from the graph and by calculation.
c) How many students will pass if the pass mark is 40?
d) If 10 % of the students had distinction, what is the lowest mark for distinction?

Solution

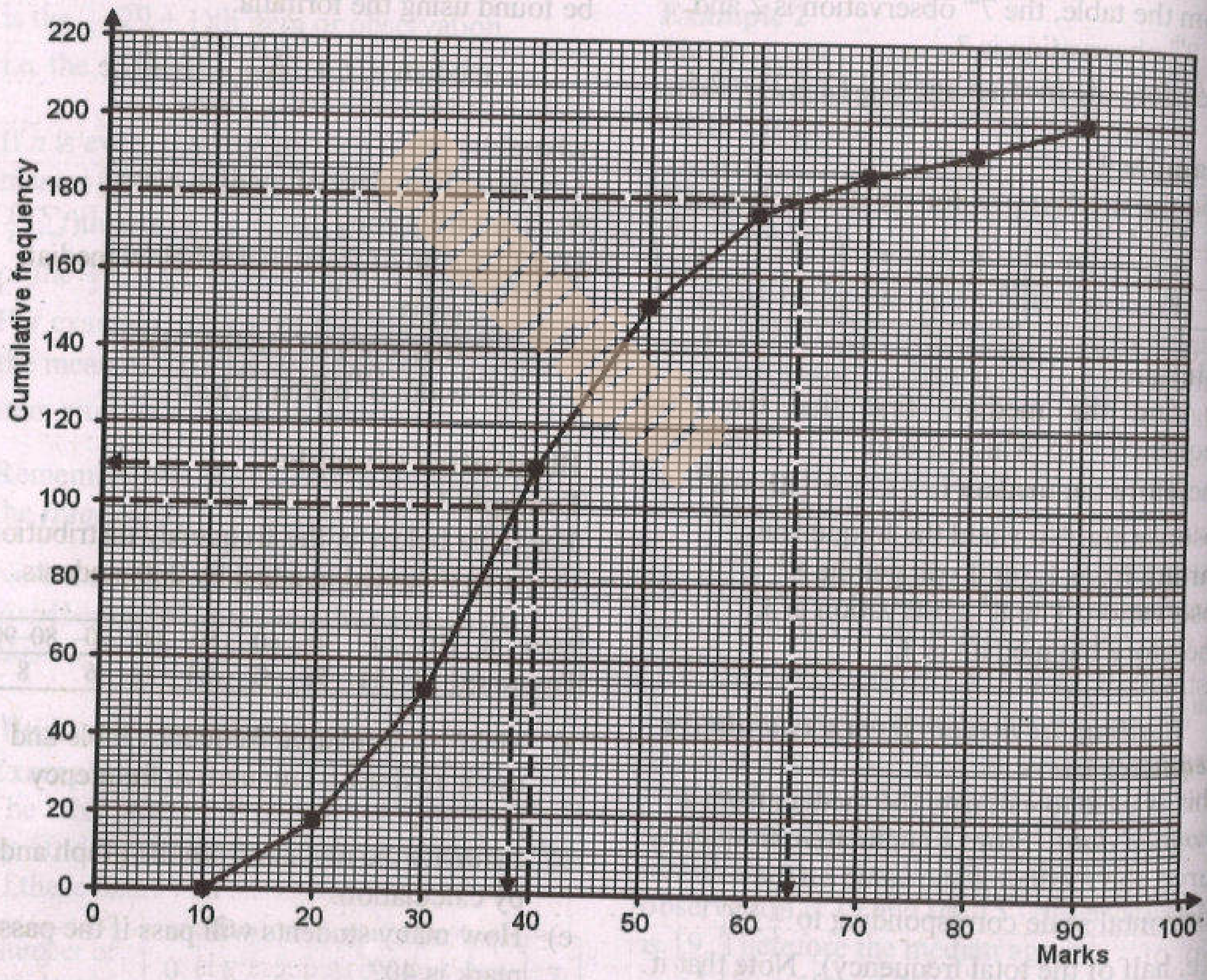
Note:

For such open classes, the marks form the class boundaries. For example, the class boundaries for the class '10–' is 10 – 20, the class boundaries of the class '20–' is 20 – 30, etc.

a) Cumulative frequency table.

Marks less than	*Cumulative frequency*
10	0
20	18
30	52
40	110
50	152
60	176
70	186
80	192
90	200

The **cumulative frequency polygon** is shown below.



b) The median is the mark corresponding to cumulative frequency $(\frac{1}{2} \times 200) = 100$. From the graph, this is 38.
Therefore the median = 38 marks

Using calculation, median

$$= L + \left(\frac{\frac{1}{2}N - \left(\sum f\right)_L}{f_{\text{median}}} \right) c$$

Where L = lower boundary of median class = 30

N = total frequency = 200

$(\sum f)_L$ = sum of frequencies up to median class = 52

f_{median} = frequency of median class = 58 (the median class is 30–)

c = width of median class = 10

$$\text{Median} = 30 + \left(\frac{\frac{1}{2}(200) - 52}{58} \right) 10 = 38.3$$

c) If the pass mark is 40, then from the graph, 40 % corresponds to 110 (i.e. 110 students failed). Therefore (200 – 110) = 90 students will pass.

d) If 10 % of students (i.e. 20 students) had distinction, then 180 students did not have distinction. From the graph, 180 on the cumulative frequency axis corresponds to 64 %. Therefore the lowest mark for distinction is 64 %.

Example 2

The table below shows the distribution of marks scored by 20 students in a test.

Mark	1	2	3	4	5	6	7	8	9	10
No. of students	2	2	3	3	4	2	2	0	1	1

i) Calculate the mean mark.
ii) Find the median mark.
iii) What is the modal mark for the distribution?

Solution

Mark (x)	*No. of students(f)*	*fx*
1	2	2
2	2	4
3	3	9
4	3	12
5	4	20
6	2	12
7	2	14
8	0	0
9	1	9
10	1	10
	$\Sigma f = 20$	$\Sigma fx = 92$

i) The mean mark, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{92}{20} = 4.6$

Note: $\Sigma fx = 92$ and $\Sigma f = 20$.

ii) The total frequency, $\Sigma f = 20$, which is even. Therefore, the median mark is the average of the $\frac{1}{2}(20) = 10^{\text{th}}$ observation and the 11th observation.
Add the frequencies, starting from 2, until you get the mark corresponding to the 10th and 11th observations. From the table, the 10th observation is 4 and the 11th observation is 5.
Therefore the median mark is $\frac{1}{2}(4 + 5) = 4.5$.

iii) The modal mark is the mark with the highest frequency. From the table, the mark with the highest frequency is 5.
Therefore the modal mark is 5.

Example 3
The table below shows the distribution of the masses of parcels in a local Post Office.

Mass(kg)	5	6	7	8	9	10
No. with this mass	3	2	11	17	6	1

i) Calculate the mean mass.
ii) Find the median mass.
iii) What is the mode of the distribution?

Solution

Mass (*x*)	*No. with this mass(f)*	*fx*
5	3	15
6	2	12
7	11	77
8	17	136
9	6	54
10	1	10
	$\Sigma f = 40$	$\Sigma fx = 304$

i) The mean mass,

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{304}{40} = 7.6\,\text{kg}$$

Note: $\Sigma fx = 304$ and $\Sigma f = 40$

ii) To find the median, first find half the total frequency $\Sigma f = 40$, which is even. Therefore, the median mark is the average of the $\frac{1}{2}(40) = 20^{\text{th}}$ observation and the 21^{st} observation. Now add the frequencies, starting from 3, until you get the mass corresponding to the 20^{th} and 21^{st} observations.

From the table, the 20^{th} observation is 8 and the 21^{st} observation is 8.
Therefore the median mass is
$\frac{1}{2}(8 + 8) = 8$ kg

iii) The mode is the mass with the highest frequency. From the table, the mass with the highest frequency is 8. Therefore the mode is 8.

Example 4
The table below shows the distribution of weekly wages of a gang of 20 labourers.

Wages (¢)	*No. of labourers*
1000	1
1100	4
1200	6
1300	4
1400	2
1500	2
1600	1

i) Calculate the mean.
ii) Find the median.
iii) What is the mode of the distribution?

Solution

Wages (¢)	*No. of labourers*	*fx*
1000	1	1000
1100	4	4400
1200	6	7200
1300	4	5200
1400	2	2800
1500	2	3000
1600	1	1600

i) The mean mark, $\bar{x} = \frac{\sum fx}{\sum f}$

$$= \frac{25200}{20} = 1260 \text{ cedis}$$

ii) To find the median, first find the total frequency $\Sigma f = 20$.
Therefore, the median mark is the average

of the $\frac{1}{2}(20) = 10^{\text{th}}$ observation and the 11th observation. Now add the frequencies, starting from 1, until you get the wage corresponding to the 10th and 11th observations. From the table, the 10th observation is 1200 and the 11th observation is 1200. Therefore the median wage is $\frac{1}{2}(1200 + 1200) =$ ¢1200.

iii) The mode is the wage with the highest frequency. From the table, the wage with the highest frequency is ¢1200. Therefore, the mode is ¢1200.

Examples 5

The frequency table below is the mark distribution of a class of 35 pupils in an oral French examination.

Marks	*Frequency*
1	2
2	4
3	5
4	6
5	10
6	1
7	2
8	3
9	1
10	1

a) Find the mean mark for the class
b) What is the median mark?
c) What is the modal mark?
d) What is the probability that a pupil chosen from the class obtained
   i) a mark less than 5?
   ii) a mark greater than or equal to 7?

Solution

a) To find the mean, form the table below.

Mark (x)	*Frequency (f)*	*fx*
1	2	2
2	4	8
3	5	15
4	6	24
5	10	50
6	1	6
7	2	14
8	3	24
9	1	9
10	1	10
	$\sum f = 35$	$\sum fx = 162$

The mean mark

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{162}{35} = 4.63 \text{ to 2dp}$$

b) From the table, $\Sigma f = 35$, which is an odd number.

Therefore, the median mark is the $\frac{1}{2}(35 + 1)$th = 18th observation.

Now add the frequencies, starting from 2, until you get the 18th observation.

The 18th observation is 5. Therefore the median is 5.

c) The modal mark is the mark with the highest frequency. From the table, the mark with the highest frequency is 5. Therefore, the mode is 5.

d) Total number of pupils = 35

i) Number of pupils who obtained a mark less than 5 = 2 + 4 + 5 + 7 = 17

$P(\text{pupil obtained marks less than } 5) = \frac{17}{35}$

ii) Number of pupils who obtained a mark greater than or equal to 7 = 2 + 3 + 1 + 1 = 7

$\therefore$ $P(\text{pupil obtained a mark greater than or equal to } 7) = \frac{7}{35}$

Example 6

In a workshop, the times taken by 120 trainees to identify a fault in a machine are shown in the table below.

Number of minutes	1	2	3	4	5	6	7	8	9	10
Number of trainees	2	3	5	10	15	30	25	15	10	5

a) Draw a frequency polygon of the distribution.

b) If a trainee is picked at random, what is the probability that the time he took to identify the fault was less than 5 minutes?

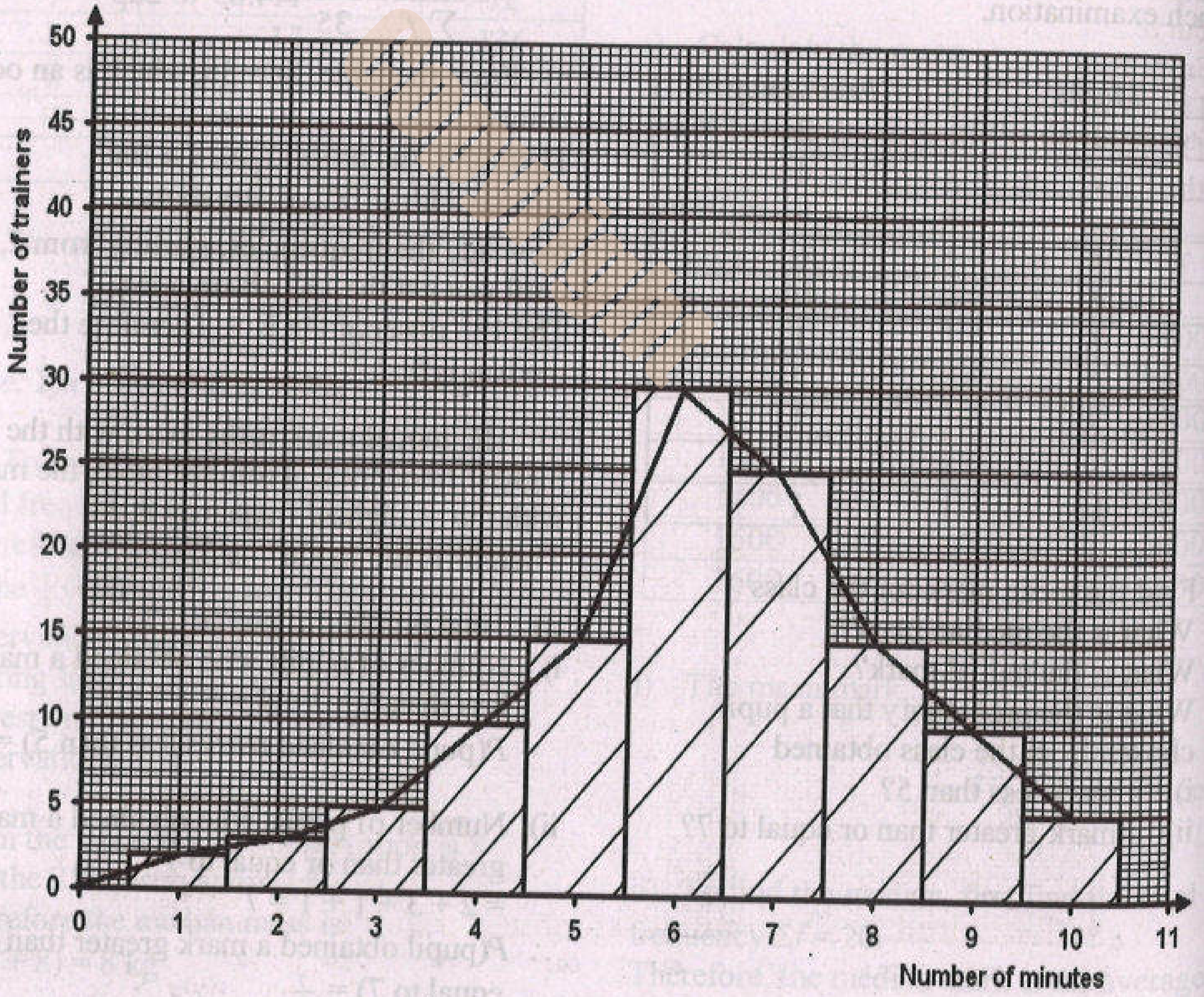
Solution

To draw the frequency polygon, first draw the histogram and join the midpoints of the tops of the rectangles by straight lines as shown below.

b) Number of trainees who took less than 5 minutes to identify the fault = 10 + 5 + 3 + 2
= 20

Total number of trainees = 120

$\therefore$ P(A trainee picked at random took less than 5 minutes to identify the fault)
$= \frac{20}{120} = \frac{1}{6}$



Example 7

The table below shows the percentage mark distribution of a number of candidates in a Physics examination.

Marks %	Frequency
0–9	2
10–19	5
20–29	11
30–39	20
40–49	34
50–59	17
60–69	6
70–79	3
80–89	1
90–99	1

a) Make a cumulative frequency table and use it to draw the cumulative frequency curve.

b) From the graph, estimate:

i) The median mark;

ii) The number of candidates that passed if the pass mark was 45.

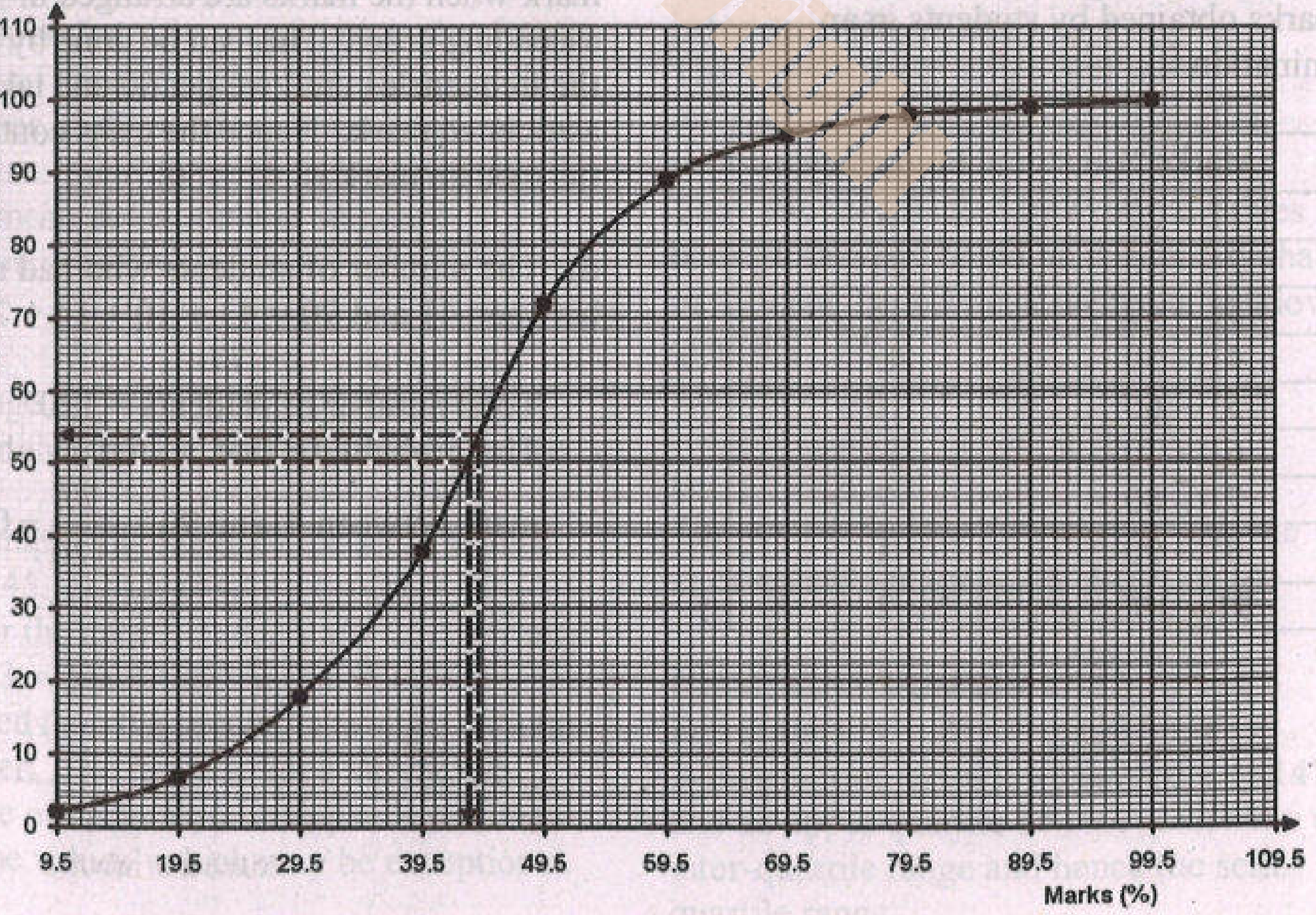
c) What is the probability that a person chosen at random from the candidates passed the examination?

Solution

a) The frequency distribution table is shown below.

Marks (%)	*Frequency (f)*	*Marks less than*	*Cumulative frequency*
0 – 9	2	9.5	2
10 – 19	5	19.5	7
20 – 29	11	29.5	18
30 – 39	20	39.5	38
40 – 49	34	49.5	72
50 – 59	17	59.5	89
60 – 69	6	69.5	95
70 – 79	3	79.5	98
80 – 89	1	89.5	99
90 – 99	1	99.5	100

The cumulative frequency curve is shown below.



b) i) To find the median mark, first find $\frac{1}{2}\Sigma f = \frac{1}{2} \times 100 = 50$ and then find its corresponding mark from the graph. Hence from the curve the median mark is 43 %.

ii) If the pass mark is 45, then from the graph, the number of students who passed $= (100 - 54) = 46$ students. Note that the mark, 45, corresponds to 54 on the cumulative frequency axis and it represents the students who failed.

c) From b(ii), the probability that a person chosen at random from the candidates who passed the examination is the number of students who passed the examination divided by the total number of students who sat for the examination. i.e. $\frac{44}{100} = 0.44$

Example 8

The table below shows the distribution of marks obtained by students in an examination.

Marks %	Frequency
11 – 20	5
21 – 30	21
31 – 40	15
41 – 50	43
51 – 60	10
61 – 70	14
71 – 80	7
81 – 90	3
91 – 100	2

a) Draw a histogram to represent the above data.

b) Use your histogram to estimate the mode.

c) Find the class that contains the median mark.

d) If a student is chosen at random, find the probability that he obtains a mark between 41and 90.

Solution

a) The graph on the next page is the histogram representing the data.

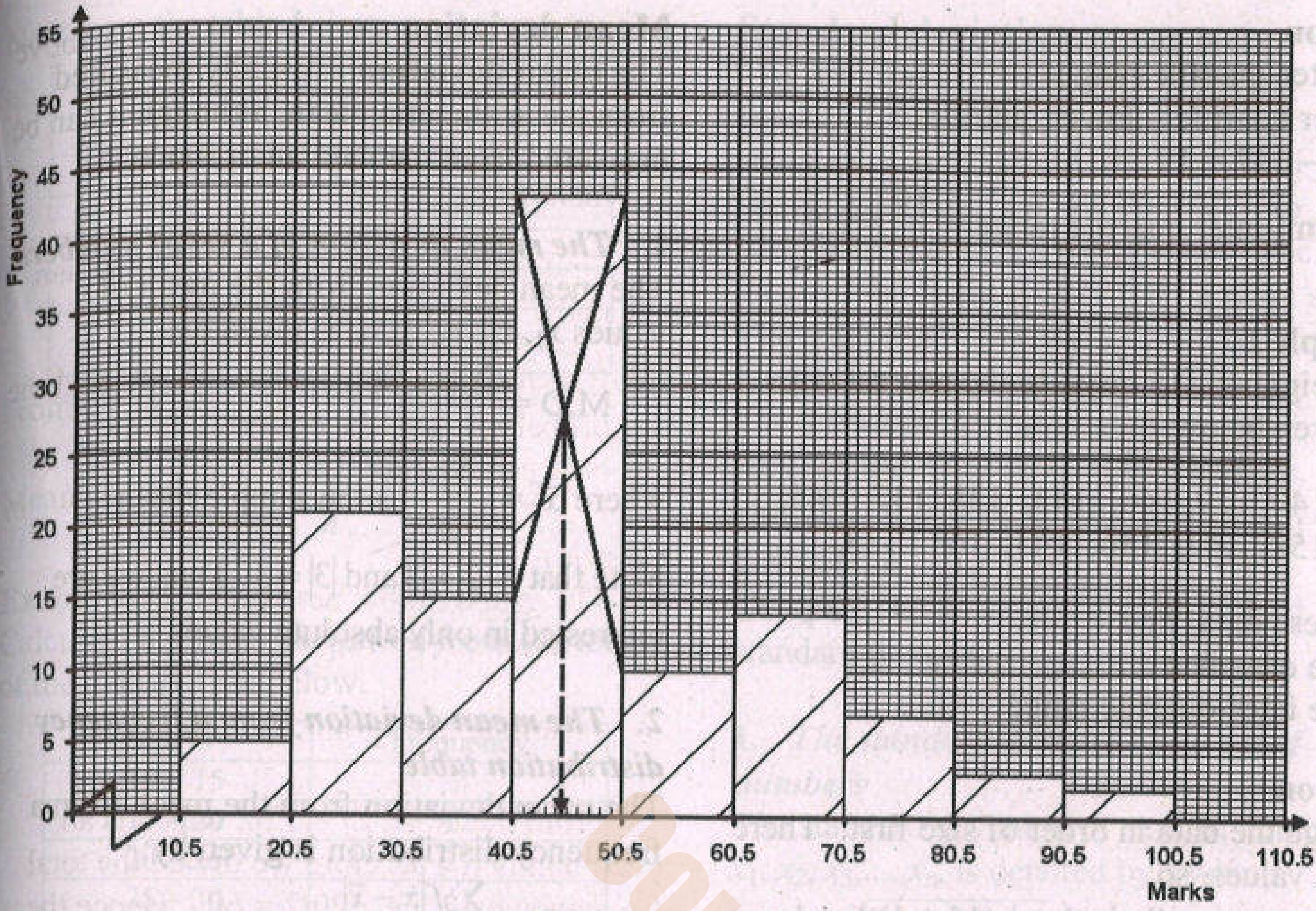
b) From the histogram, the mode is 45.

c) From the table, $\Sigma f = 120$, which is an even number. Therefore we divide the total frequency by 2. i.e. $120/2 = 60$. Hence the class containing the median mark is the 60th mark when the marks are arranged in ascending order. This can be done by adding the frequencies until we get 60 and taking the associated mark. Hence the class containing the median mark is 41 – 50.

d) The number of students who had marks between 41 and 90 $= 43 + 10 + 14 + 7 + 3$
$= 77$

Total number of students = 120

The probability that a student obtains a mark between 41and 90 $= \dfrac{77}{120} = 0.64$.



Measures of Dispersion

The **dispersion** of a set of data is the amount of *spread* of the data. The simplest **measures of dispersion** in statistics are the range, interquartile range, semi-interquartile range, mean deviation and standard deviation or variance.

Range

The range of a set of data is the *difference* between the greatest and least values in that data.

For example, the range of the set of numbers 33, 35, 45, 11, 12, is 45 – 11 = 34.

Also for the data 3, 5, 8, 11 and 25, the range is 25 – 3 = 22. Thus, the first data is more dispersed than the second data since its range is greater. The range is not a very good measure of dispersion. It depends on the two 'extreme values' which may be exceptional.

The inter–quartile range and semi-interquartile range

The inter-quartile range and *semi-interquartile range* are slightly better measures of dispersion than the range. They are not affected by extreme values because they are based on the 'middle-half' of the data, i.e. between the upper and lower quartiles.

The *inter-quartile range*

= upper quartile – lower quartile

i.e. $Q_3 - Q_1$.

The *semi-interquartile range* is half the inter-quartile range.   i.e. $\frac{1}{2}(Q_3 - Q_1)$.

Illustrative examples

Example 1

A distribution has a lower quartile of 147 and an upper quartile of 166. Calculate the inter-quartile range and hence the semi-quartile range.

Solution

The inter-quartile range
= upper quartile – lower quartile
= 166 – 147 = 19

The semi-inter quartile range = $\frac{19}{2} = 9.5$.

Example 2

The heights of 15 boys, to the nearest cm, are given below.

156, 149, 129, 140, 143, 135, 137, 145, 146, 158, 147, 136, 143, 143, 138.

For these data, find

a) the quartiles
b) the inter-quartile range

Solution

Arrange the data in order of size first. There are 15 values, so

The lower quartile is the $\frac{1}{4}(15 + 1)$th value.
i.e. the 4th value.
The median is the $\frac{1}{2}(15 + 1)$th value.
i.e. the 8th value.
The upper quartile is the $\frac{3}{4}(15 + 1)$th value.
i.e. the 12th value.

4th value = lower quartile ↗ 8th value = median ↗

129, 135, 136, **137**, 138, 140, 143, **143**, 143, 145, 146, **147**, 149, 156, 158

↓ 12th value = upper quartile

a) The first quartile or lower quartile is 137 cm
The second quartile or median is 143 cm
The third quartile or upper quartile is 147 cm

b) The inter-quartile range
= upper quartile – lower quartile
= 147 cm – 137 cm = 10 cm

Mean deviation

The **mean deviation** is usually measured from the arithmetic mean, although it can be measured from the median or mode.

1. *The mean deviation of a set of numbers*

The mean deviation from the mean of n values $x_1, x_2, x_3, \ldots, x_n$, is given by:

$$\text{M.D} = \frac{\sum|x - \bar{x}|}{n},$$

where $\bar{x} = \frac{\sum x}{n}$ is the sample mean

Note that $|-3| = 3$ and $|3| = 3$. Thus we are interested in only absolute values.

2. *The mean deviation from a frequency distribution table*

The mean deviation from the mean from a frequency distribution is given by:

$$\text{M.D} = \frac{\sum f|x - \bar{x}|}{\sum f},$$

where f = frequency and $\bar{x} = \frac{\sum fx}{\sum f}$

Note that for grouped frequency distribution, x represents the class mid-value of each class.

Illustrative examples

Example 1

Find the mean deviation of the set of numbers 11, 12, 13, 14, 15.

Solution

First find the mean,

$$\bar{x} = \frac{\sum x}{n} = \frac{11+12+13+14+15}{5}$$
$$= \frac{65}{5} = 13$$

Next form the table below.

x	$\lvert x-\bar{x}\rvert$
11	2
12	1
13	0
14	1
15	2
	$\sum\lvert x-\bar{x}\rvert = 6$

From the table, $\sum|x-\bar{x}| = 6$ and $n = 5$

$$\text{Mean deviation} = \frac{\sum|x-\bar{x}|}{n} = \frac{6}{5} = 1.2$$

Example 2

Calculate the mean deviation from the mean of the distribution below.

Class	Frequency
10 – 15	2
15 – 20	12
20 – 25	27
25 – 30	41
30 – 35	30
35 – 40	7

Solution

First calculate

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{3207.5}{119} = 26.95 \text{ (to 2 d.p.)}$$

Next form the table below.

Class interval	f	*Class midpoint* (x)	$\lvert x-\bar{x}\rvert$	$f\lvert x-\bar{x}\rvert$
10 – 15	2	12.5	14.45	28.90
15 – 20	12	17.5	9.45	113.40
20 – 25	27	22.5	4.45	120.15
25 – 30	41	27.5	0.55	22.55
30 – 35	30	32.5	5.55	166.50
35 – 40	7	37.5	10.55	73.85
	$\Sigma f =$ 119			$\sum f\lvert x-\bar{x}\rvert =$ 525.35

$$\text{Mean deviation} = \frac{\sum f|x-\bar{x}|}{\sum f} = \frac{525.35}{119}$$

$$= 4.41 \text{ (to 2 d.p.)}$$

Standard deviation

The **standard deviation** is the most widely used and important measure of dispersion in statistics. It is based on the mean and it shows you how *widely* the data are spread (dispersed) about the mean value. The *larger* its size, the more *widely* the data are spread. The standard deviation can be used to *compare* two or more sets of data.

The standard deviation is the positive square root of the variance. For example, if the variance of a data is 1.69, then the standard deviation is $\sqrt{1.69} = 1.3$.

Compare the formulas of variance and standard deviation.

1. ***The standard deviation of a set of numbers***

The standard deviation of a set of numbers $x_1, x_2, x_3, \ldots, x_n$, is denoted by s where

$$s = \sqrt{\tfrac{1}{n}\sum(x-\bar{x})^2} \quad \text{(by definition)},$$

where $\bar{x} = \dfrac{\sum x}{n}$ is the sample mean.

$$s = \sqrt{\frac{\sum x^2}{n} - \left(\bar{x}\right)^2}$$

(This is the computational form – easier for calculation).

Illustrative examples

Example 1

Find the standard deviation of the set of numbers 11, 12, 13, 14, 15.

Solution

First Method

First find the mean,

$$\bar{x} = \frac{\sum x}{n} = \frac{11+12+13+14+15}{5} = \frac{65}{5} = 13$$

Next form the table below.

x	$x-\bar{x}=x-13$	$(x-\bar{x})^2$
11	-2	4
12	-1	1
13	0	0
14	1	1
15	2	4
		$\Sigma(x-\bar{x})^2 = 10$

From the table, $\Sigma(x-\bar{x})^2 = 10$ and $n = 5$

$$\therefore\ s = \sqrt{\tfrac{1}{n}\sum(x-\bar{x})^2}$$

$$= \sqrt{\tfrac{1}{5}(10)} = \sqrt{2} = 1.41$$

Second Method

First find the mean,

$$\bar{x} = \frac{\sum x}{n} = \frac{11+12+13+14+15}{5} = \frac{65}{5} = 13$$

Next, form the table below.

x	x^2
11	121
12	144
13	169
14	196
15	225
	$\Sigma x^2 = 855$

From the table, $\Sigma x^2 = 855$ and $n = 5$

$\therefore$ The standard deviation,

$$s = \sqrt{\frac{\sum x^2}{n} - \left(\bar{x}\right)^2}$$

$$= \sqrt{\frac{855}{5} - (13)^2}$$

$$= \sqrt{171-169} = \sqrt{2} = 1.41$$

Example 2

The mean of the numbers 15, 21, 17, 18, 29 is 21. Find the variance and standard deviation of the numbers.

Solution

The mean, $\bar{x}$, has been given to be 21.

Next form the table below.

x	$x-\bar{x}=x-21$	$(x-\bar{x})^2$
15	-6	36
21	0	0
17	-4	16
26	5	25
18	-3	9
29	8	64
		$\Sigma(x-\bar{x})^2 = 150$

From the table, $\Sigma(x-\bar{x})^2 = 150$ and $n = 6$

$$\therefore\ \text{The variance, } s^2 = \frac{\sum(x-\bar{x})^2}{n} = \frac{150}{6} = 25$$

The standard deviation is the positive square root of the variance.

$\therefore$ The standard deviation, $s = \sqrt{25} = 5$.

2. ***Standard deviation from a frequency distribution table***

If the numbers $x_1, x_2, x_3 \ldots x_n$ occur with respective frequencies $f_1, f_2, f_3 \ldots f_n$ then the standard deviation is given by :

i) *First method – Ungrouped data*

$$s = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$$

where f = frequency and $\bar{x} = \dfrac{\sum fx}{\sum f}$

For an *ungrouped data, x* represents the values (i.e. the marks or ages).

Note that for grouped frequency distribution, x represents the class mid-value of each class.

ii) *Second method (Assumed mean method) – Grouped data*

If we assume or guess that the mean is say A, and the difference between the assumed mean, A and the class midpoint or marks of each class x_i is d_i (called the deviation), then the mean is given by: $\bar{x} = A + \frac{\sum fd}{\sum f}$ and the standard deviation is given by:

$$s = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2}$$

where $d_i = x_i - A$ and x_i = class midpoints.

iii) *Third method (Coding method) – Grouped data*

If the class intervals are equal, the working can be simplified further by dividing the deviations by the class width (c). This is called the ***coding method***. The mean is given by:

$\bar{x} = A + c\frac{\sum fu}{\sum f}$ and the standard deviation is given by: $s = c\sqrt{\frac{\sum fu^2}{\sum f} - \left(\frac{\sum fu}{\sum f}\right)^2}$

where $u = \frac{d}{c} = \frac{x - A}{c}$

Variance

The **variance** is the square of the standard deviation. For example, if the standard deviation of a data is 1.2, then the variance is $1.2^2 = 1.44$. Thus to find the variance, we use the formulae above for standard deviation but without the square root sign.

i.e. $s^2 = \frac{\sum x^2}{n} - (\bar{x})^2$ or

$$s^2 = \frac{\sum fx^2}{\sum f} - \bar{x}^2 \text{ or}$$

$$s^2 = \frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2$$

where the symbols denote their usual meanings.

Properties of mean and variance

1. If the mean of a set of numbers x_1, x_2, x_3 and x_4 is p then the mean of the numbers $q + x_1$, $q + x_2$, $q + x_3$ and $q + x_4$ is given by $(q + p)$.
2. If the variance of a set of numbers x_1, x_2, x_3 and x_4 is p then the variance of the numbers qx_1, qx_2, qx_3 and qx_4 is given by $(q^2 p)$.

Illustrative examples

Example 1

Find the standard deviation for the data below.

x	1	2	3	4	5	6	7	8	9	10
f	2	3	5	10	15	30	25	15	10	5

Solution

To find the standard deviation, form the table below.

x	f	fx	fx^2
1	2	2	2
2	3	6	12
3	5	15	45
4	10	40	160
5	15	75	375
6	30	180	1080
7	25	175	1225
8	15	120	960
9	10	90	810
10	5	50	500
	$\Sigma f = 120$	$\Sigma fx = 753$	$\Sigma fx^2 = 5169$

The standard deviation,

$$s = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$$

where $\bar{x} = \frac{\sum fx}{\sum f} = \frac{753}{120} = 6.275$

$$\therefore \quad s = \sqrt{\frac{5169}{120} - (6.275)^2} = \sqrt{3.7} = 1.92$$

Note: fx^2 = the fx column × the x column.

Example 2

Calculate the standard deviation of the frequency distribution below.

Age (yrs)	6	7	8	9	10	11
Frequency	4	6	10	11	8	1

Solution

To find the standard deviation, form the table below.

Age(yrs) x	Frequency (f)	fx	fx^2
6	4	24	144
7	6	42	294
8	10	80	640
9	11	99	891
10	8	80	800
11	1	11	121
	$\Sigma f = 40$	$\Sigma fx = 436$	$\Sigma fx^2 = 2890$

The standard deviation,

$$s = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$$

$$\text{where } \bar{x} = \frac{\sum fx}{\sum f} = \frac{436}{40} = 8.4$$

$$\therefore \quad s = \sqrt{\frac{2890}{40} - (8.4)^2}$$

$$= \sqrt{1.69} = 1.3 \text{ years}$$

Example 3

The table below gives the distribution of heights of 100 youths at an interview.

Heights in cm	Frequency
156 – 160	16
161 – 163	14
164 – 166	18
167 – 169	21
170 – 172	13
173 – 175	10
176 – 180	8

Using an assumed mean of 168 cm, calculate correct to one decimal place;

i) the mean height;

ii) the standard deviation of the distribution.

Solution

Using an assumed mean (A) of 168, we form the table below.

Height in (cm)	*Freq f*	*Class mid point (x)*	*Deviation* $d = (x - A)$	fd	fd^2
156-160	16	158	–10	–160	1600
161-163	14	162	–6	–84	504
164-166	18	165	–3	–54	162
167-169	21	168	0	0	0
170-172	13	171	3	39	117
173-175	10	174	6	60	369
176-180	8	178	10	80	800
	$\Sigma f =$ 100			$\Sigma fd =$ –119	$\Sigma fd^2 =$ 3543

Note that fd^2 = the fd column × the d column.

i) The mean, $\bar{x} = A + \dfrac{\sum fd}{\sum f}$,

where $d = x - A$.

From the table above,

$$\bar{x} = 168 + \frac{(-119)}{100}$$

$$= 168 - 1.19$$

$$= 166.8 \text{ to 1 d.p.}$$

ii) The standard deviation,

$$s = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2}$$

From the table, $s=\sqrt{\frac{3543}{100}-\left(\frac{-119}{100}\right)^2}$

$$=\sqrt{35.43-1.4161}$$

$$=\sqrt{34.0139}$$

$$= 5.8 \text{ to 1 d.p.}$$

Example 4

The following table gives the distribution of marks of 100 candidates in a statistics examination.

Mark Range	Frequency
1 – 20	9
21 – 30	15
31 – 40	17
41 – 50	22
51 – 60	21
61 – 70	8
71 – 85	8

Calculate the mean mark, $\bar{x}$ and the standard deviation, s giving your answer to 2 decimal places.

Solution

Using an assumed mean, $A = 45.5$, we form the table below.

Mark	*f*	*Class Mid point x*	*Deviation* $d=x-A$	*fd*	fd^2
1 – 20	9	10.5	–35	–315	11025
21 – 30	15	25.5	–20	–300	6000
31 – 40	17	35.5	–10	–170	1700
41 – 50	22	45.5	0	0	0
51 – 60	21	55.5	10	210	2100
61 – 70	8	65.5	20	160	3200
71 – 85	8	78.0	32.5	260	8450
	$\Sigma f =$ 100			$\Sigma fd =$ –155	$\Sigma fd^2 =$ 32475

Note that fd^2 = the fd column × the d column.

The mean, $\bar{x}=A+\frac{\sum fd}{\sum f}$ where $d=x-A$ and the symbols have their usual meanings.

From the table above,

$$\bar{x}=45.5+\frac{(-155)}{100}=45.5-1.55$$

$$=43.95$$

The standard deviation,

$$s=\sqrt{\frac{\sum fd^2}{\sum f}-\left(\frac{\sum fd}{\sum f}\right)^2}$$

From the table above,

$$s=\sqrt{\frac{32475}{100}-\left(\frac{-155}{100}\right)^2}$$

$$=\sqrt{324.75-(-1.55)^2}$$

$$=\sqrt{322.35}=17.78$$

Example 5

The following table shows the distribution of weekly wages earned by some construction workers in Ghana.

Wages ¢	No. of Workers
40 – 49	4
50 – 59	12
60 – 69	18
70 – 79	11
80 – 89	7
90 – 99	5
100 – 109	2
110 – 119	1

Using an assumed mean of 74.50 calculate, correct to the nearest pesewa the standard deviation.

Solution

Using an assumed mean, $A = 74.5$, we form the table below.

Wages ¢	(*f*)	*Class mid point* (*x*)	$d = x - A$	$u = \frac{d}{c}$	*fu*	fu^2
40 – 49	4	44.5	–30	–3	–12	36
50 – 59	12	54.5	–20	–2	–24	48
60 – 69	18	64.5	–10	–1	–18	18
70 – 79	11	74.5	0	0	0	0
80 – 89	7	84.5	10	1	7	7
90 – 99	5	94.5	20	2	10	20
100 – 109	2	104.5	30	3	6	18
110 – 119	1	114.5	40	4	4	16
	$\Sigma f = 60$				$\Sigma fu = -27$	$\Sigma fu^2 = 163$

The working has been simplified by dividing the deviations by the class size, $c = 10$.

This is the ***coding method***.

The standard deviation,

$$s = c\sqrt{\frac{\sum fu^2}{\sum f} - \left(\frac{\sum fu}{\sum f}\right)^2}$$

$$= 10\sqrt{\frac{163}{60} - \left(\frac{-27}{60}\right)^2}$$

$$= 10\sqrt{2.717 - 0.2025}$$

$$= 10\sqrt{2.514} = 10 \times 1.5857 = 15.857$$

$= ¢15.86$ to the nearest pesewa.

Example 6

The table shows the distribution of the ages in years of a group of participants at a seminar.

Ages in years	No. of participants
18 – 22	5
23 – 27	8
28 – 32	19
33 – 37	25
38 – 42	23
43 – 47	16
48 – 52	4

a) Using an assumed mean of 35, calculate, for the distribution,

i) the mean;

ii) the standard deviation.

Solution

a)

x	*f*	$d = x - 35$	d^2	*fd*	fd^2
20	5	–15	225	–75	1125
25	8	–10	100	–80	800
30	19	– 5	25	–95	475
35	25	0	0	0	0
40	23	5	25	115	575
45	16	10	100	160	1600
50	4	15	225	60	900
	100			85	5475

i) Mean $= A + \frac{\sum fd}{\sum f}$

$$= 35 + \frac{85}{100}$$

$$= 35 + 0.85$$

$$= 35.85$$

ii) Standard deviation $= \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2}$

$$= \sqrt{\frac{5475}{100} - \left(\frac{85}{100}\right)^2}$$

$$= \sqrt{54.75 - 0.7225}$$

$$= \sqrt{54.0275} = 7.35$$

Using the standard deviation to compare two or more sets of data

Illustrative example

The heights, in centimetres, of fifteen girls chosen at random from a class were:

129	135	136	137	138
140	143	143	143	145
146	147	149	156	158

a) Calculate the mean and standard deviation of these heights.

b) The mean and standard deviation of heights of a random sample of fifteen boys taken from the same class were found to be 143 cm and 2.68 respectively. Comment on the difference.

Solution

a) First find the mean,

$$\bar{x} = \frac{\sum x}{n} = \frac{2145}{15} = 143$$

Next, form the table.

x	$x - \bar{x} = x - 143$	$(x - \bar{x})^2$
129	−14	196
135	−8	64
136	−7	49
137	-6	36
138	−5	25
140	−3	9
143	0	0
143	0	0
143	0	0
145	2	4
146	3	9
147	4	16
149	6	36
156	13	169
158	15	225
		$(x - \bar{x})^2 = 838$

From the table,

$\Sigma(x - \bar{x})^2 = 10$ and $n = 5$

$$\therefore \ s = \sqrt{\tfrac{1}{n}\sum(x - \bar{x})^2}$$

$$s = \sqrt{\tfrac{1}{15}(838)} = \sqrt{55.87} = 7.47$$

Therefore the mean height is 143 cm

The standard deviation is 7.47 cm

b) Although the mean height of the boys is the same as that of the girls, there is less variability in the boys' heights than in the girls'. The boys' heights are more closely grouped around the mean.

Review exercise 11

1. The ages (in years) of a sample of 10 children are 2, 3, 3, 4, 5, 5, 5, 6, 7, 8.

a) Find the range of their ages.

b) Calculate the standard deviation of the ages, correct to two decimal places.

2. Twenty students score the following marks in a history test:

5, 2, 5, 3, 1, 6, 2, 2, 3, 4, 2, 1, 2, 2, 4, 3, 2, 2, 2, 3.

a) What is the mode of the distribution?

b) What is the median?

c) Find the mean.

3. The standard deviation of a distribution is 0.6, what is the variance?

4. Find the interquartile range of the following set of marks

3, 3, 1, 2, 4, 6, 2, 2, 4, 3.

5. The table below shows the distribution of marks in a Mathematics test.

Marks	1–20	21–40	41–60	61–80	81–100
Frequency	8	14	18	48	12

i) What is the median class?
ii) What is the class size of the distribution?
iii) State the modal class.
iv) Find the mean of the distribution.

6. The table below shows the distribution of marks in a Mathematics test.

Marks	10–19	20–29	30-39	40–49	50–59	60–69
Frequency	4	6	12	4	8	6

i) What is the lower class boundary of the modal class?
ii) What is the class size of the distribution?
iii) If a student is chosen at random, what is the probability that he had a mark of 50 and above?

7. The table below shows the distribution of goals scored by a football team.

Marks	1	2	3	4	5	6
Frequency	4	2	1	2	1	2

i) Calculate the mean number of goals.
ii) Calculate the standard deviation of the distribution.

8. Calculate
i) the mean
ii) the median
iii) the mode of the following set of numbers
a) 3, 4, 5, 8, 8, 4, 4, 9, 9
b) 10, 19, 17, 14, 12, 19, 18, 14, 16, 17, 19, 11
c) 4, 7, 16, 13, 2, 7, 14, 7, 20

9. The number of trains per hour through Konongo station for a period of 24 hours were as follows.

3 3 2 1 2 5 7 4 4 7 3 2
2 5 3 5 6 2 5 1 3 2 5 6

a) Construct a frequency distribution table for the data.
b) Find
i) the mean
ii) the mode
iii) the median of the distribution

10. Find the standard deviation of the following set of numbers
i) 12, 15, 19, 22, 25, 27
ii) 5, 8, 10, 11, 12, 14

11. Find the median of the following set of numbers
i) 41, 26, 27, 64, 72, 65, 85, 20, 41
ii) 32, 72, 99, 44, 57, 71

12. The following data shows the marks obtained in Chemistry by students in a class.

3 7 9 2 4 7 5 4 5 7 4 6 8 7 3
8 9 7 2 3 6 5 8 3 2 5 5 5 6 3

a) Construct a frequency distribution table for the data.
b) Find
i) the mean
ii) the mode
iii) the median of the distribution

13. If the mean of the data below is 3.8, find *m*

x	1	2	3	4	5	6
f	1	4	7	m	6	3

14. The mean of eight numbers 4, 6, 4, 7, x, $x + 1$, 8 and 2 is 5. Find x.

15. The table below gives the distribution of the annual wages (¢) of 100 junior workers in a small-scale industrial company.

Annual wages	Frequency
900 – 980	10
1000 – 1080	15
1100 – 1180	24
1200 – 1280	30
1300 – 1380	18
1400 – 1480	3

Using an assumed mean of 1,240.00, or otherwise, calculate, to the nearest cedi

i) the mean

ii) the standard deviation of the distribution.

16. The table below gives the distribution of marks of one hundred candidates in a Mathematics examination.

Mark Ranges	Frequency
1 – 20	10
21 – 30	14
31 – 40	17
41 – 50	22
51 – 60	19
61 – 70	9
71 – 85	9

a) Draw a histogram of the distribution

b) Use your histogram to estimate

i) the mode

ii) the median and lower quartile.

17. The monthly salaries in cedis, to the nearest ¢10, of a group of men are 540, 580, 620, 700, 830, 910, 620, 700, 830, 100 and 1200.

i) Find the range of the data.

ii) Using an assumed mean of ¢830 or otherwise, calculate correct to the nearest cedi, the mean and the standard deviation of the salaries.

18. The following table gives the marks scored by 278 candidates in an examination.

Marks	Number of candidates
1 – 10	2
11– 20	6
21 – 30	30
31 – 40	58
41 -50	81
51– 60	65
61 – 70	28
71 - 80	8

a) State the modal class of the distribution.

b) Construct a cumulative frequency table and draw a cumulative frequency curve.

c) Use your curve to estimate

i) the median of the distribution

ii) the 40th percentile.

d) What is the probability that a candidate chosen at random scored a mark of a least 60.5%

19. The following is the record of marks of 40 candidates in examination.

65 84 91 58 43 86
73 33 76 80 57 33
53 29 40 27 72 19
51 67 37 4 18 92
13 45 61 39 23 22
22 41 27 51 63 47
19 35 39 72

a) Using class intervals 11–20, 21–30, 31–40..., prepare for the distribution

i) a frequency table

ii) a cumulative frequency table

b) Draw a cumulative frequency curve and use it to find

i) the median

ii) the lower quartile

c) Draw a histogram and use it to estimate the mode.

20. The marks scored by 30 students in a particular subject are follows:

39	31	50	18	51	63
10	34	42	89	73	11
33	31	41	25	76	13
26	23	29	30	51	91
37	64	19	86	9	20

a) Prepare a frequency table, using class-intervals 1–20, 21–40.

b) Use the table to draw a histogram.

c) Estimate the mode from the histogram.

d) Calculate the mean mark.

21. The distribution of marks scored in a test by a group of 30 students is shown in a table below.

Marks	No. of students
10 – 14	1
15 – 19	3
20 – 24	4
25 – 29	6
30 – 34	7
35 – 39	4
40 – 44	3
45 – 49	2

Draw a histogram to represent the data

a) Using your histogram, estimate the mode of the distribution.

b) Calculate:

i) the mean of the distribution;

ii) the probability that a student selected at random scored at least 35 marks.

22. The following data shows marks obtained by students in a Mathematics test.

6	9	5	9	5	3	7	5	5	6
2	7	10	2	9	8	0	6	2	6
6	6	5	6	9	7	7	4	1	8

a) Construct a frequency distribution table for the data.

b) i) State the mode of the distribution.

ii) If a student is chosen at random, what is the probability that he had more than 5 marks?

c) Calculate the mean for the distribution.

23. The table below shows the distribution of marks in a test.

Marks	0	1	2	3	4	5
Frequency	1	4	8	10	9	2

i) Find the median of the distribution.

ii) Find the mode of the distribution?

iii) What is the mean of the distribution?

24. The following table shows the distribution of the systolic blood pressures, in millimetres of mercury of 200 industrial workers.

Blood pressure	Number of workers
90-	1
100-	6
110-	38
120-	68
130-	55
140-	20
150-	10
160-170	2

90 – means – 90 and less than 100 and so on…

i) Construct a cumulative frequency table and draw a cumulative frequency curve for the distribution.

ii) Use the curve in (i) to estimate
   a) the median
   b) the lower and the upper quartile

iii) Find also the inter-quartile range and explain briefly its significance.

25. The table below shows the marks scored by 100 candidates in an examination.

Marks	Frequency
1 – 10	4
11 – 20	6
21 – 30	10
31 – 40	20
41 – 50	30
51 – 60	12
61 – 70	8
71 – 80	6
81 – 90	3
91 – 100	1

a) Make a cumulative frequency table and use it to draw the cumulative frequency curve.

b) From the graph, estimate:
   i) The median mark;
   ii) The number of candidates who obtained marks between $\bar{x} - s$ and $\bar{x} + s$, if $\bar{x} = 44.5\%$ and $s = 18.6\%$;
   iii) The interquartile range.

26. The following table shows the distribution of the masses of parcels in a local Post Office.

Mass kg	Frequency
20 – 24	2
25 – 29	3
30 – 34	7
35 – 39	26
40 – 44	29
45 – 49	25
50 – 54	6
55 – 59	2

Using an assumed mean of 42 kg or otherwise, calculate:

i) the mean

ii) the standard deviation of the distribution

27. The table below shows the percentage marks scored by some candidates in an examination.

Marks %	Frequency
0–9	7
10–19	11
20–29	17
30–39	20
40–49	29
50–59	34
60–69	30
70–79	25
80–89	21
90–99	6

a) Construct a cumulative frequency table and use it to draw the cumulative frequency curve.

b) From the graph, estimate:
   i) The median of the distribution;
   ii) The lowest mark for a distinction if 5 % of the candidates pass with distinction;
   iii) The probability of selecting a candidate who scored between 45 % and 58 %.

28. The table gives the distribution of the ages of patients attending a clinic in a day

Age (years)	No. of patients
0–4	7
5–9	11
10–14	17
15–19	20
20–24	29
25–29	34
30–34	30
35–39	25
40–44	21
45–49	6

a) Draw a histogram for the data

b) Calculate, using the assumed mean method, the mean age of the patients

c) If a patient is selected at random, what is the probability that he or she is not less than 25 years?

29. The following table gives the distribution of the number of clips in 250 packets each of one kilogramme weight.

No. of Clips per packet	Frequency
182	6
183	10
184	17
185	22
186	27
187	38
188	48
189	32
190	21
191	16
192	13

a) Calculate the mean and the standard deviation for this distribution, giving the mean to the nearest integer and the standard deviation to one decimal place.

b) Draw a histogram for the distribution.

c) Using the histogram, calculate the percentage of packets for which the number of clips per packet lies between two standard deviations of the mean.

30. The table below gives the distribution of ages of 121 students in an institution.

Age (years)	No. of students
16 – 18	12
19 – 21	36
22 – 24	40
25 – 27	24
28 – 30	9

i) Draw a histogram for the distribution.

ii) Use your histogram to estimate the mode to the nearest whole number.

31. In an experiment to determine a physical constant, an observer made 210 observations and obtained the following results.

Value of constants	*Frequency*
138	12
139	13
140	15
141	24
142	28
143	33
144	27
145	22
146	13
147	12
148	11

a) Draw a histogram for the distribution and use it to estimate the mode of the distribution, giving your answer to two decimal places.

b) On a separate diagram, draw a cumulative frequency diagram and use it to estimate the median of the distribution giving your answer to two decimal places.

c) Calculate the mean and the standard deviation giving your answer to two decimal places.

32. In an objective test marked out of 40, the marks scored by 35 students is given in the table below:

Marks scored	No. of students
1 – 5	2
6 – 10	7
11 – 20	12
21 – 30	8
31 – 35	5
36 – 40	1

Draw a histogram for the distribution, and use it to calculate the median mark.

33. The mean of the numbers x_1, x_2, x_3 and x_4 is 6.4. Find the mean of the numbers $10 + x_1$, $10 + x_2$, $10 + x_3$ and $10 + x_4$.

34. The distribution of the lifespan of 40 batteries is shown in the table below.

Battery lifespan (days)	*Frequency*
26 – 30	4
31 – 35	7
36 – 40	13
41 – 45	8
46 – 50	6
50 – 55	2

Using an assumed mean of 43 days, calculate, correct to two significant figures, the

a) mean of the distribution.

b) variance of the distribution.

CHAPTER 12

PROBABILITY

Introduction

The words probable, likely, possibly and chance are common in our everyday conversation. For example, we say "Most probably I will get grade A". "Probably it will rain today". "It is possible that price of fuel will rise". "The chances of Ghana winning the African cup are more". In each of the above statements there is a degree of uncertainty. This degree of uncertainty can be measured numerically with the help of "Probability".

The theory of probability has its origin from the game of chance, for instance, tossing of coins, throwing of dice, drawing a card from a pack of cards, lottery etc.

Today probability is an important branch of mathematic with applications in statistics, economics, Industry, Science and Technology.

Definition of terms

Let us introduce certain terms used in the theory of probability.

Experiment

An *experiment* is any operation whose outcome cannot be predicted with certainty. The following are examples of an experiment.

1. Tossing a coin
2. Throwing of a die
3. Drawing a card from a pack of cards

In all the above examples of experiment, the outcome is not certain.

Trial

A *trial* of an experiment is a single performance of the experiment.

Outcome

The result of an experiment is called outcome.

Equally likely outcomes

In an experiment, the outcomes which have equal chance of occurring are called equally likely outcomes. For example, if we toss a coin, outcomes *Head* or *Tail* are equally likely. Also, if we throw a die, the outcomes, 1, 2, 3, 4, 5, 6 are equally likely.

Random sampling

This is choosing a sample from a population without being bias.

Sample Space

The *sample space* of an experiment is the set of all possible outcomes of a trial of the experiment. It is denoted by the letter S.

Event

An *event* is a collection of sample points with a common property. It is a subset of the sample space. For example, when a die is cast, the event E of throwing an even number is $E = \{2, 4, 6\}$. Also when two coins are tossed, the event of getting exactly one head is $E = \{HT, TH\}$.

Types of events

1. Certain events

An event which contains all the sample space is a certain event.

2. Impossible event

An event which does not contain any sample point is an impossible event.

3. Complement of an event

Let S be the sample space and A be an event of S. The set of all those sample points which are in S but not in A is called the complement of the event A and is denoted by $\overline{A}$. See fig. 23.1. Note that the complement of a certain event is an impossible event.

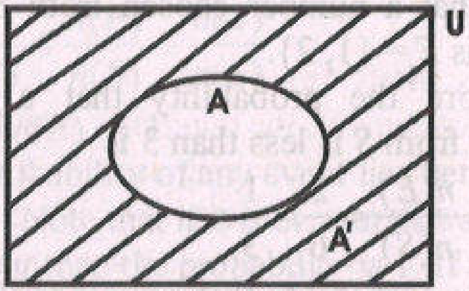


Fig. 23.1

4. Compound events

Events can be combined by the words **'or'** and **'and'**. Events which are thus combined are called compound events. The words 'or' and 'and' correspond to '$\cup$' and '$\cap$' respectively in sets. Let A and B be two events. The compound events:

1. $A \cup B$ (i.e. A or B) means either event A or event B occur or both events occur.
2. $A \cap B$ (i.e. A and B) means the events A **and** B occur together.

Note that **"or"** in probability means *addition*, **"and"** means *multiplication*.

5. Mutually exclusive events

If A and B are events such that the two events cannot occur together or at the same time, then we say that the events A and B are **mutually exclusive** events. Thus if $A \cap B = \varnothing$, then events A and B are mutually exclusive (fig 23.2). For example, when you toss a coin, you can get either a head or a tail, not both at the same time. So the events of *getting a head* and *getting a tail* are mutually exclusive.

In this case, A and B are disjoint sets.

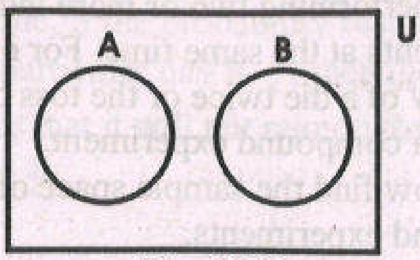


Fig. 23.2

6. Independent events

Two events A and B, are said to be **independent** if the occurrence of A does not affect the occurrence of B and vice versa. For example, if a fair die is thrown twice; let the event A = throwing a six on the first throw and B = throwing a six on the second throw. The events A and B have no influence over each other and are therefore said to be independent.

Sample space for simple experiments

Simple experiments involve performing an experiment once. For example, throwing a die once or tossing a coin once is a simple event. Let us now find the sample space of some simple experiments.

1. When we toss a fair coin once, either a *Head* (H) or a *Tail* (T) will turn up. Therefore the set of possible outcomes (sample space), $S = \{H, T\}$
2. When we throw an unbiased die, the outcomes, 1 or 2 or 3 or 4 or 5 or 6 are equally likely. Therefore the set of possible outcomes, $S = \{1, 2, 3, 4, 5, 6\}$.
3. The sample space for drawing a card from a pack of cards bearing numbers 1 to 52 is $S = \{1, 2, 3, 4, 5 \ldots 51, 52)$ Think about other simple experiments and write their sample space.

Sample space for compound experiments

Compound experiments involve the performance of the same experiment more than once or performing two or more independent experiments at the same time. For example, the throw of a die twice or the toss of a coin twice is a compound experiment.
Let us now find the sample space of some compound experiments.

1. When two fair coins are tossed
   $S = \{HH, HT, TH, TT\}$
2. When three fair coins are tossed
   $S = \{HHH, HHT, HTH, HTT, THH,$
   $T,HT\ T,TH\ TTT\}$
3. When two unbiased dice are thrown

$$S = \begin{array}{llllll} 1,1 & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 \\ 2,1 & 2,2 & 2,3 & 2,4 & 2,5 & 2,6 \\ 3,1 & 3,2 & 3,3 & 3,4 & 3,5 & 3,6 \\ 4,1 & 4,2 & 4,3 & 4,4 & 4,5 & 4,6 \\ 5,1 & 5,2 & 5,3 & 5,4 & 5,5 & 5,6 \\ 6,1 & 6,2 & 6,3 & 6,4 & 6,5 & 6,6 \end{array}$$

Note that the sample space for a die thrown twice is the same as two dice thrown simultaneously.

4. When a fair coin and an unbiased die are thrown.

$$S = \begin{array}{cccccc} H1 & H2 & H3 & H4 & H5 & H6 \\ T1 & T2 & T3 & T4 & T5 & T6 \end{array}$$

Definition of probability

As has been mentioned earlier, probability is the study of events which may or may not occur. The numerical value which is assigned to the likelihood of an event occurring is called the probability of the event.
If all the simple events in the sample space, S are equally likely, then the probability of any event E occurring is given by:

$$P(E) = \frac{\text{Number of sample points in E}}{\text{Nnumber of sample point in S}}$$

$$= \frac{n(E)}{n(S)},$$

where $n(E)$ is the number of elements in event E and $n(S)$ is the number of elements in the sample space, S or the number of possible outcomes.

For example, if $S = \{1, 2, 3, 4, 5, 6\}$ then the event that a number chosen from S is less than 3 is $E = \{1, 2\}$.
Therefore the probability that a number chosen from S is less than 3 is

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}.$$

Also the probability that a number chosen is greater than 2 is $\frac{4}{6} = \frac{2}{3}$.

The probability that a number chosen is divisible by 2 is $\frac{3}{6} = \frac{1}{2}$, since there are 3 numbers which are divisible by 2 out of a total of 6 numbers.
Similarly, in a class of 12 boys and 18 girls, the probability that a student selected from the class is a boy is $P(B) = \frac{12}{30} = \frac{2}{5}$, since the number of boys in the class is 12 and the total number of students in the class is $n(S) = 30$. Also, the probability that a student chosen at random from the class is a girl is $P(G) = \frac{18}{30} = \frac{3}{5}$, since there are 18 girls out of a total of 30 students.

Notation

If E is an event, then $P(E)$ is the probability that E will occur. $\overline{E}$ is the event that E will not occur and so $P(\overline{E})$ is the probability that E will not occur. The sample space, S, is used to denote the set of all possible outcomes.

Laws of probability

1. The probability that an event E will occur lies between 0 and 1 inclusive i.e. $0 \le P(E) \le 1$.

Proof: Since E is a subset set of S.

i.e $E \subseteq S \Rightarrow 0 \le n(E) \le n(S)$

Dividing the inequality by $n(S)$, we have

$$\frac{0}{n(S)} \le \frac{n(E)}{n(S)} \le \frac{n(S)}{n(S)}$$

$\therefore\ 0 \le P(E) \le 1.$

Thus, probability of any event lies between 0 and 1. Note that if it is certain that an event will occur then the probability is 1. If it is impossible that the event will occur then the probability is 0.

2. The probability that the sample space will occur is 1. i.e. $P(S) = 1$

Proof: $P(S) = \frac{n(S)}{n(S)} = 1$

Thus the probability of a certain event is 1.

3. $P(\varnothing) = 0$ where $\varnothing$ is the null set.

Proof: $P(\phi) = \frac{n(\phi)}{n(S)} = \frac{0}{n(S)} = 0$

Thus the probability of an impossible event is 0.

4. If the probability that an event E will occur denoted by $P(E) = r$, then the probability that the event will not occur denoted by $P(\overline{E}) = 1 - P(E) = 1 - r$.

Proof: The complement $\overline{E}$ of E contains those sample points of S which are not in E.

$\therefore\ n(\overline{E}) = n(S) - n(E)$

Dividing both sides by $n(S)$, we have

$$\frac{n(\overline{E})}{n(S)} = \frac{n(S)}{n(S)} - \frac{n(E)}{n(S)}$$

$\therefore\ P(\overline{E}) = 1 - P(E)$

For example, if the probability that it will rain on a particular day is $\frac{1}{4}$, then the probability that it will not rain on that day is $1 - \frac{1}{4} = \frac{3}{4}$.

Again, if the probability that it will rain tomorrow is 0.21, then the probability that it will not rain tomorrow is $1 - 0.21 = 0.79$.

Illustrative examples

Example 1

A number is chosen at random from the integers 5 to 25 inclusive. Find the probability that the number is a

i) multiple of 3
ii) multiple of 5.

Solution

The set of possible outcomes

$S = \{5, 6, 7, 8, 9, 10, 11, ..., 22, 23, 24, 25\}$

$n(S) = 21$

$E = \{\text{multiples of } 3\} = \{6, 9, 12, 15, 18, 21, 24\}$

$n(E) = 7$

$F = \{\text{multiples of } 5\} = \{5, 10, 15, 20, 25\}$

$n(F) = 5$

i) $P(\text{number is a multiple of } 3) = \frac{7}{21} = \frac{1}{3}$

ii) $P(\text{number is a multiple of } 5) = \frac{5}{21}$

Example 2

A number is chosen at random from the set $S = \{4, 7, 10, 13, 16, 19\}$.

What is the probability that the number is

i) even
ii) odd
iii) greater than 10
iv) less than 10
v) between 4 and 19

Solution

The set of possible outcomes

$S = \{4, 7, 10, 13, 16, 19\} \quad \Rightarrow \quad n(S) = 6$

i) $P(\text{number is even}) = \frac{3}{6} = \frac{1}{2}$

ii) $P(\text{number is odd}) = \frac{3}{6} = \frac{1}{2}$

iii) $P(\text{number is greater than } 10) = \frac{3}{6} = \frac{1}{2}$

iv) $P(\text{number is less than } 10) = \frac{2}{6} = \frac{1}{3}$

v) $P(\text{number is between 4 and } 19) = \frac{4}{6} = \frac{2}{3}$

Example 3

In a class of 20 boys and 35 girls, what is the probability that a student selected at random is a boy.

Solution

Total number of students in the class

$= 20 + 35 = 55;$

Number of boys in the class $= 20$

$\therefore \; P(\text{a boy is selected}) = \frac{20}{55} = \frac{4}{11}$

Example 4

A bag contains 50 identical balls, 20 of which are green and the rest red. Find the probability of picking at random a red ball from the bag.

Solution

Total number of balls in the bag $= 50$;

Number of red balls in the bag

$= 50 - 20 = 30$

$\therefore \; P(\text{a red ball is selected}) = \frac{30}{50} = \frac{3}{5}$

Example 5

A fair coin is tossed twice. Write down the set of possible outcomes.

What is the probability of obtaining?

i) exactly 2 heads;

ii) no head;

iii) at least one head;

iv) a head and a tail.

Solution

When a fair coin is tossed twice, the set of possible outcomes is $S = \{HH, HT, TH, TT\}$

Therefore the number of elements in the sample space, $n(S) = 4$

i) Let E denote the event of obtaining two heads, then $E = \{HH\} \quad \Rightarrow \quad n(E) = 1$

$$P(\text{exactly 2 heads}) = P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}$$

ii) Let F denote the event of obtaining no head, then $F = \{TT\} \quad \Rightarrow \quad n(F) = 1$

$$P(\text{no head}) = P(F) = \frac{n(F)}{n(S)} = \frac{1}{4}$$

iii) Let G denote the event of obtaining at least one head, then $G = \{HH, HT, TH\}$

$n(G) = 3$

$$P(\text{at least one head}) = P(G) = \frac{n(G)}{n(S)} = \frac{3}{4}$$

iv) Let H denote the event of obtaining a head and a tail, then $H = \{HT, TH\}$

$n(H) = 2$

$P(\text{a head and a tail})$

$$= P(H) = \frac{n(H)}{n(S)} = \frac{2}{4} = \frac{1}{2}$$

Note: Tossing a fair coin twice is the same as tossing two coins once.

Example 6

Three fair coins are tossed once.

a) Write down the set of possible outcomes.

b) What is the probability of obtaining

i) exactly two heads;

ii) no tail;

iii) not more than one head.

Solution

a) The set of possible outcomes,

$S = \{HHH, HHT, HTH, HTT, THH,$
$THT, TTH, TTT\}$

$n(S) = 8$

b) i) Let E denote the event of obtaining exactly two heads, then

$E = \{HHT, HTH, THH\}$

$n(E) = 3$

$P(\text{exactly two heads}) = \frac{3}{8}$

ii) Let F denote the event of obtaining no tail, then $F = \{HHH\} \Rightarrow n(F) = 1$

$P(\text{no tail}) = \frac{1}{8}$

iii) Let G denote the event of obtaining not more than one head, then

$G = \{HTT, THT, TTH, TTT\}$

$n(E) = 4.$

Therefore $P(\text{not more than one head})$

$= \frac{4}{8} = \frac{1}{2}$

Example 7

A fair coin is tossed four times, write down the set of possible outcomes. What is the probability of obtaining at least one head?

Solution

When a fair coin is tossed twice, the set of possible outcomes is $U = \{HH, HT, TH, TT\}$

Therefore when a fair coin is tossed four times, the set of possible outcomes is as shown below.

	HH	*HT*	*TH*	*TT*
HH	*HHHH*	*HHHT*	*HHTH*	*HHTT*
HT	*HTHH*	*HTHT*	*HTTH*	*HTTT*
TH	*THHH*	*THHT*	*THTH*	*THTT*
TT	*TTHH*	*TTHT*	*TTTH*	*TTTT*

The total number of outcomes =16

$P(\text{at least one head}) = \frac{15}{16}$

Alternative method 1

$P(\text{at least one head}) = 1 - P(\text{no head})$

But $P(\text{no head}) = \frac{1}{16}$

$\therefore\ P(\text{at least one head}) = 1 - P(\text{no head})$

$$= 1 - \frac{1}{16} = \frac{15}{16}$$

Alternative method 2

It can also be solved without finding the sample space.

$P(\text{at least one head}) = 1 - P(\text{no head})$

But $P(\text{no head}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$

$\therefore\ P(\text{at least one head}) = 1 - P(\text{no head})$

$$= 1 - \frac{1}{16} = \frac{15}{16}$$

Example 8

Two fair dice, A and B, each with faces numbered 1 to 6 are thrown together.

a) Construct a table showing all the equally likely outcomes.

b) From your table list the pair of numbers on the two dice for which the sum is

i) 5 ii) 10

iii) more than 10 iv) at least 10

c) Find the probability that the two dice show

i) different scores

ii) same scores

d) Find the probability that the sum of the numbers on the two dice is

i) 5 ii) 10

iii) more than 10 iv) at least 10

Solution

a) The table below shows the possible outcomes. The first column represents die A and the first row represents die B.

The total number of possible outcomes, $n(U) = 36.$

B **A**	1	2	3	4	5	6
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

b) From the table, the pair of numbers for which the sum is

- i) 5 are (1, 4), (2, 3), (3, 2) and (4, 1)
- ii) 10 are (4, 6), (5, 5) and (6, 4)
- iii) more than 10 are (5, 6), (6, 5) and (6, 6)
- iv) at least 10 are (4, 6), (5, 5), (6, 4), (5, 6), (6, 5) and (6, 6)

Note: At least 10 means 10 or more.

c) The total number of equally likely outcomes = 36

- i) $P(\text{dice show different scores}) = \frac{30}{36} = \frac{5}{6}$
- ii) $P(\text{dice show same scores}) = \frac{6}{36} = \frac{1}{6}$

d) The total number of equally likely outcomes = 36

The probability that the sum of the numbers on the two dice is

- i) 5 is $\frac{4}{36} = \frac{1}{9}$
- ii) 10 is $\frac{3}{36} = \frac{1}{12}$
- iii) more than 10 is $\frac{3}{36} = \frac{1}{12}$
- iv) at least 10 is $\frac{6}{36} = \frac{1}{6}$

Example 9

An unbiased die and a fair coin are tossed once. What is the probability of having a

- i) prime number and a head?
- ii) an even number and a head?
- iii) a score less than 3 and a tail?

Solution

This example combines a die and a coin. The table below shows the possible outcomes. The total number of possible outcomes, $n(U) = 12$.

	1	2	3	4	5	6
H	$H1$	$H2$	$H3$	$H4$	$H5$	$H6$
T	$T1$	$T2$	$T3$	$T4$	$T5$	$T6$

i) The event of obtaining a prime number and a head, $E = \{H2, H3, H5\}$

$\therefore P(\text{prime number and a head})$

$$= \frac{n(E)}{n(U)} = \frac{3}{12} = \frac{1}{4}$$

ii) The event of obtaining an even number and a head, $F = \{H2, H4, H6\}$

$\therefore P(\text{an even number and a head})$

$$= \frac{n(F)}{n(U)} = \frac{3}{12} = \frac{1}{4}$$

iii) The event of obtaining a score less than 3 and a tail, $G = \{T1, T2\}$

$\therefore P(\text{a score less than 3 and a tail})$

$$= \frac{n(G)}{n(U)} = \frac{2}{12} = \frac{1}{6}$$

Example 10

A coin is tossed twice and a die is tossed once. Find the probability of the following events:

- i) two heads and a 5;
- ii) exactly one head and a 1 or 2;
- iii) at least one head and not a 2.

Solution

The table below shows the possible outcomes. The total number of possible outcomes, $n(U) = 24$.

	1	2	3	4	5	6
HH	$HH1$	$HH2$	$HH3$	$HH4$	$HH5$	$HH6$
HT	$HT1$	$HT2$	$HT3$	$HT4$	$HT5$	$HT6$
TH	$TH1$	$TH2$	$TH3$	$TH4$	$TH5$	$TH6$
TT	$TT1$	$TT2$	$TT3$	$TT4$	$TT5$	$TT6$

i) The event of obtaining two heads and a 5, $E = \{HH5\}$

$$\therefore P(\text{two heads and a 5}) = \frac{n(E)}{n(U)} = \frac{1}{24}$$

ii) The event of obtaining exactly one head and a 1 or 2, $F = \{HT1, HT2, TH1, TH2\}$

$\therefore$ P(exactly one head and a 1 or 2)

$$= \frac{n(F)}{n(U)} = \frac{4}{24} = \frac{1}{6}$$

iii) P(at least one head and not a 2)

$$= \frac{15}{24} = \frac{5}{8}$$

Example 11

Two numbers x and y are chosen at random from the sets $A = \{1, 3, 5, 7, 9\}$ and $B = \{0, 2, 4, 6, 8\}$ such that $x \in A$ and $y \in B$.

a) Construct a table of all the possible ordered pairs (x, y) of numbers chosen.

b) What is the probability that;

i) $x + y = 7$ ii) $x \leq 7$

iii) $xy = 0$ iv) $3x + 2y = 23$.

Solution

a) The table below shows the set of possible outcomes.

From the table, the total number of possible outcomes, $n(U) = 25$.

B A	0	2	4	6	8
1	1, 0	1, 2	1, 4	1, 6	1, 8
3	3, 0	3, 2	3, 4	*3, 6	*3, 8
5	5, 0	5, 2	5, 4	*5, 6	*5, 8
7	7, 0	*7, 2	*7, 4	*7, 6	*7, 8
9	*9, 0	*9, 2	*9, 4	*9, 6	*9, 8

b) i) Let E denote the event that $x + y = 7$, then from the table $n(E) = 4$

$$\therefore \quad P(E) = \frac{n(E)}{n(U)} = \frac{4}{25}$$

ii) Let F denote the event that $x \leq 7$, then from the table $n(F) = 10$

$$\therefore \; P(F) = \frac{n(F)}{n(U)} = \frac{10}{25} = \frac{2}{5}$$

iii) Let G denote the event that $xy = 0$, then from the table $n(G) = 5$

$$\therefore \; P(G) = \frac{n(G)}{n(U)} = \frac{5}{25} = \frac{1}{5}$$

iv) Let H denote the event that $3x + 2y > 23$ then from the table above, the pairs which satisfy the inequality $3x + 2y > 23$, are marked with asterisks, $\Rightarrow \quad n(H) = 13$

$$\therefore \quad P(H) = \frac{n(H)}{n(U)} = \frac{13}{25}$$

Example 12

A box contains 4 tickets numbered 1, 2, 3 and 4. Two tickets are drawn one after the other without replacement and their numbers added. Find the probability that the sum is prime.

Solution

Let the two tickets drawn be labelled ticket A and ticket B. The first column represents ticket A drawn and the first row represents ticket B drawn. The table below shows the possible outcomes. Since the first ticket drawn is not replaced, the pairs (1, 1), (2, 2), (3, 3) and (4, 4) are not possible outcomes. The total number of possible outcomes, $n(U) = 12$.

B A	1	2	3	4
1		1, 2	1, 3	1, 4
2	2, 1		2, 3	2, 4
3	3, 1	3, 2		3, 4
4	4, 1	4, 2	4, 3	

E = event of obtaining a sum which is prime

$n(E) = 8$

$\therefore$ P(obtaining a sum which is prime)

$$= P(E) = \frac{n(E)}{n(U)} = \frac{8}{12} = \frac{2}{3}.$$

$$\therefore \quad P(H) = \frac{n(H)}{n(U)} = \frac{13}{25}$$

Example 13

A die has six faces. Two of these faces are numbered 1 and the remaining four faces 2, 3, 4 and 5. If two such dice are tossed once, find the probability of a 1 showing up on both dice.

Solution

For this particular die, the set of possible outcomes of one die is {1, 1, 2, 3, 4, 5}

Let the dice be labelled die *A* and die *B*. The first column represents die *A* and the first row represents die *B*. The table shows the possible outcomes. The total number of possible outcomes, $n(U) = 36$.

B *A*	1	1	2	3	4	5
1	1, 1	1, 1	1, 2	1, 3	1, 4	1, 5
1	1, 1	1, 1	1, 2	1, 3	1, 4	1, 5
2	2, 1	2, 1	2, 2	2, 3	2, 4	2, 5
3	3, 1	3, 1	3, 2	3, 3	3, 4	3, 5
4	4, 1	4, 1	4, 2	4, 3	4, 4	4, 5
5	5, 1	5, 1	5, 2	5, 3	5, 4	5, 5

Let *E* be the event of a 1 showing up on both dice $\Rightarrow \quad n(E) = 4$

P(a 1 showing up on both dice)

$$=\frac{n(E)}{n(U)}=\frac{4}{36}=\frac{1}{9}$$

Example 14

The table below shows the classification of students in a school.

	Day		Boarding	
	Boys	Girls	Boys	Girls
SS 1	45	40	60	55
SS 2	50	35	65	50
SS 3	70	55	45	30

a) If a student is selected at random from the school, find the probability that the student is

i) a boy   ii) a girl in SS 3

b) If a boarder is selected at random, find the probability that the one is

i) in SS 1   ii) a girl

Solution

To solve such question, we first have to find the total for each category if they are not given as shown in the table below.

	Day		Boarding		
	Boys	Girls	Boys	Girls	**Total**
SS 1	45	40	60	55	200
SS 2	50	35	65	50	200
SS 3	70	55	45	30	200
Total	165	130	170	135	*600

The grand total is 600 (marked with asterisk). This is obtained by adding all the totals of each category. Note that the horizontal totals should be equal to the vertical totals.

a) If a student is selected from the school

i) Total number of boys in the school $= 165 + 170 = 335$

P(a student selected from the school is a boy)

$$=\frac{\text{Total number of boys in the school}}{\text{Total number of students in the school}}$$

$$=\frac{335}{600}=\frac{67}{120}$$

ii) Total number of girls in SS 3 = 55 + 30 = 85

P(a student selected from the school is a girl in SS 3) =

$$\frac{\text{Total number of girls in SS 3}}{\text{Total number of students in the school}}$$

$$=\frac{85}{600}=\frac{17}{120}$$

b) If a boarder is selected, it changes the problem because the day section of the table disappears.

i) Total number of boarders in SS 1
$= 60 + 55 = 115$

Total number of boarders $= 170 + 135 = 305$

P(a student selected is a border in SS 1)

$$= \frac{\text{Total number of boarders in SS1}}{\text{Total number of boarders}} = \frac{115}{305} = \frac{23}{61}$$

ii) Total number of girls in boarding
$= 55 + 50 + 30 = 135$

P(a student selected is a girl in boarding)

$$= \frac{\text{Total number of girls in boarding}}{\text{Total number of boarders}}$$

$$= \frac{135}{305} = \frac{27}{61}$$

Example 15

The following table shows the distribution of students in a class who study Economics and Commerce.

	Economics	Commerce	Total
Girls	60	B	105
Boys	A	E	95
Total	115	D	C

a) Copy and complete the table

b) If a student is selected at random from the class, find the probability that the student is:
  i) a boy
  ii) a girl
  iii) studying economics
  iv) studying commerce
  v) a boy and studying economics
  vi) a boy and studying commerce
  vii) a girl and studying economics
  viii) a girl and studying commerce

c) If a girl is chosen at random, find the probability that she is
  i) studying economics
  ii) studying commerce

d) If a boy is chosen at random, find the probability that he is
  i) studying economics
  ii) studying commerce

e) If a student is studying economics, find the probability that the student is
  i) a boy ii) a girl

Solution

a) The missing numbers are worked out as follows:

A : Boys taking Economics $115 - 60 = 55$

B : Girls taking Commerce $105 - 60 = 45$

C : Total number of students $105 + 95 = 200$

D : Students taking Commerce $200 - 115 = 85$

E : Boys taking Commerce $85 - 45 = 40$

The completed table is shown below

	Economics	Commerce	**Total**
Girls	60	45	105
Boys	55	40	95
Total	115	85	200

b) If a student is selected at random from the class, then

  i) $P(\text{student is a boy}) = \frac{95}{200} = \frac{19}{40}$

  ii) $P(\text{student is a girl}) = \frac{105}{200} = \frac{21}{40}$

  iii) $P(\text{student is studying economics})$
  $= \frac{115}{200} = \frac{23}{40}$

  iv) $P(\text{student is studying commerce})$
  $= \frac{85}{200} = \frac{17}{40}$

  v) $P(\text{student is a boy and studying economics}) = \frac{55}{200} = \frac{11}{40}$

  vi) $P(\text{student is a boy and studying commerce}) = \frac{40}{200} = \frac{1}{5}$

  vii) $P(\text{student is a girl and studying economics}) = \frac{60}{200} = \frac{3}{10}$

  viii) $P(\text{student is a girl and studying commerce}) = \frac{45}{200} = \frac{9}{40}$

c) If a girl is chosen at random, it changes the problem because the boy section of the table disappears

i) P(she is studying economics)

$$= \frac{\text{Total number of girls studying economics}}{\text{Total number of girls}}$$

$$= \frac{60}{105} = \frac{4}{7}$$

ii) P(she is studying commerce)

$$= \frac{\text{Total number of girls studying commerce}}{\text{Total number of girls}}$$

$$= \frac{45}{105} = \frac{3}{7}$$

d) If a boy is chosen at random, it changes the problem because the girl section of the table disappears.

i) P(he is studying economics)

$$= \frac{\text{Total number of boys studying economics}}{\text{Total number of boys}}$$

$$= \frac{55}{95} = \frac{11}{19}$$

ii) P(he is studying commerce)

$$= \frac{\text{Total number of girls studying commerce}}{\text{Total number of girls}}$$

$$= \frac{40}{95} = \frac{8}{19}$$

e) If a student is studying economics, it changes the problem because the commerce section of the table disappears.

i) P(student is a boy) $= \frac{55}{115} = \frac{11}{23}$

ii) P(student is a girl) $= \frac{60}{115} = \frac{12}{23}$

Example 16

There are 100 students in a first year college intake. 36 are male and are studying accounting, 9 are male and not studying accounting, 42 are female and studying accounting, 13 are female and are not studying accounting.

a) If a student is selected at random, find the probability that the student is:
   - i) a male
   - ii) a female
   - iii) studying accounting
   - iv) not studying accounting
   - v) a male and studying accounting
   - vi) a male and not studying accounting
   - vii) a female and studying accounting
   - viii) a female and not studying accounting

b) If the student is a male, find the probability that he is
   - i) studying accounting
   - ii) not studying accounting

c) If the student is a female, find the probability that she is
   - i) studying accounting
   - ii) not studying accounting

d) If the student is studying accounting, find the probability that the student is
   - i) a male
   - ii) a female

e) If the student is not studying accounting, find the probability that the student is
   - i) a male
   - ii) a female

Solution

The most effective way to handle this problem is to draw up a table.

	Accounting	Not Accounting	**Total**
Males	36	9	45
Females	42	13	55
Total	78	22	100

a) If a student is selected at random from the first year intake, then

i) P(student is a male) $= \frac{45}{100} = 0.45$

ii) $P(\text{student is a female}) = \frac{55}{100} = 0.55$

iii) $P(\text{student is studying accounting})$
$= \frac{78}{100} = 0.78$

iv) $P(\text{student is not studying accounting})$
$= \frac{22}{100} = 0.22$

v) $P(\text{student is a male and studying accounting}) = \frac{36}{100} = 0.36$

vi) $P(\text{student is a male and not studying accounting}) = \frac{9}{100} = 0.09$

vii) $P(\text{student is a female and studying accounting}) = \frac{42}{100} = 0.42$

viii) $P(\text{student is a female and not studying accounting}) = \frac{13}{100} = 0.13$

b) If the student chosen is a male, it changes the problem because the female section of the table disappears.

i) $P(\text{he is studying accounting})$

$$= \frac{\text{Total number of males studying accounting}}{\text{Total number of males}}$$

$$= \frac{36}{45} = 0.8$$

ii) $P(\text{he is not studying accounting})$

$$= \frac{\text{Total number of males not studying accounting}}{\text{Total number of girls}}$$

$$= \frac{9}{45} = 0.2$$

c) If the student chosen is a female, it changes the problem because the male section of the table disappears

i) $P(\text{she is studying accounting})$

$$= \frac{\text{Total number of females studying accounting}}{\text{Total number of females}}$$

$$= \frac{42}{55} = 0.76$$

ii) $P(\text{she is not studying accounting})$

$$= \frac{13}{55} = 0.24$$

d) If a student is studying accounting, it changes the problem because the 'not accounting' section of the table disappears.

i) $P(\text{student is a male}) = \frac{36}{78} = 0.46$

ii) $P(\text{student is a female}) = \frac{42}{78} = 0.54$

e) If a student is not studying accounting, it changes the problem because the ' accounting' section of the table disappears.

i) $P(\text{student is a male}) = \frac{9}{22} = 0.41$

ii) $P(\text{student is a female}) = \frac{13}{22} = 0.59$

Note: In some cases it is often useful to express probabilities using **decimals**.

Relative frequency

Probabilities can also be estimated from the results or outcomes of an experiment. The **relative frequency** of an event is defined as:

$$\frac{\textit{Frequency of the event}}{\textit{Total frequency}}.$$

The relative frequency gives an estimate of the probability of an event. When the total frequency is *very large* the relative frequency provides a good estimate.

Illustrative examples

Example 1

In a survey of 150 families with four children to find the number of girls, the following results were obtained.

Number of girls	0	1	2	3	4
Frequency	8	41	55	33	13

Estimate the probability that a family with four children will have

i) two girls
ii) no girls
iii) one boy
iv) four boys
v) at least three girls.

Solution

i) $P(\text{two girls}) = \frac{55}{150} = \frac{11}{30}$

ii) $P(\text{no girls}) = \frac{8}{150} = \frac{4}{75}$

iii) $P(\text{one boy}) = P(\text{three girls}) = \frac{33}{150} = \frac{11}{50}$

iv) $P(\text{four boys}) = P(\text{no girls}) = \frac{8}{150} = \frac{4}{75}$

v) $P(\text{at least three girls})$
$= P(\text{three or more girls})$
$= \frac{33+13}{150} = \frac{46}{150} = \frac{23}{75}$

Example 2

Consider the following frequency distribution table for the ages of students in a class.

Age/yrs	11	12	13	14	15
Frequency	2	1	1	2	3

Age/yrs	16	17	18	19
Frequency	2	5	3	1

Find the probability that a student selected at random from the class is

i) 14 years old
ii) less than 14 years old
iii) greater than 16 years old
iv) between 13 and 17 years old.

Solution

Total number of students in the class = 20 (i.e. add the frequencies)

i) $P(\text{a student is 14 years old}) = \frac{2}{20} = \frac{1}{10}$

ii) $P(\text{a student is less than 14 years}) = \frac{4}{20} = \frac{1}{5}$

Note: There are 4 students with ages less than 14.

iii) $P(\text{a student is greater than 16 years old}) = \frac{9}{20}$

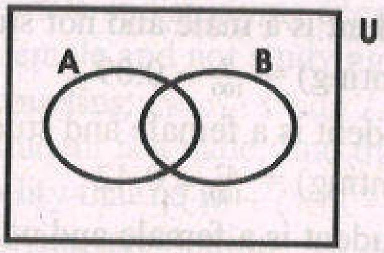
iv) $P(\text{student is between 13 and 17 yrs old}) = \frac{7}{20}$

Addition law of probability

If A and B are two events of the same experiment, then the probability of A or B or both occurring is given by:

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

This is the general addition law.



However, if events A and B are ***mutually exclusive*** events i.e. if A and B do not have anything in common, then $A \cap B = \varnothing$ and the addition law becomes:

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B).$$

This is the addition law for mutually exclusive events.

Note that in probability language the word 'or' or the symbol '$\cup$' in means *addition.*

Illustrative examples

Example 1

Given that $P(X) = 0.35$, $P(Y) = 0.45$ and $P(X \cap Y) = 0.2$, find $P(X \cup Y)$.

Solution

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y) = 0.35 + 0.45 - 0.2 = 0.6$$

Example 2

The probabilities of two events X and Y are given by $P(X) = 0.3$, $P(Y) = 0.4$ and $P(X \cap Y) = 0.12$. Find $P(X \cup Y)$.

Solution

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y) = 0.3 + 0.4 - 0.12 = 0.58$$

Example 3

If two events X and Y are mutually exclusive, $P(X) = \frac{1}{3}$, $P(Y) = \frac{2}{5}$, find:

i) $P(X \cap Y)$

ii) $P(X \cup Y)$.

Solution

i) $P(X \cap Y) = 0$ since events X and Y are mutually exclusive

ii) Since X and Y are mutually exclusive events

$P(X \cup Y) = P(X) + P(Y) = \frac{1}{3} + \frac{2}{5} = \frac{11}{15}$

Example 4

A bag contains 40 identical marbles in a box. 18 of the marbles are blue, 14 red and 8 green. What is the probability that a marble selected at random is either blue or green?

Solution

Total number of marbles in the bag = 40;
The two events are mutually exclusive.
P(a marble selected is either blue **or** green)
$= P$(blue is selected) $+ P$(green is selected)
$= \frac{18}{40} + \frac{8}{40} = \frac{26}{40} = \frac{13}{20}$

Example 5

Given that $P(X) = 0.35$, $P(Y) = 0.45$ and $P(X \cap Y) = 0.2$, find $P(X \cup Y)$.

Solution

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$$
$$= 0.35 + 0.45 - 0.2 = 0.6$$

Example 6

Given that X and Y are independent events and that $P(X) = 0.5$, $P(Y) = c$ and $P(X \cup Y) = 0.8$, find the value of c.

Solution

If X and Y are independent events, then the addition law is:

$$P(X \cup Y) = P(X) + P(Y) - P(X).P(Y)$$
$$0.8 = 0.5 + c - 0.5c$$
$$0.8 = 0.5 + 0.5c$$
$$0.5c = 0.8 - 0.5 = 0.3$$
$$c = \frac{0.3}{0.5} = 0.6$$

Example 7

The probabilities of two events X and Y are given by $P(X) = 0.3$, $P(Y) = 0.4$ and $P(X \cap Y) = 0.12$.

i) Find $P(X \cup Y)$

ii) Are events X and Y mutually exclusive or independent?

Solution

i) $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$
$= 0.3 + 0.4 - 0.12 = 0.58$

ii) Since $P(X \cap Y) = P(X) \times P(Y)$
$= 0.3 \times 0.4 = 0.12$,

It means X and Y are independent events $P(X \cap Y) = 0.12$ and therefore X and Y are not mutually exclusive.

Note that for mutually exclusive events X and Y, $P(X \cap Y) = 0$.

Multiplication Law of Probability

If A and B are two independent events then $P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B)$. This relation is called the **multiplication law** of probability. Similarly for three independent events A, B and C, then;

$P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$

Note that 'and' or $\cap$ means *multiplication*.

Illustrative examples

Example 1

X and Y are two independent events such that $P(X) = \frac{3}{7}$ and $P(X \cap Y) = \frac{2}{5}$. Find

i) $P(Y)$

ii) $P(X \cup Y)$

Solution

i) If X and Y are two independent events then $P(X \cap Y) = P(X) \times P(Y)$

$\frac{2}{5} = \frac{3}{7} \times P(Y) \quad \Rightarrow \quad P(Y) = \frac{2}{5} \times \frac{7}{3} = \frac{14}{15}$

ii) $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

$= \frac{3}{7} + \frac{14}{15} - \frac{2}{5} = \frac{45+98-42}{105} = \frac{101}{105}$

Example 2

Two independent events A and B are such that $P(A) = \frac{5}{12}$ and $P(A \cap B) = \frac{1}{8}$. Find

i) $P(B)$

ii) $P(A \cup B)$

Solution

i) If A and B are two independent events then $P(A \cap B) = P(A) \times P(B)$

$\frac{1}{8} = \frac{5}{12} \times P(B) \quad \Rightarrow \quad P(B) = \frac{1}{8} \times \frac{12}{5} = \frac{3}{10}$

ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{5}{12} + \frac{3}{10} - \frac{1}{8} = \frac{50+36-15}{120} = \frac{71}{120}$

Example 3

The events A and B are such that
$P(A) = 0.43$, $P(B) = 0.48$ and
$P(A \cup B) = 0.78$.
Show that the events A and B are neither mutually exclusive nor independent events.

Solution

A and B are mutually exclusive if
$P(A \cup B) = P(A) + P(B)$.
$P(A) + P(B) = 0.43 + 0.48$
$= 0.91 \neq P(A \cup B)$
Note that $P(A \cup B)$ is given as 0.78.
Therefore A and B are not mutually exclusive, in which case

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.43 + 0.48 - 0.78 = 0.13$$

If A and B are independent events, then
$P(A \cap B) = P(A) \times P(B)$
$P(A) \times P(B) = 0.43 \times 0.48 = 0.2064$ but
$P(A \cap B) = 0.13$
Therefore $P(A \cap B) \neq P(A) \times P(B)$ and A and B are not independent events.

Example 4

The probability that an event A occurs is $\frac{1}{5}$ and the probability that an event B occurs is $\frac{1}{3}$. Assuming that the two events are independent events, find the probability that

i) A and B both occur;

ii) Neither event occurs;

iii) At least one of the events occurs.

Solution

i) P(both A and B occur)
$= P(A \cap B) = P(A) \times P(B) = \frac{1}{5} \times \frac{1}{3} = \frac{1}{15}$

Since the two events A and B are independent

ii) The probability that the event A does not occur $= P(\overline{A}) = 1 - P(A) = 1 - \frac{1}{5} = \frac{4}{5}$

The probability that the event B does not occur $= P(\overline{B}) = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3}$

P(neither event occurs) $= P(\overline{A}) \times P(\overline{B})$
$= \frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$

iii) P(at least one of the events occurs)
$= P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= \frac{1}{5} + \frac{1}{3} - \frac{1}{15} = \frac{7}{15}$

Alternatively, P(at least one of the events occurs) $= 1 - P$(neither of the events occurs)
$= 1 - \frac{8}{15} = \frac{7}{15}$

Example 5

In a contest, Kpakpo, Osei, and Anani are asked to solve a problem, the respective probabilities that they solve the problem are $\frac{1}{6}$, $\frac{1}{8}$ and $\frac{1}{3}$. Calculate the probability that:

a) none of them solves the problem;
b) at least one of them solves the problem;
c) only one of them solves the problem.

Solution

Let K, O, and A denote the events that Kpakpo, Osei and Anani solve the problem respectively.

Then $P(K)=\frac{1}{6} \quad \Rightarrow \quad P(\overline{K})=1-\frac{1}{6}=\frac{5}{6}$

$P(O)=\frac{1}{8} \quad \Rightarrow \quad P(\overline{O})=1-\frac{1}{8}=\frac{7}{8}$

$P(A)=\frac{1}{3} \quad \Rightarrow \quad P(\overline{A})=1-\frac{1}{3}=\frac{2}{3}$

Where $\overline{K}$ denotes the event that Kpakpo cannot solve the problem etc.

Note that $P(\overline{K})=1-P(K)$ etc.

a) P(none of them solves the problem)

$= P(\overline{K}\cap\overline{O}\cap\overline{A}) = P(\overline{K})\times P(\overline{O})\times P(\overline{A})$

$= \frac{5}{6}\times\frac{7}{8}\times\frac{2}{3}=\frac{35}{72}$

b) P(at least one of them solves the problem)

$= 1 - P$(none of them solves the problem)

$= 1-\frac{35}{72}=\frac{37}{72}$

Note that this is the simplest method of working the '*at least one….*' type of problem. i.e.

P(at least one….) = 1 – P(none of them ...)

c) P(only one of them solves the problem)

$= P(K\cap\overline{O}\cap\overline{A})$ or $P(\overline{K}\cap O\cap\overline{A})$ or $P(\overline{K}\cap\overline{O}\cap A)$

$= [P(K)\times P(\overline{O})\times P(\overline{A}) + P(\overline{K})\times P(O)\times P(\overline{A})]$
$+ [P(\overline{K})\times P(\overline{O})\times P(A)]$

$= [\frac{1}{6}\times\frac{7}{8}\times\frac{2}{3}] + [\frac{5}{6}\times\frac{1}{8}\times\frac{2}{3}] + [\frac{5}{6}\times\frac{7}{8}\times\frac{1}{3}]$

$= \frac{14}{144}+\frac{10}{144}+\frac{35}{144}=\frac{59}{144}$

Example 6

An actuary found in 1970 that the probability that a certain boy will be alive in the year 2000 is 0.78 and the probability of a girl being alive in the year 2000 is 0.60.

Calculate the probability that in the year 2000:

a) both the boy and the girl will be alive;
b) neither will be alive ;
c) at least one will be alive.

Solution

Let B and G denote the events that the boy and the girl will be alive in the 2000 respectively and $\overline{B}$ and $\overline{G}$ denote the events that the boy and the girl will not be alive in the year 2000 respectively.

Then $P(B)=0.78$

$P(\overline{B})=1-0.78=0.22$

$P(G)=0.60$

$P(\overline{G})=1-0.60=0.40$

a) P(both the boy and girl will be alive)

$= P(B\cap G) = P(B)\times P(G)$

$= 0.78\times 0.60 = 0.468$

b) P(neither will be alive)

$= P(\overline{B}\cap\overline{G}) = P(\overline{B})\times P(\overline{G})$

$= 0.22\times 0.40 = 0.088$

c) P(at least one will be alive)

$= 1 - P$(neither will be alive)

$= 1 - 0.088 = 0.912$

Note that this is the simplest method of working the '*at least one….*' type of problem.

Example 7

The probability that Abu solves a particular problem is $\frac{1}{3}$ and the probability that Bello solves the same problem is $\frac{1}{5}$. If they both attempt the problem independently, what is the probability that it will be solved by at least one of them?

Solution

Let A denote the event that Abu solves the problem and $\overline{A}$ denote the event that Abu cannot solve the problem. Let B and $\overline{B}$ denote the events that Bello solves and cannot solve the problem respectively.

Then $P(A)=\frac{1}{3} \quad \Rightarrow \quad P(\overline{A})=1-\frac{1}{3}=\frac{2}{3}$

$P(B)=\frac{1}{5} \quad \Rightarrow \quad P(\overline{B})=1-\frac{1}{5}=\frac{4}{5}$

P(at least one of them solves the problem)
$= 1-P$(none of them solves the problem)

But P(none of them solves the problem)
$= P(\overline{A}\cap\overline{B}) = P(\overline{A})\times P(\overline{B}) = \frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$

Therefore P(at least one of them solves the problem) $= 1-\frac{8}{15}=\frac{7}{15}$

Note that this is the simplest method of working the '*at least one….*' type of problem.

Example 8

The probability that Kwasi, Dadzie and Ali attend school on a Friday are $\frac{2}{3}$, $\frac{3}{5}$ and $\frac{1}{4}$ respectively. Find the probability that

i) none of them attends school on a Friday
ii) at least one of them attends school on a Friday.

Solution

Let K, D and A denote the events that Kwesi, Dadzie and Ali attend school on a Friday respectively.

Then $P(K)=\frac{2}{3} \quad \Rightarrow \quad P(\overline{K})=1-\frac{2}{3}=\frac{1}{3}$

$P(D)=\frac{3}{5} \quad \Rightarrow \quad P(\overline{D})=1-\frac{3}{5}=\frac{2}{5}$

$P(A)=\frac{1}{4} \quad \Rightarrow \quad P(\overline{A})=1-\frac{1}{4}=\frac{3}{4}$

Where $\overline{K}$ denotes event that Kwesi does not attend school on a Friday etc.

i) P(none of them attends school on a Friday)
$= P(\overline{K}\cap\overline{D}\cap\overline{A})$
$= P(\overline{K})\times P(\overline{D})\times P(\overline{A})$
$= \frac{1}{3}\times\frac{2}{5}\times\frac{3}{4}=\frac{1}{10}$

ii) P(at least one of them attends school on a Friday)
$= 1-P$(none of them attends school on a Friday)
$= 1-\frac{1}{10}=\frac{9}{10}$

Note that this is the simplest method of working the '*at least one….*' type of problem.

Example 9

The probabilities of Kofi and Yaw scoring from penalties are 0.6 and 0.4 respectively. If they take a penalty kick each, what is the probability that exactly one goal will be scored?

Solution

Let K and Y denote the events that Kofi and Yaw score from penalties respectively.

Then $P(K)=0.6 \quad \Rightarrow \quad P(\overline{K})=1-0.6=0.4$

$P(Y)=0.4 \quad \Rightarrow \quad P(\overline{Y})=1-0.4=0.6$

Where $\overline{K}$ denotes event that Kofi does not score from penalties etc.

P(exactly one goal will be scored)
$= P(K\cap\overline{Y})$ or $P(\overline{K}\cap Y)$
$= [P(K)\times P(\overline{Y}))]+[P(\overline{K})\times P(Y)]$
$= [0.6\times 0.6]+[0.4\times 0.4]=0.52$

Example 10

The probability of an athlete taking part in a race is $\frac{1}{6}$. The probability that he wins the race is $\frac{3}{4}$. What is the probability that he takes part and wins the race?

Solution

P(he takes part and wins the race)

$= \frac{1}{6} \times \frac{3}{4} = \frac{1}{8}$

Example 11

When I walk to school, 3 days out of 10, I meet my friend Ade and 4 days out of 10, I meet my friend Tolu. Assuming that these events are independent, find the probability that on a particular day I shall meet

i) both Ade and Tolu.
ii) none of them.
iii) at least one of them

Solution

Let A = event that I meet my friend Ade

$P(A) = 0.3$ and $P(\overline{A}) = 1 - 0.3 = 0.7$

T = event that I meet my friend Tolu

$P(T) = 0.4$ and $P(\overline{T}) = 1 - 0.4 = 0.6$

Where $\overline{A}$ denotes event that I do not meet my friend Ade etc.

i) P(both Ade and Tolu)

$= P(A) \times P(T) = 0.3 \times 0.4 = 0.12$

ii) P(none of them)

$= P(\overline{A}) \times P(\overline{T}) = 0.7 \times 0.6 = 0.42$

iii) P(at least one of them)

$= 1 - P$(none of them) $= 1 - 0.42 = 0.58$

Example 12

The probability that Kofi goes to school on a market day is $\frac{1}{5}$ and the probability that his brother also attends school the same day is $\frac{3}{4}$. What is the probability that

i) both were absent?
ii) at least one was present?
iii) exactly one of them is present?

Solution

Let K = event that Kofi goes to school

$P(K) = \frac{1}{5} \Rightarrow P(\overline{K}) = 1 - \frac{1}{5} = \frac{4}{5}$

B = event that Kofi's brother goes to school

$P(B) = \frac{3}{4} \Rightarrow P(\overline{B}) = 1 - \frac{3}{4} = \frac{1}{4}$

Where $\overline{K}$ denotes event that Kofi does not go to school etc.

i) P(both were absent) $= P(\overline{K}) \times P(\overline{B})$

$= \frac{4}{5} \times \frac{1}{4} = \frac{1}{5}$

ii) P(at least one was present)

$= 1 - P$(both were absent)

$= 1 - \frac{1}{5} = \frac{4}{5}$

iii) P(exactly one of them is present)

$= [P(K) \times P(\overline{B})] + [P(\overline{K}) \times P(B)]$

$= \left(\frac{1}{5} \times \frac{1}{4}\right) + \left(\frac{4}{5} \times \frac{3}{4}\right) = \frac{1}{20} + \frac{12}{20} = \frac{13}{20}$

Example 13

The probability that Ato hits a target is $\frac{3}{5}$. The probability that Oko hits the target is $\frac{2}{3}$. If they both shoot together what is the probability that

i) none of them hits the target?
ii) at least one of them hits the target?
iii) one of them hits the target?
iv) both hit the target?

Solution

Let A = event that Ato hits a target

$P(A) = \frac{3}{5} \Rightarrow P(\overline{A}) = 1 - \frac{3}{5} = \frac{2}{5}$

O = event that Oko hits a target

$P(O) = \frac{2}{3} \Rightarrow P(\overline{O}) = 1 - \frac{2}{3} = \frac{1}{3}$

Where $\overline{A}$ denotes event that Ato does not hit a target etc.

i) P(none of them hits the target)

$= P(\overline{A}) \times P(\overline{O}) = \frac{2}{5} \times \frac{1}{3} = \frac{2}{15}$

ii) $P(\text{at least one of them hits the target})$
$= 1 - P(\text{none of them hits the target})$
$= 1 - \frac{2}{15} = \frac{13}{15}$

iii) $P(\text{one of them hits the target})$
$= [P(A) \times P(\overline{O}))] + [P(\overline{A}) \times P(O)]$
$= \left(\frac{3}{5} \times \frac{1}{3}\right) + \left(\frac{2}{5} \times \frac{2}{3}\right) = \frac{3}{15} + \frac{4}{15} = \frac{7}{15}$

iv) $P(\text{both hit the target}) = P(A) \times P(O)$
$= \frac{3}{5} \times \frac{2}{3} = \frac{2}{5}$

Example 14

The probability that an athlete wins a 100 m race is 0.7 and the probability that he wins a 200 m race is 0.5. What is the probability that he wins:

i) both race?
ii) only one race?

Solution

Let A = event that an athlete wins 100 m race $\Rightarrow P(A) = 0.7$ and $P(\overline{A}) = 1 - 0.7 = 0.3$

B = event that an athlete wins 200 m race

$$P(B) = 0.5 \text{ and } P(\overline{B}) = 1 - 0.5 = 0.5$$

Where $\overline{A}$ denotes event that an athlete does not win a 100 m race etc.

i) $P(\text{both race})$
$= P(A) \times P(B) = 0.7 \times 0.5 = 0.35$

ii) $P(\text{only one race})$
$= [P(A) \times P(\overline{B})] + [P(\overline{A}) \times P(B)]$
$= [0.7 \times 0.5] + [0.3 \times 0.5]$
$= 0.35 + 0.15 = 0.5$

Example 15

A car and a bus were hired for an excursion. The probability that the car fails to turn up is $\frac{1}{3}$ and the probability that the bus fails to turn up is $\frac{1}{4}$. Find the probability that at least one of the vehicles turns up?

Solution

$P(\text{at least one of the vehicles turns up})$
$= 1 - P(\text{none of the vehicles turns up})$
$P(\text{none of the vehicles turns up}) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$
$P(\text{at least one of the vehicles turns up})$
$= 1 - \frac{1}{12} = \frac{11}{12}$

Example 16

The table below shows the probabilities of selecting boys and girls from the two classes 3A and 3B in a school.

	Classes	
	3A	3B
P(Boy)	$\frac{1}{4}$	$\frac{2}{3}$
P(Girl)	$\frac{3}{4}$	$\frac{1}{3}$

If a student is selected at random from each of the classes, what is the probability that

i) both students are girls?
ii) the two students selected will be a boy and a girl?

Solution

i) $P(\text{both students are girls}) = \frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$

ii) $P(\text{the two students selected will be a boy and a girl}) = \left(\frac{3}{4} \times \frac{2}{3}\right) + \left(\frac{1}{4} \times \frac{1}{3}\right) = \frac{6}{12} + \frac{1}{12} = \frac{7}{12}$

Drawing from a box or bag with replacement

Supposing in a bag containing 3 red balls and 5 blue balls, you take a ball at random from the bag and then replace it, the result of the second draw does not depend on the ball you drew the first time. In probability problems the words **with replacement** can give you a hint that we are drawing with replacement. In such a case, the probability that the first ball drawn is blue is $\frac{5}{8}$. The probability that the second ball drawn is blue is still $\frac{5}{8}$. Also the probability that the first ball drawn is red is $\frac{3}{8}$ and the probability that the second ball drawn is red is still $\frac{3}{8}$.

Illustrative examples

Example 1

A bag contains 10 balls that differ only in colour; 4 are blue and 6 are red. Two balls are picked, one after the other, with replacement. What is the probability that:

i) both are red

ii) both are of the same colour.

Solution

Let R and B denote the events of selecting red and blue balls respectively, then from above, $n(R) = 6$, $n(B) = 4$ and total number of balls, $n(U) = 10$.

i) Let $1R$ and $2R$ denote an event that the first ball selected is red and second ball selected is red respectively.

P(both are red)

$= P$(1st ball is red and 2nd ball is red)

$= P(1R \cap 2R)$

$= P(1R) \times P(2R)$

$= \frac{6}{10} \times \frac{6}{10} = \frac{36}{100} = 0.36$

ii) P(both are of the same colour)

$= P$(both are red) or P(both are blue)

$= P(1R \cap 2R) + P(1B \cap 2B)$

$= \left(\frac{6}{10} \times \frac{6}{10}\right) + \left(\frac{4}{10} \times \frac{4}{10}\right)$

$= \frac{36}{100} + \frac{16}{100} = \frac{52}{100} = 0.52$

Example 2

A box contains 5 white balls, 3 red balls and 2 black balls of the same size. A ball is selected at random from the box and then replaced. A second ball is then selected. Find the probability of obtaining

a) two red balls

b) two white balls or two black balls

c) one black ball and one red ball in any other.

Solution

Let W, R and B denote the events of selecting white, red and black balls respectively, then from above, $n(W) = 5$, $n(R) = 3$, $n(B) = 2$ and total number of balls, $n(U) = 10$.

a) Let $1R$ and $2R$ denote an event that the first ball selected is red and second ball selected is red respectively.

P(both are red)

$= P$(1st ball is red and 2nd ball is red)

$= P(1R \cap 2R)$

$= P(1R) \times P(2R)$

$= \frac{3}{10} \times \frac{3}{10} = \frac{9}{100} = 0.09$

b) P(two white balls or two black balls)

$= P(1W \cap 2W) + P(1B \cap 2B)$

$= \left(\frac{5}{10} \times \frac{5}{10}\right) + \left(\frac{2}{10} \times \frac{2}{10}\right)$

$= \frac{25}{100} + \frac{4}{100} = \frac{29}{100} = 0.29$

c) P(one black ball and one red ball in any other)

$= P(1B \cap 2R) + P(1R \cap 2B)$

$= \left(\frac{2}{10} \times \frac{3}{10}\right) + \left(\frac{3}{10} \times \frac{2}{10}\right)$

$= \frac{6}{100} + \frac{6}{100} = \frac{12}{100} = 0.12$

Conditional probability

The probability of one event happening may **depend** upon another event having taken place. In this case the probability of the event is said to be **conditional**. The events are called **dependent events**. For example, if in a bag containing 3 red balls and 5 blue balls, you take a ball at random from the bag and then another without replacing the first, the result of the second draw depends on the ball you drew the first time. In probability problems, the **selection of objects one after the other without replacement** can give you a hint that conditional probability is involved. If any two events A and B (not necessarily from the same experiment) are such that the occurrence of A depends on that of B, then the **conditional probability** that A will occur given that B has occurred is:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{P(A \cap B)}{P(B)} = \frac{n(A \cap B)}{n(B)},$$

where $P(A|B)$ means the probability of A occurring given that B has already occurred. From the above relation,

$$P(A \cap B) = P(A|B) \times P(B)$$

Note that if A and B are **independent events** then $P(A|B) = P(A)$

$$P(A \cap B) = P(A) \times P(B)$$

Also if A and B are **mutually exclusive** then $P(A|B) = 0$.

Illustrative examples

Example 1

Find the probability of scoring a number less than five of a single throw of a die if the throw resulted in an even number.

Solution

Let $A = \{\text{scoring a number less than five}\}$
$= \{1, 2, 3, 4\}$ and
$B = \{\text{an even number}\}$
$= \{2, 4, 6\}$
$(A \cap B) = \{2, 4\}$

The required probability is $P(A|B)$

$$= \frac{P(A \cap B)}{P(B)}$$

$P(A \cap B) = \frac{2}{6}$ and $P(B) = \frac{3}{6}$

$$\therefore \; P(A|B) = \frac{\frac{2}{6}}{\frac{3}{6}} = \frac{2}{3}$$

Example 2

Find the probability of obtaining an even number of a single throw of a die if the throw resulted in a number less than five.

Solution

Let $A = \{\text{an even number}\} = \{2, 4, 6\}$ and
$B = \{\text{scoring a number less than five}\}$
$= \{1, 2, 3, 4\}$ $A \cap B = \{2, 4\}$

The required probability is $P(A|B)$

$$= \frac{P(A \cap B)}{P(B)}$$

$P(A \cap B) = \frac{2}{6}$ and $P(B) = \frac{4}{6}$

$$\therefore \; P(A|B) = \frac{\frac{2}{6}}{\frac{4}{6}} = \frac{2}{4} = \frac{1}{2}$$

Example 3

Find the probability of obtaining a total of 10 with the throw of two dice, if one die has been thrown and shows a 4.

Solution

Let $A = \{\text{a total of } 10\}$ and $B = \{\text{a four shows on one die}\}$. The required probability is $P(A|B)$.

If a 4 shows on one die, then the conditional probability, $P(A|B)$ is the probability of throwing a 6 with the other die, which is $\frac{1}{6}$.

Example 4

There are 100 students in a first year college intake. 36 are male and are studying accounting, 9 are male and not studying accounting, 42 are female and studying accounting, 13 are female and are not studying accounting.

i) If the student is a male, find the probability that he is studying accounting

ii) If the student is a male, find the probability that he is not studying accounting

iii) If the student is studying accounting, find the probability that he is a male

iv) If the student is a female, find the probability that she is studying accounting

Solution

i) $P(A|M) = \dfrac{0.36}{0.45} = 0.80$

ii) $P(\overline{A}|M) = \dfrac{0.09}{0.45} = 0.20$

iii) $P(M|A) = \dfrac{0.36}{0.78} = 0.462$

iv) $P(A|F) = \dfrac{0.42}{0.55} = 0.764$

[Work out the solution]

Example 5

The probabilities of two events X and Y are given by $P(X) = 0.30$, $P(Y) = 0.40$ and $P(X \cup Y) = 0.58$.

Find i) $P(X \cap Y)$

ii) $P(X|Y)$

iii) $P(Y|X)$

Solution

i) $P(X \cap Y) = P(X) + P(Y) - P(X \cup Y)$
$= 0.30 + 0.40 - 0.58 = 0.12$

ii) $P(X|Y) = \dfrac{P(X \cap Y)}{P(Y)} = \dfrac{0.12}{0.40} = 0.30$

iii) $P(Y|X) = \dfrac{P(X \cap Y)}{P(X)} = \dfrac{0.12}{0.30} = 0.40$

$= \frac{12}{40} = \frac{3}{10} = 0.3$

Selection without replacement (dependent events)

Let us consider a box that contains 5 red bulbs and 2 blue bulbs that are all identical except for colour. Bulbs are selected at random from the box one after the other **without replacement**. Let R and B denote the events of selecting red and blue bulbs respectively, then from above, $n(R) = 5$, $n(B) = 2$ and total number of bulbs, $n(U) = 7$. The probability that the first bulb selected is red, $P(1R) = \frac{5}{7}$.

After the first bulb is selected the number of red bulbs $n(R)$ is reduced to 4, and the total number of bulbs $n(U)$ is also reduced to 6. In such cases the event that shows up in the second selection is dependent on the first selection. Thus, if a second bulb is selected, the probability that a red bulb is selected, $P(2R) = \frac{4}{6}$ and if a third bulb is selected, the probability that a blue bulb is selected, $P(3B) = \frac{2}{5}$. The probabilities for the second bulb are dependent upon the result of the first.

These are called **conditional probabilities**. The multiplication rule can be used when combining dependent events, just as in the case of independent events. The difference is, however, that one or more of the probabilities will be conditional as in the following examples.

Illustrative examples

Example 1

A box contains 14 white balls and 6 black balls. Find the probability of first drawing a black ball and then a white ball without replacement.

Solution

Let W and B denote the events of drawing white and black balls respectively.

From the question,

$n(W) = 14$, $n(B) = 6$, total number of balls, .

$n(U) = 20$

Let $1B$ and $2W$ denote the events that the first ball selected is black and second ball selected is white respectively.

P(first ball is black and second white)

$= P(1B \cap 2W) = P(1B) \times P(2W)$

$= \frac{6}{20} \times \frac{14}{19} = \frac{21}{95}$ or 0.22

Example 2

A box contains 10 red bulbs and 15 green bulbs.

i) A bulb is selected at random from the box and is not replaced. A second bulb is then selected at random. Find the probability that :

  a) Both bulbs are red;

  b) Both bulbs are of the same colour;

  c) The first bulb is green and the second is red.

ii) If four bulbs are taken out of the box (without replacement), what is the probability that the first two are green and the last two are red?

Solution

Let R and G denote the events of drawing red and green balls respectively.

From the question, $n(R) = 10$, $n(G) = 15$, and total number of balls, $n(U) = 25$

i) Let $1R$ and $2R$ denote the events that the first bulb selected is red and second bulb selected is red respectively. Let $1G$ denote the event that the first bulb selected is green and let $2G$ denote the event that the second bulb selected is green.

a) P(both red)

$= P$(first red and second red)

$= P(1R \cap 2R)$

$= P(1R) \times P(2R) = \frac{10}{25} \times \frac{9}{24} = \frac{3}{20}$

Note that since the first red bulb was not replaced, the number of red bulbs in the box is reduced to 9 and the total number of bulbs is also reduced to 24. Therefore the probability of selecting a second red bulb in the box is $\frac{9}{24}$.

Note also that $P(1R \cap 2R)$ means the probability of selecting a first red bulb and a second red bulb.

b) Both bulbs are of the same colour means either the 2 bulbs selected are all red or the 2 bulbs selected are all green.

$\therefore$ P(both bulbs are of the same colour)

$= P$(both are red) or P(both are green)

$= P(1R \cap 2R) + P(1G \cap 2G)$

But $P(1R \cap 2R) = \frac{3}{20}$ from (a).

$P(1G \cap 2G) = P(1G) \times P(2G)$

$= \frac{15}{25} \times \frac{14}{24} = \frac{7}{20}$

$\therefore$ P(both bulbs are of the same colour)

$= \frac{3}{20} + \frac{7}{20} = \frac{10}{20} = \frac{1}{2}$

c) P(the first bulb is green and the second is red) $= P(1G \cap 2R) = P(1G) \times P(2R)$

$= \frac{15}{25} \times \frac{10}{24} = \frac{1}{4}$

Note that after the first green bulb is selected, the number of red bulbs is still 10 and the total number of bulbs is reduced to 24. Therefore the probability that the second bulb is red is $\frac{10}{24}$.

ii) If four bulbs are selected, then

P(the first two are green and the last two are red) $= P(1G \cap 2G \cap 3R \cap 4R)$

$= P(1G) \times P(2G) \times P(3R) \times P(4R)$

$= \frac{15}{25} \times \frac{14}{24} \times \frac{10}{23} \times \frac{9}{22} = \frac{63}{1012}$

Example 3

A man has 12 identical marbles in his pocket consisting of 5 greens, 2 whites, 3 blues and 2 reds. He draws out a marble at random.

a) What is the probability that it is either a green or a blue?

b) If the man makes two draws, one after the other from his pocket without replacement, what is the probability that:

   i) the first draw will be green and the second draw a blue marble

   ii) neither draw is a red marble?

Solution

Let G, W, B and R denote the events of drawing green, white, blue and red marbles respectively. From the question, $n(G) = 5$, $n(W) = 2$, $n(B) = 3$, $n(R) = 2$ and total number of balls, $n(U) = 12$.

a) P(either green or blue marble)
$= P(G \cup B) = P(G) + P(B) - P(G \cap B)$

But it is clear that $G \cap B = \varnothing$ since the marble cannot be green and at the same time blue. Hence $P(G \cap B) = 0$

$\therefore$ P(either green or blue marble)
$= P(G \cup B) = P(G) + P(B)$
$= \frac{5}{12} + \frac{3}{12} = \frac{8}{12} = \frac{2}{3}$

b) Let 1G, 1W, 1B, 1R etc. have their usual meaning as before. Note that this question is selection without replacement.

i) P(the first will be green and second blue) $= P(1G \cap 2B) = P(1G) \times P(2B)$
$= \frac{5}{12} \times \frac{3}{11} = \frac{5}{44}$

ii) P(neither draw is red)
$= P(1G \cap 2G) + P(1G \cap 2W) +$
$P(1G \cap 2B) + P(1W \cap 2G) +$
$P(1W \cap 2W) + P(1W \cap 2B) +$
$P(1B \cap 2G) + P(1B \cap 2W) +$
$P(1B \cap 2B)$

$= \left(\frac{5}{12} \times \frac{4}{11}\right) + \left(\frac{5}{12} \times \frac{2}{11}\right) + \left(\frac{5}{12} \times \frac{3}{11}\right) + \left(\frac{2}{12} \times \frac{5}{11}\right) +$
$\left(\frac{2}{12} \times \frac{1}{11}\right) + \left(\frac{2}{12} \times \frac{3}{11}\right) + \left(\frac{3}{12} \times \frac{5}{11}\right) + \left(\frac{3}{12} \times \frac{2}{11}\right) +$
$\left(\frac{3}{12} \times \frac{2}{11}\right)$
$= \frac{90}{132} = \frac{15}{22}$

This method lists all the possible outcomes but it is lengthy.

Alternative method

Neither draw is red means the first marble is not red and the second marble is also not red. But there are 10 non red marbles, i.e. $n(\overline{R}) = 10$, where $\overline{R}$ is the event of drawing a non-red marble.

$\therefore$ P(neither draw is red)
$= P(1\overline{R} \cap 2\overline{R}) = P(\overline{R}) \times P(\overline{R}) = \frac{10}{12} \times \frac{9}{11} = \frac{15}{22}$

Note: There are 10 non-red marbles out of a total of 12. It can be seen that this method is easier to use.

Example 4

A box contains 6 red balls, 4 blue balls and 3 white balls. Five balls are drawn from the box at random, one at a time without replacement. What is the probability that:

a) All five are of the same colour;

b) The first two are blue, the next two are white and then red;

c) The first three are white and the other two are not white?

Express all your answers as fractions in their lowers terms.

Solution

Let R, B and W denote the events of drawing red, blue and white balls respectively. From the question, $n(R) = 6$, $n(B) = 4$, $n(W) = 3$ and total number of balls, $n(U) = 13$

Let 1R, 1B, 1W etc. have their usual meaning as before.

a) P(all five are of the same colour)
$= P(1R \cap 2R \cap 3R \cap 4R \cap 5R)$

Since the number of blue balls are not up to 5 for which all the five drawn can be blue. The same thing applies to the white balls. (*note this carefully*)

$\therefore\ P(1R \cap 2R \cap 3R \cap 4R \cap 5R)$
$= P(1R) \times P(2R) \times P(3R) \times P(4R) \times P(5R)$
$= \frac{6}{13} \times \frac{5}{12} \times \frac{4}{11} \times \frac{3}{10} \times \frac{2}{9} = \frac{2}{429}$

b) P(the first 2 being blue, the next 2 being white and the fifth being red)
$= P(1B \cap 2B \cap 3W \cap 4W \cap 5R)$
$= P(1B) \times P(2B) \times P(3W) \times P(4W) \times P(5R)$
$= \frac{4}{13} \times \frac{3}{12} \times \frac{3}{11} \times \frac{2}{10} \times \frac{6}{9} = \frac{2}{715}$

c) P(the first three being white and the other two not white)
$= P(1W \cap 2W \cap 3W \cap 4\overline{W} \cap 5\overline{W})$
$= (1W) \times P(2W) \times P(3W) \times P(4\overline{W}) \times P(5\overline{W})$
$= \frac{3}{13} \times \frac{2}{12} \times \frac{1}{11} \times \frac{10}{10} \times \frac{9}{9} = \frac{1}{286}$

$\overline{W}$ is the event of drawing a non-white ball. Note that there are 10 non-whites.

Alternative method

We can consider all the possible outcomes P(the first three being white and the other two not white)
$= P(1W \cap 2W \cap 3W \cap 4R \cap 5R) +$
$P(1W \cap 2W \cap 3W \cap 4R \cap 5B) +$
$P(1W \cap 2W \cap 3W \cap 4B \cap 5R)) +$
$P(1W \cap 2W \cap 3W \cap 4B \cap 5B)$
$= \frac{1}{286}$

Example 5

A bag P contains 4 white and 3 black balls and a second bag Q contains 5 white and 2 black balls. A ball is drawn at random from P and put into Q. A ball is then drawn at random from Q. Find the probability that the ball drawn from Q has

i) The same colour as drawn from P
ii) A different colour from that drawn from P.

Solution

From the question, bag P contains 4 white and 3 black balls $\Rightarrow n(U) = 7$ and bag Q contains 5 white and 2 black balls $n(U) = 7$

Let W_p and B_p denote the events of drawing a white and a black ball from bag P respectively, and let W_q and B_q denote the events of drawing a white and a black ball from bag Q respectively.

Initially, bags P and Q contain 7 balls each. Note that if a white ball is drawn from bag P and placed into bag Q, the total number of white balls in bag Q increases to 6 and the total number of balls also increases to 8 and so on.

i) P(same colour as drawn from P) = P(white ball from P and white ball from Q)
or
P(black ball from P and black ball from Q)
$= P(W_p \cap W_q) + P(B_p \cap B_q)$
$= \left(\frac{4}{7} \times \frac{6}{8}\right) + \left(\frac{3}{7} \times \frac{3}{8}\right) = \frac{33}{56}$

ii) P(different colour as drawn from P)
$= P$(white ball from P and black ball from Q) or
P(black ball from P and white ball from Q)
$= P(W_p \cap B_q) + P(B_p \cap W_q)$
$= \left(\frac{4}{7} \times \frac{2}{8}\right) + \left(\frac{3}{7} \times \frac{5}{8}\right) = \frac{23}{56}$

Example 6

A box contains 9 white, 7 red and 4 blue bulbs. Three bulbs are picked at random, one after the other without replacement.

a) Find the probability that:
   i) They are all of the same colour
   ii) There is one of each colour
   iii) Two of them are red
b) If it is known that the first one picked is blue, find the probability that the rest are white.

Solution

Let W, R, and B denote the events of picking white, red and blue bulbs respectively.

From the question, $n(W) = 9$, $n(R) = 7$, $n(B) = 4$ and total number of bulbs, $n(U) = 20$

i) $P(\text{they are all of the same colour})$

$= P(\text{all white})$ or $P(\text{all red})$ or $P(\text{all blue})$

$= P(1W \cap 2W \cap 3W) + P(1R \cap 2R \cap 3R) + P(1B \cap 2B \cap 3B)$

$= \left(\frac{9}{20} \times \frac{8}{19} \times \frac{7}{18}\right) + \left(\frac{7}{20} \times \frac{6}{19} \times \frac{5}{18}\right) + \left(\frac{4}{20} \times \frac{3}{19} \times \frac{2}{18}\right)$

$= \frac{504}{6840} + \frac{210}{6840} + \frac{24}{6840} = \frac{738}{6840} = 0.107$

ii) $P(\text{there is one of each colour})$

$= P(1W \cap 2R \cap 3B) + P(1W \cap 2B \cap 3R) + P(1R \cap 2W \cap 3B) + P(1R \cap 2B \cap 3W) + P(1B \cap 2R \cap 3W) + P(1B \cap 2W \cap 3R)$

$= \left(\frac{9}{20} \times \frac{7}{19} \times \frac{4}{18}\right) + \left(\frac{9}{20} \times \frac{4}{19} \times \frac{7}{18}\right) + \left(\frac{7}{20} \times \frac{9}{19} \times \frac{4}{18}\right) + \left(\frac{7}{20} \times \frac{4}{19} \times \frac{9}{18}\right) + \left(\frac{4}{20} \times \frac{7}{19} \times \frac{9}{18}\right) + \left(\frac{4}{20} \times \frac{9}{19} \times \frac{7}{18}\right)$

$= 0.22$

iii) $P(\text{two of them are red})$

$= P(\text{two of them are red and one is non-red})$

$= P(1R \cap 2R \cap 3\overline{R}) + P(1R \cap 2\overline{R} \cap 3R) + P(1\overline{R} \cap 2R \cap 3R)$

$= \left(\frac{7}{20} \times \frac{6}{19} \times \frac{13}{18}\right) + \left(\frac{7}{20} \times \frac{13}{19} \times \frac{6}{18}\right) + \left(\frac{13}{20} \times \frac{7}{19} \times \frac{6}{18}\right)$

$= 0.239$

$\overline{R}$ is the event of drawing non-red bulbs

Note that there are 13 non-red bulbs in the box.

Alternative method

We can consider all the possible outcomes

$P(\text{two of them are red})$

$= P(1R \cap 2R \cap 3W) + P(1R \cap 2R \cap 3B) + P(1W \cap 2R \cap 3R) + P(1R \cap 2W \cap 3R) + P(1B \cap 2R \cap 3R) + P(1R \cap 2B \cap 3R)$

$= 0.239$

b) $P(\text{if first is blue then the rest are white})$

$= P(1B \cap 2W \cap 3W)$

$= \frac{4}{20} \times \frac{9}{19} \times \frac{8}{18} = 0.042$

Example 7

A bag contains 5 white and 3 red identical balls. If the balls are drawn at random, one after the other without replacement, what is the probability that:

i) The first three balls are white?

ii) The first red ball is picked at the fifth draw?

iii) A red ball is picked at the second draw?

Solution

Let W and R denote events of picking white and red balls respectively.

From the question, $n(W) = 5$, $n(R) = 3$ and total number of balls, $n(U) = 8$

i) $P(\text{first three balls are white})$

$= P(1W \cap 2W \cap 3W)$

$= P(1W) \times P(2W) \times P(3W)$

$= \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} = \frac{5}{28}$

ii) $P(\text{first red ball is picked at the fifth draw})$

$= P(1W \cap 2W \cap 3W \cap 4W \cap 5R)$

$= P(1W) \times P(2W) \times P(3W) \times P(4W) \times P(5R)$

$= \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} \times \frac{3}{4} = \frac{3}{56}$

iii) $P(\text{red ball is picked at second draw})$

$= P(1W \cap 2R) + P(1R \cap 2R)$

$= \left(\frac{5}{8} \times \frac{3}{7}\right) + \left(\frac{3}{8} \times \frac{2}{7}\right) = \frac{3}{8}$

Example 8

A box contains 3 blue and 2 red balls and another box contains 2 blue and 1 red balls. If a ball is drawn at random from each box without replacement, find the probability that

a) both balls are blue

b) one ball is blue and the other is red

Solution

From the question, let the boxes be labelled P and Q

Box P contains 3 blue and 2 red balls

$n(U) = 5$ and

Box Q contains 2 blue and 1 red ball

$n(U) = 3$

Let B_p and R_p denote the events of drawing a blue and a red ball from bag P respectively, and let B_q and R_q denote the events of drawing a blue and a red ball from bag Q respectively.

a) P(both ball are blue)

$= P$(blue ball from P and blue ball from Q)

$= P(B_p \cap B_q)$

$= \frac{3}{5} \times \frac{2}{3} = \frac{2}{5}$

b) P(one ball is blue and the other is red)

$= P$(blue ball from P and red ball from Q) or P(red ball from P and blue ball from Q)

$= P(B_p \cap R_q) + P(R_p \cap B_q)$

$= \left(\frac{3}{5} \times \frac{1}{3}\right) + \left(\frac{2}{5} \times \frac{2}{3}\right) = \frac{3}{15} + \frac{4}{15} = \frac{7}{15}$

Example 9

A carton contains 20 bulbs of which 3 are defective. A woman has agreed to buy the whole carton of bulbs, if she selects 3 bulbs at random, one after the other, without replacement and it contains at most one defective bulb. Find the probability that she does not buy the carton of bulbs.

Solution

Let D = event that bulb is defective

$n(D) = 3$ and $n(\overline{D}) = 20 - 3 = 17$

The woman buys the carton if it contains at most one defective bulb.

P(she does not buy the carton of bulbs)

$= P$(carton contains 2 or more defective bulbs)

$= P(1D \cap 2D \cap 3\overline{D}) + P(1D \cap 2\overline{D} \cap 3D) + P(1\overline{D} \cap 2D \cap 3D) + P(1D \cap 2D \cap 3D)$

$= \left(\frac{3}{20} \times \frac{2}{19} \times \frac{17}{18}\right) + \left(\frac{3}{20} \times \frac{17}{19} \times \frac{2}{18}\right) + \left(\frac{17}{20} \times \frac{3}{19} \times \frac{2}{18}\right) + \left(\frac{3}{20} \times \frac{2}{19} \times \frac{1}{18}\right) = \frac{312}{6840} = \frac{39}{855}$

Example 10

A box contains n balls, of which 3 are red and the rest white. Two balls are selected at random one after the other without replacement. If the probability of selecting 2 red balls is $\frac{3}{28}$, find the value of n.

Solution

$P(\text{2 balls are red}) = P(1R \cap 2R) = \frac{3}{28}$

But $P(1R \cap 2R) = \dfrac{3}{n} \times \dfrac{2}{n-1} = \dfrac{6}{n(n-1)}$

$$\frac{6}{n(n-1)} = \frac{3}{28} \quad \Rightarrow \quad 3n^2 - 3n = 168$$

$$n^2 - n = 56 \quad \Rightarrow \quad n^2 - n - 56 = 0$$

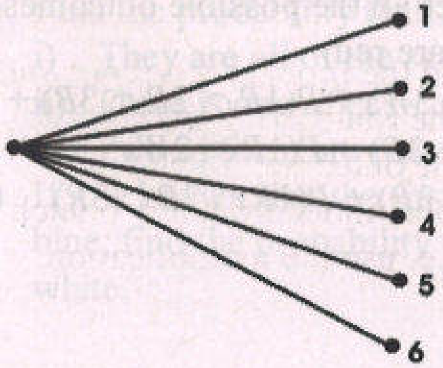
$$(n-8)(n+7) = 0 \quad \Rightarrow \quad n = 8$$

Probability tree diagrams

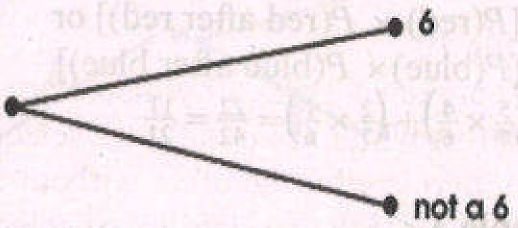
A **probability tree diagram** can be used to solve problems involving compound or combined events. Use these steps to help you.

Step 1: For each action in the problem, show the possible *outcomes* (or events) at the end of a branch of the tree.

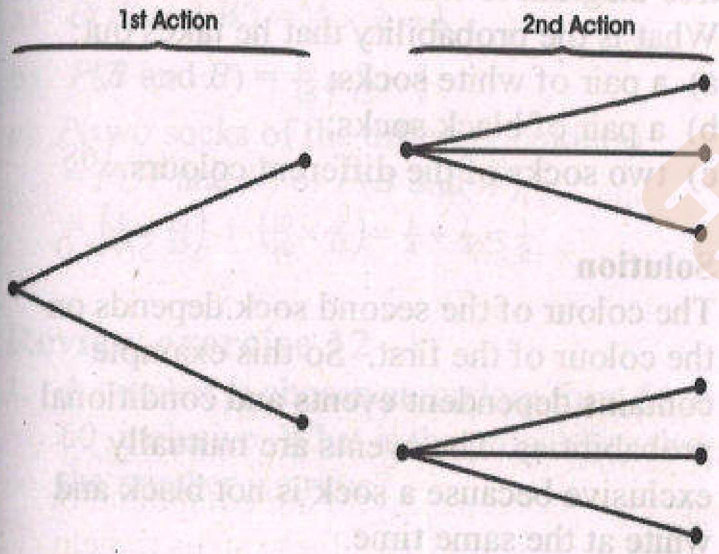
For example, when you toss a die the outcomes can be shown on the tree diagram below.



Sometimes you do not need to show all the individual outcomes. For example, you may be interested only in whether you toss a *six* or *not a six*, so you can draw this simpler tree diagram below.



Step 2: Each *action* in the problem is shown by a *stage* in the diagram. For example, for *two* actions there will be *two* stages as shown below.



Step 3: At the *end* of each route along the branches of the tree, write its final outcome.

Step 4: Work out the probability of each outcome shown by a branch on the diagram. Write each *probability* on its *branch* of the *tree*. Note that probabilities on adjacent branches of the tree must add up to 1.

Step 5: To find the probability of the *final outcome*, find the route along the branches which lead to the outcome. Multiply together the probabilities from any branch you go along (i.e. use the multiplication law).

Illustrative examples

Example 1

Each morning Jim and Jo have to catch the same bus, otherwise they are late for work. The probability that Jim catches the bus is 0.8 and the probability that Jo does is 0.9. (These probabilities are independent of each other).

Calculate the probability that on a given day:

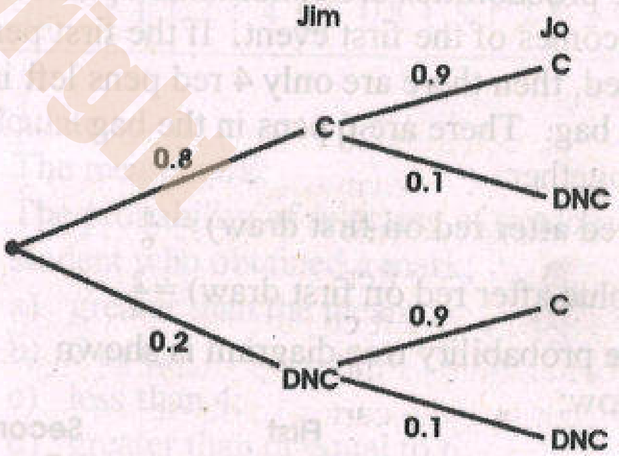
a) they both catch the bus;
b) Jim catches the bus but Jo does not;
c) neither catches the bus;
d) at least one of them catches the bus.

Solution

This is a type 4 question which can be solved using a tree diagram

C – means catches bus and *DNC* – means does not catch the bus

The probability tree diagram is shown below.



a) $P(\text{both catch the bus}) = 0.8 \times 0.9$
$= 0.72$

b) $P(\text{Jim catches the bus}) = 0.8$
$P(\text{Jo does not catch the bus}) = 0.1$
$\therefore$ $P(\text{Jim catches the bus but Jo does not})$
$= 0.8 \times 0.1 = 0.08$

c) $P(\text{neither catches the bus}) = 0.2 \times 0.1$
$= 0.02$

d) P(at least one of them catches the bus)
$= 1 - P$(neither catches the bus)
$= 1 - 0.02 = 0.98$

Example 2
There are 5 red pens and 2 blue pens in a bag. A pen is taken at random from the bag without replacement. A second pen is then taken from the bag. What is the probability that

a) both pens taken are red?
b) both pens are of the same colour?

Solution
This is a *type 6 question* which can be solved using a tree diagram

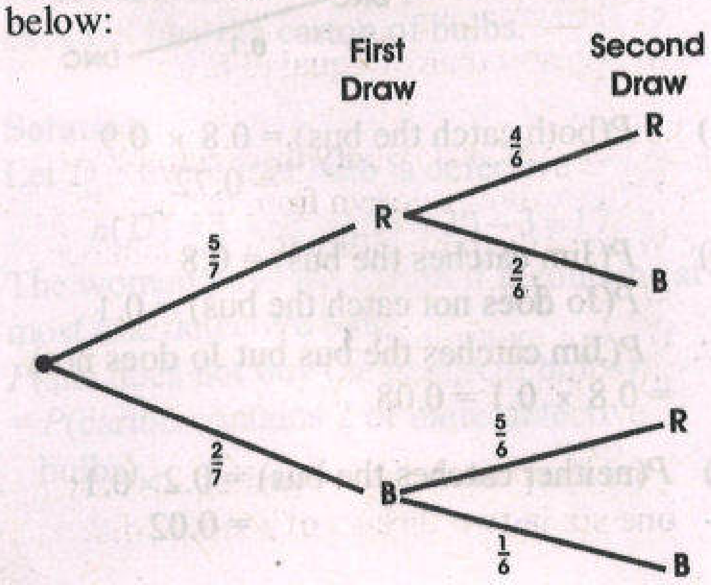
First pen:
n(red pens) = 5, n(blue pens) = 2
$n(U) = 7$; $P(\text{red}) = \frac{5}{7}$ and $P(\text{blue}) = \frac{2}{7}$

Second pen:
This event is a dependent event
The probabilities are conditional upon the outcomes of the first event. If the first pen is red, then there are only 4 red pens left in the bag. There are 6 pens in the bag altogether.

$P(\text{red after red on first draw}) = \frac{4}{6}$

$P(\text{blue after red on first draw}) = \frac{2}{6}$

The probability tree diagram is shown below:



a) P(both pens are red)
$= P(\text{red}) \times P(\text{red after red})$
$= \frac{5}{7} \times \frac{4}{6} = \frac{10}{21}$

b) P(both pens are of the same colour)
$= [P(\text{red}) \times P(\text{red after red})]$ or
$[P(\text{blue}) \times P(\text{blue after blue})]$
$= \left(\frac{5}{7} \times \frac{4}{6}\right) + \left(\frac{2}{7} \times \frac{1}{6}\right) = \frac{22}{42} = \frac{11}{21}$

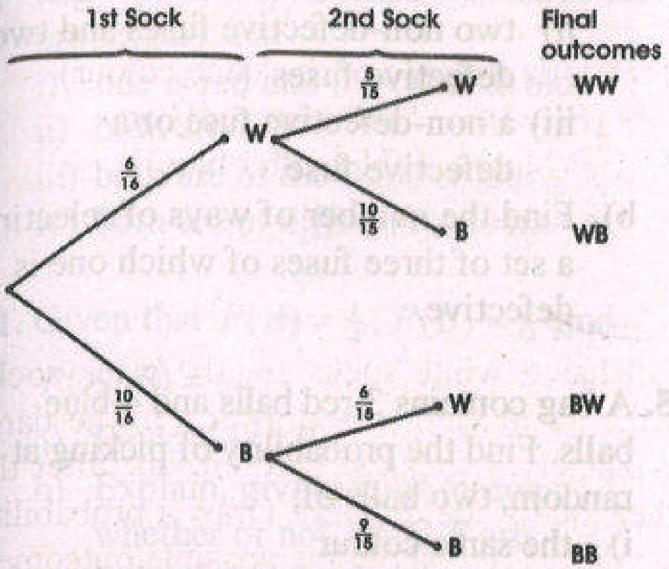
Example 3
Kofi has 6 white socks and 10 black socks of the same kind in a drawer. In the dark, he takes two socks at random, one after the other, out of the drawer. Draw a probability tree diagram to show the possible outcomes. What is the probability that he takes out:

a) a pair of white socks;
b) a pair of black socks;
c) two socks of the different colours.

Solution
The colour of the second sock depends on the colour of the first. So this example contains dependent events and conditional probabilities. The events are mutually exclusive because a sock is not black and white at the same time.
Let W stand for a white sock and B stand for a black sock. $P(W) = \frac{6}{16}$ and $P(B) = \frac{10}{16}$
The probabilities for the second sock are dependent upon the result of the first.

After a sock has been taken, 15 socks are left. After a white sock has been taken, 5 white socks and 10 black socks are left.
$P(W \text{ after } W) = \frac{5}{15}$ and $P(B \text{ after } W) = \frac{10}{15}$
Also after a black sock has been taken, 6 white socks and 9 black socks are left.
$P(W \text{ after } B) = \frac{6}{15}$ and $P(B \text{ after } B) = \frac{9}{15}$

The probability tree diagram is shown below.



a) $P(W \text{ and } W) = \frac{6}{16} \times \frac{5}{15} = \frac{1}{8}$

b) $P(B \text{ and } B) = \frac{10}{16} \times \frac{9}{15} = \frac{3}{8}$

c) $P(\text{two socks of the different colours})$
$= P(W \text{ and } B) \text{ or } P(B \text{ and } W)$
$= \left(\frac{6}{16} \times \frac{10}{15}\right) + \left(\frac{10}{16} \times \frac{6}{15}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Review exercise 12

1. A number is chosen at random from 51 to 60 inclusive. What is the probability that the number is prime?

2. A number is picked at random from the set $P = \{2, 3, 4, 5, ..., 9\}$. Find the probability that the number is prime.

3. Kweku chooses a number at random from 1 to 40 inclusive. What is the probability that the number is divisible by 4?

4. A number is chosen at random from 2 to 16 inclusive. What is the probability that the number is prime?

5. A bag contains 15 red balls and 6 blue balls. Find the probability of picking at random a blue ball from the bag.

6. In a survey on a housing estate, a group of 30 women were asked how many children they had.
The results are shown below.

No. of children	0	1	2	3	4	5	6
No. of women	2	4	5	8	7	3	1

One of the women is chosen at random from this group. What is the probability that she has

a) 4 children
b) no children
c) more than 2 children

7. The following table shows a mark out of 8 scored by a group of students on an assignment.

Marks	0	1	2	3	4	5	6	7	8
No. of students	1	2	4	6	8	5	2	2	1

Calculate:
i) The mean mark;
ii) The probability of selecting at random a student who obtained a mark;
  a) greater than the mean;
  b) 5 or 6;
  c) less than 4;
  d) greater than or equal to 6.

8. A bag contains twenty discs numbered 1-20. A disc is drawn from the bag. What is the probability that the number on the disc
i) is a multiple of 3
ii) contains a 5

9. Find the probability of getting at least one six in two throws of a single die.

10. Ali, Beme and Chukwu shoot at a single target. Their chances of hitting the target are, respectively $\frac{2}{5}$, $\frac{1}{4}$ and $\frac{2}{3}$. If they fire together, find the chance that

i) only one shot hits the target
ii) they all miss the target
iii) at least one shot hits the target.

11. Two students are chosen at random from a class consisting of eighteen boys and twelve girls. What is the probability that the two students are:

i) both boys
ii) both girls
iii) of the same sex
iv) a boy and a girl

12. In a class of 52, 24 offer mathematics and 26 offer English. 2 students offer both English and Mathematics. A student is selected at random in the class. Find the probability that he studies:

i) Mathematics only
ii) English
iii) Mathematics
iv) Neither English nor Mathematics
v) Either English or Mathematics
(Hint: You can use a Venn diagram)

13. Two fair dice, each of which has its faces numbered 1, 2, 3, 4, 5 and 6 are tossed together. Find the probability of obtaining

i) a number greater than 3 on each die
ii) a total scored of 9
iii) a number greater than 3 on each die and a total score of 9
iv) a number greater that 3 on each die or a total score of 9

14. A box contains ten identical fuses three of which are defective.

a) Find the probability of selecting at random without replacement
i) four non-defective fuses
ii) two non-defective fuses and two defective fuses
iii) a non-defective fuse or a defective fuse.

b) Find the number of ways of selecting a set of three fuses of which one is defective.

15. A bag contains 3 red balls and 4 blue balls. Find the probability of picking at random, two balls of;

i) the same colour
ii) different colours

16. A box contains 25 identical balls, 10 of them are black, 7 of them are red and the rest are white. If three balls are drawn at random from the box one after the other without replacement, find the probability that

a) one is black, one is red and one is white
b) exactly two are black
c) none of them is red

17. A fair coin is tossed three times. Write down the set of possible outcomes. What is the probability of obtaining exactly two heads?

18. The probabilities of three horses P, Q, R winning a race have been found to be 0.2, 0.5 and 0.3 respectively. What is the probability that the three horses will tie for a first position in a race?

19. Thirty cards, numbered 31 to 60 are placed in a box. If one card is drawn at random, what is the probability that the number on the card contains the digit 5?

20. A bag contains 12 blue and 8 red pens. 2 pens are chosen at random from the bag without replacement. What is the probability that
- i) one is red and the other is blue
- ii) both are red
- iii) both are of the same colour
- iv) both are of different colours.

21. Given that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{7}$ and $P(A/B) = \frac{1}{3}$
- a) Find $P(A \cup B)$.
- b) Explain, giving your reasons, whether or not A and B are independent.

22. It is known that $P(V \cup W) = \frac{2}{5}$, $P(V) = \frac{1}{10}$ and that V and W are mutually exclusive. Find:
a) $P(W)$ b) $P(V | W)$

23. A student representative council consists of 7 girls and 5 boys. If a dance committee of 4 persons is to be formed, what is the probability that the committee will consist of
- a) 2 boys and 2 girls?
- b) 3 boys and 1 girl?

24. Two coins are tossed and a die thrown. What is the probability of obtaining head, a tail and a 4?

25. 5 out of 12 articles are known to be defective. If three are picked one after the other, without replacement, find the probability that all the three articles are non-defective.

26. The probabilities that Kwasi, Kofi and Ama pass an examination are $\frac{2}{3}$, $\frac{5}{8}$ and $\frac{3}{4}$ respectively. Find the probability that
- i) all three pass the examination
- ii) none of them passes the examination
- iii) only two of them pass the examination.

27. A pair of fair dice each numbered 1 to 6 is tossed. Find the probability of getting
- i) a sum of at least 9
- ii) a sum of 7
- iii) a sum of at least 10

28. Two groups of male students cast their vote on a particular proposal. The results are as follows

	In favour	Against
Group *A*	128	32
Group *B*	96	48

- a) If a student in favour of the proposal is selected for a post, what is the probability that he is from group *A*?
- b) If a student is chosen at random, what is the probability that he is against the proposal?

29. A man has 9 identical balls in a bag. Out of these, 3 are black, 2 are blue and the remaining are red.
- a) If a ball is drawn at random, what is the probability that it is
  - i) not blue ii) not red
- b) If a ball is drawn at random, one after the other, what is the probability that both of them will be
  - i) black, if there is no replacement?
  - ii) blue, if there is a replacement?

30. Three balls are drawn one after the other with replacement, from a bag containing 5 red, 9 white and 4 blue identical balls.

What is the probability that they are one red, one white and one blue?

31. A fair die and unbaised coin are tossed once. What is the probability of obtaining 3 and a tail?

32. A box contains 10 marbles; 7 of which are black and 3 are red. Two marbles are drawn one after the other without replacement. Find the probability of getting
- i) a red, then a black marble
- ii) two black marbles.

33. A box contains 6 red, 3 white and 5 blue balls which are identical except for colour. If a ball is selected at random from each box, find the probability that it is:
- a) red
- b) white
- c) red or white
- d) not white
- e) neither white or red
- f) either white or blue.

34. The probability that Ade passes a mathematics test is p and that he passes a physics test is q. Find the probability that he passes the mathematics test and fails the physics test.

35. The probability that any of 5 telephone lines in a company is engaged at an instant is $\frac{1}{4}$. What is the probability that at least one of the lines is engaged?

36. The probabilities that a husband and his wife will be alive in 10 years time are 0.7 and 0.8 respectively. What is the probability that at that time
- a) both of them will be alive,
- b) none will be alive.
- c) at least on will be alive?

37. Every adult in a village belongs to either political party X or Y. The probability that an adult chosen at random from the village is in party X is $\frac{2}{3}$. If 6 adults are randomly chosen from the village, what is the probability that
- a) none is in X?
- b) two are in X?
- c) at least four are in Y?

38. A bag contains 6 red and 5 green identical balls. Two balls are drawn at random one after the other without replacement. What is the probability that both balls are of the same colour?

39. The probabilities that man and his wife will be alive in twenty years time are 0.6 and 0.7 respectively. What is the probability that one of them will not be alive in twenty years time?

40. A box contains 10 white and 15 green identical bulbs. Two bulbs are selected one after the other from the box without replacement. What is the probability that
- i) both bulbs are white?
- ii) the first bulb is green and the second is white?

41. The probabilities that two athletes P and Q will win a gold medal in a competition are 0.75 and 0.60 respectively. What is the probability that in the competition:
- a) both P and Q will win gold medals,
- b) neither of them will win a gold medal,
- c) at least one of them will win a gold medal?

42. An unbiased coin and a fair die are tossed. Find the probability of obtaining a tail on the coin and 1, 2, 5, or 6 the die.

43. A box contains identical balls of which 12 are red, 16 white and 8 blue. Three balls are drawn from the box one after the other without replacement.
Find the probability that
- a) three are red;
- b) the first is blue and the other two are red:
- c) two are white and one is blue.

44. The probability that two hunters P and Q hit their targets are $\frac{2}{3}$ and $\frac{3}{4}$ respectively. The two hunters aim at a target together.
- a) What is the probability that they both miss the target?
- b) If the target is hit, what is the probability that
  - i) only hunter P hits it,
  - ii) only one of them hits it,
  - iii) both hunters hit the target?

45. Two fair dice and two fair coins are tossed once. Calculate the probability that:
- a) the sum of the numbers showing on the uppermost faces of the dice is even and two tails appear on the coins;
- b) the sum of the two number showing on the uppermost faces of the dice is equal to 11 and no tail appears on the coins.

46. Two fair dice are tossed. Find the probability
- a) of getting a total of 7 or 8;
- b) of not getting a total of 9;
- c) that the two dice show the same number.

47. Two balls are drawn successively, without replacement from a bag containing 8 blue balls and 14 white balls. Find the probability that they are both blue.

48. Three balls are drawn, without replacement, from a bay containing 8 blue, 12 red and 10 yellow balls. Find the probability that
- a) the three balls are of the same colour
- b) two of the balls are red and one is blue
- c) the three balls are of different colours.

49. A box contains 2 white and 3 blue identical marbles. Two identical marbles are picked at random, one after the other, without replacement, what is the probability of picking two marbles of different colours?

50. Mrs. Jones is expecting a baby. The probability that it will be a boy is $\frac{1}{2}$ and the probability that the baby will have blue eyes is $\frac{1}{4}$. What is the probability that she will have a blue eyed boy?

51. a) A number is chosen at random from the integers 1 to 10. Find the probability that the number is
  - i) prime
  - ii) a multiple of 2
  - iii) either a prime or a multiple of 2.

b) The probability that a candidate passes an examination is $\frac{1}{5}$.
Find the probability that if
  - i) two candidates are selected at random, only one passes the examination.
  - ii) three candidates are selected at random, all pass the examination

CHAPTER 13

COORDINATE GEOMETRY 2 – THE CIRCLE

Introduction

In our school geometry, we have studied various properties of a circle. We know that the circle is the locus of a point moving in a plane such that its distance from a fixed point is always a constant.
The fixed point is called the centre, C of the circle and the fixed distance is called the radius, r of the circle.

The general form of the equation of a circle

The **general equation** of a circle is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ where $(-g, -f)$ is the centre of the circle and the radius, $r = \sqrt{g^2 + f^2 - c}$.

Note the following about the equation of a circle

1. It is a second degree equation
2. There is no term in xy
3. The coefficients of x^2 and y^2 are the same. Therefore $x^2 + y^2 + 2x - 4y - 4 = 0$ is an equation of a circle, while
$2x^2 + y^2 + 3x - 4y - 4 = 0$,
$x^2 + y^2 + 2xy - 4y - 4 = 0$ and
$x^2 - y^2 + 2x - 4y - 4 = 0$ are not equations of a circle.

To find the equation of a circle

The equation of a circle can be found if the following are given:

1. the *radius and the coordinates* of the centre.
2. *three points* on the circle
3. the coordinates of the end points of the diameter

1. To find the equation of circle given its radius and centre

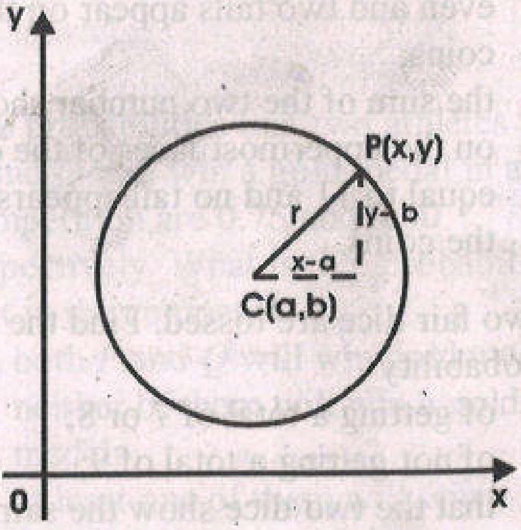
Let $P(x, y)$ be any point on a circle whose centre is $C(a, b)$ and radius r.
The equation of a circle is given by:

$$(x-a)^2 + (y-b)^2 = r^2$$

If the centre is at the origin $O(0, 0)$, then the equation becomes $(x-0)^2 + (y-0)^2 = r^2$
i.e. $x^2 + y^2 = r^2$.

Conversely, we can say that if the equation of a circle is given as $(x-a)^2 + (y-b)^2 = r^2$, then the centre of the circle is (a, b) and the radius, r.
For example, if the equation of a circle is $(x-1)^2 + (y-2)^2 = 9$ then its centre $= (1, 2)$ and radius, $r = 3$. Similarly, if the equation of a circle is $(x+2)^2 + (y-4)^2 = 16$ then its centre $= (-2, 4)$ and radius 4.

Note: The radius, r is the distance between the centre (a, b) and any point (c, d) on the circle. i.e. $r^2 = (c-a)^2 + (d-b)^2$



Illustrative examples

Example 1

Find the equation of the circle with centre (–1, 2) and radius 3.

Solution

The equation is $[x-(-1)]^2+(y-2)^2=3^2$

$$(x+1)^2+(y-2)^2=3^2$$

Expanding and simplifying

$$x^2+y^2+2x-4y-4=0\ .$$

Example 2

Find the equation of a circle with centre (–2, 3) which passes through the point (1, 2).

Solution

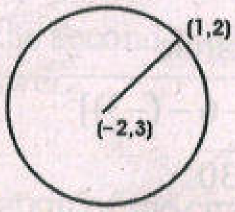
First find the radius r which is the distance between the centre (–2, 3) and the point (1, 2) i.e.

$$r=\sqrt{(-2-1)^2+(3-2)^2}=\sqrt{9+1}=\sqrt{10}$$

Therefore the equation of the circle of radius $\sqrt{10}$ and centre (–2, 3) is

$$(x+2)^2+(y-3)^2=(\sqrt{10})^2=10$$

$$x^2+y^2+4x-6y+3=0$$



Example 3

Find the equation of a circle with :

i) centre (1, 2) and radius 3

ii) centre (0, 4) and radius 1

iii) centre (–3, –7) and radius 2

iv) centre (4, 5) and radius 3

v) centre (1, 2) and radius 1

vi) centre (–1, –1) and radius 3

vii) centre $(-2, -\frac{3}{2})$ and radius $\sqrt{5}$

viii) centre (2, –3) and radius 2

ix) centre (2, 3), radius 1

x) centre (–3, 4), radius 5

xi) centre $(\frac{2}{3}, -\frac{1}{3})$, radius $\frac{2}{3}$

xii) centre (0, –5), radius 5

xiii) centre (3, 0), radius $\sqrt{2}$

xiv) centre $(-\frac{1}{4}, \frac{1}{3})$, radius $\frac{1}{2}\sqrt{2}$.

xv) centre (3, –4) and passing through the point (3, 0)

Solution

i) $x^2+y^2-2x-4y-4=0$

ii) $x^2+y^2-\ 8y+15=0$

iii) $x^2+y^2+6x+14y+54=0$

iv) $x^2+y^2-8x-10y+32=0$

v) $x^2+y^2-2x-4y+4=0$

vi) $x^2+y^2+2x+2y-7=0$

vii) $4x^2+4y^2+16x+12y+5=0$

viii) $x^2+y^2-4x+6y+9=0$

ix) $x^2+y^2-\ 4x-6y+12=0$

x) $x^2+\ y^2+6x-8y=0$

xi) $9x^2+9y^2-12x+6y+1=0$

xii) $x^2+\ y^2+10y=0$

xiii) $x^2+\ y^2-6x+7=0$

xiv) $144x^2+144y^2+72x-96y-47=0$

xv) $x^2+y^2-6x+8y+9=0$

[Work out the solution]

2. To find the equation of a circle passing through three points

To find the equation of a circle through three points A, B and C say, use any of the methods described below:

First method

1. Substitute the co-ordinates of each point in turn in the general equation of the circle
2. Solve the three simultaneous equations to find g, f and c.
3. Substitute g, f and c in the general equation

Second method

1. Find the equations of the perpendicular bisectors of any two pairs of the points, i.e. AB and BC say
2. Find the point of intersection of the two lines. This gives the centre of the circle.

3. Find the radius of the circle using this centre and any of the three points.
4. Using the centre and radius, find the equation of the circle.

Illustrative examples

Example 1

Find the equation of the circle which passes through the points $A(2, 1)$, $B(0, -3)$ and $C(4, 3)$.

Solution

First method

The general equation of a circle is:

$x^2 + y^2 + 2gx + 2fy + c = 0$

Substituting the three points into this equation, we have for $A(2, 1)$,

$(2)^2 + (1)^2 + 2g(2) + 2f(1) + c = 0$

$4 + 1 + 4g + 2f + c = 0$

$4g + 2f + c = -5$ -------(1)

For $B(0, -3)$,

$0 + 9 + 0 - 6f + c = 0$

$-6f + c = -9$ -------------(2)

For $C(4, 3)$,

$16 + 9 + 8g + 6f + c = 0$

$8g + 6f + c = -25$ -------(3)

Now solving equations (1), (2) and (3) simultaneously, we have

Eqn(1) – eqn(2)

$4g + 8f = 4$ ....(4)

Eqn(2) – Eqn(3)

$-8g - 12f = 16$ ....(5)

$2 \times$ eqn(4) + eqn(5)

$4f = 24 \Rightarrow f = 6$

Substituting $f = 6$ into eqn(4)

$4g + 48 = 4 \Rightarrow 4g = -44$

$g = -11$

Again substituting $f = 6$ and $g = -11$ into eqn(1), (2) or (3) $\Rightarrow c = 27$

$\therefore$ The equation of the circle is

$x^2 + y^2 - 22x + 12y + 27 = 0$

Second method

The gradient of $AB = 2$

$\therefore$ The gradient of the perpendicular bisector $AB = -\frac{1}{2}$

The midpoint of $AB = (1, -1)$.

Therefore the equation of the perpendicular bisector of AB is given by :

$y - (-1) = -\frac{1}{2}(x - 1)$

$2y + x = -1$ ....(1)

Similarly, the gradient of $BC = \frac{3}{2}$

$\therefore$ The gradient of the perpendicular bisector $AB = -\frac{2}{3}$

The midpoint of $BC = (2, 0)$.

Therefore the equation of the perpendicular bisector of BC is given by :

$y - 0 = -\frac{2}{3}(x - 2)$

$3y + 2x = 4$ ......(2)

Solving (1) and (2) simultaneously gives: $x = 11$ and $y = -6$.

Therefore the centre of the circle is $(11, -6)$. Using this centre and the point $B(0, -3)$ gives

The radius of the circle

$r = \sqrt{(11 - 0)^2 + [-6 - (-3)]^2}$

$= \sqrt{121 + 9} = \sqrt{130}$

The equation of the circle with centre $(11, -6)$ and radius $\sqrt{130}$ is given by;

$(x - 11)^2 + (y + 6)^2 = \left(\sqrt{130}\right)^2$

$x^2 - 22x + 121 + y^2 + 12y + 36 = 130$

$x^2 + y^2 - 22x + 12y + 27 = 0$

Example 2

If the three points $P(-2, 2)$, $Q(2, 2)$ and $R(-2, -4)$ lie on a circle. Find the equation of the circle.

Solution

Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$

Substituting the three points into this equation, we have

For $P(-2, 2)$,

$(-2)^2 + (2)^2 + 2g(-2) + 2f(2) + c = 0$

$-4g + 4f + c = -8$ ....(1)

For $Q(2, 2)$,

$(2)^2 + (2)^2 + 2g(2) + 2f(2) + c = 0$

$4g + 4f + c = -8$ ....(2)

For $R(-2, -4)$,

$(-2)^2 + (-4)^2 + 2g(-2) + 2f(-4) + c = 0$

$-4g - 8f + c = -20$ ....(3)

Solving equations (1), (2) and (3) simultaneously

Eqn(1) – eqn(2)

$-8g = 0 \Rightarrow g = 0$

Eqn(2) – eqn(3)

$8g + 12f = 12 \Rightarrow f = 1$ since $g = 0$

Substituting $g = 0$ and $f = 1$ into eqn(1)

$c = -12$

$\therefore$ The equation of the circle is

$x^2 + y^2 + 2y - 12 = 0$

Note: Use the second method to arrive at the same answer.

Example 3

Find the equation of the circle which passes through the points

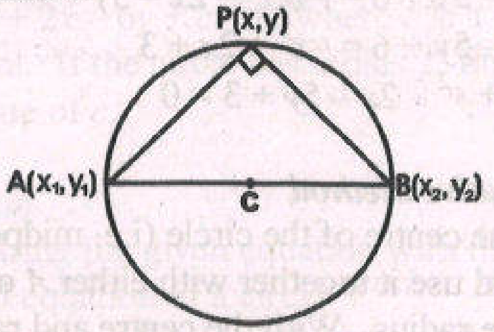
i) $A(-2, -3)$, $B(-2, 3)$ and $C(1, 2)$
ii) $A(1, 1)$, $B(2, 2)$ and $C(3, -2)$
iii) $A(0, 3)$, $B(2, 3)$ and $C(-2, 1)$
iv) $A(3, 3)$, $B(1, 3)$ and $C(1, 2)$
v) $A(0, 2)$, $B(3, 0)$ and $C(3, 2)$

Solution

i) $3x^2 + 3y^2 + 8x - 23 = 0$
ii) $5x^2 + 5y^2 - 29x - y + 20 = 0$
iii) $x^2 + y^2 - 2x - 9 = 0$
iv) $x^2 + y^2 - 4x - 5y + 9 = 0$
v) $x^2 + y^2 - 3x - 2y = 0$

[Work out the solution in each case.]

3. Equation of a circle given the ends of the diameter



Let A, B be the ends of a diameter of a circle, where $A(x_1, y_1)$ and $B(x_2, y_2)$

Let $P(x, y)$ be any point on the circle whose diameter is AB.

$\angle APB = 90°$

(slope of AP) × (slope of BP) = -1

$$\frac{y - y_1}{x - x_1} \times \frac{y - y_2}{x - x_2} = -1$$

$(y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

which is the required equation.

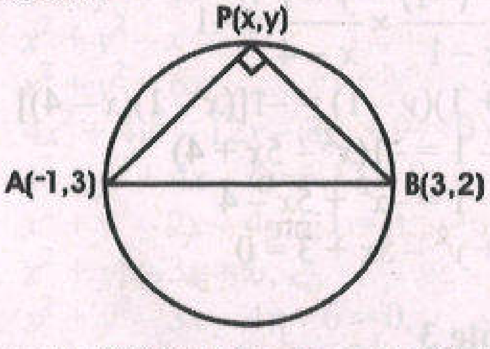
Illustrative examples

Example 1

Find the equation of a circle with AB as diameter where $A(-1, 3)$ and $B(3, 2)$.

Solution

Let $P(x, y)$ be any point on the circle as shown below.



By geometry if AB is the diameter then AP is perpendicular to PB.

$\therefore$ Gradient of AP × gradient of $PB = -1$.

i.e. $\dfrac{y - 3}{x - (-1)} \times \dfrac{y - 2}{x - 3} = -1$

$$(y-3)(y-2) = -1[(x+1)(x-3)]$$
$$y^2 - 5y + 6 = -1(x^2 - 2x - 3)$$
$$y^2 - 5y + 6 = -x^2 + 2x + 3$$
$$x^2 + y^2 - 2x - 5y + 3 = 0$$

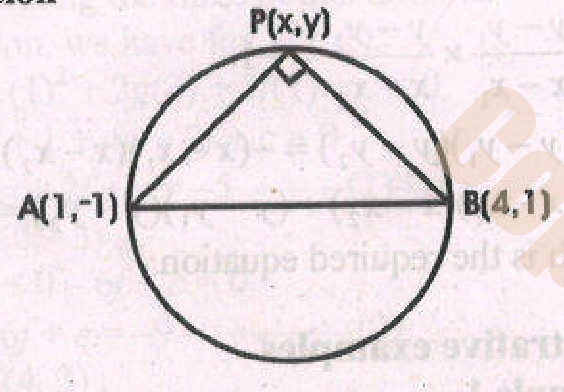
Alternative method

Find the centre of the circle (i.e. midpoint of AB) and use it together with either A or B to find the radius. With the centre and radius the equation can be formed.

Example 2

Find the equation of the circle with AB as diameter where $A(1,-1)$ and $B(4, 1)$.

Solution



Let $P(x, y)$ be any point on the circle. By geometry if AB is the diameter then the AP is perpendicular to PB as shown in the diagram below.

$\therefore$ Gradient of $AP \times$ gradient of $PB = -1$

$$\frac{y-(-1)}{x-1} \times \frac{y-1}{x-4} = -1$$
$$(y+1)(y-1) = -1[(x-1)(x-4)]$$
$$y^2 - 1 = -1(x^2 - 5x + 4)$$
$$y^2 - 1 = -x^2 + 5x - 4$$
$$x^2 + y^2 - 5x + 3 = 0$$

Example 3

Find the equation of a circle with AB as diameter where

i) $A(2, 1)$ and $B(4, 7)$
ii) $A(-3, -1)$ and $B(-5, -3)$
iii) $A(-5, 2)$ and $B(2, -6)$
iv) $A(4, -8)$ and $B(2, 4)$
v) $A(4, 5)$ and $B(9, 6)$

Solution

i) $x^2 + y^2 - 6x - 8y + 15 = 0$
ii) $x^2 + y^2 + 8x + 4y + 18 = 0$
iii) $x^2 + y^2 + 3x + 4y - 22 = 0$
iv) $x^2 + y^2 - 6x + 4y - 24 = 0$
v) $x^2 + y^2 - 13x - 11y + 66 = 0$

[Work out the solution in each case.]

To find the centre and radius of a circle given its equation

When the equation of the circle is given, the radius and centre of the circle can be found.

Illustrative examples

Example 1

Find the radius and centre of the equation $3x^2 + 3y^2 - 24x + 12y + 11 = 0$

Solution

First reduce the coefficients of x^2 and y^2 to unity

$$3x^2 + 3y^2 - 24x + 12y + 11 = 0$$
$$x^2 + y^2 - 8x + 4y + \tfrac{11}{3} = 0 \quad \text{............(1)}$$

Next compare equation (1) with the general equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$

$$2g = -8 \Rightarrow g = -4,$$
$$2f = 4 \Rightarrow f = 2 \text{ and } c = \tfrac{11}{3}$$

$\therefore$ The radius of the circle is

$$r = \sqrt{g^2 + f^2 - c} = \sqrt{(-4)^2 + 2^2 - \tfrac{11}{3}}$$
$$= \sqrt{20 - \tfrac{11}{3}} = \sqrt{\tfrac{49}{3}} = \tfrac{7}{\sqrt{3}} = \tfrac{7}{3}\sqrt{3}$$

The centre is $(-g, -f) = (4, -2)$

Alternative Method

$$3x^2 + 3y^2 - 24x + 12y + 11 = 0$$
$$x^2 + y^2 - 8x + 4y + \tfrac{11}{3} = 0$$

Now completing the squares of x and y

$$(x-4)^2 + (y+2)^2 = -\tfrac{11}{3} + 16 + 4$$

$$(x-4)^2+(y+2)^2=\tfrac{49}{3}$$

Compare this with the equation of a circle with centre (a, b) and radius r,

$(x-a)^2+(y-b)^2=r^2$

The centre of the circle is $(4, -2)$ and radius, $r=\tfrac{7}{3}\sqrt{3}$.

Example 2

Find the centre and radius of the circle $x^2+y^2-2x-6y=15$.

Solution

For this example, the coefficients of x^2 and y^2 are unity i.e. $x^2+y^2-2x-6y-15=0$

Comparing $x^2+y^2-2x-6y-15=0$ with the general equation $x^2+y^2+2gx+2fy+c=0$ gives: $2g=-2 \Rightarrow g=-1$, $2f=-6$

$f=-3$ and $c=-15$

$\therefore$ The radius of the circle is

$$r=\sqrt{g^2+f^2-c}=\sqrt{(-1)^2+(-3)^2-(-15)}$$
$$=\sqrt{10+15}=\sqrt{25}=5$$

The centre of the circle is $(-g, -f)=(1, 3)$

Example 3

Find the co-ordinates of the centre and the radius of the circle given by

$3x^2+3y^2+6x+12y+9=0$

Solution

First reduce the coefficients of x^2 and y^2 to unity

$3x^2+3y^2+6x+12y+9=0$

$x^2+y^2+2x+4y+3=0$ ....(1)

Next compare equation (1) with the general equation of a circle $x^2+y^2+2gx+2fy+c=0$

$2g=2 \Rightarrow g=1$, $2f=4 \Rightarrow f=2$ and $c=3$

$\therefore$ The radius of the circle is

$$r=\sqrt{g^2+f^2-c}=\sqrt{1^2+2^2-3}=\sqrt{2}$$

The centre is $(-g, -f)=(-1, -2)$

Example 4

The equation of a circle is of the form $x^2+y^2+2x-6y+c=0$, where c is a constant. If the circle has radius 2, find the value of c.

Solution

Comparing the given equation with the general equation of a circle

$x^2+y^2+2gx+2fy+c=0$

$2g=2 \Rightarrow g=1, 2f=-6 \Rightarrow f=-3$ and $c=c$

$\therefore$ The radius of the circle is

$$r=\sqrt{g^2+f^2-c}=\sqrt{1^2+(-3)^2-c}=2$$
$$\sqrt{10-c}=2$$
$$10-c=4 \quad\Rightarrow\quad c=6$$

Example 5

Find the centre and radius of the circle whose equation is:

i) $x^2+y^2+8x-2y-8=0$
ii) $x^2+y^2+x+3y-2=0$
iii) $x^2+y^2+6x-5=0$
iv) $2x^2+2y^2-3x+2y+1=0$
v) $x^2+y^2=4$
vi) $(x-2)^2+(y+3)^2=9$
vii) $2x+6y-x^2-y^2=1$
viii) $3x^2+3y^2+6x-3y-2=0$
ix) $x^2+y^2-2x-6y=15$
x) $x^2+y^2-6x+14y+49=0$
xi) $x^2+y^2+x+4y-2=0$
xii) $x^2+y^2-x=2$
xiii) $x^2+y^2=6x$
xiv) $4x^2+4y^2-12x-8y+9=0$
xv) $x^2+y^2+4x-6y+12=0$,
xvi) $x^2+y^2-2x-4y+1=0$,
xvii) $x^2+y^2-3x=0$,
xviii) $x^2+y^2-3x-4y-6=0$,
xix) $2x^2+2y^2+x+y=0$,
xx) $36x^2+36y^2-24x-36y-23=0$,

Solution

i) $(-4, 1)$; 5

ii) $\left(-\frac{1}{2}, -\frac{3}{2}\right)$; $\frac{3\sqrt{2}}{2}$

iii) $(-3, 0)$; $\sqrt{14}$

iv) $\left(\frac{3}{4}, -\frac{1}{2}\right)$; $\frac{\sqrt{5}}{4}$

v) $(0, 0)$; 2

vi) $(2, -3)$; 3

vii) $(1, 3)$; 3

viii) $\left(-1, \frac{1}{2}\right)$; $\frac{\sqrt{69}}{6}$

ix) $(1, 3)$; 5

x) $(3, -7)$; 3

xi) $\left(-\frac{1}{2}, -2\right)$; $\frac{5}{2}$

xii) $\left(\frac{1}{2}, 0\right)$; $\frac{3}{2}$

xiii) $(3, 0)$; 3

xiv) $\left(\frac{3}{2}, 1\right)$; 1

xv) $(-2, 3)$; 1

xvi) $(1, 2)$; 2

xvii) $\left(\frac{3}{2}, 0\right)$; $\frac{3}{2}$

xviii) $\left(\frac{3}{2}, 2\right)$; $\frac{7}{2}$

xix) $\left(-\frac{1}{4}, -\frac{1}{4}\right)$; $\frac{\sqrt{2}}{4}$

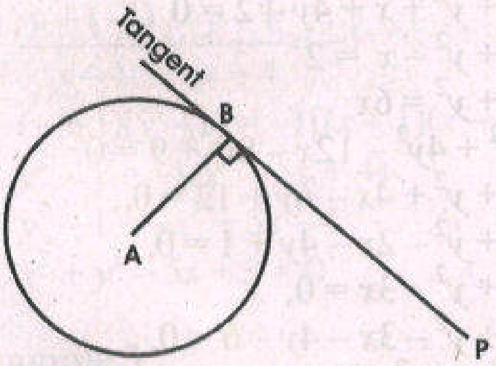
xx) $\left(\frac{1}{3}, \frac{1}{2}\right)$; 1

[Work out the solution]

Equations of a tangent and normal at a point on a circle

The tangent to a circle

The line drawn from the centre of a circle to the point of contact of the **tangent** to the circle is perpendicular to the tangent at any point on the circle. From the figure below, *AB* is perpendicular to *PB*. Therefore the gradient of the tangent *PB* is equal to the negative reciprocal of the gradient of *AB*. This condition is used to find the equation of a tangent to a circle.



Illustrative examples

Example 1

Find the equation of the tangent to the circle $x^2 + y^2 - 2x + 4y = 15$ at the point $(-1, 2)$.

Solution

First find the centre of the circle using the equation. The equation of the circle is

$x^2 + y^2 - 2x + 4y - 15 = 0$

Comparing this with the general equation

$x^2 + y^2 + 2gx + 2fy + c = 0$

$g = -1$ and $f = 2$

$\therefore$ The centre of the circle is $(-g, -f)$

i.e. $(1, -2)$

The gradient of the line joining the centre $(1, -2)$ and the point of contact $(-1, 2)$ is

$$\frac{2-(-2)}{-1-1} = -2.$$

Therefore the gradient of the tangent is $\frac{1}{2}$.

Hence the equation of the tangent with gradient $\frac{1}{2}$, and passing through $(-1, 2)$ is:

$$y - 2 = \tfrac{1}{2}(x + 1)$$

$$2y - 4 = x + 1 \Rightarrow x - 2y + 5 = 0$$

Example 2

Find the equation of the tangent to the circle $x^2 + y^2 + 4x - 10y = 12$ at the point $(3, 1)$.

Solution

The equation of the circle is

$$x^2 + y^2 + 4x - 10y - 12 = 0$$

Comparing this with the general equation

$x^2 + y^2 + 2gx + 2fy + c = 0 \Rightarrow g = 2$ and $f = -5$

$\therefore$ The centre of the circle is $(-g, -f)$

i.e. $(-2, 5)$

The gradient of the line joining the centre $(-2, 5)$ and the point of contact $(3, 1)$ is

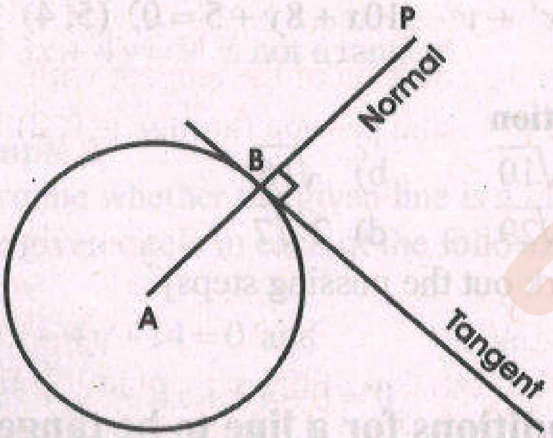
$$\frac{1-5}{3-(-2)} = -\frac{4}{5}.$$

Therefore the gradient of the tangent is $\frac{5}{4}$ (i.e. the negative reciprocal of $-\frac{4}{5}$). Hence the equation of the tangent with gradient $\frac{5}{4}$ and through $(3, 1)$ is:

$$y-1=\tfrac{5}{4}(x-3)$$
$$4y-4=5x-15$$
$$5x-4y-11=0$$

The normal to a circle

The **normal** to a circle is perpendicular to the tangent at the point of contact. The gradient of the normal to a circle is equal to the gradient of the line joining the centre to the point of contact. From the figure below, the gradient of the normal *BP* is equal to the gradient of *AB*.



Illustrative examples

Example 1

Find the equation of the normal to the circle $x^2+y^2-2x+4y=15$ at point $(-1, 2)$.

Solution

By comparing the given equation with the general equation, the centre of the circle is $(1, -2)$

The gradient of the line joining the centre $(1, -2)$ and the point of contact $(-1, 2)$ is -2.

Therefore the gradient of normal is -2.

Hence the equation of the normal with gradient -2 passing through the point $(-1, 2)$ is $y-2=-2(x+1)$

$$y-2=-2x-2$$
$$2x+y=0$$

Example 2

Find the equation of the normal to the circle $x^2+y^2+4x-10y=12$ at the point $(3, 1)$.

Solution

By comparing the given equation with the general equation, the centre of the circle is $(-2, 5)$.

The gradient of the line joining the centre $(-2, 5)$ and point of contact $(3, 1)$ is $-\frac{4}{5}$.

Therefore the gradient of the normal is $-\frac{4}{5}$

Hence the equation of the normal with gradient $-\frac{4}{5}$ and passing through $(3, 1)$ is:

$$y-1=-\tfrac{4}{5}(x-3)$$
$$5y-5=-4x+12$$
$$4x+5y-17=0$$

Example 3

Verify that the given points lie on the following circles and find the equations of the tangents and normals to the circles at these points:

i) $x^2+y^2+6x-2y=0$, $(0, 0)$;

ii) $x^2+y^2-2x-4y-8=0$, $(3, 5)$

iii) $x^2+y^2+2x+4y-12=0$, $(3, -1)$

iv) $x^2+y^2+2x-2y-8=0$, $(2, 2)$

v) $2x^2+2y^2-8x-5y-1=0$, $(1\ -1)$;

Solution

i) $3x-y=0$

ii) $x-4y+17=0$

iii) $4x+y-8=0$

iv) $3x+y-8=0$

v) $4x+9y+5=0$

[Work out the solution for the normals]

The length of a tangent from an external point to a circle

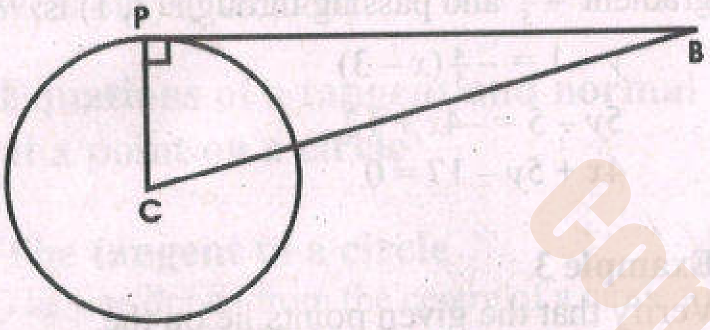
Consider the circle below. The length of the tangent at P to a point B (i.e. PB) can be found using Pythagoras theorem.

i.e. $CB^2 = CP^2 + PB^2$

$\Rightarrow PB = \sqrt{CB^2 - CP^2}$, where CB is the distance between the centre C and the given point B and CP is the radius of the circle. i.e.

The length of the tangent

$$= \sqrt{\left(\begin{array}{l}\text{distance between the}\\ \text{centre and the point}\end{array}\right)^2 - (\text{radius})^2}$$



Illustrative examples

Example 1

Find the length of the tangent from $(-5, 8)$ to the circle $x^2 + y^2 - 4x - 6y + 3 = 0$.

Solution

Comparing the given equation with the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$

$g = -2$, $f = -3$ and $c = 3$

$\therefore$ The centre of the circle is $(-g, -f)$

i.e. $(2, 3)$

The radius of the circle is

$$r = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{(-2)^2 + (-3)^2 - 3} = \sqrt{10}$$

The distance between the centre $(2, 3)$ and the point $(-5, 8)$ is

$$\sqrt{(-5-2)^2 + (8-3)^2} = \sqrt{74}$$

Therefore the length of tangent

$$= \sqrt{\left(\begin{array}{l}\text{distance between the}\\ \text{centre and the point}\end{array}\right)^2 - (\text{radius})^2}$$

$$= \sqrt{74 - 10} = \sqrt{64} = 8$$

Example 2

Find the length of the tangents from the given points to the following circles

a) $x^2 + y^2 + 4x - 6y + 10 = 0$, $(0\ 0)$

b) $x^2 + y^2 - 4x - 8y - 5 = 0$, $(8\ 2)$

c) $x^2 + y^2 + 6x + 10y - 2 = 0$, $(-2, 3)$

d) $x^2 + y^2 - 10x + 8y + 5 = 0$, $(5, 4)$

Solution

a) $\sqrt{10}$ b) $\sqrt{15}$

c) $\sqrt{29}$ d) $2\sqrt{7}$

[Work out the missing steps]

Conditions for a line to be tangential to a circle

A line is a tangent to a circle if and only if the distance of that line from the centre of the circle is equal to the radius.

Illustrative examples

Example 1

Determine whether the lines $5y = 12x - 33$ and $3x + 4y = 9$, are tangents to the circle

$$x^2 + y^2 + 2x - 8y = 8$$

Solution

First find the centre and radius of the circle.

The equation of the circle can be written as

$$(x+1)^2 + (y-4)^2 = 25$$

Therefore $C(-1, 4)$ is the centre of the circle and the radius is 5 units.

For the line $5y = 12x - 33$

i.e. $12x - 5y - 33 = 0$, the distance d_1 from the centre (–1, 4) of the circle is given by:

$$d_1 = \left|\frac{12(-1)-5(4)-33}{\sqrt{[12^2+(-5)^2]}}\right| = \left|-\tfrac{65}{13}\right| \quad \Rightarrow \quad d_1 = 5$$

Thus $5y = 12x - 33$ is a tangent

For the line $3x + 4y = 9$

i.e. $3x + 4y - 9 = 0$, the distance d_2 from the centre (–1, 4) of the circle is given by:

$$d_2 = \left|\frac{3(-1)+4(4)-9}{\sqrt{[3^2+4^2]}}\right| = \left|\tfrac{4}{5}\right| \quad \Rightarrow \quad d_2 = \tfrac{4}{5} \neq 5$$

Thus $3x + 4y = 9$ is not a tangent.

Example 2

Determine whether the given line is a tangent to the given circle in each of the following cases.

i) $3x - 4y + 14 = 0$ and
$x^2 + y^2 + 4x + 6y - 3 = 0$

ii) $5x + 12y = 4$ and
$x^2 + y^2 - 2x - 2y + 1 = 0$

iii) $x + 2y + 6 = 0$ and
$x^2 + y^2 - 6x - 4y + 8 = 0$

Solution

i) yes ii) yes iii) no

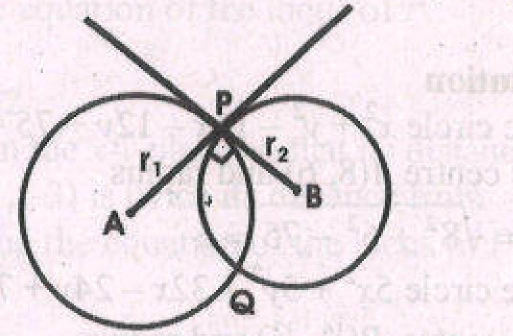
[Work out the solution in each case]

Orthogonal circles

Two circles which intersect at right angles cut *orthogonally* and are called **orthogonal circles**.

Two circles are said to be orthogonal if the square of the distance between their centres is equal to the sum of the squares of the radii. Thus, if A and B are the centres of two circles with radii r_1 and r_2, then $AB^2 = r_1^2 + r_2^2$

For example, the circles
$(x-1)^2 + (y-3)^2 = 9$
and $(x-4)^2 + (y-7)^2 = 16$ are orthogonal because $AB^2 = 3^2 + 4^2 = 25$ and $r_1^2 + r_2^2 = 9 + 16 = 25$. i.e. $AB^2 = r_1^2 + r_2^2$



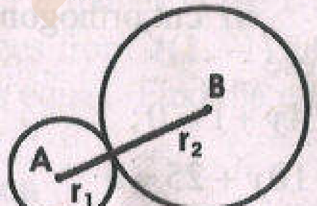
Conditions for two circles to touch each other

Two circles touch each other *externally* if the distance between their centres is the *sum* of their radii. Two circles touch each other *internally* if the distance between their centres is the *difference* of their radii.

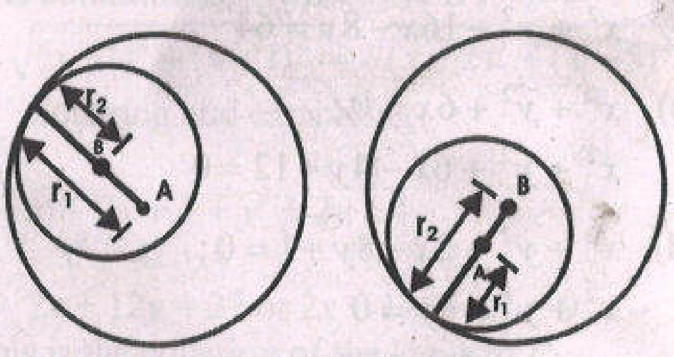
Consider two circles with centres A and B and radii r_1 and r_2. See the diagrams below.

If the circles touch externally, then
$AB = r_1 + r_2$.

If the circles touch internally, then
$AB = r_1 - r_2$.



External ⟶ $AB = r_1 + r_2$



Internal

$AB = r_1 - r_2$ or $AB = r_2 - r_1$

Illustrative examples

Example 1

Show that the circles
$x^2 + y^2 - 16x - 12y + 75 = 0$ and
$5x^2 + 5y^2 - 32x - 24y + 75 = 0$ touch externally.

Solution

The circle $x^2 + y^2 - 16x - 12y + 75 = 0$ has centre $A(8, 6)$ and radius

$$r_1 = \sqrt{8^2 + 6^2 - 75} = 5$$

The circle $5x^2 + 5y^2 - 32x - 24y + 75 = 0$ has centre $B(\frac{16}{5}, \frac{12}{5})$ and radius

$$r_2 = \sqrt{\left(\tfrac{16}{5}\right)^2 + \left(\tfrac{12}{5}\right)^2 - 15} = 1$$

The distance between the centres is given by

$$AB^2 = (8 - \tfrac{16}{5})^2 + (6 - \tfrac{12}{5})^2$$

$$= \left(\tfrac{24}{5}\right)^2 + \left(\tfrac{18}{5}\right)^2 = 36$$

$$AB = 6 \text{ and } r_1 + r_2 = 5 + 1 = 6$$

Therefore the circles touch externally.

Example 2

Some of the following pairs of circles touch and some pairs are orthogonal. Determine whether each of the pairs

i) touch　　ii) cut orthogonally
iii) do neither of these

a) $x^2 + y^2 + 2x - 4y + 1 = 0$;
$x^2 + y^2 - 6x - 10y + 25 = 0$

b) $x^2 + y^2 + 8x + 2y - 8 = 0$;
$x^2 + y^2 - 16x - 8y = 64$

c) $x^2 + y^2 + 6x = 0$;
$x^2 + y^2 + 6x - 4y + 12 = 0$

d) $x^2 + y^2 + 2x - 8y + 1 = 0$;
$x^2 + y^2 - 6y = 0$

e) $x^2 + y^2 + 2x = 3$;
$x^2 + y^2 - 6x - 3 = 0$

Solution

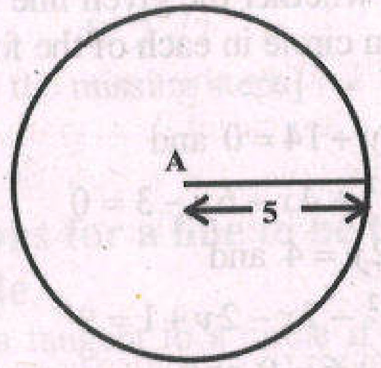
a) touch externally
b) cut orthogonally
c) touch internally
d) do neither
e) cut orthogonally

[Work out the solution in each case]

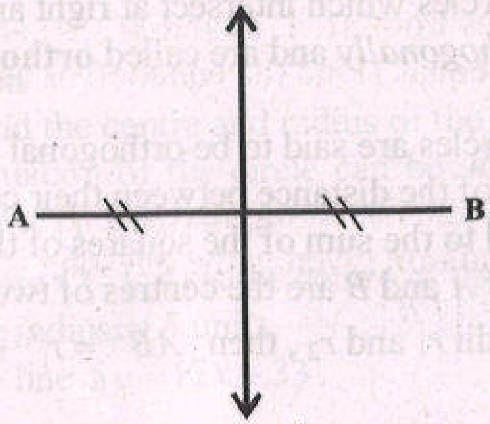
The locus of a point

When a set of points satisfy certain geometric constraints, we say that, these points represent a locus.

Consider a set of points in a plane, which are always at a distance of 5 units from a fixed point A, in the same plane. Obviously, all these points lie on the circumference of the circle with centre A and radius 5.



Suppose A and B are two fixed points and we want the set of all points which are equidistant from A and B, then obviously every point on the perpendicular bisector of AB will satisfy the given condition and hence the required set of points is the set of all points on the perpendicular bisector of AB.



Hence, given a geometrical condition, we are able to find a set of points which satisfy this condition. Such a set of points is called a locus.

Equation of a locus

The algebraic representation of a locus is called the equation of the locus.

A point *P* in co-ordinate geometry is identified by its co-ordinate (x, y).

Thus, an equation in *x, y*, satisfied by points on the locus, is called the **equation of that locus.**

Conversely, whenever a pair x_1, y_1 satisfies the equation of the locus, the corresponding point (x_1, y_1) is on the locus.

Thus, we observe that, the equation of a locus represents the locus completely.

Method of finding the equation of a locus

Step 1: Take any point P (x, y) on the locus

Step 2: Write down the geometric constraint of the locus. Note that this will be given in the question.

Step 3: Convert this geometric constraint into an algebraic equation, involving *x* and *y*.

Step 4: Simplify the equation and get the required 'equation of locus'.

Illustrative examples

Example 1

A point *P* moves in the *x-y* plane, so that its distances from $A(-1, 2)$ and $B(3, 4)$ are always equal. Find the equation of its locus.

Solution

Let (x, y) be the co-ordinates of *P*.

The condition given in the question is

$|PA| = |PB|$

$$\therefore \sqrt{(x+1)^2+(y-2)^2} = \sqrt{(x-3)^2+(y-4)^2}$$

Squaring and expanding

$$x^2+2x+1+y^2-4y+4$$
$$= x^2-6x+9+y^2-8y+16$$
$$8x+4y=20 \text{ or } 2x+y=5.$$

This is the equation of the locus of *P*.

Example 2

P moves in the *x-y* plane so that its distance from $A(-1, -3)$ is twice its distance from $B(2, 4)$, find the equation of the locus of *P*.

Solution

Let the co-ordinates of *P* be (x, y).

The condition is: $|AP| = 2|PB|$

Therefore we have

$$\sqrt{(x+1)^2+(y+3)^2} = 2\sqrt{(x-2)^2+(y-4)^2}$$

Squaring both sides

$$(x+1)^2+(y+3)^2 = 4[(x-2)^2+(y-4)^2]$$

Expanding to remove brackets

$$x^2+2x+1+y^2+6y+9$$
$$= 4x^2-16x+16+4y^2-32y+64$$
$$3x^2+3y^2-18x-38y+70=0$$

Example 3

A point *P* moves in the *x-y* plane, so that its distances from $A(1, -1)$ and $B(2, 5)$ are always equal. Find the equation of its locus.

Solution

Let (x, y) be the co-ordinates of *P*.

The condition is: $|PA| = |PB|$.

$$\therefore \sqrt{(x-1)^2+(y+1)^2} = \sqrt{(x-2)^2+(y-5)^2}$$

Squaring and expanding

$$x^2-2x+1+y^2+2y+1$$
$$= x^2-4x+4+y^2-10y+25$$
$$2x+12y=27 \text{ or } 2x+12y-27=0$$

This is the equation of the locus of *P*.

Example 4
$M(4, 3)$ and $N(-1, -3)$ are two fixed points in the x-y plane. If $P(x, y)$ is a variable point in the plane such that $PM^2 = PN^2$, find the equation of the locus of P. Describe the locus completely.

Solution
Given the points $M(4, 3)$, $N(-1, -3)$ and $P(x, y)$. The condition given in the question is $PM^2 = PN^2$

$\therefore \quad (x-4)^2 + (y-3)^2 = (x+1)^2 + (y+3)^2$

Expanding to remove brackets

$x^2 - 8x + 16 + y^2 - 6y + 9$

$= x^2 + 2x + 1 + y^2 + 6y + 9$

$-10x - 12y + 15 = 0$

which is the equation of a straight line.
To describe the line, find the gradient and the intercept on y-axis.

i.e. $-10x - 12y + 15 = 0$

$y = -\frac{10}{12}x + \frac{15}{12} \quad \Rightarrow \quad y = -\frac{5}{6}x + \frac{5}{4}$

$\therefore$ The locus is a straight line with gradient $-\frac{5}{6}$ and intercept on y-axis $(0, \frac{5}{4})$.

Example 5

a) $M(-1, 4)$ and $N(2, -3)$ are two fixed points. A variable point $S(x, y)$ moves such that $2|MS| = |NS|$. Find the equation of the locus of S. Express it in its simplest form and state its constant properties

b) Find the points in which the locus meets the y-axis.

Solution

a) Given the points $M(-1, 4)$, $N(2, -3)$ and $S = (x, y)$.
The condition is that $2|MS| = |NS|$

$\therefore \quad 2\sqrt{(x+1)^2 + (y-4)^2}$

$= \sqrt{(x-2)^2 + (y+3)^2}$

Squaring both sides

$4[(x+1)^2 + (y-4)^2] = (x-2)^2 + (y+3)^2$

Expanding to remove brackets

$4(x^2 + 2x + 1 + y^2 - 8y + 16)$

$= x^2 - 4x + 4 + y^2 + 6y + 9$

$3x^2 + 3y^2 + 12x - 38y + 55 = 0$ which is the equation of a circle.

Its constant properties are its radius and centre. To find these properties, we have

$3x^2 + 3y^2 + 12x - 38y + 55 = 0$

$x^2 + y^2 + 4x - \frac{38}{3}y + \frac{55}{3} = 0$ ....(1)

Comparing eqn(1) with the general equation of a circle, $x^2 + y^2 + 2gx + 2fy + c = 0$, we have

$2g = 4 \quad \Rightarrow \quad g = 2; \quad 2f = \frac{38}{3} \quad \Rightarrow f = \frac{19}{3}$

and $c = \frac{55}{3}$

$\therefore$ The radius, $r = \sqrt{g^2 + f^2 - c}$

$r = \sqrt{2^2 + (\frac{19}{3})^2 - \frac{55}{3}} = \frac{2}{3}\sqrt{58}$

The centre $= (-g, -f) = (-2, -\frac{19}{3})$

b) For the intercept of the locus on the y-axis, put $x = 0$ in the equation of the locus.

$\therefore$ The equation of locus becomes

$3(0)^2 + 3y^2 + 12(0) - 38y + 55 = 0$

$3y^2 - 38y + 55 = 0$

$3y^2 - 33y - 5y + 55 = 0$

$3y(y - 11) - 5(y - 11) = 0$

$(3y - 5)(y - 11) = 0 \quad \Rightarrow \quad y = \frac{5}{3}$ or 11

$\therefore$ The points of intersection with the y-axis are $(0, \frac{5}{3})$ and $(0, 11)$.

Points of intersection of a line and a circle
The **point(s) of intersection(s)** of a line and circle are obtained by solving the equations of the line and the circle simultaneously. If only one point is obtained, then it means the line touches the circle and is therefore a tangent to the circle.

Illustrative examples

Example 1

Find the points of intersection of the line $2y - x = 0$ and the circle $x^2 + y^2 - 2x - 6y - 15 = 0$

Solution

From $2y - x = 0 \Rightarrow x = 2y$

Substituting this into the equation of the circle, we have

$$(2y)^2 + y^2 - 2(2y) - 6y - 15 = 0$$
$$4y^2 + y^2 - 4y - 6y - 15 = 0$$
$$5y^2 - 10y - 15 = 0 \Rightarrow y^2 - 2y - 3 = 0$$
$$(y - 3)(y + 1) = 0 \Rightarrow y = -1 \text{ or } 3$$

Substituting into $x = 2y$

i.e. $y = -1 \Rightarrow x = -2$ and $y = 3 \Rightarrow x = 6$

$\therefore$ The points of intersection of the line and circle are $(-2, -1)$ and $(6, 3)$

Example 2

The line $2y = x + 3$ meets the circle $x^2 + y^2 - 2x + 6y - 15 = 0$ in the points M and N, where M lies in the first quadrant. Find the co-ordinates of M and N.

Solution

To find the co-ordinates of M and N, solve the equation of the line $2y = x + 3$ and the circle $x^2 + y^2 - 2x + 6y - 15 = 0$ simultaneously using the method of substitution

$2y = x + 3$ ....(1)

$x^2 + y^2 - 2x + 6y - 15 = 0$ ....(2)

From eqn(1), $x = 2y - 3$. Substituting this into equation (2)

$$(2y - 3)^2 + y^2 - 2(2y - 3) + 6y - 15 = 0$$
$$4y^2 - 12y + 9 + y^2 - 4y + 6 + 6y - 15 = 0$$
$$5y^2 - 10y = 0$$
$$5y(y - 2) = 0 \Rightarrow y = 0 \text{ or } 2$$

Substituting these values into equation (1)

$y = 0 \Rightarrow x = -3$ and

$y = 2 \Rightarrow x = 1$

$\therefore$ The points of intersection are $(-3, 0)$ and $(1, 2)$

Since M lies in the first quadrant

$M(1, 2)$ and $N(-3, 0)$

To show that a point lies on a line or circle

If a point lies on a line or circle then the co-ordinates of the point must satisfy the equation of the line or circle.

Illustrative examples

Example 1

Investigate whether the point $(1, 2)$ lies on the line $2x + y - 3 = 0$.

Solution

Putting $x = 1$ and $y = 2$ in the equation

$$2(1) + 2 - 3 = 1$$

Therefore since the point does not satisfy the equation it means it does not lie on the line.

Example 2

Show that the point $(3, 2)$ lies on the circle $x^2 + y^2 - 2x - 5y + 3 = 0$

Solution

Substituting the point $(3, 2)$ i.e. $x = 3$ and $y = 2$ into the equation.

$$(3)^2 + (2)^2 - 2(3) - 5(2) + 3 = 0$$

Therefore the point lies on the circle.

Solved examination questions

The following examples will help you gain more insight in what you have studied so far.

Example 1

Show that the points $A(-2, -2)$, $B(-1, 2)$ and $C(3, 1)$ are the vertices of an isosceles triangle. Hence or otherwise find the equation of the circle which passes through the points A, B and M, where M is the midpoint of AC.

Solution

If the triangle is isosceles then two of its sides must be equal.

$$|AB| = \sqrt{(-2+1)^2+(-2-2)^2}$$
$$= \sqrt{1+16} = \sqrt{17}$$
$$|AC| = \sqrt{(3+2)^2+(1+2)^2}$$
$$= \sqrt{25+9} = \sqrt{34}$$
$$|BC| = \sqrt{(3+1)^2+(1-2)^2}$$
$$= \sqrt{16+1} = \sqrt{17}$$

Since $|AB| = |BC|$

the triangle is isosceles.

Now if M is the midpoint of AC, then by geometry AB is the diameter of the circle passing through A, B and M. (Note: This is because ABC is an isosceles triangle with $|AB| = |BC|$)

The centre of the circle is the midpoint of AB

i.e. $(\frac{1}{2}(-2-1), \frac{1}{2}(-2+2)) = (-\frac{3}{2}, 0)$

The radius of the circle $= \frac{1}{2}|AB| = \frac{1}{2}\sqrt{17}$

$\therefore$ The equation of the circle with centre $(-\frac{3}{2}, 0)$ and radius $\frac{1}{2}\sqrt{17}$ is

$$(x+\tfrac{3}{2})^2 + (y-0)^2 = (\tfrac{1}{2}\sqrt{17})^2$$
$$x^2 + 3x + \tfrac{9}{4} + y^2 = \tfrac{17}{4}$$
$$4x^2 + 4y^2 + 12x - 8 = 0$$

Note that the midpoint of AC, i.e. $M(\frac{1}{2}, -\frac{1}{2})$ must satisfy this equation since M lies on the circle.

Alternative method

Let the equation of the circle be of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ and substitute the co-ordinates of the points A, B and M to get three equations. Solve the equations simultaneously for g, f and c.

Example 2

a) Find the equation of the line which is parallel to the line $5x + 4y = 18$ and makes an intercept of 2 units on the x-axis.

b) The co-ordinates of the points A and B are $(3, -6)$ and $(-1, 2)$ respectively. Find the equation of the circle with AB as the diameter.

Solution

a) The required line is parallel to the line with equation $5x + 4y = 18$ therefore they have the same gradient.

i.e. $5x + 4y = 18$

$$y = \tfrac{18}{4} - \tfrac{5}{4}x$$

$\therefore$ The gradient of the line $= -\frac{5}{4}$

Hence the equation of the line with gradient $-\frac{5}{4}$ and passing through $(2, 0)$ is

$$y - 0 = -\tfrac{5}{4}(x-2)$$
$$4y = -5x + 10$$
$$5x + 4y - 10 = 0$$

b) If $P(x, y)$ is any point on the circle, then by geometry AP is perpendicular to PB.

$\therefore$ Gradient $AP \times$ gradient $PB = -1$

$$\text{i.e.}\quad \frac{y-2}{x+1} \times \frac{y+6}{x-3} = -1$$
$$(y-2)(y+6) = -1[(x+1)(x-3)]$$
$$x^2 + y^2 - 2x + 4y - 15 = 0$$

Example 3

The equation of the tangent to a circle at $P(4, 2)$ is $3y = x + 2$. The line $y = 3x$ passes through the centre C of the circle. Find

a) the coordinates of C

b) the equation of the circle

c) the points T_1 and T_2 where the circle cuts the y--axis

Solution

a) $3y = x + 2$

$y = \frac{1}{3}x + \frac{2}{3} \Rightarrow$ gradient of tangent $= \frac{1}{3}$

gradient of $CP = -3$

(Since CP is perpendicular to the tangent)

CP passes through the point (4, 2), therefore

Equation of CP is $y - 2 = -3(x - 4)$

$y = -3x + 14$

∴ Equation of line CP is $y = -3x + 14$

C is the point of intersection of CP

(i.e. $y = -3x + 14$) and the line $y = 3x$

Solving the two equations simultaneously

$$-3x + 14 = 3x \quad \Rightarrow 6x = 14$$

$$x = \frac{14}{6} = \frac{7}{3} \quad \Rightarrow y = 3 \times \frac{7}{3} = 7$$

∴ Coordinates of C is $(\frac{7}{3}, 7)$

b) Now we have $P(4, 2)$ and $C(\frac{7}{3}, 7)$

radius of circle, $r^2 = (4 - \frac{7}{3})^2 + (2 - 7)^2$

$= \frac{25}{9} + 25 = \frac{250}{9}$

Equation of a circle with centre $(\frac{7}{3}, 7)$ and radius, $r^2 = \frac{250}{9}$ is given by:

$$\left(x - \tfrac{7}{3}\right)^2 + (y - 7)^2 = \left(\sqrt{\tfrac{250}{9}}\right)^2$$

$$x^2 - \frac{14}{3}x + \frac{49}{9} + y^2 - 14y + 49 = \frac{250}{9}$$

$$x^2 + y^2 - \frac{14}{3}x - 14y + \frac{49}{9} + 49 - \frac{250}{9} = 0$$

$$9x^2 + 9y^2 - 42x - 126y + 49 + 441 - 250 = 0$$

$$9x^2 + 9y^2 - 42x - 126y + 240 = 0$$

$$3x^2 + 3y^2 - 14x - 42y + 80 = 0$$

∴ The equation of the circle is

$$3x^2 + 3y^2 - 14x - 42y + 80 = 0$$

Example 4

Find the values of c such that the line $x + y = c$ shall be a tangent to the circle $x^2 + y^2 - 4x + 2 = 0$. For each value of c, find the coordinates of the points of contact.

Solution

To find the points of intersection of the line $x + y = c$ and the circle $x^2 + y^2 - 4x + 2 = 0$

put $y = c - x$ in the equation of the circle

i.e. $x^2 + (c - x)^2 - 4x + 2 = 0$

$$2x^2 - (2c + 4)x + (2 + c^2) = 0 \text{ ....(1)}$$

If $x + y = c$ is a tangent to the circle then this quadratic equation will have equal roots.

i.e. $b^2 - 4ac = 0$

$$(2c + 4)^2 - 4(2)(2 + c^2) = 0$$

$$4c^2 + 16c + 16 - 16 - 8c^2 = 0$$

$$-4c^2 + 16c = 0 \quad \Rightarrow \quad c = 0 \text{ or } c = 4$$

The equations of the tangents are: $x + y = 0$ and $x + y = 4$

Putting $c = 0$ in eqn(1) gives: $x = 1$ and putting $x = 1$ into $x + y = 1$ gives: $y = -1$

Putting $c = 4$ in eqn(1) gives: $x = 3$ and putting $x = 3$ into $x + y = 4$ gives: $y = 1$

Therefore the coordinates of the points of contact are $(1, -1)$ and $(3, 1)$.

Example 5

Find the equation of the circle with centre (4, 5) and which passes through the point where the line $5x - 2y + 6 = 0$ cuts the y-axis.

Solution

Centre of circle is (4, 5).

At the point where the line $5x - 2y + 6 = 0$ cuts the y-axis put $x = 0$

$$5(0) - 2y + 6 = 0 \quad \Rightarrow y = 3$$

∴ (0, 3) is the point of intersection of the circle and the line.

Equation of a circle is given by

$(x-a)^2+(y-b)^2=r^2$

where (a, b) is the centre and r is the radius

Note: r is the distance between (4, 5) and (0, 3).

$$r^2=(0-4)^2+(3-5)^2$$
$$r^2=16+4=20$$
$$(x-4)^2+(y-5)^2=20$$
$$x^2-8x+16+y^2-10y+25=20$$
$$x^2+y^2-8x-10y+16+25-20=0$$
$$x^2+y^2-8x-10y+21=0$$

$\therefore$ The equation of the circle is

$$x^2+y^2-8x-10y+21=0$$

Example 6

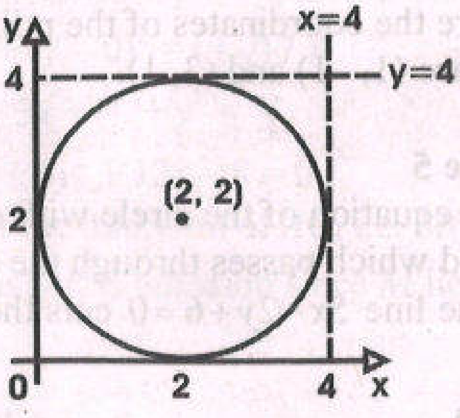
A circle touches the lines $x=0$, $x=4$, $y=0$ and $y=4$. Find the equation of

i) the circle.

ii) the line passing through (0, 4) and the centre of the circle.

Solution

i)



From the diagram, centre = (2, 2) and radius = 2

$$(x-2)^2+(y-2)^2=2^2$$
$$x^2+y^2-4x-4y+4=0$$

ii) Gradient of line $=\dfrac{2-4}{2-0}=-1$

Therefore equation of the line passing through (0, 4) is

$$y-4)=-1(x-0)$$
$$x+y-4=0$$

Example 7

A circle is defined by

$x^2+y^2+2gx+2fy-15=0$

The gradient of the tangent to the circle at the point (3, 2) is $-\frac{1}{2}$. Find the values of g and f.

Solution

The gradient of the line joining the centre $(-g,-f)$ and the point of contact (3, 2) is

$$\frac{2-(-f)}{3-(-g)}=-\frac{1}{-\frac{1}{2}}$$
$$\frac{2+f}{3+g}=2 \quad\Rightarrow 2+f=6+2g$$
$$f-2g=4 \text{ ....(1)}$$

Radius of circle $=\sqrt{g^2+f^2+15}$

Also the distance between the centre $(-g,-f)$ and point (3, 2)

$=\sqrt{(2+f)^2+(3+g)^2}$

This is the same as the radius of the circle.

$$\sqrt{g^2+f^2+15}=\sqrt{(2+f)^2+(3+g)^2}$$
$$g^2+f^2+15=(2+f)^2+(3+g)^2$$
$$15=4+4f+9+6g$$
$$2f+3g=1 \text{ ....(2)}$$

Solving (1) and (2) simultaneously gives $f=2$ and $g=-1$

Example 8

Find the centre and radius of the circles C_1 and C_2 whose equations are

$x^2+y^2-16y+32=0$ and

$x^2+y^2-18x+2y+32=0$

respectively and show that the circles touch each other externally. Find the coordinates of their point of contact.

Solution

Steps

1. Find the centre and radius of C_1: (0, 8) $4\sqrt{2}$
2. Find the centre and radius of C_2: (9, -1) $5\sqrt{2}$
3. Find the distance between the centres and show that it is equal to the sum of the radii. Thus the circles touch externally.
4. Eliminate x^2 and y^2 from the equations of C_1 and C_2 to obtain $x = y$ which is then substituted into C_1 to give a quadratic equation in x or y. This equation has equal roots, thus the coordinates of the point of contact can be found. i.e. (4, 4).
   [Work out the solution]

Review exercise 13

1. Find the radius and the centre of the following circles
   - i) $x^2+y^2-4x+6y-12=0$
   - ii) $x^2+y^2+4x-6y+4=0$
   - iii) $x^2+y^2+6x-8y+16=0$
   - iv) $x^2+y^2+x+4y-2=0$
   - v) $x^2+y^2+2x-4y+1=0$
   - vi) $x^2+y^2-6x-10y+25=0$
   - vii) $x^2+y^2+8x+2y-8=0$
   - viii) $x^2+y^2-16x-8y=64$
   - ix) $x^2+y^2+6x=0$
   - x) $x^2+y^2+6x-4y+12=0$
   - xi) $x^2+y^2+2x-8y+1=0$
   - xii) $x^2+y^2-6y=0$
   - xiii) $x^2+y^2+2x=3$
   - xiv) $x^2+y^2-6x-3=0$

2. Find the equation of the circle passing through the points
   - i) (2, 0), (3, –3) and (0, –3)
   - ii) (3, 4), (0, 5) and (4, –3)

3. The following points are the ends of a diameter of a circle, find the equation of the circles.
   - i) (1, –1) and (4, 1)
   - ii) (3, –5) and (2, 0)
   - iii) (–2, –1) and (3, 2)

4. The points P, Q and R have co-ordinates (2, 4), (8, –2) and (6, 2) respectively.
   - a) Find the equation of the straight line l which is perpendicular to the line PQ and which passes through the midpoint of PR.
   - b) The circle passing through P, Q and R has centre C. Find the co-ordinates of C and radius of the circle.

5. Find the equation of the circle C which passes through the points $O(0, 0)$, $A(3, 3)$ and $B(3, 1)$. Show that the radius of the circle is $\sqrt{5}$ and find the co-ordinates of the centre.
   Verify that the point $P(2, 4)$ lies on C. Show that the gradient of the tangent at P to C is $-\frac{1}{2}$ and hence find the equation of this tangent.

6. The equation of a circle is $x^2+y^2+14x-20y+129=0$
   - a) State the centre and radius of the circle
   - b) Verify that the point $P(-3, 12)$ lies on the circle and show that the equation of the tangent to the circle at P is $y = 6-2x$.

7. Find the equation of the tangent and normal to the circle

a) $x^2 + y^2 + 4x - 10y = 12$ at $(3, 1)$

b) $x^2 + y^2 - 6x - 3y = 16$ at $(-2, 0)$

8. Find the length of the tangent from $(-3, 2)$ to the circle $x^2 + y^2 - 2x + 2y = 3$

9. Find the length of the tangent from the origin to the circle $x^2 + y^2 - 6x - 4y + 9 = 0$

10. Find the radius and coordinates of the centre of the circle $x^2 + y^2 - 3x = 0$

11. A point $P(1, 2)$ lies on a circle with centre $C(3, 5)$

i) Find the equation of the normal at P

ii) Find the equation of the tangent at P

12. Find the Cartesian equation of the locus of the set of points P un each of the following cases :

i) P is equidistant from the point $(4, 1)$ and the line $x = -2$

ii) P is twice as far from the point $(3, 0)$ and the line $x = 5$

iii) P is equidistant from $(3, 5)$ and $(-1, 1)$

iv) P is equidistant from $(1, 2)$ and $(-2, -1)$

v) P is equidistant from the lines $3x + 4y + 5 = 0$ and $12x - 5y + 13 = 0$.

13. Two points, A and B in a plane have co-ordinates $A(4, 7)$ and $B(-2, 1)$ respectively $P(x, y)$ is a variable point in the plane such that $|AP|^2 + |BP|^2 = 40$

a) i) Obtain the equation for the locus of P

ii) Describe and sketch this locus completely.
(Hint: Find the centre and radius)

b) Find

i) the co-ordinates of D and E, the points of intersection of the y-axis with the locus.

ii) the length of the chord DE
(Hint: find the length DE)

iii) the equation of the tangent to the locus at D or E

14. Find the set of all points L, in the OXY plane, which are equidistant from $P(2, -4)$ and $R(1, -1)$

15. Find the equation of the locus of a point $P(x, y)$ in the oxy plane such that $|PA|^2 - |PB|^2 = 7$, where $A(-4, 6)$ and $B(-7, 9)$.

16. i) Find the co-ordinates of the centre and radius of the circle $4x^2 + 4y^2 - 12x + 32y = 27$

ii) the length of the chord of the circle which cuts the x-axis.
(Hint: put $y = 0$ and solve for x)

17. Find the equation of the locus of a variable point $P(x, y)$ if

a) P is equidistant from $A(-3, 1)$ and $B(4, 2)$

b) The distance of P from $A(1, 5)$ is twice its distance from $B(-3, 2)$

c) $PA^2 + PB^2 = 2PC^2$ where $A(3, 1)$, $B(1, -4)$ and $C(4, 3)$

d) $3|PA| = 2|PB|$

18. Find the equation of the locus of a point $P(x, y)$ so that the sum of the squares of its distance from the points $A(-2, 0)$ and $B(2, 0)$ is 26.
(Hint: $|PA|^2 + |PB|^2 = 26$).

19. Find the equation of the two circles which satisfy the following conditions:

i) The x-axis is a tangent to the circle

ii) The centre of the circle lies on the line $2y = x$

iii) The point (14, 2) lies on the circle.

Prove that the line $3y = 4x$ is a common tangent to these circles.

20. Find the points of intersection of the circle $x^2 + y^2 - 6x + 2y - 17 = 0$ and the line $x - y + 2 = 0$. Show that an equation of the circle which has these points as the ends of the diameter is $x^2 + y^2 - 4y - 5 = 0$. Show also that this circle touch the circle $x^2 + y^2 - 8x + 2y + 13 = 0$ externally.

21. The points P, Q and R have coordinates (2, 4), (8, –2) and (6, 2) respectively.

a) Find the equation of the straight line l which is perpendicular to the line PQ and which passes through the midpoint of PR.

b) The line l cut PQ at S. Find the ratio $PS{:}SQ$.

c) The circle passing through P, Q and R has centre C. Find the coordinates of C and the radius of the circle.

22. Obtain the equation of the circle which passes through the origin and has its centre at the point (3, –4). The line $y = x - 6$ meets this circle at the points P and Q. Find the coordinates of P and Q. Calculate the distance PQ.

23. The centre of a circle is at the point $C(3, 7)$ and the point $A(5, 3)$ is on the circumference of the circle. Find

a) the radius of the circle,

b) the equation of the line through A that is perpendicular to AC.

24. The points A and B have coordinates (8, 0) and (0, 6) respectively and O is the origin. Find the equation of;

i) the circumcircle

ii) the inscribed circle of the triangle AOB

25. Prove that the circles $x^2 + y^2 + 2x + 2y + 23 = 0$ and $x^2 + y^2 - 10x - 7y + 31 = 0$ touch each other externally, and calculate the coordinates of the point of contact.

26. A triangle ABC has vertices $A(0, 12)$, $B(-9, 0)$ and $C(16, 0)$. Find the equations of the internal bisectors of the angles ABC and ACB. Hence, or otherwise, find the equation of the inscribed circle of the triangle ABC. Find the equation of the circle passing through A, B and C.

27. Find the centre and radius of the circles C_1 and C_2 whose equations are $x^2 + y^2 + 6y + 8 = 0$ and $x^2 + y^2 - 12x - 10y - 60 = 0$ respectively and show that the circles touch each other internally. Find the coordinates of their point of contact.

28. Find the equation of the circle which touches the line $y = x$ at the point (4, 4) and whose centre lies on the line which passes through the point (–1, –3) and the origin.

29. a) The equation of a circle is $x^2 + y^2 - 4x - 6y - 12 = 0$. Find

i) the co-ordinates of its centre M,

ii) the length of its radius.

b) The circle is reflected in the line $y = x$. Find

i) the image of the centre W,

ii) the equation of the image circle.

CHAPTER 14

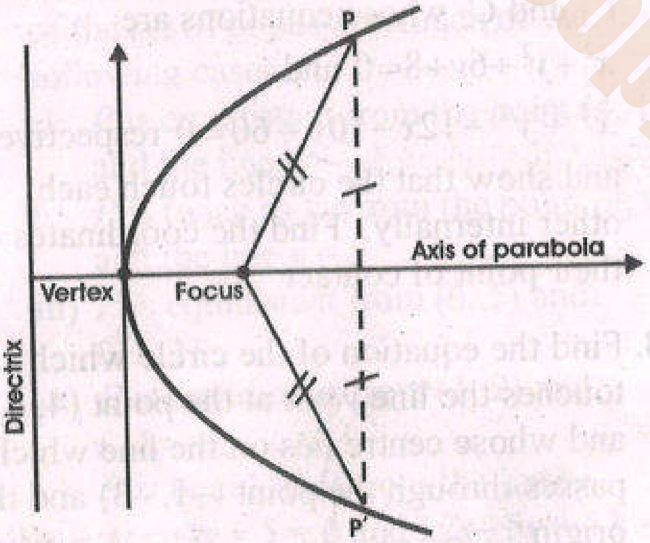
COORDINATE GEOMETRY 3 – THE PARABOLA

Introduction

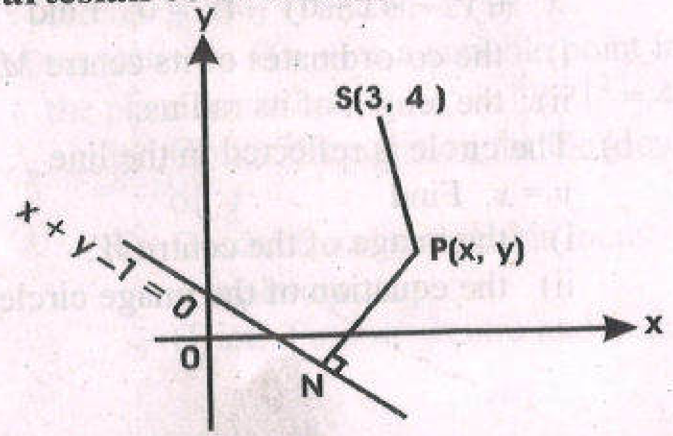
If a point is always equidistant from a fixed point (called the focus) and a fixed straight line (called the directrix), the locus of the point is called a **parabola**.

All parabolas have similar properties.

For example, every parabola is symmetrical about the line through the focus which is perpendicular to the directrix. This line is called the *axis* of the parabola. The point where a parabola crosses its axis is called the *vertex*. The distance between the vertex and the focus is called the *focal length.*



Equation of a parabola in rectangular Cartesian co-ordinates



Consider a parabola whose focus is the point $S(3, 4)$ and directrix is the line $x + y = 1$.

If $P(x, y)$ is any point on the parabola and PN is the perpendicular from P to the directrix, then by definition :

$$PS = PN \quad \text{or} \quad PS^2 = PN^2$$

But PN is the perpendicular distance from (x, y) to $x + y - 1 = 0$

i.e. $PN = \left|\dfrac{x+y-1}{\sqrt{1^2+1^2}}\right|$

and $PS^2 = (x-3)^2 + (y-4)^2$

$$(x-3)^2 + (y-4)^2 = \left(\frac{x+y-1}{\sqrt{2}}\right)^2$$

$$x^2 - 6x + 9 + y^2 - 8y + 16$$

$$= \frac{x^2 + xy - x + xy + y^2 - y - x - y + 1}{2}$$

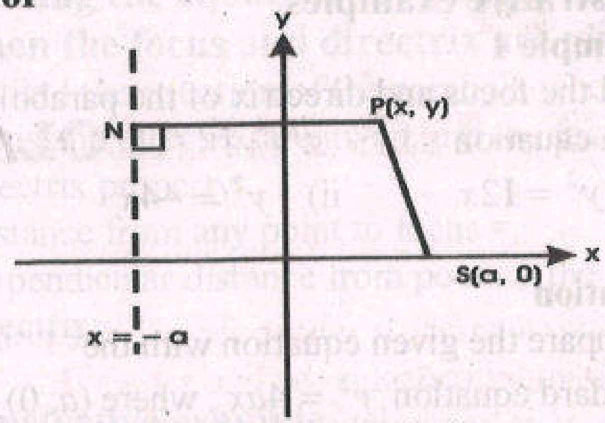
$$x^2 + y^2 - 10x - 14y - 2xy + 49 = 0$$

This is the equation of the parabola. Note that this contains a term in xy. Therefore it cannot be the equation of a circle. This analysis is cumbersome.

To make the analysis simpler, we choose the point $S(a, 0)$ for the focus of a parabola and the line $x + a = 0$ (or $x = -a$) for the directrix. This gives the equation of a parabola in rectangular cartesian coordinates as :

$$y^2 = 4ax$$

Proof



From the definition of a parabola,
The distance from the focus = the perpendicular distance from the directrix
This is called the focus–directrix property.

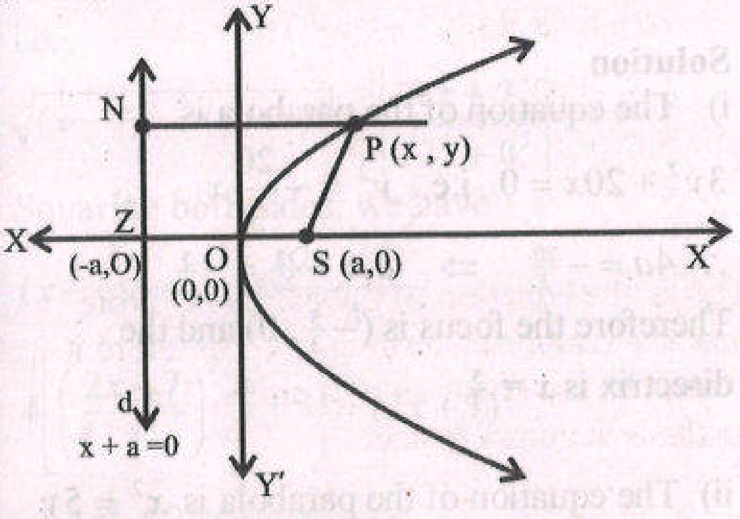
Therefore any point P on a parabola obeys the equation : $PS = PN$ or $PS^2 = PN^2$

i.e. $(x-a)^2 + y^2 = [x-0+a)]^2$

$$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$

$$y^2 = 4ax$$

Conversely, any parabola with equation $y^2 = 4ax$ has focus $(a, 0)$ and directrix $x = -a$.



Sketching a parabola

The process of sketching a parabola requires the use of the features of the standard parabola

Features of the standard parabola

1. Origin $O(0, 0)$ satisfies $y^2 = 4ax$

Therefore the parabola passes through the origin.

2. $y^2 \geq 0 \quad \therefore \quad 4ax \geq 0$ i.e. $x > 0$

Therefore the x-co-ordinate of any point on the parabola is non-negative and the parabola lies entirely in the first quadrant.

3. $y^2 = 4ax \quad \therefore \quad y = \pm 2\sqrt{ax}$

Therefore $(x, 2\sqrt{ax})$ and $(x, -2\sqrt{ax})$ lies on the parabola. The curve is symmetrical about the x-axis or the x-axis is the axis of symmetry of the parabola.

4. As $x \to \infty$, $y \to \infty^2$ or $y \to \pm\infty$

Therefore the curve extends to infinity above and below the x-axis on the right side of the y-axis

5. When $x = 0$, $y^2 = 0$ i.e. both roots are equal. This implies the y-axis meets the curve at coincident point (0, 0)

Therefore the y-axis is the tangent to the parabola at $O(0, 0)$.

6. The point of intersection of the parabola and its axis is called the vertex. Therefore $O(0, 0)$ is the vertex.

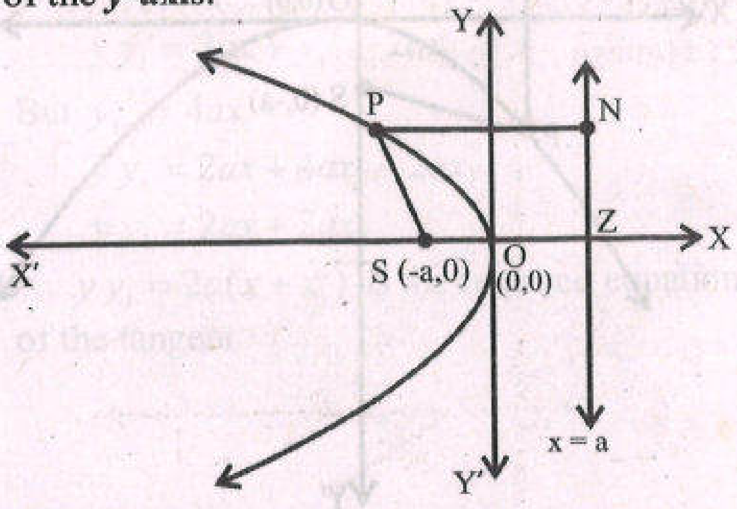
Other forms of the parabola

Let us consider three more forms of the parabola.

1. $y^2 = -4ax$

This equation represents a parabola with focus at $S(-a, 0)$ and directrix at $x = a$ or $x - a = 0$.

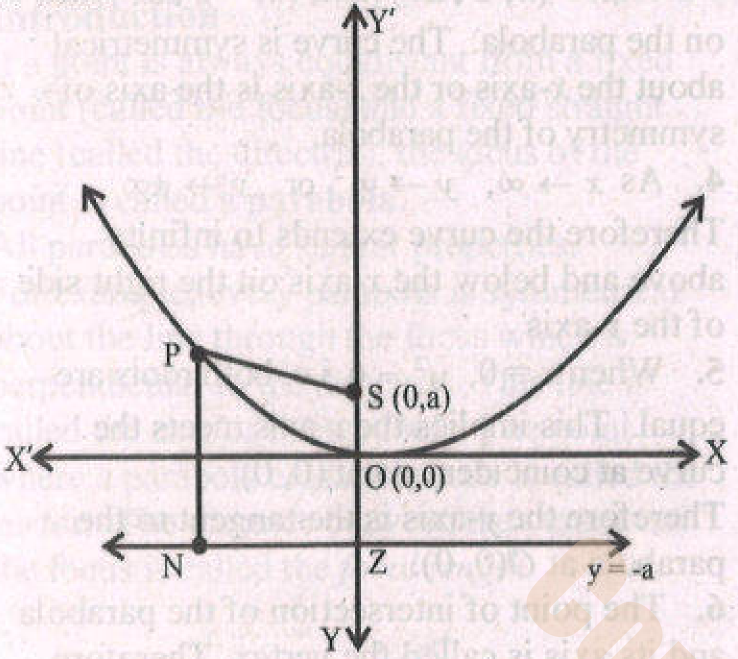
This parabola lies completely at the **left side of the *y*-axis.**



2. $x^2 = 4ay$

This equation represents a parabola with focus at $S(0, a)$, and directrix at $y = -a$ or $y + a = 0$.

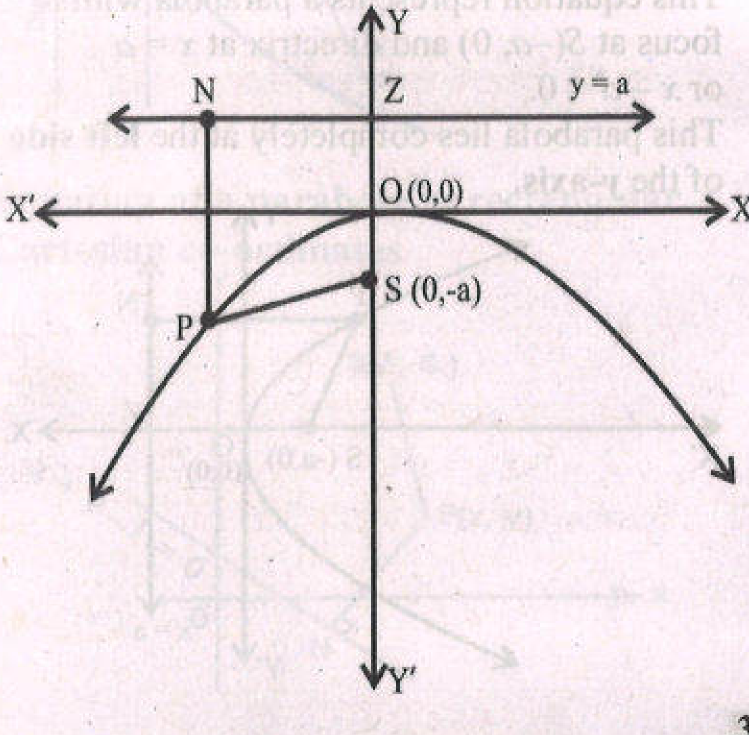
This parabola lies completely **above** the ***x*-axis.**



3. $x^2 = -4ay$

This equation represents a parabola with focus at $S(0, -a)$, and directrix at $y = a$ or $y - a = 0$.

This parabola lies completely **below** the ***x*-axis.**



Illustrative examples

Example 1

Find the focus and directrix of the parabola with equation

i) $y^2 = 12x$ ii) $y^2 = -4x$

Solution

Compare the given equation with the standard equation $y^2 = 4ax$, where $(a, 0)$ is the focus and $x = -a$ is the directrix.

i) $4a = 12 \Rightarrow a = 3$

Therefore the focus is $(3, 0)$ and the directrix is $x = -3$

ii) $4a = -4 \Rightarrow a = -1$

Therefore the focus is $(-1, 0)$ and the directrix is $x = 1$

Example 2

Find the focus and directrix of the parabola with equation

i) $3y^2 + 20x = 0$ ii) $x^2 = 5y$

Solution

i) The equation of the parabola is

$3y^2 + 20x = 0$ i.e. $y^2 = -\frac{20}{3}x$

$\therefore 4a = -\frac{20}{3} \Rightarrow a = -\frac{12}{2o} = -\frac{5}{3}$

Therefore the focus is $(-\frac{5}{3}, 0)$ and the directrix is $x = \frac{5}{3}$

ii) The equation of the parabola is $x^2 = 5y$

$\therefore 4a = 5 \Rightarrow a = \frac{5}{4}$

Therefore the focus is $(0, \frac{5}{4})$ and the directrix is $y = -\frac{5}{4}$ or $4y + 5 = 0$

Finding the equation of a parabola when the focus and directrix are given

To find the equation of the parabola when the focus and directrix are given, use the focus directrix property:
Distance from any point to focus = perpendicular distance from point to the directrix

Illustrative example

Find the equation of the parabola in the standard form whose focus is $(\frac{7}{2}, 0)$ and directrix $2x + 7 = 0$

Solution

Let $P(x, y)$ be any point on the parabola whose focus is $(\frac{7}{2}, 0)$ and directrix $2x + 7 = 0$

By definition:
Distance from any point $P(x, y)$ to focus $(\frac{7}{2}, 0)$ = Perpendicular distance from point $P(x, y)$ to the directrix.

i.e.

$$\sqrt{(x-\tfrac{7}{2})^2 + (y-0)^2} = \left|\frac{2x+7}{\sqrt{2^2+0^2}}\right|$$

Squaring both sides, we have

$$(x-\tfrac{7}{2})^2 + y^2 = \frac{(2x+7)^2}{4}$$

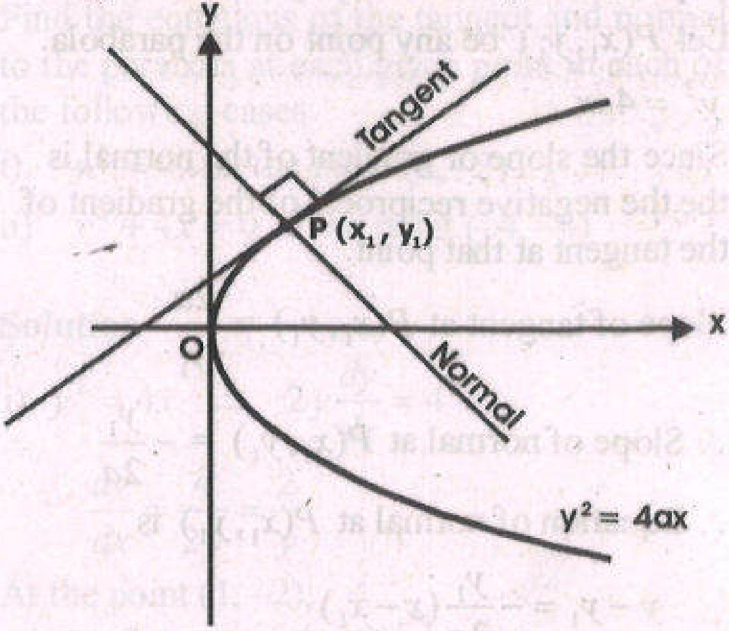
$$4\left[\left(\frac{2x-7}{2}\right)^2 + y^2\right] = (2x+7)^2$$

$$4\left[\frac{4x^2+28x+49}{4} + y^2\right] = 4x^2+28x+49$$

$$4x^2+28x+49+4y^2 = 4x^2+28x+49$$

$4y^2 = 56x$ i.e. $y^2 = 14x$ is the required equation of the parabola.

Tangent and normal to a parabola



Equation of tangent to $y^2 = 4ax$

Let $P(x_1, y_1)$ be any point on the parabola $y^2 = 4ax$

Differentiating implicitly w.r.t. x.

$$2y\frac{dy}{dx} = 4a \quad \Rightarrow \quad \frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

$$\therefore \left(\frac{dy}{dx}\right) \text{at } P(x_1, y_1) = \frac{2a}{y_1}$$

$\therefore$ Slope of tangent at $P(x_1, y_1) = \dfrac{2a}{y_1}$

The equation of the tangent at $P(x_1, y_1)$ is

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

$$yy_1 - y_1^2 = 2ax - 2ax_1$$

$$yy_1 = 2ax + y_1^2 - 2ax_1$$

But $y_1^2 = 4ax_1$

$$yy_1 = 2ax + 4ax_1 - 2ax_1$$

$$yy_1 = 2ax + 2ax_1$$

$\therefore$ $yy_1 = 2a(x + x_1)$ is the required equation of the tangent

Equation of normal to $y^2 = 4ax$

Let $P(x_1, y_1)$ be any point on the parabola $y^2 = 4ax$

Since the slope or gradient of the normal is the the negative reciprocal of the gradient of the tangent at that point.

Slope of tangent at $P(x_1, y_1) = \dfrac{2a}{y_1}$

$\therefore$ Slope of normal at $P(x_1, y_1) = -\dfrac{y_1}{2a}$

$\therefore$ Equation of normal at $P(x_1, y_1)$ is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

The equation of tangent to a parabola in terms of slope

Let the equation of the tangent be:

$y = mx + c$.......(1)

The equation of the parabola is:

$y^2 = 4ax$.........(2)

Solving equations (1) and (2) simultaneously, we have

$(mx + c)^2 = 4ax \quad \Rightarrow \quad m^2x^2 + 2mxc + c^2 = 4ax$

[i.e. putting $y = mx + c$ in (2)]

$\Rightarrow \quad m^2x^2 + (2mc - 4a)x + c^2 = 0$ ....(3)

Since the line $y = mx + c$ is a tangent to the parabola, the quadratic equation (3) must have equal roots [i.e. $b^2 - 4ac = 0$]

$\therefore \quad (2mc - 4a)^2 - 4m^2c^2 = 0$

$4m^2c^2 - 16mca + 16a^2 - 4m^2c^2 = 0$

$16a^2 = 16mca \quad \Rightarrow \quad a = mc$ or $c = \dfrac{a}{m}$

This is the **condition of tangency**.

Therefore the equation of the tangent to a parabola in the slope form is:

$$y = mx + \frac{a}{m}$$

1. To find the equation of tangent to the parabola $y^2 = 4ax$ at a given point, we use the equation $yy_1 = 2a(x + x_1)$

2. To find the equation of a tangent drawn from a point outside, we can use the slope form. In this case, two tangents will be obtained.

Illustrative examples

Example 1

Find the equation of the tangent to the parabola $3y^2 = 16x$ at (3, 4)

Solution

$3y^2 = 16x \quad \Rightarrow \quad y^2 = \frac{16}{3}x^2$

Comparing with standard equation $y^2 = 4ax$

$4a = \frac{16}{3} \quad \Rightarrow \quad a = \frac{4}{3}$

Given $(x_1, y_1) = (3, 4)$

The equation of tangent at point $P(x_1, y_1)$ is:

$yy_1 = 2a(x + x_1)$

$4y = 2(\frac{4}{3})(x + 3) \quad \Rightarrow \quad 4y = \frac{8}{3}(x + 3)$

$12y = 8x + 24 \quad \Rightarrow \quad 3y = 2x + 6$

$\therefore \; 2x - 3y + 6 = 0$ is the equation of tangent

Example 2

Find the equation of the tangent and normal to the parabola $y^2 = 8x$ at the point (2, 4).

Solution

First method – Using the condition $a = mc$

Let the equation of the tangent be $y = mx + c$.

Comparing the given equation $y^2 = 8x$ with the general equation $y^2 = 4ax$ gives: $a = 2$

Using the condition of tangency

i.e. $a = mc \quad \Rightarrow \quad 2 = mc \quad \Rightarrow \quad c = \dfrac{2}{m}$

The equation of the tangent becomes:

$y = mx + \dfrac{2}{m}$ ......(1)

At the point (2, 4) equation(1) becomes:

$$4 = 2m + \frac{2}{m}$$

$$4m = 2m^2 + 2 \quad \Rightarrow \quad 2m^2 - 4m + 2 = 0$$

$$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0$$

$m = 1$ twice

Putting $m = 1$ into eqn(1) gives the equation of the tangent as : $y = x + 2$ or $x - y + 2 = 0$

Since the gradient of the tangent at the point (2, 4) is $m = 1$, the gradient of the normal at the same point is -1. Therefore the equation of the normal is:

$$y - 4 = -1(x - 2)$$
$$y - 4 = -x + 2$$
$$x + y - 6 = 0$$

Second method – Using differentiation

$$y^2 = 8x \quad \Rightarrow \quad 2y\frac{dy}{dx} = 8$$

$$\frac{dy}{dx} = \frac{8}{2y} = \frac{4}{y}$$

At the point (2, 4), $\frac{dy}{dx} = \frac{4}{4} = 1$

i.e. substitute $y = 4$ in the expression for $\frac{dy}{dx}$

Therefore the gradient of the tangent to the parabola at the point (2, 4) is 1

The equation of the tangent is:

$$y - 4 = 1(x - 2)$$
$$y - 4 = x - 2$$
$$x - y + 2 = 0$$

The gradient of the normal to the parabola is -1.

Therefore the equation of the normal is:

$$y - 4 = -1(x - 2)$$
$$y - 4 = -x + 2$$
$$x + y - 6 = 0$$

Example 3

Find the equations of the tangent and normal to the parabola at each given point in each of the following cases:

i) $y^2 = 4x$ at the point (1, –2)

ii) $y^2 + 4x = 0$ at the point (–4, –4)

Solution

i) $y^2 = 4x \quad \Rightarrow \quad 2y\frac{dy}{dx} = 4$

$$\frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$

At the point (1, –2),

$\frac{dy}{dx} = \frac{2}{-2} = -1$ [i.e. substitute $y = -2$ in the expression for $\frac{dy}{dx}$]

Therefore the gradient of the tangent to the parabola at the point (1, –2) is –1

The equation of the tangent is:

$$y - (-2) = -1(x - 1)$$
$$y + 2 = -x + 1$$
$$x + y + 1 = 0$$

The gradient of the normal to the parabola is 1.

Therefore the equation of the normal is:

$$y - (-2) = 1(x - 1)$$
$$y + 2 = x - 1$$
$$x - y - 3 = 0$$

Note: Use the first method to arrive at the same answers

ii) tangent: $2y - x + 4 = 0$

normal: $y + 2x + 12 = 0$

[Work out the solution]

Review exercise 14

1. Find the focus and directrix of the parabola with equation :
   - i) $y^2 = 15x$
   - ii) $y^2 = -8x$
   - iii) $y^2 + 12x = 0$
   - iv) $y^2 - 6x = 0$

2 Find the equation of the parabola with focus at (3, -2) and the line $3x + 2y = 0$ as its directrix.

3 Find the equation of the parabola with focus at (-1, 3) and whose directrix is $3x + 2y = 0$.

4 Find the equation of the parabola
   - i) with focus at (7, 0) and directrix $x + 7 = 0$.
   - ii) with focus at (0, 5) and directrix $x - 2 = 0$.

5. Find the equations of the tangent and normal to the given parabola at the given points
   - i) $y^2 = 15x$ at the point (15, 15)
   - ii) $y^2 = -8x$ at the point (–2, 4)
   - iii) $y^2 + 12x = 0$ at the point (–12, 12)
   - iv) $y^2 - 6x = 0$ at the point $(\frac{3}{2}, 3)$

6. Prove that the line $y = mx + \frac{15}{4m}$ is a tangent to the parabola $y^2 = 15x$. Using this result or otherwise, find the equation of the common tangents to this parabola and the circle $x^2 + y^2 = 16$

7. The variable line $y = mx + c$ cuts the fixed parabola $y^2 = 4ax$ at two points P and Q. Show that the coordinates of the midpoint of PQ are $\left(\frac{2a - mc}{m^2}, \frac{2a}{m}\right)$

8. If the line $y = mx + c$ is a tangent to the parabola $y^2 + 4x$, show that $mc = -1$.

9. If the tangent to the parabola $y^2 = 4x$ is parallel to the line $3x + 2y = 7$. Determine the equation of the tangent.

10. Find the equations of the tangent and normal to the parabola $y^2 = 16x$ at P

11. Find the equations of the normal to the parabola $y^2 = 9x$ at $P(1, -3)$

12. Show that the line $y = x + 2$ touches the parabola $y^2 = 8x$ and find the co-ordinates of the point of contact

13. Find the equation of the tangent to $y^2 = 8x$ perpendicular to $x + 2y + 7 = 0$.

14. A parabola $y^2 = 12x$ passes through the point $P(3, 6)$. Find the
   - i) equation of the tangent;
   - ii) equation of the normal,

   to the parabola at the point P.

CHAPTER 15

SEQUENCES AND SERIES

Definitions

A **sequence** is a set of numbers (terms) written in a defined order with a rule or formula for obtaining the terms.
For example, 1, 3, 5, 7, 9, 11 ... and 2, 4, 6, 8, 10 ... are sequences. Each term is obtained by adding 2 to the preceding term.

A **series** is formed when the terms of a sequence are added. For example, $1 + 3 + 5 + 7 + 9 + \ldots$ is a series. If the series stops after a finite number of terms, it is called a **finite series.**
For example, $1 + 3 + 5 + 7 + 9$ is a finite series of five terms. However, if the series does not stop but continues indefinitely it is called an **infinite series.**
Thus $2 + 4 + 6 + 8 + 10 + \ldots$ is an infinite series. Note that the same definitions are given to finite and infinite sequences.

Types of sequences

Two types of sequences shall be considered, namely:

1. Arithmetical Progression (AP), sometimes called Linear sequence.
2. Geometrical Progression (GP), also called Exponential sequence.

Arithmetical Progression (A.P.) or Linear sequence

An **Arithmetical Progression** (A.P.) or a **linear sequence** is a sequence in which any term differs from the preceding term by a constant called the **common difference** which may be positive or negative. For example, 2, 5, 8, 11 is an Arithmetical Progression (AP) with common difference 3.
For a linear sequence, the first term is denoted by 'a' and the common difference by 'd'. The nth term is denoted by U_n, where U_1 means the first term, U_2 means the second term etc. The first four terms of an AP are:

$U_1 = a$; $U_2 = a + d$; $U_3 = a + 2d$ and $U_4 = a + 3d$.
i.e. a, $(a + d)$, $(a + 2d)$, $(a + 3d)$.

The *general term or the formula* for obtaining the terms of any linear sequence is given by:

$$\boxed{U_n = l = a + (n - 1)d}$$

where n is the number of terms and l, the last term.
This formula can be used to find any term of an AP if the first term a and the common difference d are known.
Note that $d = U_2 - U_1 = U_3 - U_2$ etc.
i.e. the difference between consecutive terms.
For example, the common difference of the AP 2, 5, 8, 11 is
$d = 5 - 2 = 8 - 5 = 11 - 8 = 3$

Illustrative examples

Example 1
Find the 11th term of a linear sequence of the form 4, 9, 14, 19, ...

Solution
From the sequence, the first term, $a = 4$ and the common difference, $d = 9 - 4 = 5$

The general formula or term of an AP is:
$U_n = a + (n-1)d$

We want to find the 11th term, $\therefore \quad n = 11$.
Substituting $a = 4,\ d = 5$ and $n = 11$ into U_n, we have

$$\begin{aligned} U_{11} &= 4 + (11-1)5 \\ &= 4 + 10(5) \\ &= 4 + 50 = 54 \end{aligned}$$

Example 2

Find the 8th term of a linear sequence of the form 47, 42, 37, 32, ...

Solution

From the sequence, the first term, $a = 47$ and the common difference, $d = 42 - 47 = -5$

The general formula or term of an AP is
$U_n = a + (n-1)d$

We want to find the 8th term, $\therefore \quad n = 8$.
Substituting $a = 47,\ d = -5$ and $n = 8$ into the formula, we have

$$\begin{aligned} U_8 &= 47 + (8-1)(-5) \\ &= 47 + 7(-5) \\ &= 47 - 35 = 12 \end{aligned}$$

Example 3

Find the 12th term of an AP of the form $7,\ 6\tfrac{1}{4},\ 5\tfrac{1}{2}, \ldots$

Solution

From the sequence, the first term, $a = 7$ and the common difference, $d = 6\tfrac{1}{4} - 7 = -\tfrac{3}{4}$

The general formula or term of an AP is
$U_n = a + (n-1)d$

We want to find the 12th term, $\therefore \quad n = 12$.
Substituting $a = 7,\ d = -\tfrac{3}{4}$ and $n = 12$ into the formula, we have

$$\begin{aligned} U_{12} &= 7 + (12-1)(-\tfrac{3}{4}) \\ &= 7 + 11(-\tfrac{3}{4}) \\ &= 7 - \tfrac{33}{4} = -\tfrac{5}{4} \end{aligned}$$

Example 4

Find the number of terms in the following linear sequences.

i) 3, 7, 11, ... ,31
ii) 2, –9, –20, ... , –141

Solution

i) From the sequence, the first term, $a = 3$ and the common difference, $d = 7 - 3 = 4$
Let n = the number of terms.
The last term $l = 31$
But $l = U_n = a + (n-1)d$

$\therefore \quad 31 = 3 + (n-1)4.$
Solving for n, we have
$31 = 3 + 4n - 4 \quad \Rightarrow \quad 4n = 32 \quad \Rightarrow \quad n = 8$

$\therefore$ The number of terms is 8.

ii) From the sequence, the first term, $a = 2$ and the common difference, $d = -9 - 2 = -11$
Let n = the number of terms.
The last term, $l = -141$
But $l = U_n = a + (n-1)d$
$-141 = 2 + (n-1)(-11)$
$-141 = 2 - 11n + 11$
$-154 = -11n \quad \Rightarrow \quad n = 14$

$\therefore$ The number of terms is 14.

Example 5

Find a formula for the nth term of the linear sequence 10, 6, 2, –2, …

Solution

From the sequence, the first term, $a = 10$ and the common difference, $d = 6 - 10 = -4$

The nth term of a linear sequence is given by

$$\begin{aligned} U_n &= a + (n-1)d \\ U_n &= 10 + (n-1)(-4) \\ U_n &= 10 - 4n + 4 \\ U_n &= 14 - 4n \end{aligned}$$

Forming the linear sequence when two terms are given

When two terms of a linear sequence are given it is possible to form the sequence by finding the first term and common difference.

Illustrative examples

Example 1

Find the linear sequence whose 8^{th} term is 38 and 22^{nd} term is 108.

Solution

The *n*th term of a linear sequence is

$U_n = a + (n-1)d$

From the question, we can write the following two equations

The 8^{th} term, $U_8 = a + 7d = 38$ ---------(1)

The 22^{nd} term, $U_{22} = a + 21d = 108$ ----(2)

Solving eqn(1) and (2) simultaneously, we have: Eqn(2) – eqn(1)

$14d = 70 \Rightarrow d = 5$

Substituting $d = 5$ into eqn(1)

$a + 7(5) = 38 \quad \Rightarrow \quad a = 3$

$\therefore$ The first term, $a = 3$ and the common difference, $d = 5$ and the sequence is 3, 8, 13, 18, 23, ...

Example 2

Find the 10^{th} term of a linear sequence whose 2^{nd} term is 28 and 17^{th} term is –2.

Solution

The *n*th term of a linear sequence is

$U_n = a + (n-1)d$

From the question, we can write the following two equations:

The second term, $U_2 = a + d = 28$ --------(1)

The 17^{th} term, $U_{17} = a + 16d = -2$ -----(2)

Solving eqn(1) and (2) simultaneously, we have

Eqn(2) – eqn(1)

$15d = -30 \Rightarrow d = -2$

$\therefore$ From eqn(1), $a = 30$

$a = 30$ and $d = -2$

Now 10th term

$U_{10} = a + 9d$

$= 30 + 9(-2) = 30 - 18 = 12$

$\therefore$ The 10^{th} term is 12.

Example 3

The fourth term of a linear sequence is 19 and the eleventh term is 54. Find the eighth term.

Solution

The *n*th term of a linear sequence is

$U_n = a + (n-1)d$

From the question, we can write the following two equations:

The 4^{th} term, $U_4 = a + 3d = 19$ -----------(1)

The 11^{th} term, $U_{11} = a + 10d = 54$ --------(2)

Solving eqn(1) and (2) simultaneously, we have

Eqn(2) – eqn(1)

$7d = 35$

$d = 5$

Substituting $d = 5$ into eqn(1)

$a + 15 = 19$

$a = 4$

$\therefore$ The eighth term,

$U_8 = a + 7d$

$= 4 + 7(5) = 39$

Example 4

The coefficient of the 5th, 6th and 7th terms in the binomial expansion of $(1+x)^n$, in ascending powers of *x*, are in linear sequence. Find the possible values of *n*.

Solution

The coefficient of the 5th, 6th, and 7th terms are:

$$\frac{n(n-1)(n-2)(n-3)}{4!}, \quad \frac{n(n-1)...(n-4)}{5!},$$

$$\frac{n(n-1)\dots(n-4)}{5!}-\frac{n(n-1)(n-2)(n-3)}{4!}$$

$$=\frac{n(n-1)\dots(n-5)}{6!}-\frac{n(n-1)\dots(n-4)}{5!}$$

$$\frac{n-4}{5!}-\frac{1}{4!}=\frac{(n-4)(n-5)}{6!}-\frac{n-4}{5!}$$

$$2(n-4)-5=\frac{(n-4)(n-5)}{6}$$

$$n^2-21n+98=0$$

$$(n-7)(n-14)=0$$

$n=7$ and 14

Sum of an AP (linear sequence)

There are two important formulae for finding the sum of an A.P or a linear sequence.
The **sum** of the first n terms of an A.P or linear sequence denoted by S_n is given by :

$$\boxed{\begin{aligned} S_n &= \tfrac{n}{2}[2a+(n-1)d] \text{ ......(1)} \\ &= \tfrac{n}{2}(a+l) \text{ ....(2)} \end{aligned}}$$

All symbols denote their usual meaning.
The first formula is used when a and d are known.
The second formula is also used when the first term a and last term l are known but not the common difference. For any arithmetic sequence we can write the following two properties:

1. $d = U_n - U_{n-1}$
where $d = U_2 - U_1 = U_3 - U_2$ etc.

2. $U_n = S_n - S_{n-1}$
where $U_2 = S_2 - S_1$
$U_3 = S_3 - S_2$ etc.

Illustrative examples

Example 1
Find the sum of the first six terms of the AP 3, 5, 7, 9, ...

Solution
From the sequence, the first term, $a = 3$ and the common difference, $d = 5 - 3 = 2$
The sum of the nth term of an AP,
$S_n = \frac{n}{2}[2a+(n-1)d]$
Hence the sum of the first six terms,
$S_6 = \frac{6}{2}[2(3)+(6-1)2] = 3[6+10]$
$= 3(16) = 48$

Example 2
Find the sum of the AP 1, 3, 5, ..., 101.

Solution
To find the sum we have to find the number of terms n first.
From the sequence, $a = 1$, $d = 3 - 1 = 2$ and the last term is $l = 101$
$l = a + (n-1)d$
$101 = 1 + (n-1)2 \Rightarrow 101 = 1 + 2n - 2$
$2n = 102 \Rightarrow n = 51$
The sum of the nth term of an AP,
$S_n = \frac{n}{2}(a+l)$. Here the first term and last term are known.
Hence the sum of the AP,
$S_{51} = \frac{51}{2}(1+101) = \frac{51}{2}(102) = 2601$
Alternatively, using $S_n = \frac{n}{2}[2a+(n-1)d]$
$S_{51} = \frac{51}{2}[2(1)+(51-1)2]$
$= \frac{51}{2}[2+100] = \frac{51}{2}(102) = 2601$

Example 3
The fifth term of an arithmetic progression is –1 and the sum of the first twenty terms is –240. Find the third term.

Solution
The 5th term, $U_5 = a + 4d = -1$ ---------(1)
The sum of the first twenty terms,
$S_{20} = \frac{20}{2}[2a+(20-1)d] = -240$
$2a + 19d = -24$ -----------(2)
Solving (1) and (2) simultaneously gives:
$a = 7$ and $d = -2$

The third term,
$U_3 = a + 2d = 7 + 2(-2) = 7 - 4 = 3$

Example 4

A sequence is such that $S_n = n^2 + 3n$.
Show that the terms of this sequence are in arithmetic progression.

Solution

$U_n = S_n - S_{n-1}$

$= (n^2 + 3n) - [(n-1)^2 + 3(n-1)]$

$= (n^2 + 3n) - [n^2 - 2n + 1 + 3n - 3]$

$= 2n + 2$

For the sequence to be an AP, $U_n - U_{n-1}$ must be a constant called the common difference, d.

$U_n - U_{n-1} = 2n + 2 - [2(n-1) + 2$

$= 2n + 2 - [2n - 2 + 2] = 2$

= (constant)

Therefore the terms of the sequence are in arithmetic progression.

Example 5

The sum of the first n terms, S_n, of an arithmetic sequence is given by $S_n = 5n^2 + 6n$. Find the 3rd term.

Solution

$U_n = S_n - S_{n-1}$

$U_3 = S_3 - S_2$

$S_n = 5n^2 + 6n$

$S_3 = 5(3)^2 + 6(3) = 45 + 18 = 63$

$S_2 = 5(2)^2 + 6(2) = 20 + 12 = 32$

$U_3 = S_3 - S_2 = 63 - 32 = 31$

Alternative method

$U_n = S_n - S_{n-1}$

$= (5n^2 + 6n) - [5(n-1)^2 + 6(n-1)]$

$= (5n^2 - 6n) - [5n^2 - 10n + 5 + 6n - 6]$

$= (5n^2 + 6n) - [5n^2 - 4n - 1] = 10n + 1$

Third term, $U_3 = 10(3) + 1 = 30 + 1 = 31$

Note: Replacing n by $n-1$ in the formulae for S_n and U_n gives S_{n-1} and U_{n-1} respectively.

Example 6

Find the sum of the first thirteen terms of a linear sequence whose 8th term is 18 and 11th term is 24.

Solution

The nth term of a linear sequence is
$U_n = a + (n-1)d$ *Note:* First find a and d
From the question, we can write the following two equations:

The 8th term is $U_8 = a + 7d = 18$ ---------(1)

The 11th term is $U_{11} = a + 10d = 24$ --------(2)

Solving eqn(1) and (2) simultaneously, we have

Eqn(2) – eqn(1)

$3d = 6 \quad \Rightarrow \quad d = 2$

Substituting $d = 5$ into eqn(1)

$a + 7(2) = 18 \quad \Rightarrow \quad a = 4$

$\therefore$ The first term, $a = 4$ and the common difference, $d = 2$

The sum of the nth term of an AP,

$S_n = \frac{n}{2}[2a + (n-1)d]$

Hence the sum of the first thirteen terms,

$S_{13} = \frac{13}{2}[2(4) + (13-1)2]$

$= \frac{13}{2}[8 + 24] = \frac{13}{2}(32) = 208$

Example 7

The sixth and eleventh terms of a linear sequence are respectively 23 and 48. Calculate the sum of the first twenty terms of the sequence.

Solution

The nth term of a linear sequence is
$U_n = a + (n-1)d$ *Note:* First find a and d
From the question, we can write the following two equations:

The 6^{th} term is $U_6 = a + 5d = 23$ --------(1)
The 11^{th} term is $U_{11} = a + 10d = 48$ -----(2)
Solving eqn (1) and (2) simultaneously, we have
Eqn(2) – eqn(1)

$5d = 25 \Rightarrow d = 5$

Substituting $d = 5$ into eqn(1)

$a + 5(5) = 23 \Rightarrow a = -2$

$\therefore$ The first term, $a = -2$ and the common difference, $d = 5$

The sum of the nth term of an AP,

$S_n = \frac{n}{2}[2a + (n-1)d]$

Hence the sum of the first twenty terms,

$S_{20} = \frac{20}{2}[2(-2) + (20-1)5]$
$= 10[-4 + 95]$
$= 10(91) = 910$

Example 8

Three consecutive terms of an A.P. have sum 15 and product 80. Find the numbers.

Solution

It is easier to let the three terms be $(a - d)$, a, $(a + d)$

Sum of terms = $(a - d) + a + (a + d) = 15$

$3a = 15 \Rightarrow a = 5$

Product of terms = $a(a-d)(a+d) = 80$

$a(a^2 - d^2) = 80$

Substituting $a = 5$

$5(5^2 - d^2) = 80 \Rightarrow 125 - 5d^2 = 80$
$45 = 5d^2 \Rightarrow d^2 = 9 \Rightarrow d = \pm 3$

Therefore the three terms are $(5 - 3)$, 5 and $(5 + 3)$ i.e. 2, 5 and 8

Example 9

Three consecutive terms of an A.P. have sum 21 and product 315. Find the numbers.

Solution

It is easier to let the three terms be $(a - d)$, a, $(a + d)$

Sum of terms = $(a - d) + a + (a + d) = 21$

$3a = 21 \Rightarrow a = 7$

Product of terms = $a(a - d)(a + d) = 315$

$a(a^2 - d^2) = 315$

Substituting $a = 7$

$7(7^2 - d^2) = 315 \Rightarrow 343 - 7d^2 = 315$
$28 = 7d^2 \Rightarrow d^2 = 4$
$d = \pm 2$

Therefore the three terms are $(7 - 2)$, 7 and $(7 + 2)$ i.e. 5, 7 and 9

Example 10

In a certain linear sequence, the sum of the first and fifth terms is 18 and fifth term is 6 more than the third term. Show that the sum of the first ten terms of the sequence is 165.

Solution

Sum of first and fifth terms:

$a + (a + 4d) = 18$

$2a + 4d = 18$ ---------(1)

The fifth term is 6 more than the third term:

$(a + 4d) - (a + 2d) = 6$

$2d = 6 \Rightarrow d = 3$

Substituting $d = 3$ into (1) $\Rightarrow a = 3$.

$\therefore$ The first term, $a = 3$ and the common difference, $d = 3$

The sum of the nth term of an A.P.,

$S_n = \frac{n}{2}[2a + (n-1)d]$

Hence the sum of the first ten terms,

$S_{20} = \frac{10}{2}[2(3) + (10-1)3]$
$= 5[6 + 27] = 5(33) = 165$

Example 11

The sum of the first n terms of an arithmetic sequence is $-45\frac{1}{2}$. If the first term is $38\frac{1}{2}$ and the common difference is -7, find the value of n.

Solution

$S_n = -45\frac{1}{2} = -\frac{91}{2}$, $a = 38\frac{1}{2} = \frac{77}{2}$, $d = -7$

$S_n = \frac{n}{2}\{2a + (n-1)d\}$

Substituting values, we have

$$-\tfrac{91}{2} = \tfrac{n}{2}\{2\times\tfrac{77}{2}+(n-1)\times -7$$

$$-91 = n(77-7n+7)$$

$$-91 = 84n-7n^2$$

$$n^2-12n-13=0$$

$$(n-13)(n+1)=0$$

$$n=13 \text{ or } n=-1$$

∴ The value of n is 13 since n cannot be negative.

Example 12

The sum of the first n terms of the series $4+7+10+\ldots$ is 209. Find n.

Solution

From the series, $a=4$ and $d=3$

$$S_n=\frac{n}{2}[2a+(n-1)d]=209$$

$$\frac{n}{2}(8+(n-1)\times 3)=209$$

$$8n+3n^2-3n=418$$

$$3n^2+5n-418=0$$

$$(3n+38)(n-11)=0$$

$n=11$ since n cannot be negative.

Example 13

The sum of the first eight terms of an arithmetic progression is 164. If the sum of the next eight terms is 333, find

i) first term;

ii) common difference.

Solution

i) $S_n=\dfrac{n}{2}[2a+(n-1)d]$

Given $S_8=164$

$$S_8=\frac{8}{2}[2a+(8-1)d]=164$$

$2a+7d=41$ ....(1)

Given $S_{16}-S_8=333$

$$S_{16}=333+164=497$$

$$8(2a+15d)=497$$

$16a+120d=497$ ....(2)

Solving eqns (1) and (2) simultaneously

$$16a+56d=328$$

$$16a+120d=497$$

$$64d=169 \Rightarrow d=\frac{169}{64}=2.64$$

Putting $d=\dfrac{169}{64}$ in eqn (1)

$$2a=41-\frac{7\times169}{64}=\frac{1441}{64}$$

$$a=\frac{721}{64}=11.26$$

Therefore, the first term is 11.26 and the common difference is 2.64.

Arithmetic mean

If three numbers a, b and c are in arithmetical progression, then b is called the **arithmetic mean** of a and c. The common difference of the progression is given by $b-a$ or $c-b$.

$\therefore\ b-a=c-b \Rightarrow 2b=a+c$

or $b=\dfrac{a+c}{2}$.

Therefore the arithmetic mean of a and c is $\frac{1}{2}(a+c)$. This is the ordinary 'average' of a and c.

Illustrative example

Find the arithmetic mean of 8 and 60.

Solution

The arithmetic mean of 8 and 60 is

$\frac{1}{2}(8+60)=34$

Illustrative examples on practical applications of Arithmetic Progression

Example 1

Mr Mensah starts a job with an annual salary of Gh¢6,400.00 which increases by Gh¢240 every year. After working for eight years Mr Mensah is promoted to a new post with an annual salary of Gh¢9500 which increases by Gh¢360 every year. Calculate

i) Mr Mensah's salary in the fifteenth year of service.

ii) His total earnings at the end of the fifteenth year of service.

Solution

This is an application of an A.P.

For the first post, let a_1 = first term and d_1 = common difference.

$\therefore \quad a_1 = 6400, d_1 = 240$ and $n = 8$

For the second post, let a_2 = first term and d_2 = common difference.

$\therefore \quad a_2 = 9500; d_2 = 360, \ n = 7$

i) In the fifteenth year he was paid using the second condition (i.e. after the promotion).

$$U_n = a_2 + (n-1)d_2$$
$$U_7 = 9500 + (7-1)\,360$$
$$U_7 = 9500 + 2160 = \text{Gh¢}11{,}660$$

His salary in the fifteenth year was Gh¢11,660

ii) Let S_1 = total earnings of the first post and S_2 = total earnings of the second post

$$\therefore \ S_1 = \tfrac{n}{2}[2a + (n-1)d_1]$$
$$= \tfrac{8}{2}[2 \times 6400 + (8-1)\,240]$$
$$= 4(12800 + 16800) = ¢57{,}920$$
$$S_2 = \tfrac{n}{2}[2a + (n-1)d_2]$$
$$= \tfrac{7}{2}[2 \times 9500 + (7-1)360]$$
$$= 3.5(1900 + 2160) = ¢74{,}060$$

$\therefore$ Total earnings at the end of the fifteenth year of service is

$S_1 + S_2 = 57{,}920 + 74{,}060 = \text{Gh¢}131{,}980$

Example 2

A man's salary increases by Gh¢8,000.00 each year. If his total salary at the end of 12 years is Gh¢18 525 000.00. Find

i) His initial salary

ii) His salary in the twelfth year

Solution

i) Let S_{12} = total salary at the end of 12 years

The common difference d = Gh¢8,000.00.

Let a be his initial salary

$$\therefore \ S_n = \tfrac{n}{2}[2a + (n-1)d] \qquad [n = 12]$$
$$18525000 = \tfrac{12}{2}[2a + (12-1)8000]$$
$$= 6(2a + 88000)$$
$$308800 = 2a + 88000$$

[Divide through by 6]

$$308800 - 88000 = 2a$$
$$2a = 3000000$$
$$a = 1500000$$

Hence his initial salary is Gh¢1 500 000.00

ii) His salary in the twelfth year is

$$U_{12} = a + (12-1)d = a + 11d$$
$$= 1500000 + 11(8000)$$
$$= 1500000 + 88000$$
$$= \text{Gh¢}1\ 588\ 000.00$$

Example 3

If a student saves Gh¢1 on the first day, Gh¢2 on the second day, Gh¢3 on the third day and so on. What will be his total savings in 365 days?

Solution

Here, the savings 1, 2, 3,…,365 are an AP with the first term, $a = 1$, number of terms, $n = 365$ and the last term, $l = 365$

$$S_n = \tfrac{n}{2}[a + l]$$

$$S_{365} = \tfrac{365}{2}[1 + 365]$$

$$= \tfrac{365}{2}[366] = 365 \times 183 = 66,795$$

Therefore the student saves Gh¢66,795 in 365 days.

Example 4

A man repays a loan of Gh¢3,250 by paying in the first month and then decreases the payment by Gh¢305 every month. How long will he take to clear his loan?

Solution

Let the time required to clear his loan be n months. The monthly payment decreases by Gh¢15. Therefore the payments are in AP with first term, $a = 305$ and common difference, $d = -15$. Also $S_n = 3250$

We have to find n using the formula for S_n.

$$S_n = \tfrac{n}{2}[2a + (n-1)d]$$

$$3250 = \tfrac{n}{2}[2(305) + (n-1)(-15)]$$

$$6500 = n[610 - 15n + 15]$$

$$6500 = n[625 - 15n\,]$$

$$6500 = 625n - 15n^2$$

$$15n^2 - 625n + 6500 = 0$$

$$3n^2 - 125n + 1300 = 0$$

$$(n-20)(3n-65) = 0$$

$$n = 20 \quad \text{or} \quad n = \tfrac{65}{3}$$

Since n is a natural number, $n = 20$

Hence the time required to clear the loan is 20 months.

Example 5

A farmer borrows Gh¢1,000 and agrees to repay with a total interest of Gh¢140 in 12 monthly instalments, each instalment being less than the preceding instalment by Gh¢10. What should be his first instalment?

Solution

Each instalment is less than the preceding instalment by Gh¢10.

Therefore the instalments are in AP with common difference, $d = -10$.

The farmer agrees to repay Gh¢1,000 with a total interest of Gh¢140 in 12 instalments.

i.e. $S_{12} = 1000 + 140 = 1140$

Here, $n = 12$, $d = -10$ and $S_{12} = 1140$

We have to find the first instalment i.e. a

$$S_n = \tfrac{n}{2}[2a + (n-1)d]$$

$$1140 = \tfrac{12}{2}[2a + (12-1)(-10)]$$

$$1140 = 6[2a + 11(-10)]$$

$$1140 = 6[2a - 110]$$

$$190 = 2a - 110$$

$$2a = 190 + 110 = 300$$

$$a = 150$$

Therefore the first instalment is Gh¢150.

Geometrical progression (GP) or exponential sequence

A **Geometrical Progression (GP)** or **exponential sequence** is a sequence where each term is obtained from the preceding term by multiplying by a constant factor. This constant factor is called the **common ratio**, denoted by r.

For example, the sequence 6, 12, 24, 48 is a G.P with common ratio 2.

The general term or formula of a GP with first term a and common ratio r is given by :

$$\boxed{U_n = ar^{n-1}}$$

where n is the number of terms.

The first four terms of a GP are:

$U_1 = a$; $U_2 = ar$; $U_3 = ar^2$ and $U_4 = ar^3$

i.e. $a, ar; ar^2, ar^3 \ldots$

Note that the common ratio, r is given by:

$$r = \frac{U_2}{U_1} = \frac{U_3}{U_2} = \frac{U_n}{U_{n-1}}.$$

For the sequence, 6, 12, 24, 48, the common ratio $r = \frac{12}{6} = \frac{24}{12} = 2$

Illustrative examples

Example 1

Find the 7th term of an exponential sequence of the form 5, 10, 20, 40, ...

Solution

From the sequence, the first term, $a = 5$ and the common ratio, $r = \frac{10}{5} = 2$

The general formula or term of an exponential sequence is $U_n = ar^{n-1}$

We want to find the 7th term, $\therefore\ n = 7$.

Substituting $a = 5$, $r = 2$ and $n = 7$ into the formula, the 7th term,

$$U_7 = 5 \times (2)^{7-1} = 5 \times (2)^6$$
$$= 5 \times 64 = 320$$

Example 2

Find the 11th term of the exponential sequence of the form 324, 108, 36, 12, ...

Solution

From the sequence, the first term, $a = 324$ and the common ratio, $r = \frac{108}{324} = \frac{36}{108} = \frac{1}{3}$

The general formula or term of an exponential sequence is $U_n = ar^{n-1}$

We want to find the 11th term, $\therefore\ n = 11$.

Substituting $a = 324$, $r = \frac{1}{3}$ and $n = 11$ into the formula, the 11th term,

$$U_{11} = 324 \times (\tfrac{1}{3})^{11-1} = 324 \times (\tfrac{1}{3})^{10}$$

Example 3

Find a formula in terms of n of the G.P 2, 4, 8, 16, ...

Solution

From the sequence, the first term, $a = 2$ and the common ratio, $r = \frac{4}{2} = 2$

The general formula or term of an exponential sequence is $U_n = ar^{n-1}$

Substituting $a = 2$ and $r = 2$ into the formula,

The formula in terms of n is

$$U_n = 2 \times (2)^{n-1} = 2^n$$

Note: The general term is a formula in terms of n

Example 4

Find the 12th term of the G.P 128, 64, 32 ...

Solution

From the sequence, the first term, $a = 128$ and the common ratio, $r = \frac{64}{128} = \frac{1}{2}$

The general formula or term of an exponential sequence is $U_n = ar^{n-1}$

We want to find the 12th term, $\therefore\ n = 12$.

Substituting $a = 128$, $r = \frac{1}{2}$ and $n = 12$ into the formula, the 12th term ,

$$U_{12} = 128 \times (\tfrac{1}{2})^{12-1}$$
$$= 128 \times (\tfrac{1}{2})^{11}$$
$$= 2^7 \times 2^{-11} = 2^{-4} = \tfrac{1}{16}$$

Example 5

An exponential sequence is given by $\frac{3}{4}, \frac{9}{2}, 27, ...$ find the common ratio.

Solution

From the sequence, the common ratio,

$$r = \frac{9/2}{3/4} = \frac{9 \times 4}{3 \times 2} = 6$$

Example 6

Find the nth term of the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$

Solution

The sequence is a G.P with common ratio,

$$r = \frac{1/4}{1/2} = \tfrac{1}{2} \text{ and first term} = \tfrac{1}{2}$$

$\therefore$ The nth term is

$U_n = ar^{n-1} = \frac{1}{2} \times (\frac{1}{2})^{n-1} = (\frac{1}{2})^n$

Forming the exponential sequence when two terms are given

When any two terms of a GP is given, the sequence can be formed by finding the first term, 'a' and common ratio, 'r'.

Illustrative examples

Example 1

The 4th and 8th terms of a GP are 24 and $\frac{8}{27}$ respectively. Find the two possible values of the first term a and the common ratio r.

Solution

The general formula or term of an exponential sequence is $U_n = ar^{n-1}$

The 4th term, $U_4 = ar^3 = 24$ -----(1)

The 8th term, $U_8 = ar^7 = \frac{8}{27}$ ---- (2)

Dividing eqn(2) by eqn(1)

$$\frac{ar^7}{ar^3} = \frac{\frac{8}{27}}{24} \quad \Rightarrow \quad r^4 = \frac{1}{81}$$

$$r = \pm\frac{1}{3}$$

From eqn(1), $r = \pm\frac{1}{3} \Rightarrow a = \pm 648$

$\therefore$ The GP is either 648, 216, 72, … or −648, 216, −72, ...

Example 2

The 3rd and 6th terms of an exponential sequence (GP) are $\frac{1}{4}$ and $\frac{1}{32}$ respectively.

Find

a) the first term and the common ratio.

b) the 8th term in the sequence

Solution

a) The general formula or term of an exponential sequence is $U_n = ar^{n-1}$

The 3rd term, $U_3 = ar^2 = \frac{1}{4}$ ------(1)

The 6th term, $U_6 = ar^5 = \frac{1}{32}$ ---- (2)

Dividing eqn(2) by eqn(1)

$$\frac{ar^5}{ar^2} = \frac{\frac{1}{32}}{\frac{1}{4}} = \frac{1}{32} \times \frac{4}{1} \quad \Rightarrow r^3 = \frac{1}{8}$$

$$r = \frac{1}{2}$$

From eqn(1), $r = \frac{1}{2} \quad \Rightarrow \quad a = 1$

$\therefore$ The first term is 1 and the common ratio is $\frac{1}{2}$

b) The 8th term, $U_8 = ar^7 = 1 \times (\frac{1}{2})^7 = \frac{1}{128}$

Example 3

Three consecutive terms of a geometric series have product 343 and sum $\frac{49}{2}$. Find the numbers.

Solution

It is easier to let the three terms be $\frac{a}{r}$, a, ar

Product of terms: $\frac{a}{r} . a. ar = 343$

$$a^3 = 343 \quad \Rightarrow \quad a = 7$$

Sum of terms: $\frac{a}{r} + a + ar = \frac{49}{2}$

(i.e. putting $a = 7$)

$$\frac{7}{r} + 7 + 7r = \frac{49}{2}$$

$$2r^2 - 5r + 2 = 0 \quad \Rightarrow \quad r = \frac{1}{2} \text{ or } 2$$

Therefore the numbers are $\frac{7}{2}$, 7, 14.

Example 4

The third term of a geometrical progression is 10, and the sixth is 80. Find the common ratio, the first term and the sum of the first six terms.

Solution

$r = 2$; $a = 2\frac{1}{2}$; $S_6 = 157\frac{1}{2}$

Example 5

The third term of a geometrical progression is 2, and the fifth is 18. Find two possible values of the common ratio, and the second term in each case

Solution

$r = \pm 3$; $\pm\frac{2}{3}$

Sum of a GP or exponential sequence

The general formula of the **sum** of the first n terms of an exponential sequence is given by:

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$$

This formula is used when for $r < 1$ i.e. when the common ratio is less than unity.

When $r > 1$ i.e. when the common ratio is greater than 1, the sum of the first n terms is given by:

$$\boxed{S_n = \frac{a(r^n-1)}{r-1}}$$

For any geometric sequence, we can write the following two properties:

1. $r = \dfrac{U_n}{U_{n-1}}$
2. $U_n = S_n - S_{n-1}$

Illustrative examples

Example 1

Find the 8th term and sum of the first eight terms of the exponential sequence $\frac{1}{2}, 1, 2, 4, \ldots$

Solution

From the sequence, the first term $a = \frac{1}{2}$ and the common ratio $r = 1 \div \frac{1}{2} = 1 \times \frac{2}{1} = 2$

The general formula or term of an exponential sequence is $U_n = ar^{n-1}$

We want to find the 8th term, $\therefore \; n = 8$.

The 8th term ,

$$U_8 = \tfrac{1}{2} \times (2)^{8-1} = \tfrac{1}{2} \times (2)^7$$
$$= \tfrac{1}{2} \times 128 = 64$$

Also the sum of the first n terms is given by

$$S_n = \frac{a(r^n-1)}{r-1} \quad \text{since } r > 1 \text{ i.e. } r = 2 > 1$$

$\therefore$ Sum of the first 8 terms,

$$S_8 = \frac{\frac{1}{2}(2^8-1)}{2-1} = \tfrac{1}{2}(256-1) = 127.5$$

Example 2

Find the sum of the first thirteen terms of the G.P $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ leaving your answer in the exponent form.

Solution

From the sequence,

$a = \frac{1}{2}$; $r = \frac{1}{4} \div \frac{1}{2} = \frac{1}{4} \times \frac{2}{1} = \frac{1}{2}$ and $n = 13$

The sum of the first n terms is given by:

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{since } r < 1 \text{ i.e. } r = \tfrac{1}{2} < 1$$

$\therefore$ Sum of the first 13 terms,

$$S_{13} = \frac{\frac{1}{2}(1-(\frac{1}{2})^{13})}{1-\frac{1}{2}} = 1 - (\tfrac{1}{2})^{13}$$

Example 3

The third term of a G.P is 10 and the sixth term is 80. Find the common ratio, the first term and the sum of the first six terms.

Solution

The general formula or term of an exponential sequence is $U_n = ar^{n-1}$

The 3rd term, $U_3 = ar^2 = 10$ -----(1)

The 6th term, $U_6 = ar^5 = 80$ ---- (2)

Dividing eqn(2) by eqn(1)

$$\frac{ar^5}{ar^2} = \frac{80}{10} \quad \Rightarrow \quad r^3 = 8 \quad \Rightarrow \quad r = 2$$

From eqn(1), $r = 2 \;\; \Rightarrow \;\; a = 2.5$

Also the sum of the first n terms is given by

$$S_n = \frac{a(r^n-1)}{r-1}, \text{ since } r > 1 \text{ i.e. } r = 2 > 1$$

∴ Sum of the first 6 terms,

$$S_6 = \frac{2.5(2^6 - 1)}{2 - 1} = 2.5(64 - 1) = 157.5$$

Example 4

The second and fourth terms of an exponential sequence (GP) of positive terms are 9 and 4 respectively. Find:

a) the common ratio;
b) the first term;
c) the sum of the first n terms of the sequence.

Solution

a, b) Let a and r be the first term and common ratio respectively.

The general formula or term of an exponential sequence is $U_n = ar^{n-1}$

The 2nd term, $U_2 = ar = 9$ ------(1)

The 4th term, $U_4 = ar^3 = 4$ ----- (2)

Dividing eqn(2) by eqn(1)

$$\frac{ar^3}{ar} = \frac{4}{9} \quad \Rightarrow \quad r^2 = \tfrac{4}{9} \quad \Rightarrow \quad r = \tfrac{2}{3}$$

From eqn(1), $r = \frac{2}{3} \quad \Rightarrow \quad a = 13.5$

c) The sum of the first n terms is given by

$$S_n = \frac{a(1 - r^n)}{1 - r}, \text{ since } r < 1$$

i.e. $r = \frac{2}{3} < 1$.

∴ Sum of the first n terms,

$$S_n = \frac{13.5(1 - (\frac{2}{3})^n)}{1 - \frac{2}{3}} = 40.5[1 - (\tfrac{2}{3})^n]$$

Example 5

The first three terms of a GP are $(k - 3)$, $(2k - 4)$, $(4k - 3)$, in that order. Find the value of k and the sum of the first eight terms

Solution

Since $(k - 3)$, $(2k - 4)$, $(4k - 3)$ are three consecutive terms of a GP

$$\frac{2k - 4}{k - 3} = \frac{4k - 3}{2k - 4}$$

$$(2k - 4)^2 = (k - 3)(4k - 3)$$

$$4k^2 - 16k + 16 = 4k^2 - 15k + 9$$

$$k = 7$$

The first three terms of the GP are 4, 10, 25

i.e. $a = 4$ and $r = \frac{5}{2}$

∴ Sum of the first 8 terms,

$$S_8 = \frac{4[(\frac{5}{2})^8 - 1]}{\frac{5}{2} - 1} = 4066.34 \text{ to 2 d.p.}$$

Example 6

If S_n is the sum of the first n terms of the sequence $1, (1 + x), (1 + x)^2, \ldots, (1 + x)^{n-1}$, show that

$$S_n = n + \tfrac{1}{2}n(n - 1)x + \tfrac{1}{6}(n - 1)(n - 2)x^2$$

neglecting all terms in x^3 and higher powers of x. If $n = 20$ and $x = 0.1$, calculate the approximate value of S_n.

Solution

The sequence is a GP with the first term, $a = 1$ and common ratio, $r = 1 + x = (1 + x)$

The sum of the first n terms is given by

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\therefore \quad S_n = \frac{1((1 + x)^n - 1)}{(1 + x) - 1} = \frac{(1 + x)^n - 1}{x}$$

From the binomial theorem;

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ..$$

neglecting all terms in x^3 and higher powers of x

$$S_n = \tfrac{1}{x}[1 + nx + \tfrac{1}{2}n(n - 1)x^2 + \tfrac{1}{6}n(n-1)(n-2)x^3 - 1]$$

$$= \tfrac{1}{x}[nx + \tfrac{1}{2}n(n - 1)x^2 + \tfrac{1}{6}n(n - 1)(n - 2)x^3]$$

$$= n + \tfrac{1}{2}n(n - 1)x + \tfrac{1}{6}n(n - 1)(n - 2)x^2$$

Now if $n = 20$ and $x = 0.1$, then

$S_{20}=20+\frac{1}{2}(20)(19)(0.1)+\frac{1}{6}(20)(19)(18)(0.1)^2$

$= 20 + 19 + 11.4 = 50.4$

Example 6

In an exponential sequence, the 6th term is 8 times the 3rd term and the sum of the 7th and 8th terms is 192. Find:

a) the common ratio;
b) the first term;
c) the sum of the 5th to 11th terms, inclusive.

Solution

a) Let a and r be the first term and common ratio respectively.

The general formula or term of an exponential sequence is $U_n = ar^{n-1}$

The 3rd term, $U_3 = ar^2$ and the 6th term, $U_6 = ar^5$

The 6th term is 8 times the 3rd

$ar^5 = 8\,ar^2$

$r^3 = 8$

$r = 2$

Therefore the common ratio is 2.

b) The 7th term, $U_7 = ar^6$ and the 8th term, $U_8 = ar^7$

The sum of the 7th and 8th terms is 192

$ar^6 + ar^7 = 192$

$64a + 128a = 192$

$192a = 192$

$a = 1$

Therefore the first term is 1.

c) The sum of the 5th to 11th terms, inclusive = sum of the first 11 terms minus sum of the first 4 terms

i.e. $S_{11} - S_4 = \frac{1(2^{11}-1)}{2-1} - \frac{1(2^4-1)}{2-1}$

$= (2^{11}-1)-(2^4-1)$

$= 2^{11} - 2^4 = 2048 - 16 = 2032$

Example 7

What is the smallest number of terms of the geometrical progression, 8 + 24 + 72 + …, that will give a total greater than 6 000 000?

Solution

13 (Work out the solution)

Sum to infinity of a geometrical progression

When the common ratio, r of a GP is less than unity, the sum to n terms is given by of

a G.P is given by : $S_n = \frac{a(1-r^n)}{1-r} = \frac{a-ar^n}{1-r}$

which may be written as $S_n = \frac{a}{1-r} - \frac{ar^n}{1-r}$.

Since $r < 1$, as $n \to \infty$ (i.e. as n becomes large), $r^n \to 0$.

Hence $\frac{ar^n}{1-r} \to 0$ as $n \to \infty$.

Thus $S_n \to \frac{a}{1-r}$ as as $n \to \infty$.

The quantity $\frac{a}{1-r}$ is called the **sum to infinity**, denoted by S_∞.

Hence the sum to infinity of a GP is:

$$\boxed{S_\infty = \frac{a}{1-r}}$$

which is valid when $-1 < r < 1$.

Illustrative examples

Example 1

Find the sum to infinity of the following exponential sequence

i) $1, \frac{4}{5}, (\frac{4}{5})^2, \ldots$ ii) $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \frac{5}{16}, \ldots$

Solution

i) $a = 1$ and $r = \frac{4}{5} \div 1 = \frac{4}{5}$

$$\therefore \; S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{4}{5}} = \frac{1}{\frac{1}{5}} = 5$$

ii) $a = \frac{5}{2}$ and $r = \frac{5}{4} \div \frac{5}{2} = \frac{1}{2}$

$$\therefore \; S_\infty = \frac{a}{1-r} = \frac{\frac{5}{2}}{1-\frac{1}{2}} = \frac{\frac{5}{2}}{\frac{1}{2}} = 5$$

Example 2

Find the sum to infinity of $\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \ldots$, giving your answer in the form $\frac{p}{q}$, where p and q are positive integers.

Solution

$\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \ldots$ is a geometric series with $a = \frac{7}{10}$, and $r = \frac{1}{10}$

$$\therefore \; S_\infty = \frac{a}{1-r} = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{\frac{7}{10}}{\frac{9}{10}} = \frac{7}{9}$$

Example 3

Find the sum to infinity of the exponential sequence 72, 24, 8, ...

Solution

$a = 72$ and $r = \frac{24}{72} = \frac{1}{3}$

$$\therefore \; S_\infty = \frac{a}{1-r} = \frac{72}{1-\frac{1}{3}} = \frac{72}{\frac{2}{3}} = 72 \times \frac{3}{2} = 108$$

Example 4

An exponential sequence (GP) is given by 8, 2, $\frac{1}{2}$, ... Find:

a) the n^{th} term;

b) the sum of the first n terms;

c) the sum, S for large values of n, of the sequence.

Solution

a) The GP is 8, 2, $\frac{1}{2}$, ...

$\therefore$ The first term, $a = 8$ and the common ratio, $r = \frac{2}{8} = \frac{1}{4}$. Therefore the n^{th} term is

$$U_n = ar^{n-1} = 8 \times (\tfrac{1}{4})^{n-1} = 8 \times (2^{-2})^{n-1}$$
$$= 2^3 \times 2^{2-2n} = 2^{5-2n}$$

b) The sum of the first n terms is given by

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{since } r < 1$$

$$\therefore \; S_n = \frac{8[1-(\frac{1}{4})^n]}{1-\frac{1}{4}}$$

$$= \frac{8[1-(\frac{1}{4})^n]}{\frac{3}{4}} = \tfrac{32}{3}[1-(\tfrac{1}{4})^n]$$

c) The sum S, for large values of n is the sum to infinity

$$\therefore \; S = \frac{a}{1-r} = \frac{8}{1-\frac{1}{4}} = \frac{8}{\frac{3}{4}} = 8 \times \tfrac{4}{3} = \tfrac{32}{3}$$

Alternatively, from (b), as $n \to \infty$, $(\frac{1}{4})^n \to 0$

$$S = \tfrac{32}{3}[1-0] = \tfrac{32}{3}$$

Example 5

a) The n^{th} term of an exponential sequence (GP) is 2^{1-n}. Calculate:

  i) the sum S_4, of the first four terms of the sequence,

  ii) the sum of the sequence for large positive integral values of n.

b) The sum of the first n terms of a series is given by $S_n = n(n+2)$. Find an expression for the nth term.

Solution

i) Given the nth term, $U_n = 2^{1-n}$

When $n = 1 \;\Rightarrow\; U_1 = 2^0 = 1$;

$n = 2 \;\Rightarrow\; U_2 = 2^{-1} = \frac{1}{2}$

$n = 3 \;\Rightarrow\; U_3 = 2^{-2} = \frac{1}{4}$;

$n = 4 \;\Rightarrow\; U_4 = 2^{-3} = \frac{1}{8}$

$\therefore$ The GP is 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$

Hence $S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$

Alternatively, $S_n = \dfrac{a(1-r^n)}{1-r}$ since $r < 1$,

where $r = \frac{1}{2}$, $a = 1$ and $n = 4$

$$\therefore\quad S_4 = \frac{1(1-(\frac{1}{2})^4)}{1-\frac{1}{2}} = \frac{1-\frac{1}{16}}{\frac{1}{2}} = \frac{15}{16} \times \frac{2}{1} = \frac{15}{8}$$

b) Given that sum of the first n terms of a series is given by $S_n = n(n+2)$

When $n = 1 \quad\Rightarrow\quad S_1 = 1(1+2) = 3$;

$n = 2 \quad\Rightarrow\quad S_2 = 2(2+2) = 8$ and

$n = 3 \quad\Rightarrow\quad S_3 = 3(3+2) = 15$

$\therefore$ The sequence is 3, (8 – 3), (15 – 8), ...
i.e. 3, 5, 7, ... This is an AP with first term, $a = 3$ and common difference, $d = 2$

$\therefore$ The nth term $U_n = a + (n-1)d$

$= 3 + (n-1)2 = 2n + 1$

Alternative method

$$\begin{aligned} U_n &= S_n - S_{n-1} \\ &= (n^2+2n) - [(n-1)^2 + 2(n-1)] \\ &= (n^2+2n) - [n^2 - 2n + 1 + 2n - 2] \\ &= 2n + 1 \end{aligned}$$

Example 6

a) If $\log x$, $\log y$ and $\log z$ are consecutive terms of a linear sequence (AP).
Show that x, y and z are in exponential sequence (GP).

b) The sum of the first two terms of an exponential sequence (GP) with positive common ratio is 12. If the sum of this sequence, as the number of terms increases indefinitely is 16, find,
   i) the first three terms of the sequence,
   ii) the common difference of the linear sequence (AP) obtained by taking the logarithms of the terms in b(i)

Solution

a) Since $\log x$, $\log y$ and $\log z$ are consecutive terms of a linear sequence

$\log y - \log x = d$ ......(1) and

$\log z - \log y = d$ ......(2)

where d is the common difference

From eqn(1), $\log \frac{y}{x} = d \quad\Rightarrow\quad \frac{y}{x} = 10^d$

Also from eqn(2), $\log \frac{z}{y} = d \quad\Rightarrow\quad \frac{z}{y} = 10^d$.

Since $\frac{y}{x} = 10^d$ and $\frac{z}{y} = 10^d$

x, y and z form an exponential sequence with common ratio 10^d.

b) The first two terms of a GP with first term a and common ratio r are a and ar

$\therefore\quad a + ar = 12$ ....(1)

Also the sum to infinity (i.e. the sum of a GP as n is large)

$$S = \frac{a}{1-r} = 16$$

$a = 16 - 16r$ .... (2)

Solving eqns(1) and (2) simultaneously,
Eqn(1) becomes $16 - 16r + r(16 - 16r) = 12$
(i.e. putting $a = 16 - 16r$ into eqn(1))

$16 - 16r + 16r - 16r^2 = 12$

$16 - 16r^2 = 12$

$16r^2 = 4 \quad\Rightarrow\quad r^2 = \frac{1}{4}$

$r = \frac{1}{2}$ (Taking only the positive value)

From eqn(2), $a = 16 - 16(\frac{1}{2}) = 8$

i) The first three terms of the GP are 8, 4, 2

ii) The AP formed by taking the logarithms of 8, 4 and 2 are log 8, log 4, log 2, ...
The common difference,

$d = \log 4 - \log 8$

$= \log \frac{4}{8} = \log 0.5 = -0.301$

Geometric mean

If three numbers a, b and c are in geometrical progression, then b is called the **geometric mean** of a and c. The common ratio of the progression is given by b/a or c/b.

$$\therefore \quad \frac{b}{a} = \frac{c}{a} \quad \Rightarrow \quad b^2 = ac$$

or $b = \sqrt{ac}$.

Therefore the geometric mean of a and c is $\sqrt{ac}$.

Illustrative example

Find the arithmetic mean of 5 and 125.

Solution

The geometric mean of 5 and 125 is

$\sqrt{5 \times 125} = \sqrt{625} = 25$

Illustrative examples on practical applications of Geometric Progression

Example 1

Ali bought a new car for Gh¢50,000.00. It is reckoned that the value of the depreciation per annum is 20 % of its value at the beginning of the year. Calculate the number of years after which the car is worth Gh¢13,110.00

Solution

This is an application of a GP. If the car depreciates by 20 % each year, then the values of the car for successive years form an exponential series , $a, ar, ar^2, ..., ar^n$

The first term, a = Gh¢50,000.00 and the common ratio, $r = 80\ \% = 0.8$

Note: Value decreases

$\therefore \quad r = (100 - 20)\% = 80\%$

Hence the second term is

$ar = (50,000)(0.8)$ = Gh¢40,000 which is the value after 1 year

The third term is $ar^2 = (50,000)(0.8)^2$

= Gh¢32,000

which is the value after 2 years, and so on.

When Gh¢13,110 has been reached,

$13,110 = ar^n$

$$13,110 = 50,000(0.8)^n$$

$$\frac{13,110}{50,000} = (1.08)^n$$

$$0.2622 = (1.08)^n$$

Taking logarithms to base 10 on both sides.

$$\log 0.2622 = \log (0.8)^n$$

$$\log 0.2622 = n\log (0.8)$$

$$n = \frac{\log 0.2622}{\log 0.8} = 5.999$$

Hence it will take 6 years to reach more than Gh¢13,110.00

Example 2

If Gh¢100 000.00 is invested at a compound interest of 8 % per annum determine

a) the value after 10 years

b) the time, correct to the nearest year, it takes to reach more than Gh¢300,000.00

Solution

a) The amount at the end of each year forms a GP, $a, ar, ar^2, ..., ar^n$

The first term, a = Gh¢100000 and the common ratio, $r = 1.08$

Note: Value increases

$\therefore r = (100 + 80)\% = 108\%$

Hence the second term is

$ar = (100\ 000)(1.08)$ = Gh¢108 000,

which is the value after 1 year

The third term is

$ar^2 = (100\ 000)(1.08)^2$ = ¢116 640

which is the value after 2 years, and so on.

$\therefore$ The value after 10 years

$= ar^{10} = (100\ 000)(1.08)^{10}$ = Gh¢215,890

b) When Gh¢300 000 has been reached,

$$300\ 000 = ar^n$$

$$300\ 000 = 100000(1.08)^n$$

$$3 = (1.08)^n$$

Taking logarithms to base 10 on both sides.

$$\log 3 = \log (1.08)^n$$
$$\log 3 = n\log (1.08)$$
$$n = \frac{\log 3}{\log 1.08} = 14.3$$

Combined AP and GP

Some questions are such that some terms of an AP form consecutive terms of a GP as in the examples below.

Illustrative example

Example 1

The first, third and ninth terms of a linear sequence are the first three terms of an exponential sequence. If the seventh term of the linear sequence is 14, calculate

i) the 20th term of the linear sequence

ii) the sum of the first twelve terms of the exponential sequence.

Solution

Let d = common difference of linear sequence and a = first term of both the A.P and G.P.

For the linear sequence, $U_n = a + (n-1)d$

$\therefore\ U_1 = a,\ U_3 = a + 2d$ and $U_9 = a + 8d$

$\therefore$ The exponential sequence is $a, (a + 2d), (a + 8d)$.

The common ratio $= \dfrac{a+2d}{a} = \dfrac{a+8d}{a+2d}$

$$(a + 2d)(a + 2d) = a(a + 8d)$$
$$a^2 + 4ad + 4d^2 = a^2 + 8ad$$
$$4d^2 = 4ad$$
$$d^2 = ad \Rightarrow d = a \text{ ------(1)}$$

Now the seventh term of linear sequence is 14.

Thus $a + 6d = 14 \Rightarrow 7d = 14 \Rightarrow d = 2$

But $a = d \Rightarrow a = 2$

i) For the A.P, $a = 2$ and $d = 2$
Hence the 20th term,

$$U_{20} = a + 19d = 2 + 19(2) = 40.$$

ii) The exponential sequence

$= a,\ a + 2d,\ a + 8d$

$= 2, (2 + 4), (2 + 16) = 2, 6, 18$

$\therefore$ The first term, $a = 2$ and common ratio, $r = \frac{6}{2} = 3$.

The sum of the first n terms of a GP is given by $S_n = \dfrac{a(r^n - 1)}{r - 1}$ since $r > 1$

$$\therefore\ S_{12} = \frac{2(3^{12} - 1)}{3 - 1} = 3^{12} - 1$$

Example 2

The 5th, 9th and 16th terms of a linear sequence (AP) are consecutive terms of an exponential sequence (GP).

i) Find the common difference of the linear sequence in terms of the first term.

ii) Show that the 21st, 31st and 65th terms of the linear sequence are consecutive terms of the exponential sequence whose common ratio is $\frac{7}{4}$.

Solution

i) Let a = first term and d = common difference.

The nth term of a linear sequence;

$$U_n = a + (n-1)d$$

$\therefore\ U_5 = a + 4d;\quad U_9 = a + 8d$ and $U_{16} = a + 15d$

The exponential sequence is $(a + 4d), (a + 8d), (a + 15d)$

The common ratio $= \dfrac{a+8d}{a+4d} = \dfrac{a+15d}{a+8d}$

$$(a + 8d)(a + 8d) = (a + 4d)(a + 15d)$$
$$a^2 + 8ad + 8ad + 64d$$
$$= a^2 + 15ad + 4ad + 60d^2$$
$$4d^2 = 3ad \Rightarrow d = \tfrac{3}{4}a$$

ii) For the A.P, $U_{21} = a + 20d$,
$U_{37} = a + 36d$ and $U_{65} = a + 64d$

The exponential sequence is $(a + 20d)$, $(a + 36d)$, $(a + 64d)$ or $16a$, $28a$, $49a$ since $d = \frac{3}{4}a$

The common ratio, $r = \dfrac{28a}{16a} = \dfrac{49a}{28a}$

$r = \frac{7}{4}$

Hence U_{21}, U_{37} and U_{65} form an exponential sequence with common ratio $\frac{7}{4}$.

Example 3

The second, fourth and eighth terms of an arithmetic sequence (A.P) form three consecutive terms of a geometric sequence. The sum of the third and fifth terms of the A.P. is 20. Find

i) the first four terms;

ii) sum of the first ten terms of the arithmetic sequence.

Solution

For the AP, $U_n = a + (n-1)d$

$U_2 = a + d$, $U_3 = a + 2d$, $U_4 = a + 3d$, $U_5 = a + 4d$ and $U_8 = a + 7d$

But $U_3 + U_5 = 20$

$a + 2d + a + 4d = 20$

$a + 3d = 10 \ldots\ldots\ldots\ldots\ldots\ldots(1)$

The exponential sequence is: U_2, U_4, U_8

i.e. $(a + d)$, $(a + 3d)$, $(a + 7d)$

The common ratio of the *G.P.*,

$$r = \frac{U_4}{U_2} = \frac{U_8}{U_4}$$

$$\frac{a+3d}{a+d} = \frac{a+7d}{a+3d}$$

$(a + 3d)(a + 3d) = (a + d)(a + 7d)$

$a^2 + 6ad + 9d^2 = a^2 + 8ad + 7d^2$

$2d^2 = 2ad \quad \Rightarrow \quad d = a \ldots\ldots\ldots(2)$

Putting $d = a$ in equation 1

$a + 3a = 10 \quad \Rightarrow \quad a = \frac{10}{4} = \frac{5}{2} \quad \Rightarrow \quad d = \frac{5}{2}$

i) For the *A.P.*

$U_1 = \frac{5}{2}$, $U_2 = \frac{5}{2} + \frac{5}{2} = 5$, $U_3 = 5 + \frac{5}{2} = \frac{15}{2}$

$U_4 = \frac{15}{2} + \frac{5}{2} = 10$

$\therefore$ The first four terms of the A.P. are $\frac{5}{2}$, 5, $\frac{15}{2}$, 10

ii) $S_n = \frac{n}{2}[2a + (n-1)d]$, $n = 10$

$S_{10} = \frac{10}{2}[2 \times \frac{5}{2} + (10-1) \times \frac{5}{2}]$

$S_{10} = 5[5 + \frac{45}{2}] = \frac{275}{2} = 137\frac{1}{2}$

$\therefore$ The sum of the first ten terms of the AP is $137\frac{1}{2}$.

Example 4

The first, second and fourth terms of an arithmetic progression are also consecutive terms in a geometric progression. Given that the ninth term of the arithmetic progression is 12, find the common difference of the arithmetic progression and the common ratio of the geometric progression.

Solution

The common difference, $d = \frac{4}{3}$ and the common ratio, $r = 2$. [Work out the solution]

Example 5

The three consecutive terms of an exponential sequence (G.P.) are the second, third and sixth terms of a linear sequence (A.P.). Find the common ratio of the exponential sequence.

Solution

Common ratio = 3 (Work it out)

The meaning of the sigma notation (Σ)

$\sum_{r=a}^{b} u_r$ means the **sum** of all terms such as u_r, where r takes all integral values from a to b inclusive. It is read as 'sigma from $r = a$ to b of u_r'. For example,

$$\sum_{r=1}^{n} r = 1 + 2 + 3 + \ldots + n \ ;$$

$$\sum_{r=3}^{7} r^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2 \ ;$$

$$\sum_{r=2}^{5} r = 2 + 3 + 4 + 5. \ ;$$

$$\sum_{r=1}^{\infty} r^3 = 1^3 + 2^3 + 3^3 + 4^3 + \ldots$$

This is a sum to infinity.

Illustrative example

Example 1

Evaluate i) $\sum_{n=0}^{5} 2(n-3)$

ii) $\sum_{n=0}^{5} n(n+1)$

Solution

i) $\sum_{n=0}^{5} 2(n-3)$

$= 2(-3) + 2(-2) + 2(-1) + 2(0) + 2(1) + 2(2)$

$= (-6) + (-4) + (-2) + 0 + 2 + 4 = -6$

ii) $\sum_{n=0}^{5} n(n+1)$

$= 0(1) + 1(2) + 2(3) + 3(4) + 4(5) + 5(6)$

$= 0 + 2 + 6 + 12 + 20 + 30 = 70$

Example 2

For the following series, write down the term indicated, and the number of terms in the series

i) $\sum_{r=3}^{10} (2r+3)$ 4th term

ii) $\sum_{r=-1}^{8} (2r+3)$ 5th term

iii) $\sum_{r=-6}^{-1} \frac{1}{(2r+1)}$ last term

iv) $\sum_{r=0}^{\infty} \frac{1}{(r^2+1)}$ 3rd term

Solution

i) 17 ; 8 ii) 9 ; 10 iii) -1 ; 6

iv) $\frac{1}{5}$; ∞

Example 3

Find the sum of the series $\sum_{r=1}^{8} \left(2 - \frac{2}{3}r\right)$

Solution

$\sum_{r=1}^{8} \left(2 - \frac{2}{3}r\right) = \frac{4}{3} + \frac{2}{3} + 0 - \frac{2}{3} - \ldots \frac{10}{3}$

This is an AP with 8 terms where

$a = \frac{4}{3}, \ d = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}, \ n = 8$

Using, $S_n = \frac{n}{2}[2a + (n-1)d]$ gives

$S_8 = \frac{8}{2}\left[\frac{8}{3} + 7\left(-\frac{2}{3}\right)\right] = -8$

Example 4

Find the sum of the series $\sum_{r=1}^{\infty} \frac{1}{2^r}$

Solution

$\sum_{r=1}^{\infty} \frac{1}{2^r} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

This is a GP where $a = \frac{1}{2}, \ r = \frac{1}{2}$

We are finding the sum to infinity,

Using $S_\infty = \frac{a}{1-r}$ gives $S_\infty = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

Recurrence sequence

Some series (or sequences) are given by defining a relation between U_{n+1} and U_n etc. together with the first term. For example, $U_{n+1} = 3U_n$; $n \geq 1$ and $U_1 = 3$ defines the sequence 3, 9, 27, ...

Such a relation is referred to as **recurrent relation**. Note that the resultant sequence will be either an A.P or a G.P.

Illustrative examples

Example 1

A sequence of numbers $U_1, U_2, U_3 ..., U_n$ satisfies the relation $U_{n+1} + n^2 = nU_n + 2$ for all integers $n \geq 1$. If $U_1 = 2$, find:

a) the values of U_2, U_3 and U_4

b) an expression for U_n in terms of n.

c) the sum of the first n terms of the sequence

d) the value of $U_n + U_{n+1} + U_{n+2}$ when $n = 20$.

Solution

a) The relation is given by the formula

$U_{n+1} + n^2 = nU_n + 2$; $n \geq 1$ and $U_1 = 2$

When $n = 1$; $U_2 + (1)^2 = (1)U_1 + 2$

$U_2 + 1 = U_1 + 2$

But $U_1 = 2 \Rightarrow U_2 + 1 = 2 + 2 \Rightarrow U_2 = 3$

When $n = 2$; $U_3 + (2)^2 = 2U_2 + 2$

$U_3 = 2U_2 - 2$

But $U_2 = 3 \Rightarrow U_3 = 2(3) - 2 = 4$

$\therefore U_3 = 4$

Similarly when $n = 3$; $U_4 + 9 = 3U_3 + 2$

$U_4 = 3U_3 - 7 = 3(4) - 7 = 5 \therefore U_4 = 5$

$\therefore$ The sequence U_1, U_2, U_3, U_4 ... is 2, 3, 4, 5 ...

b) The sequence is an AP with first term, $a = 2$ and common difference, $d = 1$

$\therefore U_n = a + (n-1)d = 2 + (n-1)(1)$
$= 2 + n - 1 = n + 1$

c) The sum of the nth term of an AP,

$S_n = \frac{n}{2}[2a + (n-1)d]$
$= \frac{n}{2}[2(2) + (n-1)1] = \frac{n}{2}(3 + n)$

d) From $U_n = n + 1$, when $n = 20$

$U_n = U_{20} = 20 + 1 = 21$
$U_{n+1} = U_{21} = 21 + 1 = 22$
$U_{n+2} = U_{22} = 22 + 1 = 23$
$\therefore U_n + U_{n+2} + U_{n+2} = 21 + 22 + 23 = 66$

Example 2

A sequence is given by the relation $U_{n+1} = U_n + 2$, given that $U_1 = 1$ and $n \geq 1$, find U_2, U_3 and U_4.

Solution

Given $U_{n+1} = U_n + 2$; $U_1 = 1$ and $n \geq 1$

When $n = 1$; $U_2 = U_1 + 2 \Rightarrow U_2 = 1 + 2 = 3$
since $U_1 = 1$

$n = 2$; $U_3 = U_2 + 2 \Rightarrow U_3 = 1 + 2 = 3$
since $U_2 = 3$

$n = 3$; $U_4 = U_3 + 2 \Rightarrow U_4 = 3 + 2 = 5$
since $U_3 = 5$

$\therefore$ The sequence is 1, 3, 5, 7 ... which is an AP

Note that the resultant sequence will be either an A.P or a G.P.

Example 3

A sequence of numbers U_1, U_2, U_3 ... satisfies the relation $(3n-2)U_{n+1} = (3n+1)U_n$ for all positive integers n. If $U_1 = 1$, find:

i) U_3 and U_4

ii) an expression for U_n

iii) the sum of the first n terms of the sequence.

Solution

i) Given that $(3n-2)U_{n+1} = (3n+1)U_n$ and $U_1 = 1$

When $n = 1$; $(3-2)U_2 = (3+1)U_1$

$U_2 = 4U_1$

But $U_1 = 1 \Rightarrow U_2 = 4(1) = 4 \therefore U_2 = 4$

When $n = 2$; $(6-2)U_3 = (6+1)U_2$

$4U_3 = 7U_2$

But $U_2 = 4 \Rightarrow 4U_3 = 7(4) \Rightarrow U_3 = 7$

When $n = 3$ $\quad (9-2)U_4 = (9+1)U_3$

$7U_4 = 10U_3$

But $U_3 = 7 \Rightarrow 7U_4 = 10(7) \Rightarrow U_4 = 10$

$\therefore U_3 = 7$ and $U_4 = 10$.

ii) The sequence is U_1, U_2, U_3, U_4, or 1, 4, 7, 10 ...

This is an AP with first term, $a = 1$ and common difference, $d = 3$

$$\therefore U_n = a + (n-1)d = 1 + (n-1)3$$
$$= 1 + 3n - 3 = 3n - 2.$$

iii) The sum of the nth term of an AP,

$$S_n = \tfrac{n}{2}[2a + (n-1)d]$$
$$= \tfrac{n}{2}[2 + (n-1)3] = \tfrac{n}{2}(3n-1)$$

Example 4

The n^{th} term, U_n of a sequence is given by $U_n = 3 + 5 + 7 + 9 + ... + (2n+1)$.

a) Express U_n in the form $U_n = pn(qn + r)$ where p, q and r are constants. Hence find U_9, U_{10}, U_{11}, U_{12} and U_{13}

b) Find a_1, a_2, a_3, a_4 given that $a_n = U_{n+9} - U_{n+8}$. Deduce an expression for a_n in terms of n and hence find a_{20}.

Solution

a) Given $U_n = 3 + 5 + 7 + 9 + ... + (2n+1)$.
Note that U_n is the sum of the sequence 3, 5, 7, 9, ..
The sequence is an AP with first term, $a = 3$ and common difference, $d = 2$

$$\therefore U_n = \tfrac{n}{2}[2a + (n-1)d]$$
$$= \tfrac{n}{2}[2(3) + (n-1)2]$$
$$= \tfrac{n}{2}[6 + 2n - 2] = \tfrac{n}{2}(2n+4)$$
$$= n(n+2)$$
$$= pn(qn+r)$$

where $p = 1$, $q = 1$ and $r = 2$

$U_9 = 9(9+2) = 9(11) = 99;$

$U_{10} = 10(10+2) = 10(12) = 120$

$U_{11} = 11(11+2) = 11(13) = 143$

$U_{12} = 12(12+2) = 12(14) = 168$

$U_{13} = 13(13+2) = 13(15) = 195$

b) Given $a_n = U_{n+9} - U_{n+8}$

$a_1 = U_{10} - U_9 = 120 - 99 = 21$

$a_2 = U_{11} - U_{10} = 143 - 120 = 23$

$a_3 = U_{12} - U_{11} = 168 - 143 = 25$

$a_4 = U_{13} - U_{12} = 195 - 168 = 27$

$\therefore$ The sequence is 21, 23, 25, 27 ... which is an AP.

The first term, $a = 21$ and the common difference $d = 2$

$\therefore a_n = a + (n-1)d$

(The nth term of an A.P)

$$a_n = 21 + (n-1)2$$
$$= 21 + 2n - 2 = 2n + 19$$

Hence $a_{20} = 2(20) + 19 = 40 + 19 = 59$.

Example 5

A sequence U_n is defined by the relation $U_{n+1} = 2U_n - 1$, $n = 1, 2, 3$, and $U_1 = 1$

i) Find the first five terms

ii) Show that $U_{n+1} = 3U_n - 2U_{n-1}$

Solution

i) Given that $U_{n+1} = 2U_n - 1$, $n = 1, 2, 3$, and $U_1 = 1$

When $n = 1$, $U_2 = 2U_1 - 1 = 2(2) - 1 = 3$

When $n = 2$, $U_3 = 2U_2 - 1 = 2(3) - 1 = 5$

When $n = 3$, $U_4 = 2U_3 - 1 = 2(5) - 1 = 9$

When $n = 4$, $U_5 = 2U_4 - 1 = 2(9) - 1 = 17$

Therefore the first five terms are 1, 3, 5, 9, 17

ii) $U_{n+1} = 2U_n - 1$ -----------(1)

Also putting $n = n - 1$

$U_n = 2U_{n-1} - 1$ -------(2)

Relation (1) – (2)

$U_{n+1} - U_n = 2U_n - 2U_{n-1}$

$U_{n+1} = 3U_n - 2U_{n-1}$

Example 6

A sequence of numbers $a_1, a_2, a_3 ..., a_n$ satisfies the relation $a_n = a_{n-1} + 4$, for all integers $n \geq 1$. If $a_1 = 25$, find:

a) the values of a_2, a_3 and a_4

b) an expression for a_n in terms of n.

c) the sum of the first n terms of the sequence S_n. Hence find S_{10}

Solution

a) $a_2 = 29$, $a_3 = 33$ and $a_4 = 37$

b) $a_n = 21 + 4n$

c) $S_n = n(23 + 2n)$; $S_{10} = 430$

[Work out the missing steps]

Evaluation of the limit of U_n or S_n when $n \to \infty$ (or n is large)

When evaluating the limit of any function as $n \to \infty$ (or n is large), the following must be noted

1. As $n \to \infty$; $\frac{1}{n} \to 0$
2. As $n \to \infty$; $a^n \to 0$ if $a < 1$.

   For example, as $n \to \infty$; $\left(\frac{1}{2}\right)^n \to 0$
3. As $n \to \infty$; $a^n \to \infty$ if $a > 1$.

   For example, as $n \to \infty$; $2^n \to \infty$

Note: A series is said to **converge** when its sum to infinity exist.

Illustrative examples

Example 1

A sequence of numbers $U_1, U_2, U_3 ..., U_n$ satisfies the relation $4U_{n+1} - U_n = 12$, for all integers $n \geq 1$. Find the value of the constant k such that $U_{n+1} - k = \frac{1}{4}(U_n - k)$ for all positive integer of n. If $U_1 = 5$, obtain an expression for U_n all positive integer of n. Hence find the limiting value of U_n as n increase indefinitely.

Solution

The relation given is:

$4U_{n+1} - U_n = 12$ ....(1)

Also we have: $U_{n+1} - k = \frac{1}{4}(U_n - k)$

$4U_{n+1} - 4k = U_n - k$

$4U_{n+1} - U_n = 3k$ ....(2)

Comparing relation (1) and (2) gives:

$3k = 12 \quad \Rightarrow \quad k = 4,$

From $U_{n+1} - k = \frac{1}{4}(U_n - k)$ and substituting $k = 4$, we have

$U_{n+1} - 4 = \frac{1}{4}(U_n - 4)$

When $n = 1$; $U_2 - 4 = \frac{1}{4}(U_1 - 4)$

$= \frac{1}{4}(5 - 4) = \frac{1}{4}$ [Given that $U_1 = 5$]

When $n = 2$;

$U_3 - 4 = \frac{1}{4}(U_2 - 4) = \frac{1}{4} \times \frac{1}{4} = \left(\frac{1}{4}\right)^2$

When $n = 3$;

$U_4 - 4 = \frac{1}{4}(U_3 - 4) = \frac{1}{4} \times \left(\frac{1}{4}\right)^2 = \left(\frac{1}{4}\right)^3$

A pattern has been developed looking at the values at the RHS with general term $\left(\frac{1}{4}\right)^{n-1}$.

i.e. $U_n - 4 = \left(\frac{1}{4}\right)^{n-1} \quad \Rightarrow \quad U_n = 4 + \left(\frac{1}{4}\right)^{n-1}$

Since $\left(\frac{1}{4}\right)$ is between -1 and 1, as n increase indefinitely, $\left(\frac{1}{4}\right)^{n-1}$ approaches 0.

Hence the limiting value of U_n as n increase indefinitely is 4.

Example 2

A sequence of numbers $U_1, U_2, U_3 ..., U_n$ satisfies the relation $U_{n+1} = U_n + \left(\frac{1}{2}\right)^n$ for all integers $n \geq 1$. If $U_1 = 1$, find:

a) the values of U_2, U_3 and U_4

b) an expression for U_n in terms of n.

c) the value of U_n for large values of n

Solution

a) $U_2 = \frac{3}{2}$, $U_3 = \frac{7}{4}$ and $U_4 = \frac{15}{8}$

b) $U_n = 2\left[1 - \left(\frac{1}{2}\right)^n\right]$ c) 2

[Work out the missing steps]

Example 3

A sequence of numbers $U_1, U_2, U_3 \ldots U_n$ satisfies the relation $U_n = U_{n-1} + \left(\frac{1}{5}\right)^{n-1}$ for all integers $n \geq 1$. If $U_1 = 1$, find:

a) the values of U_2, U_3 and U_4

b) an expression for U_n in terms of n.

c) the value of U_n for large values of n.

Solution

a) $U_2 = \frac{3}{2}$, $a_3 = \frac{7}{4}$ and $U_4 = \frac{15}{8}$

b) $U_n = \frac{5}{4}\left[1 - \left(\frac{1}{5}\right)^n\right]$ c) $\frac{5}{4}$

[Work out the missing steps]

Example 4

The nth term of a sequence is given by $U_n = 3 \times 2^{n-1}$

i) Write down the first four terms of the sequence.

ii) Calculate the least value of n for which $U_n > 9000$

Solution

i) Given that $U_n = 3 \times 2^{n-1}$ and $n \geq 1$

When $n = 1$; $U_1 = 3 \times 2^0 = 3$;

$n = 2$; $U_2 = 3 \times 2^1 = 6$;

$n = 3$; $U_3 = 3 \times 2^2 = 12$;

$n = 4$; $U_4 = 3 \times 2^3 = 24$

$\therefore$ The sequence is 3, 6, 12, 24.

ii) $U_n > 9000 \quad \Rightarrow 3 \times 2^{n-1} > 9000$

$2^{n-1} > 3000 \quad \Rightarrow 2^n > 3000$

$2^n > 6000$

Taking logarithms of both sides to base 10

$\log_{10} 2^n > \log_{10} 6000$

$n\log 2 > \log 6000$

$$n > \frac{\log 6000}{\log 2} \Rightarrow n > 12.6 \Rightarrow n \geq 13$$

Note that log2 is positive so the sign remains unchanged

$\therefore$ The least value of n for which $U_n > 9000$ is 13.

Recurring decimals

Recurring decimals (i.e. decimals which do not terminate and the same digits or group of digits are repeated indefinitely) can be written as an infinite exponential series. For example, 0.9999… can be written as an exponential series as

$$0.9 + 0.09 + 0.009 + 0.0009 + \ldots$$

with a common ratio, 0.1. Similarly, 0.333333… can be written as an exponential series as: $0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + 0.000003 + \ldots$ with a common ratio 0.1

Such recurring decimals can be expressed as fractions by writing them as exponential series and finding their sum as in the examples below.

Illustrative examples

Example 1

Write 0.9999 as exponential series and show that the sum to infinity equals 1.

Solution

$0.9999 = 0.9 + 0.09 + 0.009 + 0.0009$ with first term, $a = 0.9$ and common ratio, $r = 0.1$

The sum to infinity,

$$S = \frac{a}{1-r} = \frac{0.9}{1-0.1} = \frac{0.9}{0.9} = 1$$

Example 2

a) Write 0.16666 as an infinite geometric series

b) Find the sum of the geometric series in (a)

Solution

a) 0.16666

$= 0.1 + 0.06 + 0.006 + 0.0006 + 0.00006 + \ldots$

b) $0.06 + 0.006 + 0.0006 + 0.00006$ is a G.P. with first term, $a = 0.06$ and common ratio, $r = 0.1$

The sum to infinity

$$= \frac{a}{1-r} = \frac{0.06}{1-0.1} = \frac{0.06}{0.9} = \frac{6}{90} = \tfrac{1}{15}$$

Therefore the sum of

$0.1 + 0.06 + 0.006 + 0.0006 + 0.00006 + \ldots$

$= 0.1 + \tfrac{1}{15} = \tfrac{1}{10} + \tfrac{1}{15} = \tfrac{1}{6}$

Example 3

Express the recurring decimal $0.1\dot{5}7\dot{6}$ as a fraction in its lowest term.

Solution

$0.1\dot{5}7\dot{6} = 0.1\,\overline{576}\,\overline{576}\,\overline{576}$

$= 0.1 + 0.0576 + 0.0000576 + 0.000000576 + \ldots$

$$= \frac{1}{10} + \frac{576}{10^4} + \frac{576}{10^7} + \frac{576}{10^{10}} + \ldots$$

$$= \frac{1}{10} + \frac{576}{10^4}\left[1 + \frac{1}{10^3} + \frac{1}{10^6} + \ldots\right]$$

$$= \frac{1}{10} + \frac{576}{10^4}\left[1 + \frac{1}{10^3} + \left(\frac{1}{10^3}\right)^2 + \ldots\right]$$

The series in bracket is a GP whose first term is 1 and common ratio is $\frac{1}{10^3}$.

Hence it has a sum to infinity of

$$\frac{01}{1-0.001} = \frac{1}{0.999} = \frac{10^3}{999}$$

$$0.1\dot{5}7\dot{6} = \frac{1}{10} + \frac{576}{10^4} \times \frac{10^3}{999}$$

$$= \frac{1}{10} + \frac{576}{9990} = \frac{1575}{9990} = \frac{35}{222}$$

Example 4

Express the following recurring decimals as a fraction in its lowest term.

a) $0.1\dot{6}\dot{2}$ b) $0.\dot{3}\dot{4}$ c) $0.0\dot{2}\dot{1}$

d) $0.0\dot{5}\dot{1}$ e) $0.\dot{1}\dot{0}$

Solution

a) $\frac{161}{990}$ b) $\frac{34}{99}$ c) $\frac{7}{330}$

d) $\frac{17}{330}$ e) $\frac{10}{99}$

Work out the solution in each case.

Review exercise 15

1. Find the 8th term of the following linear sequences.
   - i) 2, 4, 6, 8, ...
   - ii) −7, −4, −1, 2, ...
   - iii) $0, -\frac{1}{2}, -1, -1\frac{1}{2}, \ldots$
   - iv) 12, 5, −2, ...
   - v) 8, 14, 20, …
   - vi) 63, 58, 53, …

2. Find a formula for the *n*th term of the following linear sequences
   - i) 12, 5, −2 ... ii) 10, 6, 2, −2,...

3. The 3rd term of an AP is 15 and the 15th term is 21. Find the common difference, first term and the sum of the first ten terms.

4. Find the 8th term of the GP
   - i) 16, 8, 4, ... ii) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$
   - iii) 1, −2, 4 −8, ... iv) $\frac{1}{2}, \frac{3}{4}, \frac{9}{8} \ldots$

5. The 3rd and 6th term of a GP. are 108 and −32 respectively. Find the sum of the first seven terms.

6. For the sequence given by $U_{n+1} = 3 + \frac{1}{4}U_n$ $(n \geq 1)$ and $U_1 = 3$, find
   - i) U_2, U_3 and U_4
   - ii) a formula for U_n in terms of *n*.
   - iii) the sum of the first *n* terms.

7. Find the sum to infinity of the GP

i) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ ii) $1, \frac{1}{2}, \frac{1}{4}, \ldots$

iii) $30, 15, 7\frac{1}{2}, \ldots$ iv) $4, 2, 1, \ldots$

v) $2\frac{1}{2}, -1\frac{1}{4}, \frac{5}{8}, \ldots$ vi) $3, 1, \frac{1}{3}, \ldots$

8. If a ball is dropped from a height h meters it bounces to r % of the height. Write down a sequence for the distances which the ball travels as it falls and rises up to the fourth time.

9. Find the number of terms of the arithmetical progression $4, 6\frac{1}{2}, 9, 11\frac{1}{2}, \ldots$ needed to make a total of 126.

10. The first term of an exponential sequence is 3 and the 6th term is $\frac{3}{32}$. Find the common ratio.

11. The nth term of a sequence is $U_n = an + b (n \geq 1)$ where a and b are constants. $U_1 = 4$ and $U_3 = 10$. Find

a) $a^2 - b^2$ b) U_4

12. Find the number of terms of the sequence 5, 8, 11, …of which the sum is 1025.

13. Which term of the series 2187, 729, 243, ... is $\frac{1}{9}$.

14. The first, twelfth and last terms of an AP are 4, $31\frac{1}{2}$ and $376\frac{1}{2}$ respectively. Determine

a) the number of terms in the sequence

b) the sum of all the terms

c) the 80th term

15. Find the sum of the first 12 terms of the sequence 5, 9, 13, 17, ...

16. The sum of the first 7 terms of an AP is 35 and the common difference is 1.2. Determine the first term

17. The 6th term of an AP is 17 and 13th term is 38. Determine the 19th term

18. Determine the number of the term whose value is 22 in the sequence $2\frac{1}{2}, 4, 5\frac{1}{2}, 7, \ldots$

19. The first term of a GP is 4 and the 6th term is 128. Determine the 8th and 11th terms.

20. On commencing employment a man is paid a salary of Gh¢7,200.00 per annum and receives annual increments of Gh¢350.00. Determine his salary in the 9th year and calculate the total salary he will receive in the first 12 years.

21. The value of a lathe originally valued at Gh¢30,000.00 depreciates 15 % per annum. Calculate its value after 5 years. The machine is sold when its value is less than Gh¢5,500.00. After how many years is the lathe sold?

22. If Gh¢250,000 is invested at compound interest of 6 % per annum, determine

a) the value after 15 years

b) the time, correct to the nearest year, it takes to reach Gh¢750,000.

23. 100 grammes of a radioactive substance disintegrate at a rate of 3 % per annum. How much of the substance is left after 12 years?

24. In a geometric progression of positive terms, the 5th term is 9 times the 3rd term and the sum of the 6th and 7th terms is 1944. Determine

a) the common ratio
b) the first term
c) the sum of the 4th to 10th terms inclusive.

25. The first three terms of a GP are $3(q+5)$, $3(q+3)$ and $(q+7)$ respectively.

a) Calculate the possible values of q
b) For each possible value of q, find the common ratio of the GP.

26. The sum of the first ten terms of an arithmetic series is 60. The sum of the first twenty-two terms is 220. Find the common difference and the first term of the series.

27. The sum of the first two terms of a GP is 24 and the sum to infinity is 27.

a) Show that the common ratio is $\pm\frac{1}{3}$
b) Find the two possible values of the first term

28. Three consecutive terms of a GP are c, $(c+4)$ and $(c+6)$, in that order.

a) Calculate the value of c
b) The value of the common ratio of the GP.

29. A man invests Gh¢100,000 at the beginning of each year for 10 years. The rate of compound interest is 9 % per annum. Calculate the total value of the investment at the end of the 10 full years.

30. The sum of the first twenty terms of an AP is 45, and the sum of the first forty terms is 290. Find the first term and the common difference

31. Evaluate

a) $\sum_{r=1}^{6} r^2$ b) $\sum_{n=0}^{5}(n-3)$

c) $\sum_{n=2}^{8} 2(n-3)$

32. i) Write the following as geometric series

a) 0.090 909 09
b) 0.431 043 104 31
c) 0.432 121 1211

ii) Find the sum of the geometric series in (i).

33. The first term of an AP is 11 and the fifteenth term is 21. Find the common difference, the first term and the nth term.

34. The nth term of an AP is $12-4n$. Find the first term and the common difference.

35. The 9th term of an AP is 8 and the 4th term is 20. Find the common difference, the first term and the nth term.

36. The nth term of an AP is $\frac{1}{2}(3-n)$. Write down the first three terms and 20th term.

37. In an arithmetic progression, the 8th term is twice the 4th term and the 20th term is 40. Find the first term, the common difference and the sum of the terms from the 8th term to the 20th term inclusive

38. In an arithmetic progression, the 6th term is twice the 4th term and the 3rd term is 15. Find the first term, the common difference

39. Write down the fifth term and nth term of the following GPs:

i) 2, 4, 8, ...

ii) 2, 1, $\frac{1}{2}$,....

iii) 3,–6, 12, ...

iv) first term 8, common ratio $-\frac{1}{2}$

iv) first term 3, common ratio $\frac{1}{81}$

40. The 6th term of a GP is 16 and the 3rd term is 2. Find the common ratio and the first term and the nth term.

41. The sum of the first 3 terms of a GP is 14. If the first term is 2, find the possible values of the sum of the first 5 terms.

42. a) A mortgage is taken out for Gh¢100,000.00 and is repaid by annual instalments of Gh¢20,000.00. Interest is charged on outstanding debt at 10% annually. If the first repayment is made one year after the mortgage is taken out find the number of years it takes for the mortgage to be repaid.

b) A bank loan of Gh¢5,000,000.00 is arranged to be repaid in two years by equal monthly instalments. Interest, calculated monthly, is charged on outstanding debt at 11% annually. Calculate the monthly repayment if the first repayment is to be made one month after the loan is granted.

43. A sequence of numbers $U_1, U_2, U_3 ..., U_n$ satisfies the relation $U_n = U_{n-1} + \left(\frac{1}{2}\right)^{n-1}$ for all integers $n \geq 1$. If $U_1 = 1$, find:

a) the values of U_2, U_3 and U_4

b) an expression for U_n in terms of n.

c) the value of U_n for large values of n

44. The sum of the first twelve terms of an AP is 168. If the third term is 7, find the value of

i) the common difference,

ii) the first term.

45. A sequence of numbers $U_1, U_2, U_3 ..., U_n$ satisfies the relation $3U_{n+1} - U_n = 15$, for all integers $n \geq 1$. Find the value of the constant k such that $U_{n+1} - k = \frac{1}{3}(U_n - k)$ for all positive integer of n. If $U_1 = 9$, obtain an expression for U_n all positive integer of n. Hence find the limiting value of U_n as n increase indefinitely.

46. A sequence of numbers $U_1, U_2, U_3 ..., U_n$ satisfies the relation $3U_{n+1} - 2U_n = 4$, for all integers $n \geq 1$. Find the value of the constant k such that $U_{n+1} - k = \frac{2}{3}(U_n - k)$ for all positive integer of n. If $U_1 = 5$, obtain an expression for U_n all positive integer of n. Hence find the limiting value of U_n as n increase indefinitely.

47. The first three terms of a geometric progression are also the first, ninth and eleventh terms respectively, of an arithmetic progression. Given that the terms of the geometric progression are all different, find the common ratio r. If the sum to infinity of the geometric progression is 8, find the first term and common difference of the arithmetic progression.

48. If £100 is invested at compound interest of 8% per annum determine

a) the value after 10 years,

b) the time, correct to the nearest year, it takes to reach more than £300.

49. On commencing employment a man is paid a salary of £7200 per annum and receives annual increments of £350. Determine his salary in the 9th year and calculate the total he will have received in the first 12 years.

50. The value of a lathe originally valued at £3000 depreciates 15% per annum. Calculate its value after 5 years. The machine is sold when its value is less than £550. After how many years is the lathe sold?

51. Express the following recurring decimals as a fraction in its lowest term.

a) $0.4\dot{0}\dot{7}$ b) $0.\dot{2}\dot{3}$ c) $0.3\dot{4}\dot{2}\dot{5}$

d) $0.\dot{7}$ e) $0.\dot{2}\dot{0}\dot{3}$

52. The sum of the first n terms of a series is given by S_n where $S_n = n(3n-4)$. Show that the terms of the series are in arithmetic progression.

53. How many terms of the AP, $1 + 3 + 5\ldots$ are required to make a sum of 1521?

54. Find the least number of terms of the AP, $1 + 3 + 5 \ldots$ that are required to make a sum exceeding 4000.

55. If the sum of the first n terms of a series is S_n where $S_n = 2n^2 - n$,

a) prove that the series is an AP, stating the first term and the common difference,

b) find the sum of the terms from the 3rd to the 12th inclusive.

56. The 5th term of a GP is 8, the third term is 4, and the sum of the first ten terms is positive. Find the first term, the common ratio, and the sum of the first ten terms.

57. Write down the fifth term and the nth term of the following GPs:

a) $2, 4, 8, \ldots$ b) $2, 1, \frac{1}{2}, \ldots$

c) $3, -6, 12, \ldots$

d) first term 8, common ratio $-\frac{1}{2}$

e) first term 3, last term $\frac{1}{81}$, 6 terms

58. Find the sum, to the number of terms given, of the following GPs:

a) $3 + 6 + \ldots$, 6 terms

b) $3 - 6 + \ldots$, 8 terms

c) $1 + \frac{1}{2} + \frac{1}{4} + \ldots$, 20 terms.

d) first term 5, common ratio $\frac{1}{5}$, 5 terms

e) first term $\frac{1}{2}$, common ratio $-\frac{1}{2}$, 10 terms

59. The 6th term of a GP is 16 and the 3rd term is 2. Find the first term and the common ratio.

60. Find the common ratio, given that it is negative, of a GP whose first term is 8 and whose 5th term is $\frac{1}{2}$.

61. The nth term of a GP is $(-\frac{1}{2})^n$. Write down the first term and the 10th term. Evaluate $\sum_{r=1}^{10}(1.05)^r$

62. Find the sum of the first n terms of the GP $2 + \frac{1}{2} + \frac{1}{8} + \ldots$ and find the least value of n for which this sum exceeds 2.65.

63. The sum to infinity of a GP is twice the first term. Find the common ratio.

64. The sum to infinity of a GP is 16 and the sum of the first 4 terms is 15. Find the first four terms.

65. The sum of the first n terms is a series is n^3. Write down the first four terms and the n th term of the series.

66. The fourth term an AP is 8 and the sum of the first ten terms is 40. Find the first term and the tenth term.

67. The sum of the first three terms of a geometric sequence is 8 and the sum of the first six terms is 12. Find the common ratio.

68. The second term of a GP is $\frac{1}{2}$ and the sum to infinity of the series is 4. Find the first term and the common ratio of the series.

69. A sequence is defined by $2+\sqrt{3}, 5+\sqrt{3}, 8+\sqrt{3},\ldots$ Find the n^{th} term of the sequence.

70. A geometric sequence is defined by $(\sqrt{2}-1),(3-2\sqrt{2})$,…
Find the common ratio.

71. The 4th and the 13th terms of an A.P. are 5 and –1 respectively, find its 8th term.

72. In a Geometric Progression, the fourth term exceeds the third term by 72 and the third term exceeds the second term by 24, Calculate the first four terms of the progression and their sum.

73. The sum of the first 9 terms of an A.P. is 72 and the sum of the next 4 terms is 71, find the first term and common difference.

74. Given that the 6th term of a GP is 1215 and the 3rd term is 360, calculate
i) the first term
ii) the common ratio
iii) the sum of the first four terms

75. If the second and fourth terms of a G.P. are 8 and 32 respectively, what is the sum of the first four terms?

76. In an A.P. the first term is 2, and the sum of the 1st and 6th terms is $16\frac{1}{2}$, what is the eleventh term?

77. a) In an A.P., the difference between the 8th and 4th terms is 20 and the 8th term is $1\frac{1}{2}$ times the fourth term, what is the:
i) common difference
ii) first term of the sequence

b) The value of a machine depreciates each year by 5% of its value at the beginning of that year. If its value when new on 1st January 1980 was £10,250.00, what was its value in January 1989 when it was 9 years old? Give your answer correct to three significant figures.

78. Ten wooden poles are to be used for pillars and the lengths of the poles form an Arithmetic Progression (A.P.). If the second pole is 2 m and the sixth pole is 5 m, give the lengths of the poles in order.

79. The sum of the first n terms of a series is given by $S_n = n^2 + n$. Find the nth term and the first three terms of the series.

80. The nth term of a sequence is given by: $(-1)^{n-2}2^{n-1}$. Find the sum of the second and third terms.

81. a) What is the 25th term of 5, 9, 13, …?

b) Find the 5th term if the 3rd and 6th terms of a Geometric Progression (G.P.) are 48 and $14\frac{2}{9}$ respectively, Write down the first four terms of the G.P.

82. The 4th term of an arithmetic progression is 27 and the sum of the 1st and 4th terms is 42. What is the third term?

83. The first and third terms of a G.P. are 5 and 80 respectively. What is the 4th term?

84. The third term of a linear sequence (A.P.) is 16 and its 6th term is 34, find the second term.

85. a) The nth term of a sequence of numbers $U_1, U_2, U_3, \ldots$ is given by $U_n = 6 + 2^n$

i) Show that for all values of $n \geq 1$ $U_{n+1} = 2(U_n - 3)$

ii) Obtain and simplify the sum of the first n terms of the sequence.

b) The sum S of the first n terms of a series is given by $S_n = 4 - (\frac{1}{2})^{n-1}$ Find the smallest n such that $S_n > 3.999$

86. If the 2nd and 5th terms of G.P. are −6 and 48 respectively, find the sum of the first four terms.

87. The fourth term of an A.P. is 37 and the 6th term is 12 more than the fourth term. Find the first and seventh terms.

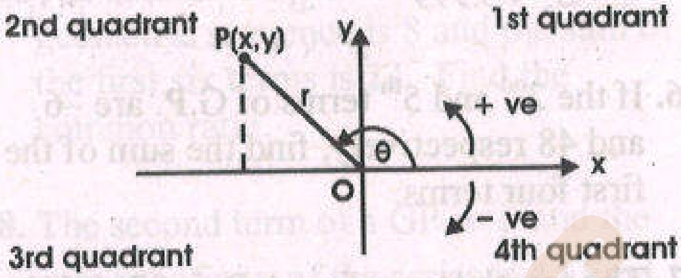
88. How many terms are there in the series $3 + 6 + 12 + \ldots + 384$

CHAPTER 16

TRIGONOMETRIC RATIOS AND RULES

Basic trigonometric ratios and their reciprocals

Let Ox, Oy be the usual perpendicular axes, with the same scale. See the figure below. If P is the point (x, y), then $OP = r$ (positive) and $\angle POX = \theta$ (anticlockwise positive).


2nd quadrant
P(x,y)
Y
1st quadrant
+ ve
θ
X
O
– ve
3rd quadrant
4th quadrant

From the diagram above, the definitions of the three main trigonometric ratios when θ is the general angle are:

1. sine $\theta = \sin\theta = \dfrac{y}{r}$
2. cosine $\theta = \cos\theta = \dfrac{x}{r}$
3. tangent $\theta = \tan\theta = \dfrac{y}{x}$

The three other trigonometric ratios are the reciprocal of the three main ratios. They are:

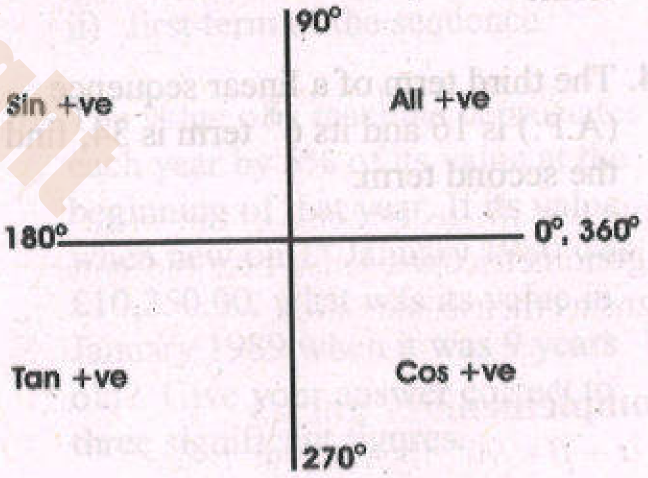
4. cosecant $\theta = \operatorname{cosec}\theta = \dfrac{1}{\sin\theta} = \dfrac{r}{y}$
5. secant $\theta = \sec\theta = \dfrac{1}{\cos\theta} = \dfrac{r}{x}$
6. cotangent $\theta = \cot\theta = \dfrac{1}{\tan\theta} = \dfrac{x}{y}$

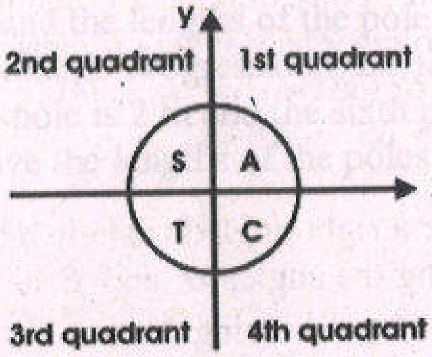
Note that these functions are less commonly used. Therefore they should always be replaced by their reciprocals.

The signs of the trigonometric ratios

The trigonometric ratios are defined in terms of x, y and r where (x, y) are the coordinates of point P on the final arm of the angle θ. The signs of the trigonometric ratios depend only on the signs of x and y. They may be positive or negative depending on the quadrant in which they lie and their values are related to the corresponding acute angles with the x-axis. The signs of the ratios can be remembered using the **CAST** diagram below.

Note : In trigonometry, positive angles are measured in anticlockwise direction from the positive x-direction, starting from 0° to 360°.


90°
Sin +ve
All +ve
180°
0°, 360°
Tan +ve
Cos +ve
270°


Y
2nd quadrant
1st quadrant
S
A
X
T
C
3rd quadrant
4th quadrant

1. If $P(x, y)$ is in the first quadrant, then both x and y are positive. Hence all the trigonometric ratios are positive.

2. If $P(x, y)$ the second quadrant, x is negative and y is positive.

   $\therefore\ \sin\theta = \frac{y}{r}$ is positive, whereas

   $\cos\theta = \frac{x}{r}$ and $\tan\theta = \frac{y}{x}$ are negative

3. If $P(x, y)$ is in the third quadrant, both x and y are negative.

   $\therefore\ \tan\theta = \frac{y}{x}$ is positive, whereas

   $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$ are negative

4. If $P(x, y)$ is in the fourth quadrant, x is positive but y is negative.

   $\therefore\ \cos\theta = \frac{x}{r}$ is positive, whereas

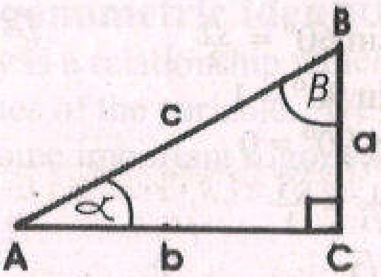
   $\sin\theta = \frac{y}{r}$ and $\tan\theta = \frac{y}{x}$ are negative

Note that the reciprocals of the basic trigonometric ratios also take the same signs as their basic ratios.

Complementary angles

If $\alpha + \beta = 90^\circ$, then α and β are complementary angles because their sum is 90°. For example, 25° and 65° are complementary angles because the sum of the two angles is 90°.
i.e. $25^\circ + 65^\circ = 90^\circ$.

Consider a right-angled triangle ABC containing the angles α and β as shown in the following diagram.



$$\sin\alpha = \frac{a}{c} = \cos\beta\ ; \qquad \cos\alpha = \frac{b}{c} = \sin\beta$$

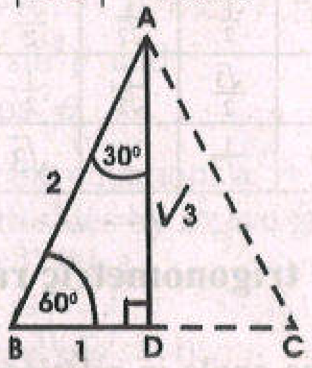
$$\tan\alpha = \frac{a}{b} = \cot\beta \qquad \cot\alpha = \frac{b}{a} = \tan\beta$$

In general, if α and β are complementary angles then we can write the following:
i) $\cos\alpha = \sin\beta$ or $\cos\alpha = \sin(90 - \alpha)$
For example, $\cos 30^\circ = \sin 60^\circ$ and
$\sin 20^\circ = \cos 70^\circ$

ii) $\tan\alpha = \cot\beta$ or $\tan\alpha = \cot(90 - \alpha)$
For example, $\tan 30^\circ = \cot 60^\circ$

Trigonometric ratios of special angles

The trigonometric ratios for the angles 0°, 30°, 45°, 60° and 90° (and other angles for which these are the associated acute angles, e.g. 120°, 150°, 225°, 300 etc.) are used frequently in problems so it is very useful to remember their values.
First, consider an equilateral triangle ABC of sides 2 units.
From Pythagoras theorem, $|AD| = \sqrt{3}$.
Also $|BD| = |DC| = 1$ unit.



From triangle ABC:
$\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$

$\cos 30^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}$

$\cos 0^\circ = \sin 90^\circ = 1$

$\sin 0^\circ = \cos 90^\circ = 0$

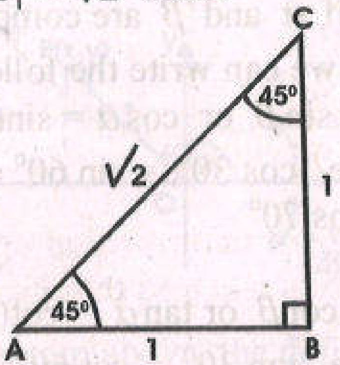
$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\tan 60^\circ = \sqrt{3}$

$\tan 90^\circ = \infty$

$\tan 0^\circ = 0$

Again consider a right-angled isosceles triangle ABC with the two sides $|AB| = |BC| = 1$ unit. From Pythagoras theorem, $|AC| = \sqrt{2}$ units.



From triangle ABC:

$\tan 45^\circ = 1$

$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Note: $\cos 45^\circ = \sin 45^\circ = \frac{\sqrt{2}}{2}$

The table below gives a summary of the ratios of the special angles. These ratios should be memorized.

	0°	30°	45°	60°	90°
sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

To find the trigonometric ratio of any angle

1. Sketch the angle in relation to the x and y axes;
2. Find the 'associated acute angle' i.e. the acute angle the radius makes with the x-axis;
3. Find the required ratio of this angle;
4. Find the appropriate sign using the **CAST** diagram. This shows which ratios are positive in each quadrant.

Note : In trigonometry, angles are measured in the anticlockwise direction from the positive x-axis.

Illustrative example

Find the values of:

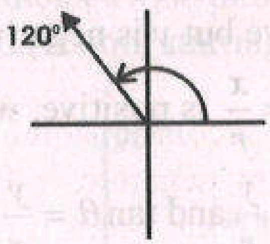
i) sin 120° ii) cos 240°

iii) tan 315° iv) sec 120°

v) sin 210°

Solution

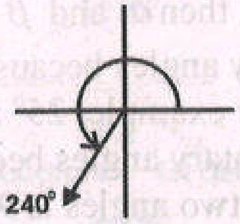
i)



120° is in the second quadrant, therefore its sine ratio is positive. The associated acute angle with the x-axis is 60°

$\therefore \sin 120^\circ = +\sin 60^\circ = \frac{\sqrt{3}}{2}$

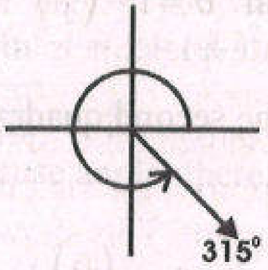
ii)



240° is in the third quadrant, therefore its cosine ratio is negative. The associated acute angle with the x-axis is 60°

$\therefore \cos 240^\circ = -\cos 60^\circ = -\frac{1}{2}$

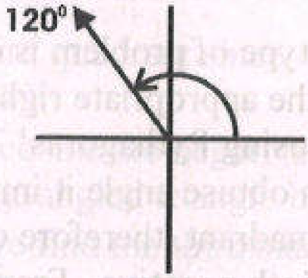
iii)



315° is in the fourth quadrant, therefore its tangent ratio is negative. The associated acute angle with the x-axis is 45°

$\therefore\ \tan 315^\circ = -\tan 45^\circ = -1$

iv) $\sec 120^\circ = \dfrac{1}{\cos 120^\circ}$

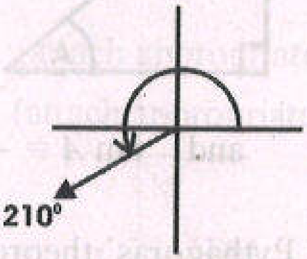


120° is in the second quadrant, therefore its cosine ratio is negative. The associated acute angle with the x-axis is 60°

$\cos 120^\circ = -\cos 60^\circ = -\frac{1}{2}$

$$\therefore\quad \sec 120^\circ = \frac{1}{\cos 120^\circ} = \frac{1}{-\frac{1}{2}} = -2$$

v)

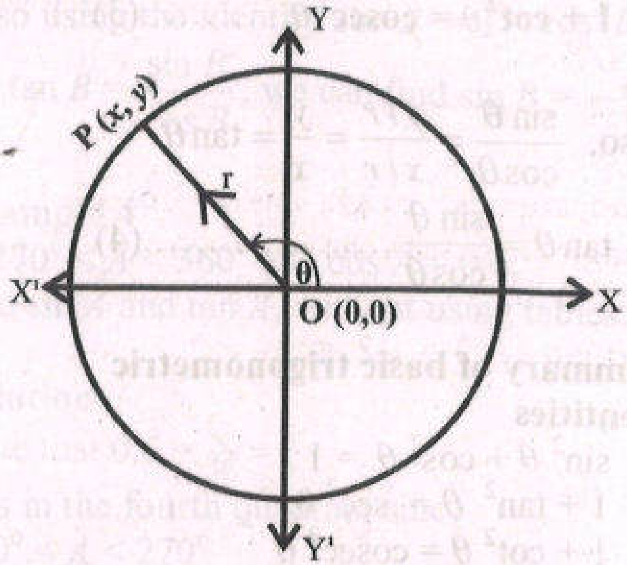


sin 210° is in the third quadrant, therefore its sine ratio is negative. The associated acute angle with the x-axis is 30°

$\therefore\ \sin 210^\circ = -\sin 30^\circ = -\frac{1}{2}$

Basic trigonometric identities

An identity is a relationship which is true for all values of the variable. We shall consider some important trigonometric identities.



Draw a circle of radius, r. Let O be the centre which is taken as the origin.
Draw the X and Y axes.
If P is the point (x, y) on the circle, then the equation of the circle with centre (0, 0) and radius r is:

$$x^2 + y^2 = r^2 \ \ldots\ldots\ldots\ldots\ldots\ldots (A)$$

But we know that $\cos\theta = x/r$ and $\sin\theta = y/r$ so we divide both sides of the equation of the circle by r^2 to obtain

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2} \quad \Rightarrow \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$\therefore\ \mathbf{\cos^2\theta + \sin^2\theta \ \ = 1} \ldots\ldots\ldots\ldots (1)$$

From identity 1, we can also write the following:

$\cos^2\theta = 1 - \sin^2\theta$ and

$\sin^2\theta = 1 - \cos^2\theta$

Again, from the equation, $x^2 + y^2 = r^2$
Dividing both sides by x^2, we get

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$$

$\Rightarrow\ 1 + (\tan\theta)^2 = (\sec\theta)^2$

$$\therefore\ \mathbf{1 + \tan^2\theta \ \ = \sec^2\theta} \ldots\ldots\ldots\ldots (2)$$

Similarly, if we divide the equation (A) by y^2, we get

$$\left(\frac{x}{y}\right)^2+1=\left(\frac{r}{y}\right)^2 \Rightarrow (\cot\theta)^2+1=(\operatorname{cosec}\theta)^2$$

$$\therefore\ \mathbf{1+\cot^2\theta=\operatorname{cosec}^2\theta}\ \ldots\ldots\ldots.(3)$$

Also, $\dfrac{\sin\theta}{\cos\theta}=\dfrac{y/r}{x/r}=\dfrac{y}{x}=\tan\theta$

$$\therefore\ \tan\theta=\frac{\sin\theta}{\cos\theta}\ \ldots\ldots\ldots\ldots\ldots\ldots.(4)$$

Summary of basic trigonometric identities

1. $\sin^2\theta+\cos^2\theta.=1$
2. $1+\tan^2\ \theta=\ \sec^2\theta$
3. $1+\cot^2\theta=\operatorname{cosec}^2\theta$
4. $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$

Note 1: $\sin^2\theta$ means $(\sin\theta)^2$ and so on.

Note 2: Identities (1) to (4) can be used to convert a given trigonometric ratio to another. They can also be used to simplify or establish trigonometric identities and solve trigonometric equations.

Converting one trigonometric ratio to another

Illustrative examples

Example 1

If $\sin\theta=\dfrac{\sqrt{3}}{2}$ and θ lies in the second quadrant, find $\cos\theta$ and $\tan\theta$.

Solution

We have $\sin\theta=\dfrac{\sqrt{3}}{2}$

$$\cos^2\theta=1-\sin^2\theta=1-\left(\tfrac{\sqrt{3}}{2}\right)^2=1-\tfrac{3}{4}=\tfrac{1}{4}$$

$$\therefore\ \cos\theta=\pm\tfrac{1}{2}$$

But θ lies in the second quadrant, therefore θ is negative.

$$\therefore\ \cos\theta=-\tfrac{1}{2}$$

Also, $\tan\theta=\dfrac{\sin\theta}{\cos\theta}=\dfrac{\left(\frac{\sqrt{3}}{2}\right)}{\left(-\frac{1}{2}\right)}=-\sqrt{3}$

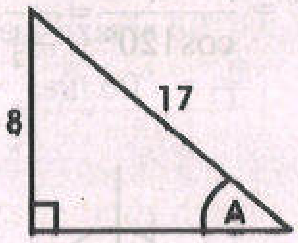
Example 2

Given that $\sin A=\frac{8}{17}$ and that A is an obtuse angle, find $\cos A$ and $\tan A$ without using tables.

Solution

Note: This type of problem is often solved by drawing the appropriate right-angled triangle and using Pythagoras' Theorem. Since A is an obtuse angle it implies A is in the second quadrant, therefore cosine and tangent are both negative. From the right-angled triangle (see figure below) drawn for $\sin A=\frac{8}{17}$, we can find the third side (i.e. the adjacent). i.e. the third side

$$=\sqrt{17^2-8^2}=\sqrt{289-64}=\sqrt{225}=15$$



$\cos A=-\frac{15}{17}$ and $\tan A=-\frac{8}{15}$

Note 1: Use Pythagoras' theorem to find the third side of the right-angled triangle.

Note 2: Attach the appropriate sign (In this case it is negative).

Alternatively

$\cos^2 A = 1 - \sin^2 A = 1 - (\frac{8}{17})^2 = 1 - \frac{64}{289} = \frac{225}{289}$

$\Rightarrow \cos A = \sqrt{\frac{225}{289}} = \pm\frac{15}{17}$.

But A is an obtuse angle, therefore

$\cos A = -\frac{15}{17}$

Using identity (1),

$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{8}{17}}{-\frac{15}{17}} = -\frac{8}{15}$$

Example 3

If $180^\circ < B < 270^\circ$ and $\tan B = \frac{12}{5}$, find $\sin B$ and $\cos B$, without using tables.

Solution

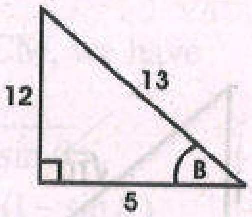
B is in the third quadrant since $180^\circ < B < 270^\circ$

$\therefore$ sine and cosine are negative.

From the right-angled triangle below for $\tan B = \frac{12}{5}$; we find the third side (i.e. the hypotenuse)

i.e. the third side $= \sqrt{12^2 + 5^2}$

$= \sqrt{144 + 25} = \sqrt{169} = 13$



$\sin B = -\frac{12}{13}$ (attach appropriate sign)

$\cos B = -\frac{5}{13}$ (attach appropriate sign)

Alternatively

From identity 5, $1 + \tan^2 B = \sec^2 B$

$1 + \left(\frac{12}{5}\right)^2 = \sec^2 B \Rightarrow 1 + \frac{144}{25} = \sec^2 B$

$\sec^2 B = \frac{169}{25}$

$\sec B = \sqrt{\frac{169}{25}} = \pm\frac{13}{5}$

But $\cos B = \dfrac{1}{\sec B} = \pm\frac{5}{13}$

Since B is in the third quadrant, cosine is negative $\Rightarrow \cos B = -\frac{5}{13}$

Also using the identity $\sin^2 B = 1 - \cos^2 B$ or tan $B = \dfrac{\sin B}{\cos B}$, we can find $\sin B = -\frac{12}{13}$

Example 4

If $270^\circ < A < 360^\circ$ and $\cos A = 0.6$, find $\sin A$ and $\tan A$, without using tables.

Solution

Note that $0.6 = \frac{6}{10} = \frac{3}{5}$

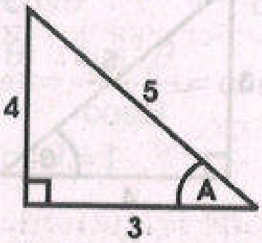
A is in the fourth quadrant since $180^\circ < A < 270^\circ$

$\therefore$ sine and tan are negative.

From the right-angled triangle below for $\cos A = \frac{3}{5}$; we can find the third side (i.e. the opposite).

i.e. the third side $= \sqrt{5^2 - 3^2}$

$= \sqrt{25 - 9} = \sqrt{16} = 4$



$\sin A = -\frac{4}{5}$ (attach appropriate sign)

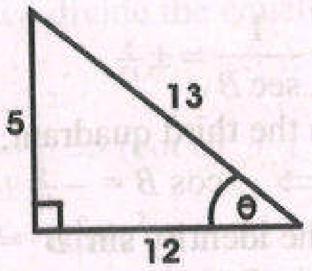
$\tan A = -\frac{4}{3}$ (attach appropriate sign)

Example 5

Find, without using tables, the values of $\cos\theta$ and $\tan\theta$, if $\sin\theta = \frac{5}{13}$.

Solution

Since $\sin\theta$ is positive, θ may be in the first or second quadrant.



If θ is in the first quadrant, then
$\cos\theta = \frac{12}{13}$ and $\tan\theta = \frac{5}{12}$

If θ is in the second quadrant, then
$\cos\theta = -\frac{12}{13}$ and $\tan\theta = -\frac{5}{12}$

Note : In this example, θ is not bounded.

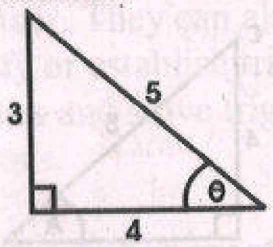
Example 6
Find, without using tables, the values of

i) $\sin\theta$ and $\cos\theta$, if $\tan\theta = \frac{3}{4}$

ii) $\tan\theta$ and $\sin\theta$, if $\cos\theta = \frac{4}{5}$

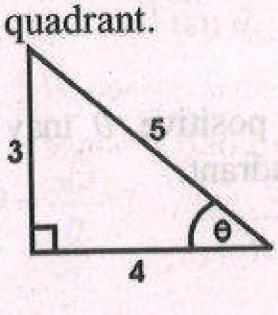
iii) $\cos\theta$ and $\tan\theta$, if $\sin\theta = -\frac{24}{25}$

Solution
i) Since $\tan\theta$ is positive, θ may be in the first or third quadrant.



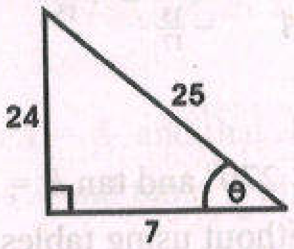
If θ is in the first quadrant, then
$\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$
If θ is in the third quadrant, then
$\sin\theta = -\frac{3}{5}$ and $\cos\theta = -\frac{4}{5}$

ii) Since $\cos\theta$ is positive, θ may be in the first or fourth quadrant.



If θ is in the first quadrant, then
$\tan\theta = \frac{3}{4}$ and $\sin\theta = \frac{3}{5}$
If θ is in the fourth quadrant, then
$\tan\theta = -\frac{3}{4}$ and $\sin\theta = -\frac{3}{5}$

iii) Since $\sin\theta$ is negative, θ may be in the third or fourth quadrant.

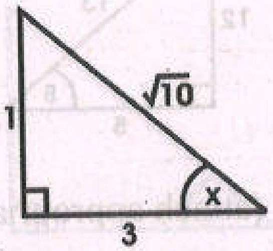


If θ is in the third quadrant, then
$\cos\theta = -\frac{7}{25}$ and $\tan\theta = \frac{24}{7}$
If θ is in the fourth quadrant, then
$\cos\theta = \frac{7}{25}$ and $\tan\theta = -\frac{24}{7}$

Note : In these examples θ is not bounded.

Example 7
Find, without using tables, the values of $\sin x$ and $\cos x$, if $\tan x = \frac{1}{3}$ and x is acute.

Solution



If x is an acute angle, then
$\sin x = \frac{1}{\sqrt{10}}$ and
$\cos x = \frac{3}{\sqrt{10}}$

Note: Draw the right-angled triangle and use it to find the third side.

Simplifying or establishing an identity

Illustrative examples

Example 1

Simplify $\dfrac{1}{1-\cos\theta}+\dfrac{1}{1+\cos\theta}$

Solution

Finding the LCM, we have

$$\frac{1}{1-\cos\theta}+\frac{1}{1+\cos\theta}$$

$$=\frac{(1+\cos\theta)+(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$$

$$=\frac{2}{1-\cos^2\theta}=\frac{2}{\sin^2\theta}=2\text{cosec}^2\theta$$

Note: $1-\cos^2\theta=\sin^2\theta$ and

$$\frac{1}{\sin^2\theta}=\frac{1}{(\sin\theta)^2}=\text{cosec}^2\theta$$

Example 2

Simplify $\dfrac{1}{1-\sin\theta}+\dfrac{1}{1+\sin\theta}$

Solution

Finding the LCM, we have

$$\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}$$

$$=\frac{(1+\sin\theta)+(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$$

$$=\frac{2}{1-\sin^2\theta}$$

$$=\frac{2}{\cos^2\theta}$$

$$=2\sec^2\theta$$

Note: $1-\sin^2\theta=\cos^2\theta$ and

$$\frac{1}{\cos^2\theta}=\frac{1}{(\cos\theta)^2}=\sec^2\theta$$

Example 3

Simplify the following:

i) $(1-\cos^2\theta)\sec^2\theta$

ii) $\dfrac{\sec\theta}{\cos\theta}-\dfrac{\tan\theta}{\cot\theta}$

iii) $\sec\theta-\sin\theta\tan\theta$

Solution

i) $(1-\cos^2\theta)\sec^2\theta$

$$=\sin^2\theta\times\frac{1}{\cos^2\theta}=\left(\frac{\sin\theta}{\cos\theta}\right)^2=\tan^2\theta$$

ii) $\dfrac{\sec\theta}{\cos\theta}-\dfrac{\tan\theta}{\cot\theta}$

$$=\sec\theta\times\frac{1}{\cos\theta}-\tan\theta\times\frac{1}{\cot\theta}$$

$$=\sec^2\theta-\tan^2\theta=1$$

iii) $\sec\theta-\sin\theta\tan\theta$

$$=\sec\theta-\sin\theta\frac{\sin\theta}{\cos\theta}$$

$$=\sec\theta-\frac{\sin^2\theta}{\cos\theta}=\frac{\sec\theta\cos\theta-\sin^2\theta}{\cos\theta}$$

$$=\frac{1-\sin^2\theta}{\cos\theta}=\frac{\cos^2\theta}{\cos\theta}=\cos\theta$$

Note: $\sec\theta\cos\theta=1$

Example 4

Show that

i) $\cot\theta+\dfrac{\sin\theta}{1+\cos\theta}=\text{cosec}\,\theta$

ii) $\tan\theta+\cot\theta=\dfrac{1}{\sin\theta\cos\theta}$

Solution

i) $\cot\theta+\dfrac{\sin\theta}{1+\cos\theta}=\dfrac{\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{1+\cos\theta}$

$$\cot\theta+\frac{\sin\theta}{1+\cos\theta}=$$

$$= \frac{\cos\theta(1+\cos\theta)+\sin^2\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{\cos\theta+\cos^2\theta+\sin^2\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{(\cos\theta+1)}{\sin\theta(1+\cos\theta)} = \frac{1}{\sin\theta} = \operatorname{cosec}\theta$$

ii) $\tan\theta+\cot\theta = \dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}$

$$= \frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta}$$

Note: Change the left hand side of the identity, step by step, using the standard identities.

Example 5

Eliminate θ from the equations $x = 3\cos\theta$ and $y = 2\sin\theta$

Solution

$x = 3\cos\theta \quad \Rightarrow \quad \dfrac{x}{3} = \cos\theta$ ....(1) and

$y = 2\sin\theta \quad \Rightarrow \quad \dfrac{y}{2} = \sin\theta$ ...(2)

$(1)^2 + (2)^2$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = \cos^2\theta+\sin^2\theta$$

$$\frac{x^2}{9}+\frac{y^2}{4} = 1$$

Example 6

Prove that $\dfrac{\tan\theta}{1+\tan^2\theta} = \sin\theta\cos\theta$

Solution

In such examples, we start with either the L.H.S or the R.H.S and simplify it so as to obtain the other side.

We notice that we have only $\sin\theta$ and $\cos\theta$ in the R.H.S. Hence, we shall replace $\tan\theta$ by $\dfrac{\sin\theta}{\cos\theta}$ in L.H.S.

Also note that $1 + \tan^2\theta = \sec^2\theta$ which in turn is equal to $\dfrac{1}{\cos^2\theta}$.

$$\text{L. H. S} = \frac{\tan\theta}{1+\tan^2\theta}$$

$$= \frac{\tan\theta}{\sec^2\theta} = \left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\sec^2\theta}\right)$$

$$= \left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{\cos^2\theta}{1}\right)$$

$$= \sin\theta\cos\theta = \text{R.H.S.}$$

Example 7

Prove that $\dfrac{\cot A-1}{\cot A+1} = \dfrac{1-\tan A}{1+\tan A}$

Solution

$$\text{L.H.S.} = \frac{\cot A-1}{\cot A+1} = \frac{\dfrac{1}{\tan A}-1}{\dfrac{1}{\tan A}+1}$$

$$= \frac{\dfrac{1-\tan A}{\tan A}}{\dfrac{1+\tan A}{\tan A}} = \frac{1-\tan A}{1+\tan A} = \text{R.H.S}$$

Example 8

Prove that $\sqrt{\dfrac{1-\sin A}{1+\sin A}} = \sec A - \tan A$

Solution

We multiply the numerator and denominator of the L.H.S. by a suitable factor so as to remove the square root sign. For this, we will use: $1 - \sin^2 A = \cos^2 A$ and obtain $\cos A$

in the denominator of the L.H.S, as $\cos A$ is required in the denominator in R.H.S.

$$\text{L.H.S} = \sqrt{\frac{1-\sin A}{1+\sin A}} = \sqrt{\frac{1-\sin A}{1+\sin A} \times \frac{1-\sin A}{1-\sin A}}$$

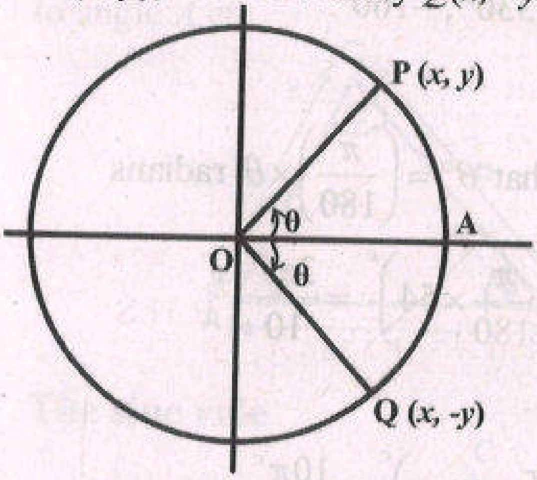
$$= \sqrt{\frac{(1-\sin A)^2}{1+\sin^2 A}} = \sqrt{\frac{(1-\sin A)^2}{\cos^2 A}}$$

$$= \frac{1-\sin A}{\cos A} = \frac{1}{\cos A} - \frac{\sin A}{\cos A}$$

$$= \sec A - \tan A = \text{R.H.S}$$

Negative angles

Angles measured in the **clockwise direction** with respect to the positive x-axis are **negative**. We now establish a relation between the trigonometric ratios of θ and $-\theta$. Consider a circle of radius r ($r > 0$). Let OA rotate around O, reaching OP and forming an angle θ, which is positive. Also, rotate OA in clockwise direction, to reach OQ and form an angle $-\theta$ or $(360^\circ - \theta)$.
If $P(x, y)$, then obviously $Q(x, -y)$.



i) $\sin\theta = \dfrac{y}{r}$ and $\sin(-\theta) = \dfrac{-y}{r}$

$\therefore \sin(-\theta) = \sin(360^\circ - \theta) = -\sin\theta$

ii) $\cos\theta = \dfrac{x}{r}$ and $\cos(-\theta) = \dfrac{x}{r}$

$\therefore \cos(-\theta) = \cos(360^\circ - \theta) = \cos\theta$

iii) Also $\tan(-\theta) = \tan(360^\circ - \theta) = -\tan\theta$

Proof:

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

For example;

$\sin(-30^\circ) = -\sin 30^\circ = -\frac{1}{2}$

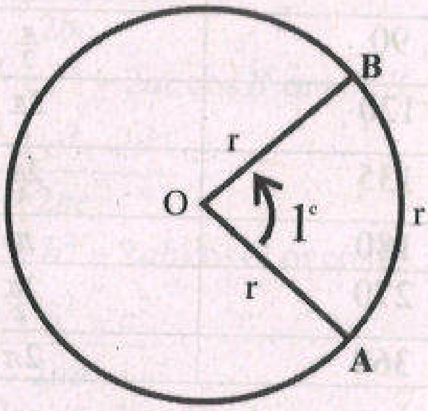
$\cos(-60^\circ) = \cos 60^\circ = \frac{1}{2}$

$\tan(-45^\circ) = -\tan 45^\circ = -1$

Note: The relations $\cot(-\theta) = -\cot\theta$ $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$ and $\sec(-\theta) = \sec\theta$, follow automatically.

Circular measure

Unlike sexagesimal system, where the angles are measured in degrees, in the circular measure system the unit of measurement is a **'radian'**
To define a radian, consider a circle with centre at O and radius r ($r > 0$). Let B be a point on the circumference of this circle obtained by rotating the initial arm OA in the anti-clockwise direction so that the length of the arc AB is also r. The circular measure of the angle AOB thus formed is called 'one radian' and is denoted by 1^c.



The angle of 1^c does not depend on the radius of the circle, i.e. whatever the value of the radius may be; the only requirement to get an angle of one radian is that the arc AB must have the length r.

Hence, we can define a radian as follows;

Definition: One radian is the angle subtended at the centre of a circle, by an arc of the circle, the length of the arc being equal to the radius of the circle. One radian is denoted as 1^c.

Relation between the circular measure and the sexagesimal measure

Since the circumference of a circle is 2π times the radius, there are 2π arc lengths equal to the radius in a circumference. Hence 2π radians are measured by the circumference of a circle or

2π radians = 360°. From which, π = 180°

Therefore, $1° = \frac{\pi^c}{180}$.

Below are some useful common equivalence.

Angle in degrees	Radian equivalent
0	0
30	$\frac{\pi}{6}$
45	$\frac{\pi}{4}$
60	$\frac{\pi}{3}$
90	$\frac{\pi}{2}$
120	$\frac{2\pi}{3}$
135	$\frac{3\pi}{4}$
180	π
270	$\frac{3\pi}{2}$
360	2π

Degrees can be converted to radians by the formula: $\theta° = \left(\frac{\pi}{180}\right) \times \theta$ radians

Also radians can be converted to degrees by the formula: $R = \left(\frac{180}{\pi}\right) \times R$ degrees.

Illustrative examples

Example 1

Convert the following angles into degrees.

$\frac{3\pi^c}{5}, \frac{5\pi^c}{9}, \frac{21\pi^c}{5}, \frac{-2\pi}{5}$

Solution

$$\theta = \left(\frac{180}{\pi}\right) \times R \quad \therefore \quad \theta = \left(\frac{180}{\pi} \times \frac{3\pi}{5}\right)^{\circ} = 108°$$

Similarly,

$$\frac{5\pi^c}{9} = \left(\frac{180}{\pi} \times \frac{5\pi}{9}\right)^{\circ} = 100°$$

$$\frac{21\pi^c}{5} = \left(\frac{180}{\pi} \times \frac{21\pi}{5}\right)^{\circ} = 756°$$

$$\frac{-2\pi^c}{5} = \left(\frac{180}{\pi} \times \frac{-2\pi}{5}\right)^{\circ} = -72°$$

Example 2

Convert the following angles into radians.

54°, 200°, 330°, −160°.

Solution

We know that $\theta° = \left(\frac{\pi}{180}\right) \times \theta$ radians

$$\therefore \quad 54° = \left(\frac{\pi}{180} \times 54\right)^c = \frac{3\pi^c}{10}$$

Similarly,

$$200° = \left(\frac{\pi}{180} \times 200\right)^c = \frac{10\pi^c}{9}$$

$$330° = \left(\frac{\pi}{180} \times 330\right)^c = \frac{11\pi^c}{6}$$

$$-160° = \left(\frac{\pi}{180} \times (-160)\right)^c = -\frac{8\pi^c}{9}$$

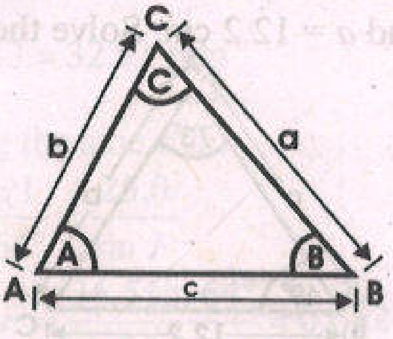
Cosine and sine rules – Solutions of plane triangles and their areas

Triangles are involved in many practical measurements (e.g. in surveying).

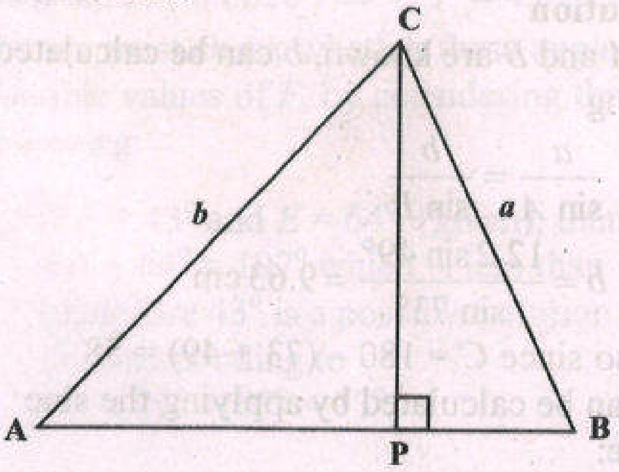
A triangle has three sides and three interior angles. The sum of the three angles of a triangle is 180°. To **solve a triangle** means to find the values of unknown sides and angles in that triangle. If the triangle is **right-angled** or **isosceles**, it can be solved using **Pythagoras theorem** and/or the basic trigonometric ratios. However, if the triangle is a **non-right-angled triangle**, Pythagoras theorem and trigonometric ratios **cannot** be used. Instead two rules, called cosine and sine rules are used. To solve a triangle, first sketch the triangle and mark in the given data. Then use the appropriate formula(e).

Standard notation

In a triangle *ABC*, the angles are *A*, *B*, *C* and the sides are *a*, *b*, *c*. Thus side *a* is opposite to angle *A* etc.



The sine rule



In triangle *ABC*, *CP* is perpendicular to *AB*. The length of the altitude *CP* is equal to $b\sin A$ (from triangle *APC*) and it is also equal to $a\sin B$ (from triangle *BPC*).

Equating these expressions, we have $a\sin B = b\sin A$ and hence

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Applying the same argument to the line from *A*, perpendicular to *BC*, we could obtain

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Putting these expressions together, we have,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The rule may be used only when

1. One side and any two angles are initially given, or
2. Two sides and an angle (not the included angle) are initially given.

The cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos A \text{ or } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$b^2 = a^2 + c^2 - 2ac\cos B \text{ or } \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \text{ or } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

The scalar product has been used to proof the cosine rule under in *chapter 24*.

The rule may be used only when

1. Two sides and the included angle are initially given, or
2. Three sides are initially given.

Note that for any triangle, the longest side is opposite the largest angle and the smallest side is opposite the smallest angle. Therefore, it is always worth checking that the longest side is opposite the largest angle and vice-versa. Also, the sum of the three angles is 180°.
i.e. $A + B + C = 180^\circ$

Applications of the sine and cosine rules

1. **Given three sides :** find the largest angle using the cosine rule, the second angle by the cosine or sine rule and the third angle by the 'angle sum'.

2. **Given two sides and the included angle :** find the third side by the cosine rule, the smaller angle by the sine rule and the third angle by the 'angle sum'.

3. **Given two sides and a non-included angle (the ambiguous case)**
   Try to find an angle using the sine rule. This can give two, one or no possible solutions.
   Find the third angle by the angle sum and the third side by the sine rule.

4. **Given one side and two angles :** Find the third angle by the 'angle sum' and the other two sides by the sine rule.

5. **Given two or three angles only :** The sides cannot be found. The sine rule gives the ratios between sides.

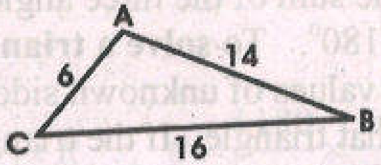
Note that the calculation involved in using the cosine rule is less straightforward than that required in using the sine formula. Consequently, the cosine rule is used only when the sine rule is inapplicable.

Illustrative examples

Example 1

Solve the triangle with sides 6, 14 and 16 units.

Solution



The largest angle is opposite to the 16 units side.

$$\cos A = \frac{14^2 + 6^2 - 16^2}{2 \times 14 \times 6} = -0.1429$$

$$A = 98.2^\circ$$

(obtuse because cos$A$ is negative)

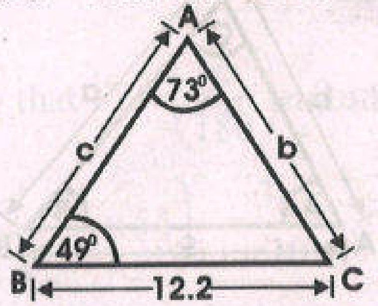
$$\cos C = \frac{16^2 + 6^2 - 14^2}{2 \times 16 \times 6} = 0.5 \;\Rightarrow\; C = 60^\circ$$

$$B = 180 - (98.2 + 60) = 28.8^\circ$$

(angle of a triangle sum)

Example 2

In a triangle ABC, it is known that $A = 73^\circ$, $B = 49^\circ$ and $a = 12.2$ cm. Solve the triangle.



Solution

a, A and B are known, b can be calculated using

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$b = \frac{12.2 \sin 49^\circ}{\sin 73^\circ} = 9.63 \text{ cm}$$

Also since $C = 180 - (73 + 49) = 58$

c can be calculated by applying the sine rule:

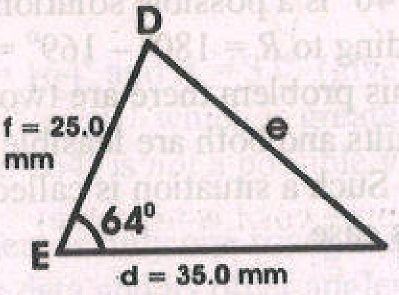
$$\frac{a}{\sin A}=\frac{c}{\sin C}$$

$$c=\frac{12.2\sin 58^\circ}{\sin 73^\circ}=10.82\text{ cm}$$

Example 3

Solve the triangle *DEF* and find its area given that $E = 64^\circ$, $EF = 35.0$ mm and $DE = 25.0$ mm.

Solution



Triangle *DEF* is shown above.

Applying the cosine rule:

$$e^2 = d^2 + f^2 - 2df\cos E$$

$$e^2 = (35.0)^2 + (25.0)^2 - 2(35.0)(25.0)\cos 64^\circ$$

$$=1083$$

$$e=\sqrt{1083}=32.91\text{ mm}$$

Applying the sine rule:

$$\frac{32.91}{\sin 64^\circ}=\frac{25.0}{\sin F}$$

$$\sin F=\frac{36.5\sin 64^\circ}{32.91}=0.6828$$

Now $\sin F = 0.6828 \Rightarrow F = 43^\circ \text{ or } 136^\circ$

But we must check whether these angles are possible values of *F*, by considering the following:

a) If $F = 43^\circ$ and $E = 64^\circ$ (given), then $43^\circ + 64^\circ = 107^\circ$ which is less than 180°
Therefore 43° is a possible solution for *F* corresponding to
$D = 180^\circ - 107^\circ = 73^\circ$

b) If $F = 136^\circ$ and $E = 64^\circ$ (given), then $136^\circ + 64^\circ = 200^\circ$ which is greater than 180°
Therefore 136° is *not* a possible value for *C*.
Hence there is only one triangle defined by the given data and its other angles are $F = 43^\circ$ and $D = 73^\circ$
The area of any triangle *DEF* is given by

$$\Delta=\frac{1}{2}\times df\sin E=\frac{1}{2}(35.0)(25.0)\sin 64^\circ$$

$$=393.2\text{ mm}^2$$

The ambiguous case

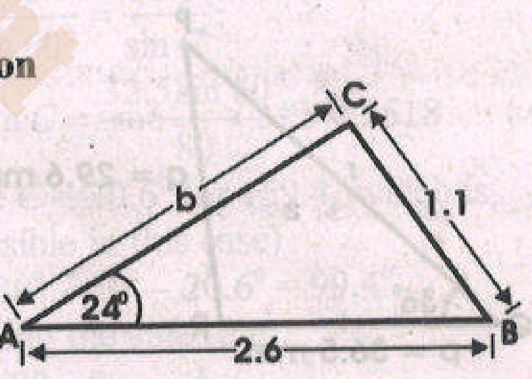
Sometimes, when two sides and one angle are specified, two separate sets of results can be found from the given data and both are feasible solutions. Such a situation is called the **ambiguous case.**

Illustrative examples

Example 1

Solve the triangle *ABC* in which $A = 24^\circ$, $c = 2.6$ cm and $a = 1.1$ cm

Solution



From the diagram, we can solve for *C* by applying the sine rule

$$\frac{a}{\sin A}=\frac{c}{\sin C}\quad\Rightarrow\quad\frac{1.1}{\sin 24^\circ}=\frac{2.6}{\sin C}$$

$$\sin C=\frac{2.6\sin 24^\circ}{1.1}=0.9614$$

Now $\sin C = 0.9614 \Rightarrow C = 74^\circ \text{ or } 106^\circ$

But we must check whether these angles are possible values of C, by considering the following:

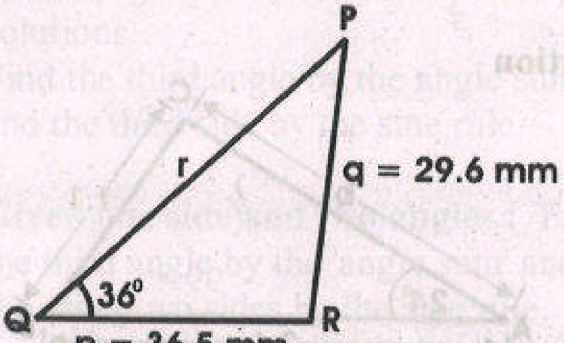
a) If $C = 74^\circ$ and $A = 24^\circ$ (given), then $74^\circ + 24^\circ = 98^\circ$ which is less than 180°
Therefore 74° is a possible solution for C corresponding to $B = 180^\circ - (74^\circ + 24^\circ) = 82^\circ$

b) If $C = 106^\circ$ and $A = 24^\circ$ (given), then $106^\circ + 24^\circ = 130^\circ$ which is less than 180°
Therefore 74° is also a possible solution for C corresponding to
$B = 180^\circ - (106^\circ + 24^\circ) = 50^\circ$

Thus, in this problem there are two separate sets of results and both are feasible solutions. Such a situation is called the ambiguous case.

Example 2
Solve the triangle PQR and find its area given that $Q = 36^\circ$, $QR = 36.5$ mm and $PR = 29.6$ mm

Solution



From the diagram, we can solve for P by using the sine rule

$$\frac{q}{\sin Q} = \frac{p}{\sin P} \quad \Rightarrow \quad \frac{29.6}{\sin 36^\circ} = \frac{36.5}{\sin P}$$

$$\sin P = \frac{36.5 \sin 36^\circ}{29.6} = 0.7248$$

Now $\sin P = 0.7248 \Rightarrow P = 46^\circ \text{ or } 133^\circ$

But we must check whether these angles are possible values of P, by considering the following:

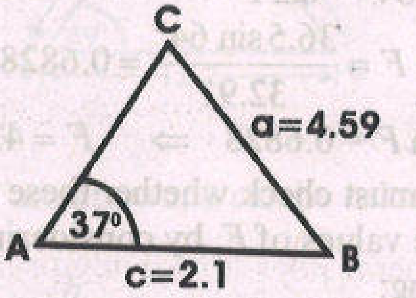
a) If $P = 46^\circ$ and $Q = 36^\circ$ (given), then $46^\circ + 36^\circ = 82^\circ$ which is less than 180°
Therefore 46° is a possible solution for P corresponding to $R = 180^\circ - 82^\circ = 98^\circ$

b) If $P = 133^\circ$ and $Q = 36^\circ$ (given), then $133^\circ + 36^\circ = 169^\circ$ which is less than 180°
Therefore 46° is a possible solution for P corresponding to $R = 180^\circ - 169^\circ = 11^\circ$
Thus, in this problem there are two separate sets of results and both are feasible solutions. Such a situation is called the ambiguous case.

However, it must not be thought that there are always two possible triangles when one angle and two sides are given. The following example shows that this is not necessarily so.

Example 3
If $A = 37^\circ$, $c = 2.1$ cm and $a = 4.59$ cm, show that there is only one possible triangle ABC and find the remaining angles.

Solution



From the diagram, we can solve for C by using the sine rule

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{4.59}{\sin 37^\circ} = \frac{2.1}{\sin C}$$

$$\sin C = \frac{2.1 \sin 37^\circ}{4.59} = 0.2753$$

Now $\sin C = 0.2753 \;\Rightarrow\; C = 16^\circ$ or 164°

But we must check whether these angles are possible values of C, by considering the following:

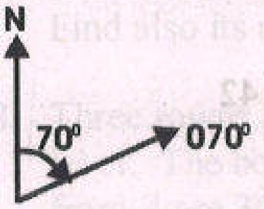
a) If $C = 16^\circ$ and $A = 37^\circ$ (given), then $16^\circ + 37^\circ = 53^\circ$ which is less than 180°
Therefore 16° is a possible solution for C corresponding to $B = 180^\circ - 53^\circ = 127^\circ$

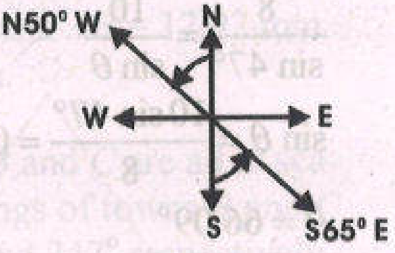
b) If $C = 164^\circ$ and $A = 37^\circ$ (given), then $164^\circ + 37^\circ = 201^\circ$ which is greater than 180°
Therefore 164° is *not* a possible value for C.

Hence there is only one triangle defined by the given data and its other angles are
$C = 16^\circ$ and $B = 127^\circ$

Application of sine and cosine rules to Bearings

Bearings are measured as clockwise angles from the North and given as three-digit numbers. Compass directions are measured as acute angles from *N* or *S* to *E* or *W*.
See the diagrams below.





The general steps for solving problems involving bearings are:

1. Draw a rough diagram;
2. Form triangles from the diagram;
3. Apply the sine and cosine rules.

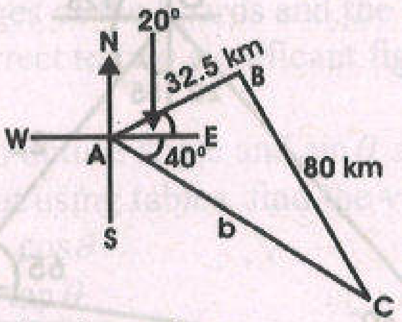
Illustrative examples

Example 1

A man leaves a point *A* walking at 6.5 km/h in a direction of 070°. A cyclist leaves the same point at the same time in a direction 130° travelling at a constant speed. Find the average speed of the cyclist if the walker and the cyclist are 80 km apart after 5 hours.

Solution

After 5 hours the walker has travelled $5 \times 6.5 = 32.5$ km (shown as *AB* in the figure below)
If *AC* is the distance the cyclist travels in 5 hours then $BC = 80$ km.



Applying the sine rule:

$$\frac{80}{\sin 60^\circ} = \frac{32.5}{\sin C}$$

$$\sin C = \frac{32.5 \sin 60^\circ}{80} = 0.3518$$

Hence $C = 20.6^\circ$ (or 159.4° which is impossible in this case)
$B = 180^\circ - 60^\circ - 20.6^\circ = 99.4^\circ$
Applying the sine rule:

$$\frac{80}{\sin 60^\circ} = \frac{b}{\sin 99.4^\circ}$$

$$b = \frac{80 \sin 99.4^\circ}{\sin 60} = 91.14 \text{ km}$$

Since the cyclist travels 91.14 km in 5 hours, Average speed $= \dfrac{\text{distance}}{\text{time}}$

$$= \frac{91.14}{5} = 18.23 \text{ km/h}$$

Example 2

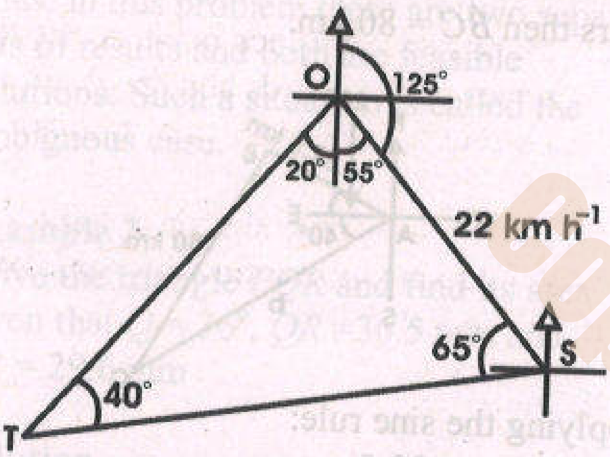
Two ships S and T leave a port O at the same time. Ship S moves at 22 km h^{-1} on a bearing of 125° and T on a bearing of 200°. After $1\frac{1}{2}$ hrs, the bearing of ship T from S is 240°. Calculate, correct to one decimal place,

i) the initial speed of the ship T;

ii) the distance between the positions of the ships at that instance.

Solution

The diagram shows the above information.



Angle at $O = 20^\circ + 55^\circ = 75^\circ$
Angle at $S = 35^\circ + 30^\circ = 65^\circ$
Angle at $T = 180^\circ - 75^\circ - 65^\circ = 40^\circ$
Using sine rule

$$\frac{22}{\sin 40^\circ} = \frac{OS}{\sin 65^\circ}$$

$$OT = \frac{22 \times \sin 65^\circ}{\sin 40^\circ} = 31\,\text{kmh}^{-1}$$

i) The initial speed of the ship T is 31 km h^{-1}

ii) Again applying the sine rule to the diagram

$$\frac{22}{\sin 40} = \frac{TS}{\sin 75}$$

$$TS = \frac{22 \times \sin 75}{\sin 40} = 33\,\text{kmh}^{-1}$$

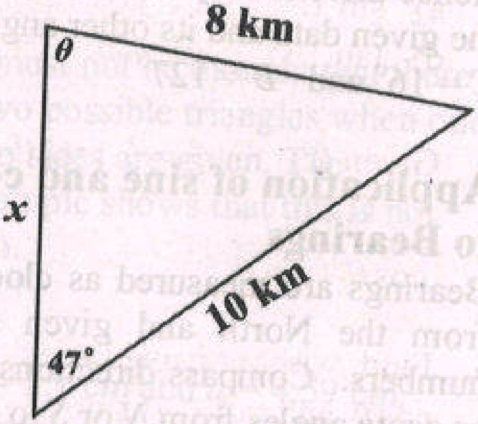
After $1\frac{1}{2}$ hrs the distance between S and $T = 33\,\text{kmh}^{-1} \times \frac{3}{2}\text{h} = 49.5\,\text{km}$

Example 3

Two vehicles P and Q set out from a station at the same time. P moves up north and Q on a bearing of 047°. When Q had travelled 10 km, the vehicles were 8 km apart. How far was P from the station?

Solution

a) The problem is represented by the diagram below.



Using the sine rule

$$\frac{8}{\sin 47^\circ} = \frac{10}{\sin \theta}$$

$$\sin \theta = \frac{10 \sin 47^\circ}{8} = 0.9142$$

$$\theta = 66.09^\circ$$

Using the sine rule again

$$\frac{\sin 66.91^\circ}{x} = \frac{\sin 47^\circ}{8}$$

$$x = \frac{8 \sin 66.91^\circ}{\sin 47^\circ} = 10.06\,\text{km}$$

Review exercise 16

1. Find, without using tables, the values of
    i) $\sin\theta$ and $\cos\theta$, if $\tan\theta = -\frac{3}{4}$
    ii) $\tan\theta$ and $\sin\theta$, if $\cos\theta = -\frac{4}{5}$
    iii) $\cos\theta$ and $\tan\theta$, if $\sin\theta = \frac{24}{25}$

2. Find $\sin A$ and $\cos A$ if $\tan A = \frac{3}{4}$ and A is acute.

3. Find $\sin A$ and $\tan A$, if $\cos A = -\frac{4}{5}$ and $180^\circ < A < 270^\circ$

4. Using compound angles, find the values of the following in surd form.
    a) $\sin 15^\circ$ b) $\cos 15^\circ$ c) $\sin 105^\circ$
    d) $\tan 105^\circ$ e) $\sin\frac{5\pi}{6}$

5. Given that $\tan\theta = \frac{3}{4}$ find the value of
$$\frac{\sin\theta}{\cos\theta + \sin\theta}$$

6. Find, without using tables, the values $\cos\theta$ and $\tan\theta$, if $\sin\theta = \frac{5}{13}$

7. Solve the triangle ABC given $B = 79^\circ$, $AC = 22.31$ mm and $AB = 17.92$ mm. Find also its area.

8. Three towns A, B and C are all at sea level. The bearings of towns B and C from A are 36° and 247° respectively. If B is 120 km from A and C is 234km from A, calculate the distance BC and bearing of town C and B.

9. A vertical tower AB of height 40 m is observed from two points C and D in the same horizontal plane as B, the foot of the tower. The points B, C and D lie in a straight line and $BC = CD$. Given that the angle of elevation of A from D is 60°, calculate:
    i) the distance of C from the foot of the tower;
    ii) the angle of elevation of A from C.

10. The top edges of two boards are inclined to each other at 30°. If the boards are 6 m and 5 m respectively, find:
    a) the distance between the bottom edges of the boards, correct to three significant figures;
    b) the angle the longer board makes with the floor;
    c) the shortest distance between the top edges of the boards and the floor, correct to two significant figures.

11. θ is an obtuse angle and $\sin\theta = 0.6$. Without using tables, find the value of:
    i) $1 - \cos\theta$
    ii) $\dfrac{2\tan\theta}{1 - \tan^2\theta}$

12. The sides of a triangle are 3, 7 and 8 units respectively. Prove that one of the angles is 60° and calculate the other two angles to the nearest degree.

13. In triangle ABC, $AB = 12$ cm, $BC = 10$ cm and the angle $CAB = 45^\circ$. Find, to the nearest degree, the two possible values of angle BCA. Find the corresponding lengths of the side AC.

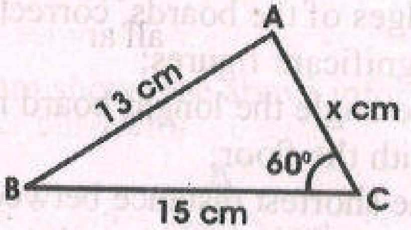
14. In triangle ABC, $BC = 8$ cm, $AC = 5$ cm and the angle $ABC = 30^\circ$. Find, to the nearest degree, the two possible values of angle BAC.

15. In triangle ABC, $AB = 12$ cm, $BC = 6\sqrt{3}$ cm and the angle $ABC = 150^\circ$. Calculate

i) AC

ii) angle ACB.

16. The figure below shows triangle ABC, $AB = 13$ cm, $BC = 15$ cm and the angle $ACB = 60^\circ$. Give that $AC = x$ cm, calculate the possible values of x.



17. Evaluate the following

i) $\lim\limits_{\theta \to 0}\left(\dfrac{\tan\theta + \sin\theta}{\theta}\right)$ ii) $\lim\limits_{\theta \to 0}\left(\dfrac{3\sin\theta - \theta}{\tan 2\theta}\right)$

iii) $\lim\limits_{\theta \to 0}\left(\dfrac{\sin 3\theta}{\tan 2\theta}\right)$ iv) $\lim\limits_{x \to 0}\left(\dfrac{\tan 4x}{x}\right)$

v) $\lim\limits_{x \to 0}\left(\dfrac{\cot x}{\operatorname{cosec} 2x}\right)$

18. Solve the following trigonometric ratios without tables.

i) $\dfrac{\sin 60^\circ}{\cos 60^\circ}$ ii) $\dfrac{\sin 30^\circ \ \tan 45^\circ}{\tan 30^\circ}$

iii) $\dfrac{\cos 45^\circ}{\sin 45^\circ}$ iv) $\dfrac{\tan 60^\circ \ \sin 60^\circ}{\cos 30^\circ \ \frac{1}{\tan 30^\circ}}$

19. A vertical flagstaff 15 m high stands on a horizontal plane. Find the angles of elevation of the top and middle point of the flagstaff from a point on the horizontal plane 4.5 m from the foot of the flagstaff.

20. A torpedo boat passes at a distance of 30 m from a fort the guns of which are 30 m above sea level; to what angle should the guns be depressed so that they point straight at the torpedo boat?

21. Prove that

i) $\dfrac{1}{\sec A - \tan A} = \sec A + \tan A$

ii) $\dfrac{\cos A}{1 - \sin A} = \dfrac{1 + \sin A}{\cos A}$.

22. Convert the following angles into degrees

$\dfrac{4\pi^c}{5}, \dfrac{8\pi^c}{3}, \dfrac{11\pi^c}{9}, \dfrac{-27\pi}{5}$

23. Convert the following angles into radians.

15°, 36°, 340°, –520°

CHAPTER 17

COMPOUND AND MULTIPLE ANGLES

Compound angle identities

It is often useful to express the trigonometric ratios of the sum and difference of two angles such as $A + B$ or $A - B$ in terms of the trigonometric ratios of A and of B. Angles such as $(A + B)$ or $(A - B)$ are called **compound angles** since they are the sum or difference of two angles, A and B.

At first sight you might think, for example that $\sin(A + B)$ is $\sin A + \sin B$.

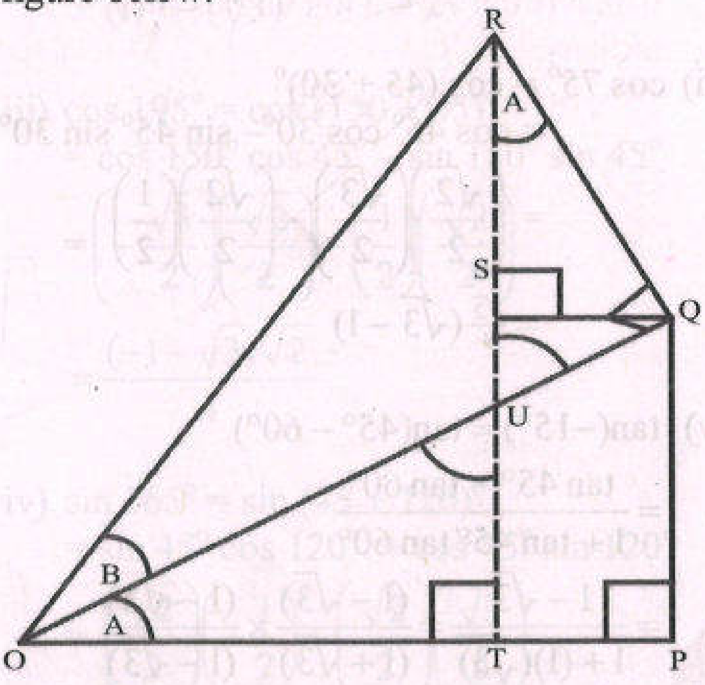
This is false since for example,

$\sin(45° + 45°) = \sin 90° = 1$, whereas

$\sin 45° + \sin 45° = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} \neq 1$

So we see that the sine function is *not distributive*. The correct expression is $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

This formula can be proved geometrically when A an B are both acute, by using the figure below.



The right-angled triangles OPQ and OQR contain angles A and B as shown above.

$$\sin(A + B) = \frac{TR}{OR} = \frac{TS + SR}{OR} = \frac{PQ + SR}{OR}$$

$$= \frac{PQ}{OQ} \times \frac{OQ}{OR} + \frac{SR}{QR} \times \frac{QR}{OR}$$

$\sin(A + B) = \sin A \cos B + \cos A \sin B$….(1)

This formula is valid for all angles.

We can also prove the following identities (*refer to chapter 25*)

$\sin(A - B) = \sin A \cos B - \cos A \sin B$.....(2)

$\cos(A + B) = \cos A \cos B - \sin A \sin B$…(3)

$\cos(A - B) = \cos A \cos B + \sin A \sin B$...(4)

Note that $\sin(A + B)$ *is not equal to* $(\sin A + \sin B)$ and so on.

The identities for tangent of compound angles can be derived from the above four identities.

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Dividing the numerator and denominator by $\cos A \cos B$, we have

$$\tan(A + B) = \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$\tan(A + B) = \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$……………(5)

Similarly,

$$\tan(A-B)=\tan[A+(-B)]=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

Summary of compound angle identities

1. $\sin(A+B)=\sin A\cos B+\cos A\sin B$
2. $\sin(A-B)=\sin A\cos B-\cos A\sin B$
3. $\cos(A+B)=\cos A\cos B-\sin A\sin B$
4. $\cos(A-B)=\cos A\cos B+\sin A\sin B$
5. $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$
6. $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

To find the values of trigonometric ratios of compound angles

Illustrative examples

Example 1

Without using tables, find the values of the following:

i) $\sin 36°\cos 54°+\cos 36°\sin 54°$

ii) $\sin\frac{\pi}{3}\cos\frac{\pi}{6}+\cos\frac{\pi}{3}\sin\frac{\pi}{6}$

iii) $\cos 42°\cos 18°-\sin 42°\sin 18°$

iv) $\dfrac{\tan 62°-\tan 17°}{1+\tan 62°\tan 17°}$

Solution

i) $\sin 36^\circ\cos 54^\circ+\cos 36^\circ\sin 54^\circ$

$=\sin(36°+54°)$

$=\sin 90°=1$

ii) $\sin\frac{\pi}{3}\cos\frac{\pi}{6}+\cos\frac{\pi}{3}\sin\frac{\pi}{6}$

$=\sin(\frac{\pi}{3}+\frac{\pi}{6})=\sin\frac{\pi}{2}=1$

iii) $\cos 42°\cos 18°-\sin 42°\sin 18°$

$=\cos(42°+18°)=\cos 60°=\frac{1}{2}$

iv) $\dfrac{\tan 62°-\tan 17°}{1+\tan 62°\tan 17°}=\tan(62°-17°)$

$=\tan 45°=1$

Example 2

Find without using tables the values of:

i) sin 75°   ii) cos 105°

iii) cos 75°   iv) tan (–15°),

leaving your answer in the surd form.

Solution

For such questions, it is advisable to break the angle into two angles whose ratios are commonly known i.e. 30°, 45°, 60°, 90°, etc.

i) $\sin 75^\circ=\sin(45+30)^\circ$

$=\sin 45^\circ\cos 30^\circ+\cos 45^\circ\sin 30^\circ$

$=\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$=\frac{\sqrt{2}}{4}(\sqrt{3}+1)$

ii) $\cos 105^\circ=\cos(60+45)^\circ$

$=\cos 60^\circ\cos 45^\circ-\sin 60^\circ\sin 45^\circ$

$=\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$=\dfrac{(1-\sqrt{3})\sqrt{2}}{4}$

iii) $\cos 75^\circ=\cos(45+30)^\circ$

$=\cos 45^\circ\cos 30^\circ-\sin 45^\circ\sin 30^\circ$

$=\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$=\frac{\sqrt{2}}{4}(\sqrt{3}-1)$

iv) $\tan(-15°)=\tan(45°-60°)$

$=\dfrac{\tan 45°-\tan 60°}{1+\tan 45°\tan 60°}$

$=\dfrac{1-\sqrt{3}}{1+(1)(\sqrt{3})}=\dfrac{(1-\sqrt{3})}{(1+\sqrt{3})}\times\dfrac{(1-\sqrt{3})}{(1-\sqrt{3})}$

$=\dfrac{1-\sqrt{3}-\sqrt{3}+3}{-2}=\dfrac{4-2\sqrt{3}}{-2}=\sqrt{3}-2$

Example 3

Find the following in the surd form

i) $\tan 15^o$

ii) $\tan 105^o$

iii) $\cos 195^o$

iv) $\sin 165^o$

Solution

i) $\tan(15°) = \tan(60° - 45°)$

$$= \frac{\tan 60° - \tan 45°}{1 + \tan 60° \tan 45°}$$

$$= \frac{\sqrt{3} - 1}{1 + (\sqrt{3})(1)} = \frac{(\sqrt{3} - 1)}{(1 + \sqrt{3})} \times \frac{(1 - \sqrt{3})}{(1 - \sqrt{3})}$$

$$= \frac{\sqrt{3} - 3 - 1 + \sqrt{3}}{-2} = \frac{2\sqrt{3} - 4}{-2} = 2 - \sqrt{3}$$

ii) $\tan(105°) = \tan(45° + 60°)$

$$= \frac{\tan 45° + \tan 60°}{1 - \tan 45° \tan 60°}$$

$$= \frac{1 + \sqrt{3}}{1 - (1)(\sqrt{3})} = \frac{(1 + \sqrt{3})}{(1 - \sqrt{3})} \times \frac{(1 + \sqrt{3})}{(1 + \sqrt{3})}$$

$$= \frac{1 + 2\sqrt{3} + 3}{(1)^2 - (\sqrt{3})^2} = \frac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3}$$

iii) $\cos 195^o = \cos(150 + 45)^o$

$$= \cos 150^o \cos 45^o - \sin 150^o \sin 45^o$$

$$= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{(-1 - \sqrt{3})\sqrt{2}}{4}$$

iv) $\sin 165^o = \sin(45 + 120)^o$

$$= \sin 45^o \cos 120^o + \cos 45^o \sin 120^o$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \tfrac{\sqrt{2}}{4}(\sqrt{3} - 1)$$

Example 4

If $\sin A = \frac{3}{5}$ and $\cos B = \frac{15}{17}$, where A is obtuse and B is acute, find without using tables, the value of:

i) $\sin(A + B)$

ii) $\cos(A - B)$

Solution

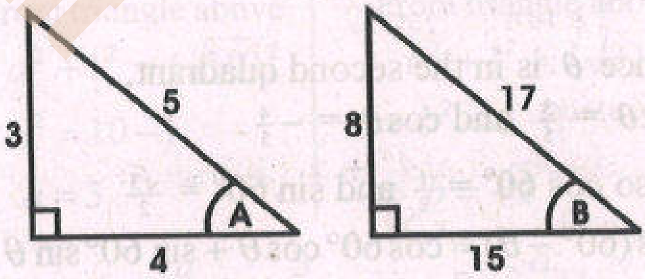
i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Therefore it is necessary to find the values of $\cos A$ and $\sin B$. Consider the figures representing $\sin A$ and $\cos B$. Note that $\cos A$ is negative since A is an obtuse angle and lies in the second quadrant and $\sin B$ is positive since B is an acute angle and lies in the first quadrant. From the right-angled triangle for $\sin A = \frac{3}{5}$, we can find the third side (i.e. the adjacent).

The third side $= \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4$

Also, from the right-angled triangle below for $\cos B = \frac{15}{17}$, we can find the third side (i.e. the opposite).

Third side $= \sqrt{17^2 - 15^2} = \sqrt{289 - 225} = \sqrt{64} = 8$



$\therefore$ $\cos A = -\frac{4}{5}$ and $\sin B = \frac{8}{17}$

From the identity above

i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$= \left(\frac{3}{5}\right)\left(\frac{15}{17}\right) + \left(-\frac{4}{5}\right)\left(\frac{8}{17}\right)$$

$$= \frac{45}{85} - \frac{32}{85} = \frac{13}{85}$$

ii) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$= \left(-\frac{4}{5}\right)\left(\frac{15}{17}\right) + \left(\frac{3}{5}\right)\left(\frac{8}{17}\right)$$

$$= -\frac{60}{85} + \frac{24}{85} = -\frac{36}{85}$$

Example 5

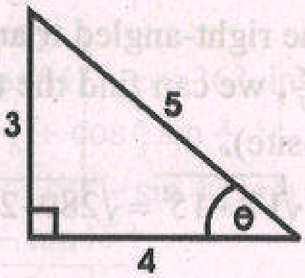
Find, without using tables, the values of $\cos(60° - \theta)$, if $\tan\theta = -\frac{3}{4}$, where $0° < \theta < 180°$

Solution

Since $\tan\theta$ is negative and θ is obtuse, it is in the second quadrant.

From the right-angled triangle below for $\tan\theta = -\frac{3}{4}$, we can find the third side (i.e. the hypotenuse).

Third side $= \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$



Since θ is in the second quadrant, $\sin\theta = \frac{3}{5}$ and $\cos\theta = -\frac{4}{5}$.

Also $\cos 60° = \frac{1}{2}$ and $\sin 60° = \frac{\sqrt{3}}{2}$

$$\cos(60° - \theta) = \cos 60°\cos\theta + \sin 60°\sin\theta$$

$$= \left(\tfrac{1}{2}\right)\left(-\tfrac{4}{5}\right) + \left(\tfrac{\sqrt{3}}{2}\right)\left(\tfrac{3}{5}\right) = -\tfrac{2}{5} + \tfrac{3\sqrt{3}}{10}$$

Example 6

If $\sin A = \frac{3}{5}$ and $\sin B = \frac{12}{13}$ where A is obtuse and B is acute, find without using tables, the value of

i) $\cos(A+B)$

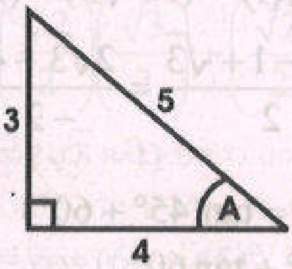
ii) $\tan(A-B)$.

Solution

i) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

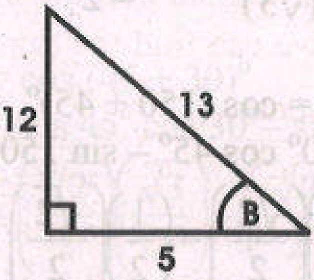
Therefore it is necessary to find the values of $\cos A$ and $\sin B$. Consider the figures representing $\sin A$ and $\cos B$. Note that $\cos A$ is negative since A is an obtuse angle and lies in the second quadrant and $\sin B$ is positive since B is an acute angle and lies in the first quadrant. From the right-angled triangle below for $\sin A = \frac{3}{5}$, we can find the third side (i.e. the adjacent).

Third side $= \sqrt{5^2 - 3^2} = \sqrt{25-9} = \sqrt{16} = 4$



Also, from the right-angled triangle below for $\sin B = \frac{12}{13}$, we can find the third side (i.e. the adjacent).

Third side $= \sqrt{13^2 - 12^2} = \sqrt{169-144} = \sqrt{25} = 5$



$\therefore\ \cos A = -\frac{4}{5}$ and $\cos B = \frac{5}{13}$

From the identity above

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$$= \left(-\frac{4}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(\frac{12}{13}\right)$$

$$= -\frac{20}{65} - \frac{36}{65} = -\frac{56}{65}$$

ii) Again from the diagrams, $\tan A = -\frac{3}{4}$ and $\tan B = \frac{12}{5}$

$$\therefore \quad \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$$

$$= \frac{\left(-\frac{3}{4}\right)-\left(\frac{12}{5}\right)}{1+\left(-\frac{3}{4}\right)\left(\frac{12}{5}\right)} = \frac{63}{16}$$

Example 7

Given that α and β are acute angles so that $\sin\alpha = \frac{1}{\sqrt{10}}$ and $\sin\beta = \frac{1}{\sqrt{5}}$, prove, without using tables or calculator that $\sin(\alpha+\beta) = \frac{1}{\sqrt{2}}$.

Solution

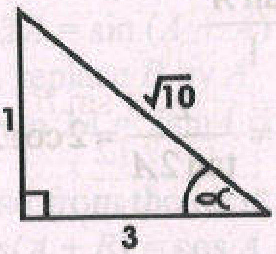
$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

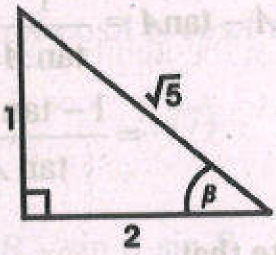
Therefore it is necessary to find the values of $\cos\alpha$ and $\cos\beta$. Consider the figures representing $\sin\alpha$ and $\sin\beta$. Note that $\cos\alpha$ and $\cos\beta$ are positive since α and β are acute angles and lie in the first quadrant.

From the right-angled triangle below for $\sin\alpha = \frac{1}{\sqrt{10}}$, we can find the third side (i.e. the adjacent).

Third side $= \sqrt{(\sqrt{10})^2 - 1^2} = \sqrt{10-1} = \sqrt{9} = 3$

Also, from the right-angled triangle below for $\sin\beta = \frac{1}{\sqrt{5}}$, we can find the third side (i.e. the adjacent).





Third side $= \sqrt{(\sqrt{5})^2 - 1^2} = \sqrt{5-1} = \sqrt{4} = 2$

$\therefore \quad \cos\alpha = \frac{3}{\sqrt{10}}$ and $\cos\beta = \frac{2}{\sqrt{5}}$

$$\therefore \quad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \left(\frac{1}{\sqrt{10}}\right)\left(\frac{2}{\sqrt{5}}\right) + \left(\frac{3}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{5}}\right)$$

$$= \frac{2}{\sqrt{50}} + \frac{3}{\sqrt{50}}$$

$$= \frac{5}{\sqrt{50}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Example 8

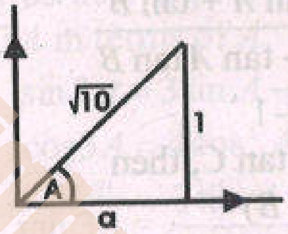
Given that $\sin A = \frac{1}{\sqrt{10}}$ and $\sin B = \frac{1}{\sqrt{3}}$, where A and B are acute, find $\sin(A+B)$, leaving your answer in surd form.

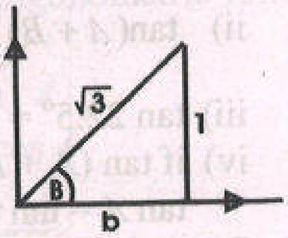
Solution

a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin A = \frac{1}{\sqrt{10}}$ and $\sin B = \frac{1}{\sqrt{3}}$,

Consider the following diagrams.





From triangle above

$a^2 + 1^2 = (\sqrt{10})^2$

$a^2 = 10 - 1$

$a = 3$

From triangle above

$b^2 + 1^2 = (\sqrt{3})^2$

$\Rightarrow b^2 = 3 - 1$

$\Rightarrow b = \sqrt{2}$

$$\therefore \quad \cos A = \frac{a}{\sqrt{10}} = \frac{3}{\sqrt{10}}; \quad \cos B = \frac{b}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\sin(A+B) = \tfrac{1}{\sqrt{10}} \times \tfrac{\sqrt{2}}{\sqrt{3}} + \tfrac{3}{\sqrt{10}} \times \tfrac{1}{\sqrt{3}}$$

$$\sin(A+B) = \tfrac{\sqrt{2}}{\sqrt{30}} + \tfrac{3}{\sqrt{30}}$$

$$= \tfrac{3+\sqrt{2}}{\sqrt{30}} = \tfrac{3+\sqrt{2}}{\sqrt{30}} \times \tfrac{\sqrt{30}}{\sqrt{30}}$$

$$\therefore \quad \sin(A+B) = \tfrac{3\sqrt{30}+\sqrt{60}}{30} = \tfrac{1}{10}\sqrt{30} + \tfrac{1}{15}\sqrt{15}$$

Example 9

Find tan A if $\tan(A-45^\circ)=\frac{1}{3}$

Solution

$$\tan(A-45^\circ)=\frac{\tan A-\tan 45^\circ}{1+\tan A\tan 45^\circ}$$

$$=\frac{\tan A-1}{1+\tan A}=\frac{1}{3}$$

Note : $\tan 45^\circ = 1$

$3(\tan A-1)=1+\tan A$

$3\tan A-3=1+\tan A$

$2\tan A=4 \Rightarrow \tan A=2$

Example 10

a) Use the formulae $\sin(A+B)=\sin A\cos B+\cos A\sin B$ and $\cos(A+B)=\cos A\cos B-\sin A\sin B$ to show that

i) $\sin 75^\circ=\frac{1}{4}(\sqrt{6}+\sqrt{2})$

ii) $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

iii) $\tan 22.5^\circ=\sqrt{2}-1$

iv) if $\tan(A+B)=\tan C$, then $\tan A=\tan(C-B)$

b) Prove the identity

$\cot A-\tan A=2\cot 2A$

Solution

For i) and ii) refer to notes on compound angles.

iii) $\tan 22.5^\circ=\tan(45-22.5)$.

Using $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

$$\tan 22.5^\circ=\tan(45^\circ-22.5^\circ)$$

$$=\frac{\tan 45^\circ-\tan 22.5^\circ}{1+\tan 45^\circ\tan 22.5^\circ}$$

$$=\frac{1-\tan 22.5^\circ}{1+\tan 22.5^\circ}\ \text{[since } \tan 45^\circ=1\text{]}$$

Let $\tan 22.5^\circ=y$

$$y=\frac{1-y}{1+y}$$

$$y+y^2=1-y$$

$$y^2+2y-1=0$$

Using the formula for solving quadratic equations, i.e. $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, we have

$$y=\tfrac{1}{2}\left[-2\pm(\sqrt{4+4})\right]$$

$$=\tfrac{1}{2}\left[-2\pm\sqrt{8}\right]=\tfrac{1}{2}\left[-2\pm 2\sqrt{2}\right]=-1\pm\sqrt{2}$$

$$y=-1+\sqrt{2}$$

(i.e. taking the positive sign since the angle is acute and lies in the first quadrant)

But $y=\tan 22.5 \Rightarrow \tan 22.5=\sqrt{2}-1$ as required.

iv) If $\tan(A+B)=\tan C$

$A+B=C \Rightarrow A=C-B$.

Taking tangent of both sides

$\tan A=\tan(C-B)$ as required.

b) $$\cot A-\tan A=\frac{\cos A}{\sin A}-\frac{\sin A}{\cos A}$$

$$=\frac{\cos^2 A-\sin^2 A}{\sin A\cos A}=\frac{\cos 2A}{\frac{1}{2}\sin 2A}$$

$$=\frac{2\cos 2A}{\sin 2A}=2\cot 2A$$

Alternatively

$$\cot A-\tan A=\frac{1}{\tan A}-\frac{\tan A}{1}$$

$$=\frac{1-\tan^2 A}{\tan A}=\frac{2}{\tan 2A}=2\cot 2A$$

Note that

$$\frac{1}{\tan 2A}=\cot 2A \text{ and } \tan 2A=\frac{2\tan A}{1-\tan^2 A}$$

Example 11

Given that $\tan A = -\frac{3}{4}$, where $90^\circ \le A \le 180^\circ$ and $\sin B = -\frac{3}{5}$, where $270^\circ \le B \le 360^\circ$, evaluate $\cos(A+B)$.

Solution

$\tan A = -\frac{3}{4}$ for $90^\circ \le A \le 180^\circ$

$\cos A = -\frac{4}{5}$ and $\sin A = \frac{3}{5}$

Also $\sin B = -\frac{3}{5}$ for $270^\circ \le B \le 360^\circ$,

$\cos B = \frac{4}{5}$

Note: In all cases draw the right angled triagle for the trigonometric ratios given and deduce the other ratios needed.

$$\begin{aligned}\therefore\ \cos(A+B) &= \cos A\cos B - \sin A\sin B\\ &= \left(-\tfrac{4}{5}\right)\left(\tfrac{4}{5}\right) - \left(\tfrac{3}{5}\right)\left(-\tfrac{3}{5}\right)\\ &= -\tfrac{16}{25} + \tfrac{9}{25}\\ &= -\tfrac{7}{25} = -0.28\end{aligned}$$

Trigonometric ratios of multiple angles

With the help of the compound angles formulae studied in the previous section, we will find the trigonometric ratios of multiple angles. Angles such as $2A$ or $3A$ are called multiple angles.

Double angle formulae

From the identity;

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\sin 2A = \sin(A + A) = \sin A \cos A + \cos A \sin A$

i.e. replace B by A

$\therefore \sin 2A = 2\sin A \cos A$ ……… …. (7)

Also from the identity

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos 2A = \cos(A + A) = \cos A\cos A - \sin A\sin A$

i.e. replace B by A

$\therefore \cos 2A = \cos^2 A - \sin^2 A$ … …(8)

From identity (13), if we put $\cos^2 A = 1 - \sin^2 A$, we have

$\cos 2A = 1 - \sin^2 A - \sin^2 A$

$\cos 2A = 1 - 2\sin^2 A$ …… ..(9)

Also putting $\sin^2 A = 1 - \cos^2 A$ into identity (8), we have

$\cos 2A = 2\cos^2 A - 1$ …. … ……(10)

From identity (5),

i.e. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan 2A = \tan(A + A) = \dfrac{\tan A + \tan A}{1 - \tan A \tan A}$

i.e. replace B by A

$\therefore\ \tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$ ………..(11)

Trigonometric ratios of 3A in terms of A

Let us now express the trigonometric ratios of $3A$ in terms of A.

1. $\sin 3A = 3\sin A - 4\sin^3 A$
2. $\cos 3A = 4\cos^3 A - 3\cos A$
3. $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Proof

1. The left-hand side of the identity (i.e. $\sin 3A$) may be written as $\sin(A + 2A)$, so by using the formula for $\sin(A + B)$, we have

$$\sin(A + 2A) = \sin A \cos 2A + \cos A \sin 2A$$

But the right-hand side of the identity to be proved is in terms of $\sin A$, and this suggests that $\cos 2A$ should be expressed in terms of $\sin A$. (We have only one formula for $\sin 2A$, so it must be used.) Therefore

$$\begin{aligned}\sin 3A &= \sin A(1 - 2\sin^2 A) + \cos A(2\sin A\cos A)\\ &= \sin A - 2\sin^3 A + 2\sin A\cos^2 A\end{aligned}$$

Now $\cos^2 A$ must be expressed in terms of $\sin A$ by means of the identity $\cos^2 A = 1 - \sin^2 A$, therefore

$$\sin 3A = \sin A - 2\sin^3 A + 2\sin A(1-\sin^2 A)$$
$$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A$$
$$\therefore \quad \mathbf{\sin 3A = 3\sin A - 4\sin^3 A}$$

2. $\cos 3A = \cos(2A + A)$

$$= \cos 2A \cos A - \sin 2A \sin A$$
$$= (2\cos^2 A - 1)\cos A - (2\sin A \cos A)\sin A$$
$$= 2\cos^3 A - \cos A - 2\sin^2 A \cos A$$
$$= 2\cos^3 A - \cos A - 2(1-\cos^2 A)\cos A$$
$$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$$
$$\therefore \quad \mathbf{\cos 3A = 4\cos^3 A - 3\cos A}$$

3. $\tan 3A = \tan 3A = \tan(2A + A)$

$$= \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$
$$= \frac{\dfrac{2\tan A}{1-\tan^2 A} + \tan A}{1 - \left[\dfrac{2\tan A}{1-\tan^2 A}\right].\tan A}$$
$$= \frac{2\tan A + \tan A(1-\tan^2 A)}{(1-\tan^2 A) - (2\tan A)\tan A}$$
$$= \frac{2\tan A + \tan A - \tan^3 A}{1 - \tan A^2 - 2\tan^2 A}$$
$$\therefore \quad \mathbf{\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}}$$

Illustrative examples

Example 1

Without using tables, find the values of the following:

i) $2\sin 15° \cos 15°$
ii) $\sin 165° \cos 165°$
iii) $1 - 2\sin^2 15°$
iv) $2\cos^2 \frac{\pi}{8} - 1$
v) $\cos^2 105° - \sin^2 105°$
vi) $\sin\frac{3}{8}\pi \cos\frac{3}{8}\pi$
vii) $\sin^2 22\frac{1}{2}° - \cos^2 22\frac{1}{2}°$
viii) $\dfrac{2\tan\frac{3}{8}\pi}{1-\tan^2\frac{3}{8}\pi}$

Solution

i) $2\sin 15°\cos 15° = \sin(2\times 15°)$
$$= \sin 30° = \tfrac{1}{2}$$

ii) $2\sin 165°\cos 165° = \sin(2\times 165)$
$$= \sin 330° = -\tfrac{1}{2}$$
$$\therefore \quad \sin 165°\cos 165° = \tfrac{1}{2}\left(-\tfrac{1}{2}\right) = -\tfrac{1}{4}$$

iii) $1 - 2\sin^2 15° = \cos(2\times 15°)$
$$= \cos 30° = \tfrac{\sqrt{3}}{2}$$

iv) $2\cos^2\frac{\pi}{8} - 1 = \cos(2\times\frac{\pi}{8}) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

v) $\cos^2 105° - \sin^2 105° = \cos(2\times 105°)$
$$= \cos 210° = -\cos 30° = -\tfrac{\sqrt{3}}{2}$$

vi) $\sin\frac{3}{8}\pi\cos\frac{3}{8}\pi = \frac{1}{2}\sin(2\times\frac{3}{8}\pi)$
$$= \tfrac{1}{2}\sin\tfrac{3}{4}\pi = \tfrac{1}{2}\left(\tfrac{\sqrt{2}}{2}\right) = \tfrac{\sqrt{2}}{4}$$

vii) $\sin^2 22\frac{1}{2}° - \cos^2 22\frac{1}{2}°$
$$= -(\cos^2 22\tfrac{1}{2}° - \sin^2 22\tfrac{1}{2}°)$$
$$= -\cos(2\times 22\tfrac{1}{2}°) = -\cos 45° = -\tfrac{\sqrt{2}}{2}$$

viii) $\dfrac{2\tan\frac{3}{8}\pi}{1-\tan^2\frac{3}{8}\pi} = \tan(2\times\frac{3}{8}\pi)$
$$= \tan\tfrac{3}{4}\pi = -\tan\tfrac{1}{4}\pi = -1$$

Example 2

If $\cos 2x = \frac{4}{5}$, find $\cos x$, $\sin x$ and $\tan x$, where x is an acute angle.

Solution

$$\cos 2x = 2\cos^2 x - 1$$

$$\tfrac{4}{5} = 2\cos^2 x - 1$$

$$\cos^2 x = \tfrac{9}{10}$$

$$\cos x = \tfrac{3}{\sqrt{10}}$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\tfrac{4}{5} = 1 - 2\sin^2 x$$

$$\sin^2 x = \tfrac{1}{10}$$

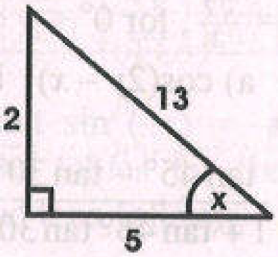
$$\sin x = \tfrac{1}{\sqrt{10}}$$

$$\therefore \quad \tan x = \frac{\sin x}{\cos x} = \frac{\frac{1}{\sqrt{10}}}{\frac{3}{\sqrt{10}}} = \tfrac{1}{3}$$

Example 3

Find without using tables the values of $\sin 2x$ and $\cos 2x$, if $\sin x = \frac{12}{13}$

Solution



Note: the third side is 5

From the right-angled triangle above

$\sin x = \frac{12}{13}$ and $\cos x = \frac{5}{13}$

$$\sin 2x = 2\sin x\cos x = 2\left(\tfrac{12}{13}\right)\left(\tfrac{5}{13}\right) = \tfrac{120}{169}$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= \left(\tfrac{5}{13}\right)^2 - \left(\tfrac{12}{13}\right)^2 = \tfrac{25}{169} - \tfrac{144}{169} = -\tfrac{119}{169}$$

or

$$\cos 2x = 1 - 2\sin^2 x = 1 - 2\left(\tfrac{12}{13}\right)^2$$

$$= 1 - \tfrac{288}{169} = -\tfrac{119}{169}$$

or

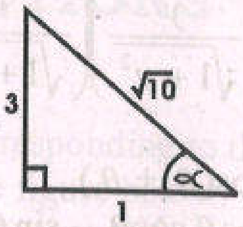
$$\cos 2x = 2\cos^2 x - 1$$

$$= 2\left(\tfrac{5}{13}\right)^2 - 1 = \tfrac{50}{169} - 1 = -\tfrac{119}{169}$$

Example 4

If $\sin\alpha = \frac{3}{\sqrt{10}}$, for $0 < \alpha < \frac{\pi}{2}$, find $\cos 2\alpha$

Solution



α is an acute angle and $\sin\alpha$ is positive means it is in the first quadrant.

Therefore $\cos\alpha = \frac{1}{\sqrt{10}}$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$= \left(\tfrac{1}{\sqrt{10}}\right)^2 - \left(\tfrac{3}{\sqrt{10}}\right)^2 = \tfrac{1}{10} - \tfrac{9}{10} = -\tfrac{8}{10} = -\tfrac{4}{5}$$

Alternatively

$$\cos 2\alpha = 1 - 2\sin^2\alpha$$

$$= 1 - 2\left(\tfrac{3}{\sqrt{10}}\right)^2 = 1 - \tfrac{18}{10} = -\tfrac{8}{10} = -\tfrac{4}{5} \text{ or}$$

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

$$= 2\left(\tfrac{1}{\sqrt{10}}\right)^2 - 1 = \tfrac{2}{10} - 1 = -\tfrac{8}{10} = -\tfrac{4}{5}$$

Example 5

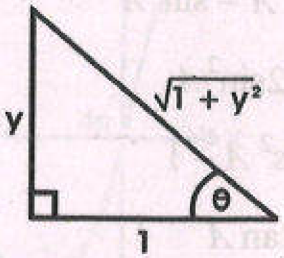
Given $\sin(A+B) = \sin A\cos B + \cos A\sin B$, $\cos(A+B) = \cos A\cos B - \sin A\sin B$ and $\tan\theta = y$, express in terms of y.

a) $\sin 2\theta$

b) $\cos 2\theta$

Solution

If $\tan\theta = y$, then from the right-angled triangle below (i.e. the third side is $\sqrt{(1+y^2)}$).



$$\cos\theta = \frac{1}{\sqrt{1+y^2}} \quad \text{and} \quad \sin\theta = \frac{y}{\sqrt{1+y^2}}$$

a) $$\begin{aligned}\sin 2\theta &= \sin(\theta+\theta)\\ &= \sin\theta\cos\theta + \cos\theta\sin\theta\\ &= 2\sin\theta\cos\theta\\ &= 2\left(\frac{y}{\sqrt{1+y^2}}\right)\left(\frac{1}{\sqrt{1+y^2}}\right)=\frac{2y}{1+y^2}\end{aligned}$$

b) $$\begin{aligned}\cos 2\theta &= \cos(\theta+\theta)\\ &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ &= \cos^2\theta - \sin^2\theta\\ &= \left(\frac{1}{\sqrt{1+y^2}}\right)^2 - \left(\frac{y}{\sqrt{1+y^2}}\right)^2\\ &= \frac{1}{1+y^2} - \frac{y^2}{1+y^2} = \frac{1-y^2}{1+y^2}\end{aligned}$$

Summary of trigonometric identities

Compound angles

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$$

Multiple angles

$\sin 2A = 2\sin A\cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 2A = 2\cos^2 A - 1$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$\sin 3A = 3\sin A - 4\sin^3 A$

$\cos 3A = 4\cos^3 A - 3\cos A$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$$

Review exercise 17

1. Given that $\sin A = \frac{1}{\sqrt{10}}$ and $\sin B = \frac{1}{\sqrt{3}}$, where A and B are acute, find sin (A+B), leaving your answer in surd form.

2. Solve the following equations for values of θ, if $0^\circ \le \theta \le 360^\circ$.
   i) $\cos(\theta^\circ + 30^\circ) + \cos(\theta^\circ - 30^\circ) = \cos 30^\circ$
   ii) $\cos 2\theta - \cos\theta = 0$

3. If $\dfrac{\tan\theta + \tan 30^\circ}{1-\tan\theta\tan 30^\circ} = 1$, find $\tan\theta$, leaving your answer in surd form.

4. Given that $\sin x = \frac{\sqrt{3}}{2}$, for $90^\circ < x < 180^\circ$ and $\cos y = \frac{\sqrt{2}}{2}$, for $0^\circ < y < 90^\circ$
   Evaluate a) $\cos(2y - x)$ b) $\sin(x + 2y)$

5. Evaluate $\dfrac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ\tan 30^\circ}$

6. If $\sin\theta = \frac{5}{13}$, $0^\circ \le \theta \le 90^\circ$, find $\cos 2\theta$.

7. Using the identities
   $\sin(A+B) = \sin A\cos B + \cos A\sin B$
   and
   $\cos(A+B) = \cos A\cos B - \sin A\sin B$,
   express $\sin 2A$ and $\cos 2A$ in terms of $\sin A$ and $\cos A$. Also show that
   $\cos 3A = 4\cos^3 A - 3\cos A$

8. Given that $\sin A = \frac{3}{4}$, where $0 < A < \frac{\pi}{2}$, find $\sin 2A$.

CHAPTER 18

TRIGONOMETRIC FUNCTIONS

Graphs of the trigonometric functions

The graphs of the three major circular functions are particularly important and these will be examined to show the behaviour of each function as x varies. We will draw the graphs of sine, cosine and tangent functions.

Graph of $y = \sin x$

We will first draw the graph of $y = \sin x$
$-\pi \le x \le \pi$
Let us write the corresponding values of x and y is the following table.

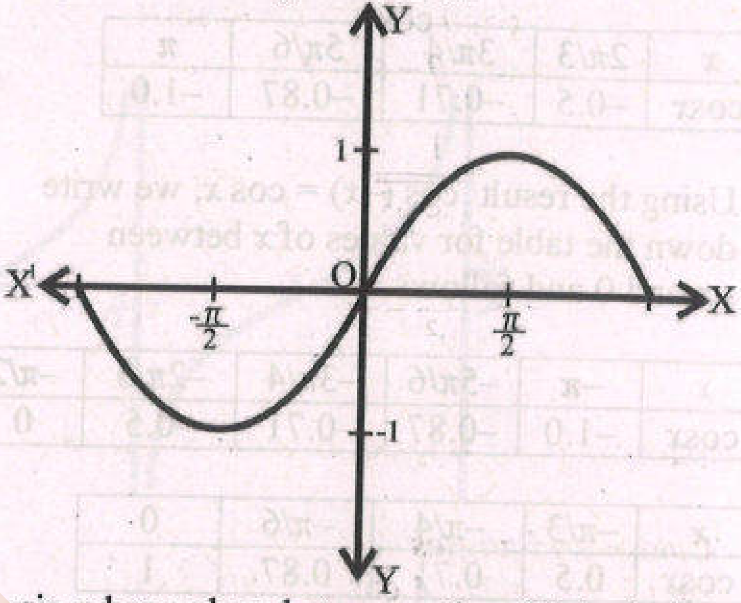
x	0	$\pi/6$	$\pi/4$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin x$	0	0.5	0.71	1.0	0.87	0.71	0.5	0

Using the result $\sin(-x) = -\sin x$, we can write down the tables for the values of x between $-\pi$ and 0 as follows

x	$-\pi$	$-5\pi/6$	$-3\pi/4$	$-2\pi/3$	$-\pi/2$
$\sin x$	0	-0.5	-0.71	-0.87	-1.0

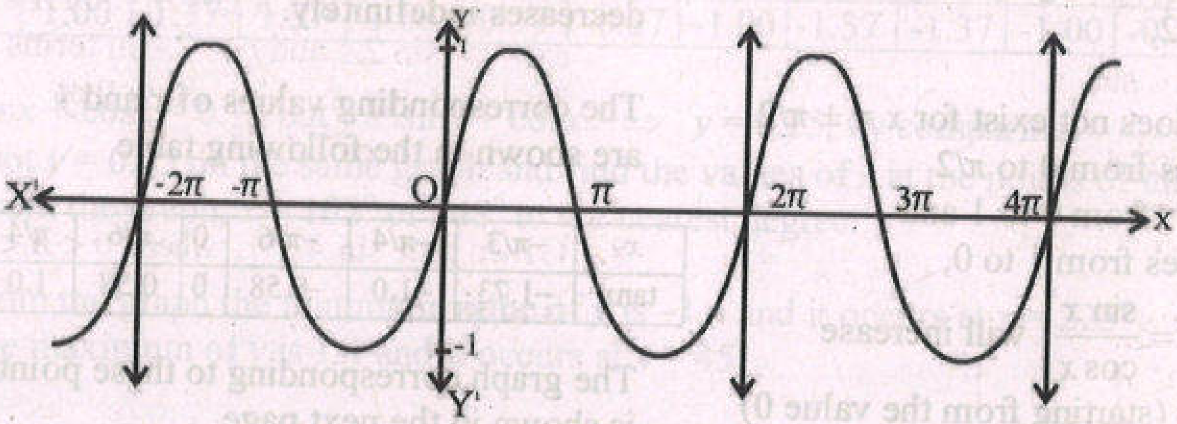
x	$-\pi/3$	$-\pi/4$	$-\pi/6$	0
$\sin x$	0	-0.5	-0.71	0

The graph corresponding to these points is as given in the figure below.



$\sin x$ has values between -1 and 1 inclusive

Extension of the graph of $\sin x$
Since $\sin(\pi + x) = -\sin x$
The graph drawn in the figure above can be extended to the interval $(\pi, 2\pi)$. Similar extensions are possible beyond $x = 2\pi$ and for negative values of x before $x = -\pi$.
The graph will be as shown in the figure below.



Graph of $y = \cos x$

As in the previous case, we will first sketch the graph of $y = \cos x$ for $-\pi \le x \le \pi$. The corresponding values of x and y are given in the following table.

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\cos x$	1.0	0.87	0.71	0.5	0

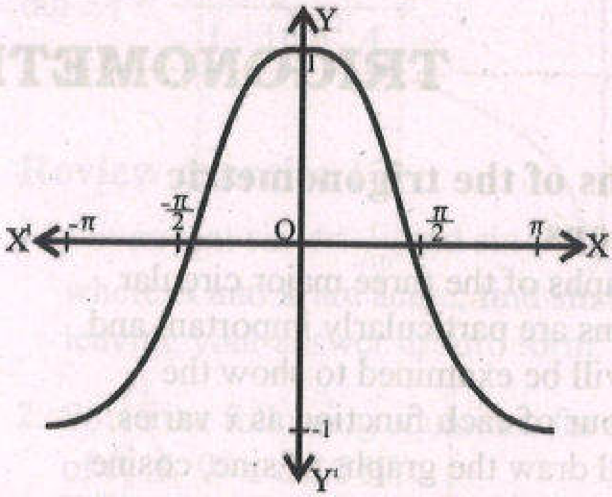
x	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\cos x$	–0.5	–0.71	–0.87	–1.0

Using the result $\cos(-x) = \cos x$, we write down the table for values of x between $-\pi$ and 0 and follows

x	$-\pi$	$-5\pi/6$	$-3\pi/4$	$-2\pi/3$	$-\pi/2$
$\cos x$	–1.0	–0.87	–0.71	–0.5	0

x	$-\pi/3$	$-\pi/4$	$-\pi/6$	0
$\cos x$	0.5	0.71	0.87	1

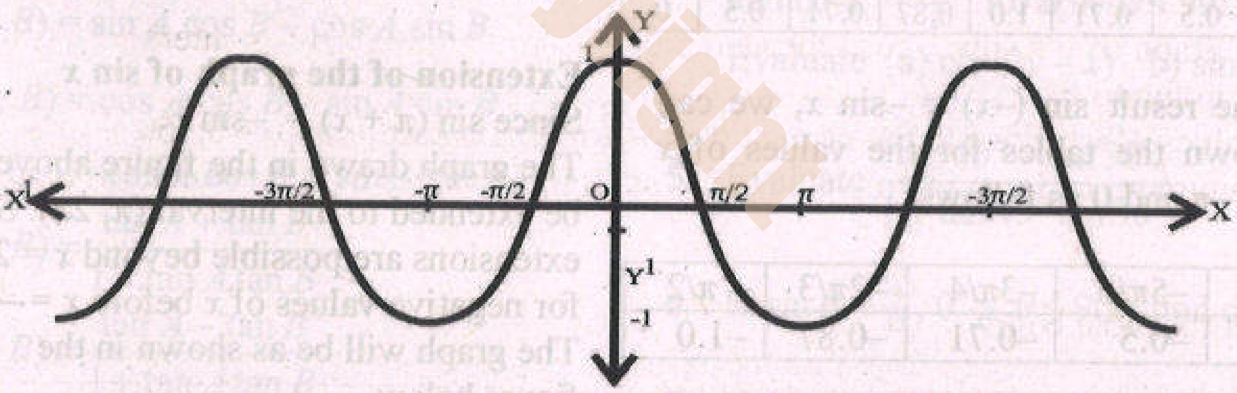
The graph corresponding to these points is as given in the figure below.



$\cos\theta$ has values between –1 and 1 inclusive

Extension of the graph of cos x

As explained in the case of the graph of the $\sin x$, the graph of $\cos x$ will also be extended and this is shown in the figure below.



Graph of $y = \tan x$

We will first sketch the graph of $y = \tan x$ for $-\pi/2 < x < \pi/2$.

Note that $\tan x$ does not exist for $x = \pm\,\pi/2$ but as x increases from 0 to $\pi/2$

i) $\sin x$ increases from 0 to 1 and

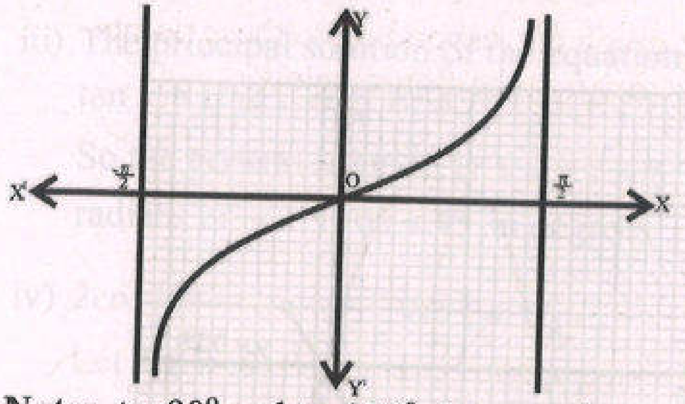
ii) $\cos x$ decreases from 1 to 0,

Therefore $\tan x = \dfrac{\sin x}{\cos x}$ will increase indefinitely as x (starting from the value 0) approaches $\pi/2$. Similarly, as x (starting from the value 0) approaches $-\pi/2$, $\tan x$ decreases indefinitely.

The corresponding values of x and y are shown in the following table

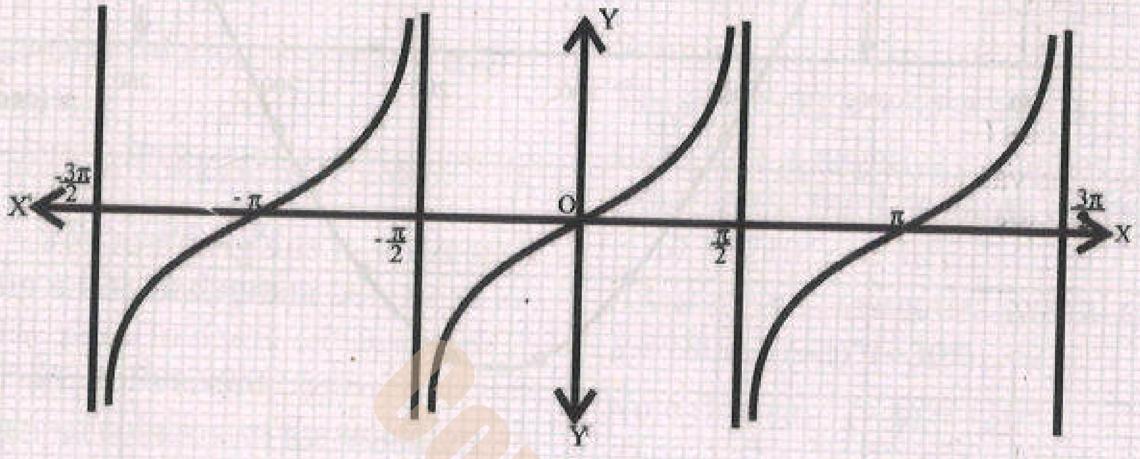
x	$-\pi/3$	$-\pi/4$	$-\pi/6$	0	$\pi/6$	$\pi/4$	$\pi/3$
$\tan x$	–1.73	–1.0	–0.58	0	0.58	1.0	1.73

The graph corresponding to these points is shown in the next page.



Note: tan90° and tan270° do not exist

Extension of the graph of tan *x*
When the graph of tan *x* is extended to values beyond π/2, the entire branch of the curve shown in the figure above repeats completely for the intervals (π/2, 3π/2), (3π/2, 5π/2), etc. as well as for intervals (–3π/2, –π/2), (–5π/2, –3π/2)……etc. The extended graph is shown below.



Illustrative example

Draw the graph of $y = \sin x + \cos x$ for values of x from 0° to 360° using intervals of 30°. Use your graph to find:

a) the values of x correct to the nearest degree for which $\sin x + \cos x = 0.75$;

b) the minimum and maximum values of y, stating the values of x for which they occur.

Solution

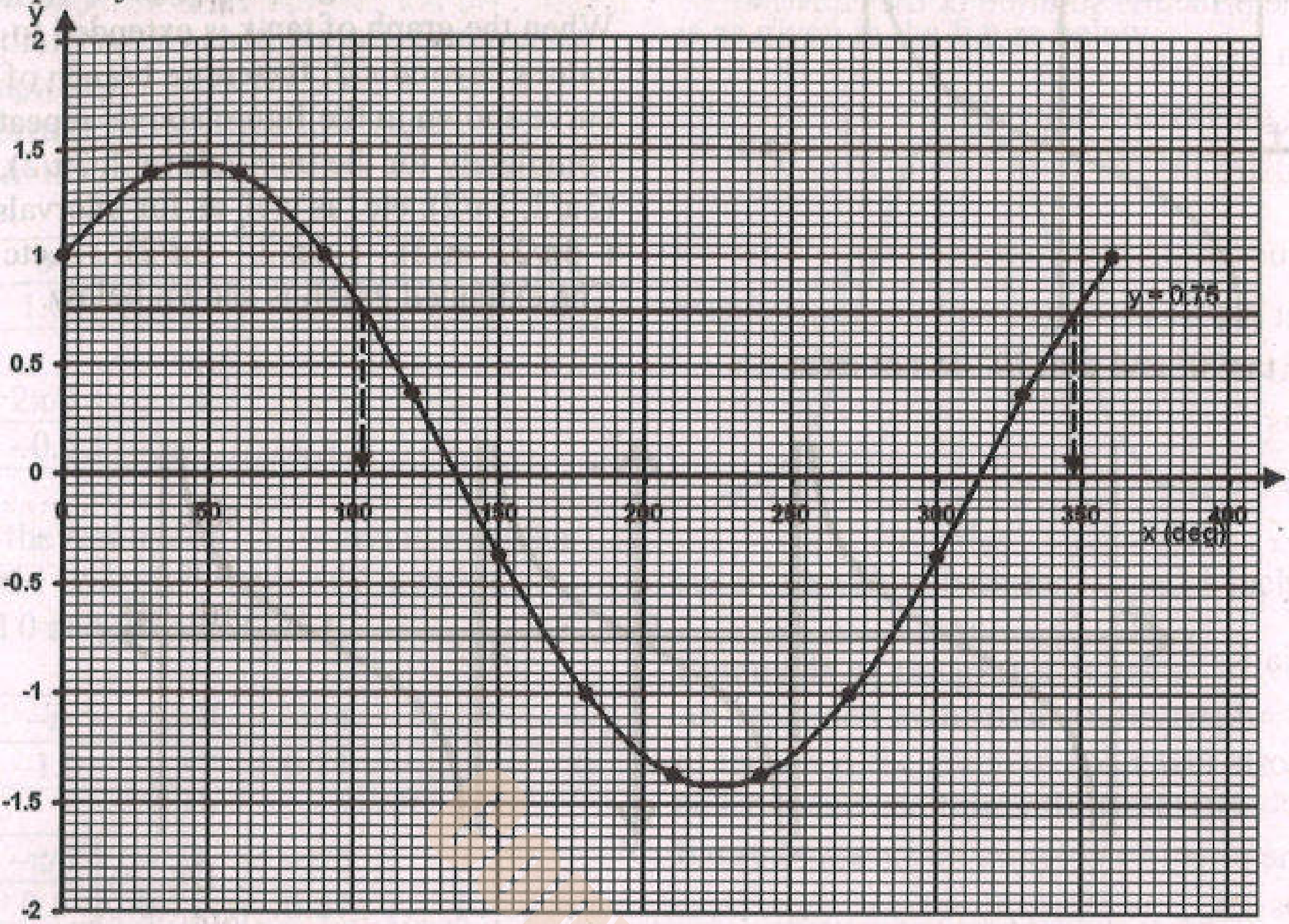
Construct the table below.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
sin$x$	0.00	0.50	0.87	1.00	0.87	0.50	0.00	-0.50	-0.87	-1.00	-0.87	-0.50	0.00
cos$x$	1.00	0.87	0.50	0.00	-0.50	-0.87	-1.00	-0.87	-0.50	0.00	0.50	0.87	1.00
y	1.00	1.37	1.37	1.00	0.37	-0.37	-1.00	-1.37	-1.37	-1.00	-0.37	0.37	1.00

a) $\sin x + \cos x = 0.75$ but $y = \sin x + \cos x \Rightarrow y = 0.75$, by comparing.
∴ Plot $y = 0.75$ on the same graph and find the values of x at the points of intersections.
From the graph, $x = 102°$ or $348°$ to the nearest degree.

b) From the graph the minimum value of y is –1.4 and it occurs at $x = 225°$
The maximum of y is 1.4 and it occurs at $x = 45°$

The graph of *f* is shown below.



General solutions of trigonometric equations

The **general solution** of $\sin\theta = a$ is

$\theta = n\pi + (-1)^n\alpha$,

where $\alpha = \sin^{-1} a$ in radians.

$\theta = 180n + (-1)^n\alpha$,

where $\alpha = \sin^{-1} a$ in degrees.

The **general solution** of $\cos\theta = b$ is

$\theta = 2n\pi \pm \beta$,

where $\beta = \cos^{-1} b$ is in radians.

$\theta = 360n \pm \beta$,

where $\beta = \cos^{-1} b$ is in degrees.

The **general solution** of $\tan\theta = c$ is

$\theta = n\pi + \gamma$, where $\gamma = \tan^{-1} c$ is in radians.

$\theta = 180n + \gamma$,

where $\gamma = \tan^{-1} c$ is in degrees.

Note that $n = 0, 1, 2, 3, \ldots$

Illustrative example

Find the general solutions of the following equations.

i) $\sin\theta = 1$ ii) $\cos\theta = \frac{1}{2}$

iii) $\tan x = 1$ iv) $2\cos\frac{1}{2}x = 1$

v) $\cos 2x = \frac{1}{2}$

Solution

i) The principal solution of the equation $\sin\theta = 1$ is $\theta = \frac{\pi}{2}$, or 90°

So the general solution is

$\theta = n\pi + (-1)^n\frac{\pi}{2}$ in radians or

$180n + (-1)^n 90$ in degrees.

ii) The principal solution of the equation $\cos\theta = \frac{1}{2}$ is $\theta = \frac{\pi}{3}$ or 60°

So the general solution is $\theta = 2n\pi \pm \frac{\pi}{3}$ in radians or $\theta = 360n \pm 60$ in degrees.

iii) The principal solution of the equation $\tan x = 1$ is $x = \frac{\pi}{4}$ or 45°

So the general solution is $x = n\pi + \frac{\pi}{4}$ in radians or $x = 180n + 45$ in degrees.

iv) $2\cos\frac{1}{2}x = 1 \quad \Rightarrow \quad \cos\frac{1}{2}x = \frac{1}{2}$.

Let $\frac{1}{2}x = \theta$

The principal solution of the equation $\cos\theta = \frac{1}{2}$ is $\theta = \frac{\pi}{3}$ or 60°

So the general solution is $\theta = 2n\pi \pm \frac{\pi}{3}$

$\frac{1}{2}x = 2n\pi \pm \frac{\pi}{3}$

Hence the general solution is $x = 4n\pi \pm \frac{2\pi}{3}$ in radians or $\frac{1}{2}x = 360n \pm 60$

$x = 720n \pm 120$ in degrees

v) Let $\theta = 2x$ so that $\cos\theta = \frac{1}{2}$

The principal solution of the equation $\cos\theta = \frac{1}{2}$ is $\theta = \frac{\pi}{3}$ or 60°

So the general solution is

$\theta = 2n\pi \pm \frac{\pi}{3} \quad \Rightarrow \quad 2x = 2n\pi \pm \frac{\pi}{3}$

$\Rightarrow \quad x = n\pi \pm \frac{\pi}{6}$

Hence the general solution is $x = n\pi \pm \frac{\pi}{6}$ in radians or $2x = 360n \pm 60$

$x = 180n \pm 30$ in degrees.

Solving trigonometric equations involving the basic identities

Some trigonometric equations may be solved as an algebraic equation. Use the basic trigonometric identities below to express the equation in terms of one ratio only. The equation can then be solved in the same way as an algebraic equation.

1. $\sin^2\theta + \cos^2\theta = 1$
2. $1 + \tan^2\theta = \sec^2\theta$
3. $1 + \cot^2\theta = \operatorname{cosec}^2\theta$
4. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Note: Do not cancel a ratio from an equation since one of the solutions will be lost. The following examples show how this is done.

Illustrative examples

Example 1

Solve the equation

$\sin\theta = -\frac{1}{2}$ for $0^\circ < \theta < 360^\circ$

Solution

The acute angle whose sine is $\frac{1}{2}$ is 30°, but since sine is negative it implies the angle is either in the third or fourth quadrant.

$\therefore \ \theta$ is 210° or 330°.

Alternatively, using the general solution,

$\theta = 180n + (-1)^n(-30^\circ)$.

Note that $\sin^{-1}(-\frac{1}{2}) = -30^\circ$

When $n = 0$, $\theta = -30^\circ$. Discard this solution because it lies outside the range given.

When $n = 1$,

$\theta = 180 + (-1)(-30^\circ) = 180 + 30 = 210^\circ$

When $n = 2$,

$\theta = 360 + (1)(-30^\circ) = 360 - 30 = 330^\circ$

$\therefore \ \theta$ is 210° or 330°.

Note: $(-1)^n$ is equal to $(+1)$ if n is even and equal to (-1) if n is odd.

Example 2

Solve the equation $\sin(\theta - 15^\circ) = \frac{1}{2}$ for $0 < \theta < 180^\circ$

Solution

The acute angle whose sine is $\frac{1}{2}$ is 30°, but since sine is positive it implies the angle is either in the first or second quadrant.

$\therefore \ (\theta - 15^\circ) = 30^\circ$ or 150°

$\theta = 45^\circ$ or 165°

Alternatively, using the general solution,
$(\theta - 15) = 180n + (-1)^n(30°)$.

Note that $\sin^{-1}(\frac{1}{2}) = 30°$.

When $n = 0$, $(\theta - 15) = 30° \Rightarrow \quad \theta = 45°$.

When $n = 1$,
$(\theta - 15) = 180 + (-1)(30°) = 180 - 30 = 150°$

$$\theta = 150° + 15° = 165°$$

When $n = 2$,
$\theta = 360 + (1)(-30°) = 360 - 30 = 330° > 180°$

$\therefore \ \theta$ is 45° or 165°.

Example 3
Find the possible values of x from 0° to 360°, when:

i) $\sin^2 x = 0.75$
ii) $5\sin x = 3\cos x$
iii) $\sec 2x = 2$

Solution

i) $\sin^2 x = 0.75 = \frac{3}{4} \quad \Rightarrow \quad \sin x = \pm\frac{\sqrt{3}}{2}$

$\sin x = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad x = 60°, 120°$ and

$\sin x = -\frac{\sqrt{3}}{2} \quad \Rightarrow \quad x = 240°, 300°$

The possible values of x are 60°, 120°, 240° and 300°.

ii) $5\sin x = 3\cos x$

$$\frac{\sin x}{\cos x} = \frac{3}{5}$$

$$\tan x = 0.6$$

$$x = 31° \text{ and } 211°$$

The possible values of x are 31° and 211°.

iii) $\sec 2x = 2$

$$\frac{1}{\cos 2x} = 2$$

$$\cos 2x = \tfrac{1}{2}$$

$$2x = 60° \text{ and } 300°$$

$$x = 30° \text{ and } 150°.$$

Example 4
Solve the equation $2\sin^2\theta = \sin\theta$ for $0° \le \theta \le 360°$.

Solution

$$2\sin^2\theta = \sin\theta$$

$$2\sin^2\theta - \sin\theta = 0$$

$$\sin\theta(2\sin\theta - 1) = 0 \quad \text{(Factorizing)}$$

$$\sin\theta = 0 \text{ or } 2\sin\theta - 1 = 0$$

$$\sin\theta = 0 \text{ or } \tfrac{1}{2}$$

If $\sin\theta = 0$

$\theta = 0°, 180°, 360°$ (in the interval given)

Also if $\sin\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = 30°, 150°$

The values of θ in the interval $0 \le \theta \le 360°$ are 0°, 30°, 150°, 180°, and 360°.

Example 5
Find the truth set of the equation $3 - 3\cos\theta = 2\sin^2\theta$, for values of θ in the interval $0° \le \theta \le 360°$

Solution

$3 - 3\cos\theta = 2\sin^2\theta$ (cosine in the equation suggests a quadratic equation in $\cos\theta$)

But $\sin^2\theta = 1 - \cos^2\theta$

$$3 - 3\cos\theta = 2(1 - \cos^2\theta)$$

$$3 - 3\cos\theta = 2 - 2\cos^2\theta$$

$$2\cos^2\theta - 3\cos\theta + 1 = 0$$

Now let $y = \cos\theta$

$2y^2 - 3y + 1 = 0$ Factorizing

$(y - 1)(2y - 1) \Rightarrow y = 1$ or $\frac{1}{2}$.

But $y = \cos\theta \Rightarrow \cos\theta = 1$ or $\frac{1}{2}$

If $\cos\theta = 1 \quad \Rightarrow \quad \theta = 0°$ and $360°$ and

If $\cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = 60°$ and $300°$

The truth set of the equation is $\{\theta : \theta = 0°, 60°, 300° \text{ and } 360°\}$.

Alternative method

Using the general solution, if $\cos\theta = 1$

$\theta = 360n$. Note that $\cos^{-1}(1) = 0°$

When $n = 0$, $\theta = 0°$.

When $n = 1$, $\theta = 360(1) = 360°$.

When $n = 2$, $\theta = 360(2) = 720° > 360°$

Similarly, if $\cos\theta = \frac{1}{2}$,

$\theta = 360n \pm 60°$.

Note that $\cos^{-1}(\frac{1}{2}) = 60°$

When $n = 0$, $\theta = 60°$.

When $n = 1$, $\theta = 360 \pm 60 = 300°$ or $420°$ taking each sign. Discard 420° since it is outside the required range.

$\therefore$ The truth set of the equation is

$\{\theta : \theta = 0°, 60°, 300° \text{ and } 360°\}$

Example 6

Find the truth set of the equation

$1 + \sin x = 2\cos^2 x$, for $0° < x < 360°$

Solution

$1 + \sin x = 2\cos^2 x$ (sine suggests a quadratic equation in $\sin x$)

But $\cos^2 x = 1 - \sin^2 x$

$1 + \sin x = 2(1 - \sin^2 x)$

$1 + \sin x = 2 - 2\sin^2 x$

$2\sin^2 x + \sin x - 1 = 0$

Let $\sin x = y$

$2y^2 + y - 1 = 0$ Factorizing

$(y + 1)(2y - 1) = 0$

$y = -1$ or $\frac{1}{2}$

But $\sin x = y$

$\sin x = -1$ or $\frac{1}{2}$

If $\sin x = -1 \Rightarrow x = 270°$ and

If $\sin x = \frac{1}{2} \Rightarrow x = 30°, 150°$

The truth set is $\{x: x = 30°, 150° \text{ and } 270°\}$

Example 7

Solve $\cos x = 2\sin^2 x - 1$, for $0° < x < 360°$

Solution

Since $\sin^2 x = 1 - \cos^2 x$, the equation becomes $\cos x = 2(1 - \cos^2 x) - 1$

$2\cos^2 x + \cos x - 1 = 0$

Factorizing, we have

$(2\cos x - 1)(\cos x + 1) = 0$

$(2\cos x - 1) = 0$ or $(\cos x + 1) = 0$

$\cos x = \frac{1}{2}$ or $\cos x = -1$

If $\cos x = \frac{1}{2} \Rightarrow x = 60°$ and $300°$ and

If $\cos x = -1 \Rightarrow x = 180°$

$\therefore$ $x = 60°$, $180°$ and $300°$ in the given interval

Example 8

a) Find the truth set of the equation $2\cos^2 x + 3\cos x - 2 = 0$ for values of x in the interval $0° < x < 360°$

b) If $\sin\theta = \frac{12}{13}$, where θ is an obtuse angle, find the value of $\dfrac{\tan\theta}{2\sin\theta - 3\cos\theta}$.

c) Find the largest domain of the function

$$f:\theta \to \frac{\tan\theta}{2\sin\theta - 3\cos\theta}$$

Solution

a) $2\cos^2 x + 3\cos x - 2 = 0$.

This is a quadratic equation in $\cos x$

Let $\cos x = y$

$2y^2 + 3y - 2 = 0$

$(2y - 1)(y + 2) = 0 \Rightarrow y = \frac{1}{2}$ or -2

$\cos x = \frac{1}{2}$ or -2.

But $\cos x \neq -2$ since $\cos x$ has values between -1 and 1 inclusive.

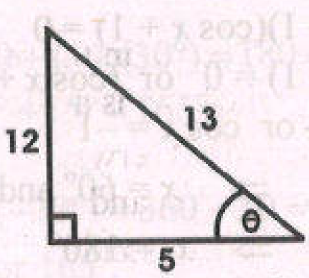
$\cos x = \frac{1}{2} \Rightarrow x = 60°$ and $300°$

$\therefore \{x : x = 60°, 300°\}$

b) If $\sin\theta = \frac{12}{13}$, where θ is obtuse, then $\cos\theta$ and $\tan\theta$ have negative values, since in the second quadrant cosine and tangent are negative.

From the right-angled triangle below drawn for $\sin\theta = \frac{12}{13}$, the third side (i.e. adjacent) is 5.

$\cos\theta = -\frac{5}{13}$ and $\tan\theta = -\frac{12}{5}$



$$\therefore \quad \frac{\tan\theta}{2\sin\theta - 3\cos\theta} = \frac{-\frac{12}{5}}{2\left(\frac{12}{13}\right) - 3\left(-\frac{5}{13}\right)}$$

$$= \frac{-\frac{12}{5}}{\frac{24}{13}+\frac{15}{13}} = \frac{-\frac{12}{5}}{3} = -\frac{12}{15} = -\frac{4}{5}$$

c) $f:\theta \to \dfrac{\tan\theta}{2\sin\theta - 3\cos\theta}$

The largest domain is the possible values of θ for which $f(\theta)$ is defined.

f is defined for all values of θ except

$2\sin\theta - 3\cos\theta = 0 \Rightarrow 2\sin\theta = 3\cos\theta$

$\frac{\sin\theta}{\cos\theta} = \frac{3}{2} \quad \Rightarrow \quad \tan\theta = \frac{3}{2}$

(Note: $\tan\theta = \sin\theta/\cos\theta$)

$\theta = 56^\circ$ and $(180 + 56)^\circ = 56^\circ$ and 236°

The domain of f

$= (\theta : 0^\circ < \theta < 360^\circ$; except $56^\circ, 236^\circ)$

Maximum and minimum points of trigonometric functions of the form $f(\theta) = a\cos\theta + b\cos\theta$

The trigonometric function $f(x) = a\cos\theta + b\sin\theta$ can be reduced to one of the standard forms, $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \pm \alpha)$.

If $f(x) = R\sin(\theta \pm \alpha)$ or $f(x) = R\cos(\theta \pm \alpha)$, then the maximum value of $f(x)$ is R and this occurs when $\sin(\theta \pm \alpha) = 1$ or $\cos(\theta \pm \alpha) = 1$. The minimum value of $f(x)$ is $-R$ and this occurs when $\sin(\theta \pm \alpha) = -1$ or $\cos(\theta \pm \alpha) = -1$.

Illustrative examples

Example 1

Given that $f(x) = 3\cos x + 2\sin x$, where $0 < x < 360^\circ$

i) What is the maximum value of $f(x)$?

ii) For what value of x is $f(x)$ maximum?

Solution

i) $f(x) = 3\cos x + 2\sin x$

$= R\cos(x - \alpha)$

$= R(\cos x\cos\alpha + \sin x\sin\alpha)$

$= R\cos x\cos\alpha + R\sin x\sin\alpha$

Equating coefficients of $\cos x$ and $\sin x$, we have

$R\cos\alpha = 3$ ……….(1) and

$R\sin\alpha = 2$ ……….(2)

To find R, square both equations and add them.

$(1)^2 + (2)^2$

$\Rightarrow R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 2^2$

$\Rightarrow R^2(\cos^2\alpha + \sin^2\alpha) = 9 + 4 = 13$

$\Rightarrow R^2 = 13 \qquad \Rightarrow \qquad R = \sqrt{13}$

Note that $\cos^2\alpha + \sin^2\alpha = 1$

To find the angle α, divide equation (2) by equation (1)

$(2) \div (1)$

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{2}{3}$

$\tan\alpha = \frac{2}{3} = 0.6667 \quad \Rightarrow \quad \alpha = 33.7^\circ$

Note that $\frac{\sin\alpha}{\cos\alpha} = \tan\alpha$

Hence $f(x) = 3\cos x + 2\sin x$ becomes

$f(x) = \sqrt{13}\cos(x - 33.7^\circ)$

Therefore the maximum value of $f(x)$ is $\sqrt{13}$.

ii) This occurs when $\cos(x-33.7^\circ)=1$

$$x-33.7^\circ=0^\circ \quad \therefore \quad x=33.7^\circ$$

Example 2

Given that $4\sin x-\cos x=R\sin(x-\theta)$, where $R>0$ and $0^\circ<\theta<90^\circ$, find the values of R and θ. Hence find the maximum value of $y=4\sin x-\cos x$ and the value of x between 0° and 360° for which the curve y has a turning point.

Solution

$$\begin{aligned}4\sin x-\cos x &= R\sin(x-\theta)\\ &= R(\sin x\cos\theta-\cos x\sin\theta)\\ &= R\sin x\cos\theta-R\cos x\sin\theta\end{aligned}$$

Equating coefficients of $\cos x$ and $\sin x$, we have

$R\cos\theta=4$ ....(1) and

$R\sin\theta=1$ ....(2)

To find R, square both equations and add them.

$(1)^2+(2)^2$

$$R^2\cos^2\theta+R^2\sin^2\theta=4^2+1^2$$

$$R^2(\cos^2\theta+\sin^2\theta)=16+1=17$$

$$R^2=17 \quad\Rightarrow\quad R=\sqrt{17}$$

Note that $\cos^2\theta+\sin^2\theta=1$

To find the angle α, divide equation (2) by equation (1)

$(2)\div(1)$

$$\frac{R\sin\theta}{R\cos\theta}=\frac{1}{4}$$

$$\tan\theta=\tfrac{1}{4}=0.25 \quad\Rightarrow\quad \theta=14^\circ$$

Note that $\dfrac{\sin\theta}{\cos\theta}=\tan\theta$

Hence $4\sin x-\cos x$ becomes $\sqrt{17}\sin(x-14^\circ)$

Therefore the maximum value of $y=4\sin x-\cos x$ is $\sqrt{17}$

This occurs when $\sin(x-14^\circ)=1$

$$x-14^\circ=90^\circ \quad\therefore\quad x=104^\circ$$

Therefore y has a turning point when $x=104^\circ$

Example 3

Express $\sin x-2\cos x$ in the form $R\sin(x-\alpha)$, where α is acute and R is positive. Hence, or otherwise

i) find the maximum and minimum points of $y=\sin x-2\cos x$

ii) solve the equation $\sin x-2\cos x=1$ for $0<x<360^\circ$

Solution

$$\begin{aligned}\sin x-2\cos x &= R\sin(x-\alpha)\\ &= R(\sin x\cos\alpha-\cos x\sin\alpha)\\ &= R\sin x\cos\alpha-R\cos x\sin\alpha\end{aligned}$$

Equating coefficients of $\cos x$ and $\sin x$, we have

$R\cos\alpha=1$ ....(1) and

$R\sin\alpha=2$ ....(2)

To find R, square both equations and add them.

$(1)^2+(2)^2$

$$R^2\cos^2\alpha+R^2\sin^2\alpha=1^2+2^2$$

$$R^2(\cos^2\alpha+\sin^2\alpha)=1+4=5$$

$$R^2=5 \quad\Rightarrow\quad R=\sqrt{5}$$

Note that $\cos^2\alpha+\sin^2\alpha=1$

To find the angle α, divide equation (2) by equation (1)

$(2)\div(1)$

$$\frac{R\sin\alpha}{R\cos\alpha}=\frac{2}{1}$$

$$\tan\alpha=2 \quad\Rightarrow\quad \alpha=63.4^\circ$$

Note that $\dfrac{\sin\alpha}{\cos\alpha}=\tan\alpha$

Hence $\sin x-2\cos x$ becomes $\sqrt{5}\sin(x-63.4^\circ)$

i) Therefore the maximum value of $y = 4\sin x - \cos x$ is $\sqrt{5}$

This occurs when $\sin(x - 63.4°) = 1$

$$x - 63.4° = 90° \quad \therefore \quad x = 153.4°$$

Therefore the minimum value of $y = 4\sin x - \cos x$ is $-\sqrt{5}$

This occurs when $\sin(x - 63.4°) = -1$

$$x - 63.4° = 270° \quad \therefore \quad x = 333.4°$$

Therefore y has maximum and minimum points at $(\sqrt{5}, 153.4°)$ and $(-\sqrt{5}, 333.4°)$ respectively

ii) Since $\sin x - 2\cos x$ becomes $\sqrt{5}\sin(x - 63.4°)$, $\sin x - 2\cos x = 1$ becomes $\sqrt{5}\sin(x - 63.4°) = 1$

$$\sin(x - 63.4°) = \tfrac{1}{\sqrt{5}} = 0.4472$$

The principal solution of $\sin(x - 63.4°) = 0.4472$ is $26.6°$

Therefore the general solution is

$$x - 63.4° = 180n + (-1)^n 26.6°$$

When $n = 0$

$$x - 63.4° = 26.6° \quad \Rightarrow \quad x = 90°$$

When $n = 1$

$$x - 63.4° = 180 - 26.6°$$

$$x - 63.4° = 153.4° \quad \Rightarrow \quad x = 216.8°$$

Example 4

Given that $3\cos\theta - 4\sin\theta = R\cos(\theta + \alpha)$, obtain the values of α and R, where α is acute and R is a positive constant. Hence, or otherwise

i) find the maximum and minimum values of $3\cos\theta - 4\sin\theta$ and find the corresponding value of θ corresponding to each.

ii) find the angles between 0° and 360° for which $3\cos\theta - 4\sin\theta = 2$

Solution

$$\begin{aligned} 3\cos\theta - 4\sin\theta &= R\cos(\theta + \alpha) \\ &= R(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \\ &= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha \end{aligned}$$

Equating coefficients of $\cos\theta$ and $\sin\theta$, we have

$R\cos\alpha = 3$ ………(1) and

$R\sin\alpha = 4$ ………(2)

To find R, square both equations and add them.

$(1)^2 + (2)^2$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2$$

$$R^2(\cos^2\alpha + \sin^2\alpha) = 9 + 16 = 25$$

$$R^2 = 25 \quad \Rightarrow \quad R = \sqrt{25} = 5$$

Note that $\cos^2\alpha + \sin^2\alpha = 1$

To find the angle α, divide equation (2) by equation (1)

$(2) \div (1)$

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{4}{3}$$

$$\tan\alpha = \tfrac{4}{3} = 1.3333 \quad \Rightarrow \quad \alpha = 53°$$

Note that $\dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha$

Hence $3\cos\theta - 4\sin\theta$ becomes $5\cos(\theta + 53°)$

i) Therefore the maximum value of $3\cos\theta - 4\sin\theta$ is 5

This occurs when $\cos(\theta + 53°) = 1$

$$\theta + 53° = 360° \quad \therefore \quad \theta = 307°$$

Therefore the minimum value of $3\cos\theta - 4\sin\theta$ is -5

This occurs when $\cos(\theta + 53°) = -1$

$$\theta + 53° = 180° \quad \therefore \quad \theta = 127°$$

Therefore the maximum and minimum points occur at $(5, 307°)$ and $(-5, 127°)$ respectively.

ii) Since $3\cos\theta - 4\sin\theta$ becomes $5\cos(\theta + 53°)$

$3\cos\theta - 4\sin\theta = 2$ becomes

$5\cos(\theta + 53°) = 2$

$\cos(\theta + 53°) = \frac{2}{5} = 0.4000$

The principal solution of $\cos(\theta + 53°) = 0.4000$ is $66.4°$

Therefore $\theta + 53° = 360n \pm 66.4°$

When $n = 0$,

$\theta + 53° = 66.4°$

$\theta = 66.4 - 53 = 13.4°$

When $n = 1$,

$\theta + 53° = 360 - 66.4$

$\theta = 293.4 - 53 = 240.6°$

i.e. $\theta = 13.4°$ or $240.6°$

Example 5

Solve $12\cos\theta + 5\sin\theta = 3$ for $0 < \theta < 360°$

Solution

$12\cos\theta + 5\sin\theta$

$= R\cos(\theta - \alpha)$

$= R(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

$= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

Equating coefficients of $\cos\theta$ and $\sin\theta$, we have

$R\cos\alpha = 12$ ……….(1) and

$R\sin\alpha = 5$ ………..(2)

To find R, square both equations and add them.

$(1)^2 + (2)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 12^2 + 5^2$

$R^2(\cos^2\alpha + \sin^2\alpha) = 144 + 25 = 169$

$R^2 = 169 \quad \Rightarrow \quad R = \sqrt{169} = 13$

Note that $\cos^2\alpha + \sin^2\alpha = 1$

To find the angle α, divide equation (2) by equation (1)

$(2) \div (1)$

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{5}{12}$$

$\tan\alpha = \frac{5}{12} = 0.4167 \quad \Rightarrow \quad \alpha = 23°$

Note that $\dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha$

Hence $12\cos\theta + 5\sin\theta = 3$ becomes $13\cos(\theta - 23°) = 3$

$\cos(\theta - 23°) = \frac{3}{13} = 0.2308$

Therefore $\theta - 23° = 360n \pm 77°$

When $n = 0$,

$\theta - 23° = 77° \quad \Rightarrow \quad \theta = 77 + 23 = 100°$

When $n = 1$,

$\theta - 23° = 360 - 77$

$\theta = 283 + 23 = 306°$

i.e. $\theta = 100°$ or $306°$

Review exercise 18

1. a) Draw the graph of $y = 3\sin x + 4\cos x$ for values of x from 0° to 90°. Take x at intervals of 10°.
   b) Use your graph to find the truth set of the equation:
      i) $6\sin x + 8\cos x - 9 = 0$ (Hint: Divide through by 2 and plot $y = 4.5$);
      ii) $36\sin x + 48\cos x - x = 0$ (Hint: Divide through by 12 and plot $y = \frac{x}{12}$).

2. a) Draw the graph of the function $f: x \rightarrow 2\sin x + \cos x$ for $0° < x < 180°$ at intervals of 20°.
   b) Use your graph to estimate in the given interval:
      i) the maximum value of f correct to two decimal place;
      ii) the truth set of the equation $\sin x + \frac{1}{2}\cos x = 0$ correct to the nearest degree.

3. i) Draw a graph of the function $f: x \to 2\cos x - \sin x$ for values of x from 0° to 180° at intervals of 20°.
   ii) Use your graph to find, in the given interval:
      a) the minimum value of f, correct to one decimal place, and the value of x for which it occurs, correct to the nearest degree;
      b) The solution set of the equation $2\cos x - \sin x = 0$, giving your answer correct to the nearest degree.

4. i) Draw a graph of the function $f: x \to \sin x + 2\cos x$ for values of x from 0° to 180° at intervals of 20°.
   ii) Use your graph to estimate, correct to the nearest degree, the truth set of the equation $40\sin x + 80\cos x = 100 - x$

5. Find the truth set of the following equations for $0 < x < 360$.
   i) $2\sin x + 8\cos^2 x = 5$
   ii) $3\sin x + 2\cos^2 x = 0$
   iii) $2\sin^2 x - 2\sin x - 3 = 0$

6. Find the possible values of x from 0° to 360°, when:
   i) $\cos^2 x = 0.75$
   ii) $\sin x = \cos x$
   iii) $\operatorname{cosec} 2x = 2$
   iv) $3\sin^2 x + 2\cos x = 2$
   v) $\cot^2 x = \operatorname{cosec}^2 x$
   vi) $\sec^2 x + \tan^2 x = 6$
   vii) $5\cos x - 4\sin^2 x = 2$
   viii) $4\sec^2 x - 3\tan x = 5$
   ix) $3\sin^2 x + \cos x = 1$
   x) $\cos x \sin x = 2\cos x$
   xi) $\cos^2 x = 2 - 2\sin x$
   xii) $2\cos x - 4\sin^2 x + 2 = 0$
   xiii) $\sin^2 x = \frac{1}{4}$

7. Find the maximum and minimum values of the following functions
   i) $\sqrt{3}\cos x - \sin x$
   ii) $\cos x - 3\sin x$
   iii) $4\sin x - 3\cos x$
   iv) $\cos x - \sqrt{3}\sin x$
   v) $7\cos x - 24\sin x$
   vi) $7\cos x + 6\sin x$
   vii) $2\cos x - \sin x$
   viii) $3\cos x - 4\sin x$

8. i) Construct a table for the mapping $f: x \to \cos x - 3\sin x$ for values of x from 0° to 180° at intervals of 20°.
   ii) Using a scale of 2 cm to 20 on the x-axis and 2cm to 1 unit on the y-axis, draw the graph of f.
   iii) Use your graph to estimate, in the given intervals, the truth sets of the equations
      a) $3\tan x = 1$
      b) $2 + \cos x = 3\sin x$.
   Giving your answer to the nearest degree.

9. Plot a graph of $f(x) = \sin x + 2\cos x$ for values of x from 0° to 180° at intervals of 30°. Use your graph to find in the given interval the truth set of the equation.
   i) $\sin x = 2.1 - 2\cos x$
   ii) $\sin x + 2\cos x = 0$

CHAPTER 19

DIFFERENTIATION

Limit of a function

The concept of limit of a function is very important in the subject of calculus. In fact, the subject of calculus was evolved through the process of finding a limit. Let us now understand the concept of limit.

Meaning of $x \to a$

For a given function *f*(*x*), $x \to a$ means the behaviour of *f*(*x*) for values of *x* around *a*. Let us discuss the meaning of $x \to a$ using the following example.

Let us discuss the behaviour of the function $f(x) = 2x + 3$ around $x = 4$, i.e. let us give values which are near 4 to find the corresponding values of *f*. We can give values which are greater than 4 as well as values which are less than 4. Hence we form two tables as follows

x	4.5	4.2	4.1	4.01	4.001
f(*x*)	12	11.4	11.2	11.02	11.002

x	3.4	3.7	3.9	3.99	3.999
f(*x*)	9.8	10.4	10.8	10.98	10.998

Studying the corresponding values of *f*, we see in the first table that, $f(x)$ starts at 12 and reduces progressively to reach 11.002 and in the second table, we see that $f(x)$ starts at 9.8 and increases to reach 10.998 finally. In both cases, we see that:
As *x* is brought near 4, $f(x)$ also approaches near and near 11. In fact, we can say that $f(x)$ can be brought as near 11 as we like by bringing *x* sufficiently near 4.

Therefore as *x* is brought near 4, from either side of 4, we say that *x* approaches 4 or *x* tends to 4. This is denoted by $x \to 4$
In this case, as *x* approaches 4, we see that $f(x)$ approaches 11 and this can be written symbolically as:
As $x \to 4$, $f(x) \to 11$

Definition of limit of a function

From the above example, we see that $f(x) \to 11$, when $x \to 4$. The number 11, is called the **limit** of $f(x)$ as $x \to 4$.
This is written in notation as $\lim_{x \to 4} f(x) = 11$
or $\lim_{x \to 4}(2x + 3) = 11$

The limiting value of a function *f*(*x*) as *x* approaches a value '*a*' (i.e. $x \to a$) means that *f*(*x*) can be brought as near '*l*' as we like by bringing *x* sufficiently near *a* without making $x = a$.
This is denoted as $\lim_{x \to a} f(x) = l$.

Standard limits

If we are asked to find $\lim_{x \to a} f(x)$, we cannot follow the procedure of making tables and checking the limit every time as it will be very difficult to calculate the various values of *f*. Hence we assume the following results as standard formulae.

1. *Limits of constant functions*

If *f*(*x*) = *k*, where *k* is a constant, then $\lim_{x \to a} k = k$ i.e. if *f*(*x*) = 2, then $\lim_{x \to a} 2 = 2$
Thus the limit of a constant is equal to the constant.

2. *Limits of the function x^k*

$$\lim_{x\to a} x^k = a^k$$

Example: $\lim_{x\to 2} x^3 = 2^3 = 8$;

$\lim_{x\to 3} x^4 = 3^4 = 81$

3. *Limits of the function x*

$$\lim_{x\to a} x = a$$

Example: $\lim_{x\to 2} x = 2, \ \lim_{x\to -3} x = -3$

4. *Limits of the function kx*

$$\lim_{x\to a} kx = ka$$

Example: $\lim_{x\to 5} 3x = 3(5) = 15$

Illustrative example

Evaluate the following:

i) $\lim_{x\to 3}(x+2)$ ii) $\lim_{x\to -2}(x^2-1)$

iii) $\lim_{x\to 2}(x^2-4x+2)$

Solution

i) $\lim_{x\to 3}(x+2) = 3+2 = 5$

ii) $\lim_{x\to -2}(x^2-1) = (-2)^2 - 1 = 4 - 1 = 3$

iii) $\lim_{x\to 2}(x^2-4x+2) = (2)^2 - 4(2) + 2$

$= 4 - 8 + 2 = -2$

5. *Limits of the functions f(x).g(x)*

$$\lim_{x\to a} f(x).g(x) = \lim_{x\to a} f(x).\ \lim_{x\to a} g(x)$$

$$= f(a).\ g(a)$$

Illustrative example

Find $\lim_{x\to 2}(x+3)(x^2-5)$

Solution

$\lim_{x\to 2}(x+3)(x^2-5)$

$= \lim_{x\to 2}(x+3)\lim_{x\to 2}(x^2-5)$

$= (2+3)\ (2^2-5) = -5$

6. *Limits of rational functions*

If $f(x)$ is a rational function,

i.e. if $f(x) = \dfrac{g(x)}{h(x)}$, then

$\lim_{x\to a} f(x) = \dfrac{\underset{x\to a}{\text{limit}}\ g(x)}{\underset{x\to a}{\text{limit}}\ h(x)} = \dfrac{g(a)}{h(a)}$, provided

either $\lim_{x\to a} h(x) \neq 0$ or both $\lim_{x\to a} h(x) \neq 0$ and $\lim_{x\to a} g(x) \neq 0$.

Note: If $\lim_{x\to a} h(x) = 0$ or both $\lim_{x\to a} h(x) = 0$ and $\lim_{x\to a} g(x) = 0$, reduce the rational function to its *simplest form.*

Illustrative example

Evaluate the following:

i) $\lim_{x\to -1}\left\{\dfrac{x^2-1}{x+1}\right\}$ ii) $\lim_{x\to 2}\left\{\dfrac{x^2-4}{x-2}\right\}$

iii) $\lim_{x\to 2}\left\{\dfrac{x^2-7x+10}{x^2-4}\right\}$ iv) $\lim_{x\to 1}\left\{\dfrac{x^3-1}{x-1}\right\}$

v) $\lim_{x\to 9}\left\{\dfrac{9-x}{3-\sqrt{x}}\right\}$ vi) $\lim_{x\to 3}\left\{\dfrac{9-x^2}{4-\sqrt{x^2+7}}\right\}$

Solution

i) Substituting $x = -1$ into the expression will give $\frac{0}{0}$. Therefore we have to first reduce the expression to its simplest form.

i) $\lim_{x\to -1}\left\{\dfrac{x^2-1}{x+1}\right\} = \lim_{x\to -1}\left\{\dfrac{(x-1)(x+1)}{x+1}\right\}$

$= \lim_{x\to -1}\{x-1\} = -1-1 = -2$

ii) $\lim_{x\to 2}\left\{\dfrac{x^2-4}{x-2}\right\} = \lim_{x\to 2}\left\{\dfrac{(x-2)(x+2)}{x-2}\right\}$

$= \lim_{x\to 2}\{x+2\} = 2+2 = 4$

iii) $\lim\limits_{x\to2}\left\{\dfrac{x^2-7x+10}{x^2-4}\right\}$

$$=\lim_{x\to2}\left\{\frac{(x-5)(x-2)}{(x+2)(x-2)}\right\}$$

$$=\lim_{x\to2}\left\{\frac{x-5}{x+2}\right\}=\frac{2-5}{2+2}=-\tfrac{3}{4}$$

iv) $\lim\limits_{x\to1}\left\{\dfrac{x^3-1}{x-1}\right\}=\lim\limits_{x\to1}\left\{\dfrac{(x-1)(x^2+x+1)}{x-1}\right\}$

$$=\lim_{x\to1}\{x^2+x+1\}=1^2+1+1=3$$

v) $\lim\limits_{x\to9}\left\{\dfrac{9-x}{3-\sqrt{x}}\right\}=\lim\limits_{x\to9}\left\{\dfrac{(9-x)(3+\sqrt{x})}{(3-\sqrt{x})(3+\sqrt{x})}\right\}$

$$=\lim_{x\to9}\left\{\frac{(9-x)(3+\sqrt{x})}{(9-x)}\right\}$$

$$=\lim_{x\to9}\{3+\sqrt{x}\}=3+\sqrt{9}$$

$$=3+3=6$$

Note: Rationalise the denominator first.

vi) $\lim\limits_{x\to3}\left\{\dfrac{9-x^2}{4-\sqrt{x^2+7}}\right\}$

$$=\lim_{x\to3}\left\{\frac{(9-x^2)(4+\sqrt{x^2+7})}{(4-\sqrt{x^2+7})(4+\sqrt{x^2+7})}\right\}$$

$$=\lim_{x\to3}\left\{\frac{(9-x^2)(4+\sqrt{x^2+7})}{16-(x^2+7)}\right\}$$

$$=\lim_{x\to3}\left\{\frac{(9-x^2)(4+\sqrt{x^2+7})}{(9-x^2)}\right\}$$

$$=\lim_{x\to3}\{4+\sqrt{x^2+7}\}=4+\sqrt{3^2+7}$$

$$=4+\sqrt{3^2+7}=4+\sqrt{16}=4+4=8$$

Note: Rationalise the denominator first.

7: *Limits involving infinity*

The limits of a function $f(x)$ as x approaches infinity or becomes very large is denoted as $\lim\limits_{x\to\infty} f(x)$.

i) $\lim\limits_{x\to\infty}\dfrac{1}{x^n}=0$, if the index n is positive.

Note that $\dfrac{1}{\infty}=\dfrac{1}{\infty^n}=0$.

For example, $\lim\limits_{x\to\infty}\dfrac{1}{2^x}=0$

ii) $\lim\limits_{x\to\infty} x^n$, does not exist if the index n is positive.

iii) If $f(x)=\dfrac{g(x)}{h(x)}$, then $\lim\limits_{x\to\infty} f(x)$ can be found by first dividing the numerator and denominator by the variable with the highest index or power of x.

Illustrative example

Evaluate the following:

i) $\lim\limits_{x\to\infty}\left(\dfrac{5x^2-1}{2x^2+1}\right)$ ii) $\lim\limits_{x\to\infty}\left(\dfrac{7x}{x-4}\right)$

iii) $\lim\limits_{x\to\infty}\left(\dfrac{2x^3-7x+3}{x^3+2}\right)$ iv) $\lim\limits_{x\to\infty}\left(\dfrac{3x}{x+2}\right)$

Solution

i) Dividing the numerator and denominator by the highest power of x i.e. x^2

$$\lim_{x\to\infty}\left(\frac{5x^2-1}{2x^2+1}\right)=\lim_{x\to\infty}\left(\frac{\frac{5x^2}{x^2}-\frac{1}{x^2}}{\frac{2x^2}{x^2}+\frac{1}{x^2}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{5-\frac{1}{x^2}}{2+\frac{1}{x^2}}\right)=\frac{5+0}{2+0}=\tfrac{5}{2}$$

ii) $\lim\limits_{x\to\infty}\left(\frac{7x}{x-4}\right)=\lim\limits_{x\to\infty}\left(\frac{\frac{7x}{x}}{\frac{x}{x}-\frac{4}{x}}\right)$

$$=\lim_{x\to\infty}\left(\frac{7}{1-\frac{4}{x}}\right)=\frac{7}{1-0}=7$$

iii) $\lim\limits_{x\to\infty}\left(\frac{2x^3-7x+3}{x^3+2}\right)=\lim\limits_{x\to\infty}\left(\frac{\frac{2x^3}{x^3}-\frac{7x}{x^3}+\frac{3}{x^3}}{\frac{x^3}{x^3}+\frac{2}{x^3}}\right)$

$$=\lim_{x\to\infty}\left(\frac{2-\frac{7}{x^2}+\frac{3}{x^3}}{1+\frac{2}{x^3}}\right)$$

$$=\frac{2-0+0}{1+0}=2$$

iv) $\lim\limits_{x\to\infty}\left(\frac{3x}{x+2}\right)=\lim\limits_{x\to\infty}\left(\frac{\frac{3x}{x}}{\frac{x}{x}+\frac{2}{x}}\right)$

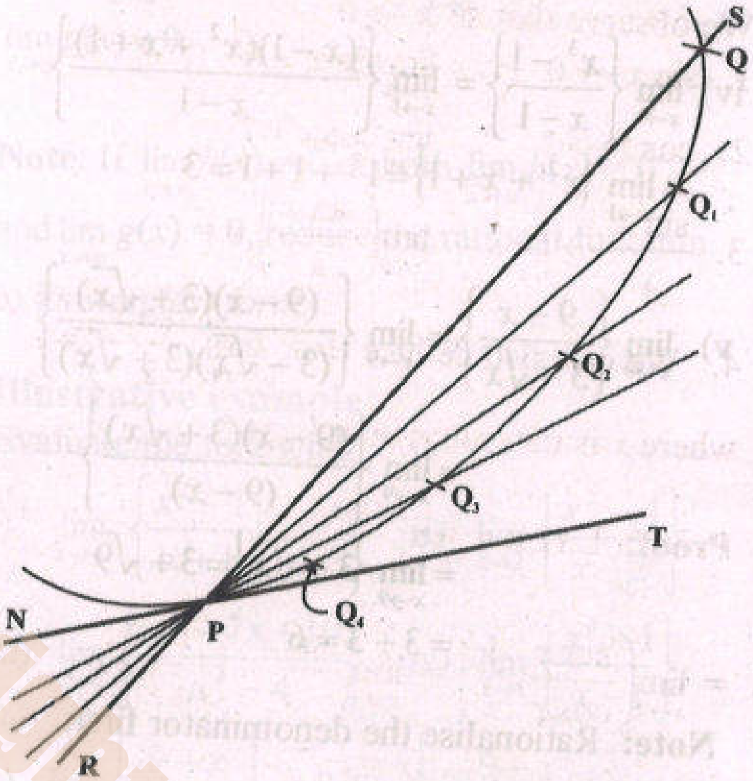
$$=\lim_{x\to\infty}\left(\frac{3}{1+\frac{2}{x}}\right)=\frac{3}{1+0}=3$$

The concept of limits and the gradient of a point on a curve

Draw a curve as shown in the figure. Draw chord PQ and extend it in both directions to get straight line $RPQS$. Now let the point Q approach P by drawing chords PQ_1, PQ_2, PQ_3, PQ_4, PQ_5. We can see that the nearer the point Q gets to P, the nearer does the gradient of the chord approach the gradient of the tangent NPT.

Thus as $Q \to P$ along the curve

1. chord $PQ \to$ the tangent at P
2. the gradient of the chord $PQ \to$ the gradient of the tangent at P
3. the tangent at P is called the **limit** of the gradient of chord PQ
4. the gradient of the curve at P is the limit of the gradient of chord PQ



Limit of simple trigonometric functions

Let us discuss the behaviour of $\sin x$, $\cos x$ and $\frac{\sin x}{x}$ (where x is measured in radians), as x approaches 0 from values which are less than $\frac{\pi}{2}$.

Look at the tables below.

1. $f(x) = \sin x$

x	1.5	0.8	0.2	0.1	0.01	0.001
sinx	.9975	.7174	.1987	.0998	.009	.0009

2. $f(x) = \cos x$

x	1.5	0.8	0.2	0.1	0.01	0.001
cosx	.0707	.6967	.9801	.995	.999	.9999

3. $f(x) = \dfrac{\sin x}{x}$

x	1.5	0.8	0.2	0.1	0.01	0.001
$\dfrac{\sin x}{x}$	.665	.896	.993	.998	.999	.9999

What happens when x takes values closer to zero?

We observe that as $x \to 0$

1. $\sin x \to 0 \quad \therefore \quad \lim\limits_{x\to 0} \sin x = 0$
2. $\cos x \to 1 \quad \therefore \quad \lim\limits_{x\to 0} \cos x = 1$
3. $\dfrac{\sin x}{x} \to 1 \quad \therefore \quad \lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right) = 1$
4. We can also deduce that $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right) = 1$

where x is measured in radians

Proof: L.H.S $= \lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)$

$$= \lim_{x\to 0}\left(\frac{\sin x}{\cos x}\cdot\frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\cdot\lim_{x\to 0}\left(\frac{1}{\cos x}\right) = 1\times\frac{1}{\cos 0} = 1$$

The limiting values of other simple trigonometric functions can be found from the above relations.

For example, $\lim\limits_{x\to 0}\sec x = \lim\limits_{\theta\to 0}\left(\dfrac{1}{\cos x}\right) = 1$

Note:

While solving problems using

$\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right) = 1$

We must take care to see that the following conditions are satisfied.

1. The angle x in $\sin x$ must be measured in radians.
2. This angle must tend to zero
3. This angle must be **fully repeated** in the denominator.

i.e. to find $\lim\limits_{x\to 0}\left(\dfrac{\sin 5x}{x}\right)$, we must adjust x in the denominator.

i.e. $\lim\limits_{x\to 0}\dfrac{\sin 5x}{5x}.5 = 1\times 5 = 5$

Illustrative examples

Example 1

Evaluate the following:

i) $\lim\limits_{x\to 0}(\cos x + \sin x)$

ii) $\lim\limits_{x\to 0}\left(\dfrac{2\tan x - x}{\sin 2x}\right)$

iii) $\lim\limits_{x\to 0}\left(\dfrac{\sqrt{2} - \sin x}{\cos x}\right)$

iv) $\lim\limits_{x\to 0}\left(\dfrac{\sin 2x}{x}\right)$

v) $\lim\limits_{x\to 0}\left(\dfrac{\cot x}{\cot 2x}\right)$

Solution

i) $\lim\limits_{x\to 0}(\cos x + \sin x) = \lim\limits_{x\to 0}\cos x + \lim\limits_{x\to 0}\sin x$

$= 1 + 0 = 1$

ii) $\lim\limits_{x\to 0}\left(\dfrac{2\tan x - x}{\sin 2x}\right)$

$$= \lim_{x\to 0}\left(\frac{2\tan x - x}{2\cos x\sin x}\right) \quad [\text{i.e. } \sin 2x = 2\cos x\sin x]$$

$$= \lim_{x\to 0}\left(\frac{\dfrac{2\tan x}{x} - \dfrac{x}{x}}{\dfrac{2\cos x\sin x}{x}}\right)$$

$$= \lim_{x\to 0}\left(\frac{\dfrac{2\tan x}{x}-1}{2\cos x.\dfrac{\sin x}{x}}\right)\left[\begin{array}{l}\text{i.e. dividing top}\\ \text{and bottom by } x\end{array}\right]$$

$$= \frac{2(1)-1}{2(1)} = \frac{1}{2}$$

$$\left[\text{since } \lim_{x\to 0}\left(\frac{\tan x}{x}\right) = 1;\right.$$

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1 \text{ and}$$

$$\left.\lim_{x\to 0}\cos x = 1\right]$$

iii) $\lim_{x\to 0}\left(\dfrac{\sqrt{2}-\sin x}{\cos x}\right) = \dfrac{\sqrt{2}-0}{1} = \sqrt{2}$

[since $\lim_{x\to 0}\sin x = 0$ and $\lim_{x\to 0}\cos x = 1$]

iv) $\lim_{x\to 0}\left(\dfrac{\sin 2x}{x}\right)$

$$= \lim_{x\to 0}\left(\frac{2\sin 2x}{2x}\right)$$

$$= \lim_{x\to 0}\left(2.\frac{\sin 2x}{2x}\right) = 2$$

[since $\lim_{x\to 0}\left(\dfrac{\sin 2x}{2x}\right) = 1$]

v) $\lim_{x\to 0}\left(\dfrac{\cot x}{\cot 2x}\right) = 2$ [work it out]

Example 2

Evaluate the following:

i) $\lim_{x\to 0}\left(\dfrac{\sin 6x-\sin 9x}{x}\right)$

ii) $\lim_{x\to 0}\left(\dfrac{x^2+\sin 3x}{2x-\sin x}\right)$

Solution

i) $\lim_{x\to 0}\left(\dfrac{\sin 6x-\sin 9x}{x}\right)$

$$= \lim_{x\to 0}\left(\frac{\sin 6x}{x}-\frac{\sin 9x}{x}\right)$$

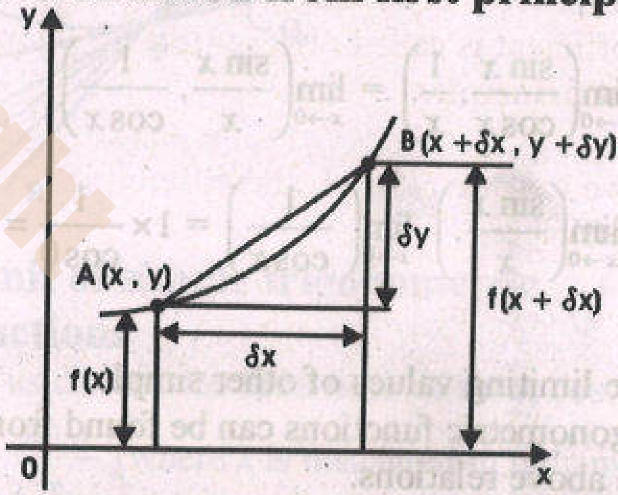
$$= \lim_{x\to 0}\left(\frac{\sin 6x}{6x}.6-\frac{\sin 9x}{9x}.9\right)$$

$$= 1(6)-1(9) = -3$$

ii) $\lim_{x\to 0}\left(\dfrac{x^2+\sin 3x}{2x-\sin x}\right) = \lim_{x\to 0}\left(\dfrac{x+\dfrac{\sin 3x}{x}}{2-\dfrac{\sin x}{x}}\right)$

$$= \lim_{x\to 0}\left(\frac{x+\dfrac{\sin 3x}{3x}.3}{2-\dfrac{\sin x}{x}}\right) = \frac{0+1(3)}{2-1} = 3$$

Differentiation from first principles



In the figure above, A and B are two points very close together on a curve, δx (delta x) and δy (delta y) representing small increments in the directions of x and y respectively.

Gradient of chord $AB = \dfrac{\delta y}{\delta x}$

However, $\delta y = f(x+\delta x)-f(x)$

Hence $\dfrac{\delta y}{\delta x}=\dfrac{f(x+\delta x)-f(x)}{\delta x}$

As δx approaches zero, $\dfrac{\delta y}{\delta x}$ approaches a limiting value and the gradient of the chord approaches the gradient of the tangent at A. Thus the gradient of the curve at A can be written as

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left(\frac{f(x+\delta x)-f(x)}{\delta x}\right)$$

In **Leibniz notation,**

$$\frac{dy}{dx}=\lim_{\delta x\to 0}\left(\frac{f(x+\delta x)-f(x)}{\delta x}\right)$$

In **functional notation,**

$$f'(x)=\lim_{\delta x\to 0}\left(\frac{f(x+\delta x)-f(x)}{\delta x}\right)$$

$\dfrac{dy}{dx}$ is the same as $f'(x)$ and is called the **differential coefficient** or the **derivative**. This process of finding the differential coefficient is called **differentiating from first principles**.

If we replace δx by h, the differential coefficient,

$$\frac{dy}{dx}=f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

Illustrative examples

Example 1

Obtain from first principles, the derivative of $f(x)=x^2$.

Solution

To 'differentiate from first principles' means to find $f'(x)$ by using the expression

$$\frac{dy}{dx}=f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$f(x)=x^2$, substituting $(x+h)$ for x gives
$f(x+h)=(x+h)^2=x^2+2xh+h^2$

Hence

$$f'(x)=\lim_{h\to 0}\left\{\frac{(x^2+2xh+h^2)-(x^2)}{h}\right\}$$

$$=\lim_{h\to 0}\left\{\frac{2xh+h^2}{h}\right\}=\lim_{h\to 0}\{2x+h\}$$

$$=2x$$

as $h\to 0$, $\{2x+h\}\to\{2x+0\}$.

Thus $f'(x)=2x$ i.e. the differential coefficient of x^2 is $2x$.

Example 2

Find from first principles, the derivative with respect to x of the function $y=5x$.

Solution

Let $y=f(x)=5x$

$f(x+h)=5(x+h)=5x+5h$

Hence $\dfrac{dy}{dx}=\lim_{h\to 0}\left(\dfrac{f(x+h)-f(x)}{h}\right)$

$$=\lim_{h\to 0}\left\{\frac{(5x+5h)-(5x)}{h}\right\}$$

$$=\lim_{h\to 0}\left\{\frac{5h}{h}\right\}=\lim_{h\to 0}\{5\}=5$$

Therefore $\dfrac{dy}{dx}=5$

Example 3

Find from first principles, the derivative with respect to x of the function $y=8$.

Solution

Let $y=f(x)=8 \;\Rightarrow\; f(x+h)=8$ since there are no x-values in the original equation.

Hence $\dfrac{dy}{dx}=\lim_{h\to 0}\left(\dfrac{f(x+h)-f(x)}{h}\right)$

$$=\lim_{h\to 0}\left\{\frac{8-8}{h}\right\}=\lim_{h\to 0}\left\{\frac{0}{h}\right\}=0$$

Therefore $\frac{dy}{dx} = 0$. Thus the derivative of a constant is zero.

Example 4

Differentiate from first principles $f(x) = 2x^3 + 3$.

Solution

$f(x) = 2x^3 + 3$, substituting $(x + h)$ for x gives

$$\begin{aligned} f(x+h) &= 2(x+h)^3 + 3 \\ &= 2(x^3 + 3x^2h + 3xh^2 + h^3) + 3 \\ &= 2x^3 + 6x^2h + 6xh^2 + 2h^3 + 3 \end{aligned}$$

$$f(x+h) - f(x)$$
$$= (2x^3 + 6x^2h + 6xh^2 + 2h^3 + 3) - (2x^3 + 3)$$
$$= 6x^2h + 6xh^2 + 2h^3$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{6x^2h + 6xh^2 + 2h^3}{h}\right) \\ &= \lim_{h\to 0}\left(6x^2 + 6xh + 2h^2\right) = 6x^2 \end{aligned}$$

Hence $\frac{dy}{dx} = 6x^2$ i.e. the differential coefficient of $(2x^3 + 3)$ is $6x^2$.

Example 5

From first principles, find the derivative of $2x^2 + 4x - 3$.

Solution

Let $f(x) = 2x^2 + 4x - 3$

$$f(x+h) = 2(x+h)^2 + 4(x+h) - 3$$

Note: We have to find an expression for $f(x + h) - f(x)$ and then find the limit of this expression as h approaches zero which is the same as differentiating $f(x)$ from first principles.

$$f(x+h) - f(x)$$
$$= [2(x+h)^2 + 4(x+h) - 3\,] - [2x^2 + 4x - 3]$$
$$= 2(x^2 + 2xh + h^2) + 4(x+h) - 3 - 2x^2 - 4x + 3$$
$$= 2x^2 + 4xh + 2h^2 + 4x + 4h - 3 - 2x^2 - 4x + 3$$
$$= 4xh + 2h^2 + 4h$$

$$\begin{aligned} \therefore\ f'(x) &= \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{4xh + 2h^2 + 4h}{h}\right) \\ &= \lim_{h\to 0}\left(4x + 2h + 4\right) = 4x + 4 \end{aligned}$$

Hence $\frac{dy}{dx} = f'(x) = 4x + 4$

i.e. the differential coefficient of $(2x^2 + 4x - 3)$ is $4x + 4$.

Example 6

Obtain from first principles, the derivative of $f(x) = 3x^2 + 1$.

Solution

To 'differentiate from first principles' means to find $f'(x)$ by using the expression

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$f(x) = 3x^2 + 1$, substituting $(x + h)$ for x gives

$f(x + h) = 3(x + h)^2 + 1 = 3x^2 + 6xh + 3h^2 + 1$

Hence

$$\frac{dy}{dx} = f'(x)\lim_{h\to 0}$$
$$= \left(\frac{f(x+h) - f(x)}{h}\right)$$
$$= \lim_{h\to 0}\left\{\frac{(3x^2 + 6xh + 3h^2 + 1) - (3x^2 + 1)}{h}\right\}$$
$$= \lim_{h\to 0}\left\{\frac{6xh + 3h^2}{h}\right\}$$
$$= \lim_{h\to 0}\{6x + 3h\} = 6x$$

Hence $\frac{dy}{dx} = 6x$ i.e. the differential coefficient of $(3x^2 + 1)$ is $6x$.

Example 7

Find from first principles, the derivative with respect to x of the function

$$f : x \to 3x + \frac{1}{x}.$$

Solution

$$f(x) = 3x + \frac{1}{x}$$

$$\Rightarrow \quad f(x+h) = 3(x+h) + \frac{1}{(x+h)}$$

$$= \frac{3(x+h)^2 + 1}{(x+h)}$$

$f(x+h) - f(x)$

$$= \frac{3(x+h)^2 + 1}{(x+h)} - \left(3x + \frac{1}{x}\right)$$

$$= \frac{3x^2 + 6xh + 3h^2 + 1}{(x+h)} - \left(\frac{3x^2 + 1}{x}\right)$$

$$= \frac{x[3x^2 + 6xh + 3h^2 + 1] - [(x+h)(3x^2 + 1)]}{x(x+h)}$$

$$= \frac{3x^3 + 6x^2h + 3xh^2 + x - 3x^3 - x - 3x^2h - h}{x^2 + xh}$$

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{3x^2h + 3xh^2 - h}{h(x^2 + xh)}\right)$$

$$= \lim_{h \to 0}\left(\frac{3x^2 + 3xh - 1}{(x^2 + xh)}\right) = \frac{3x^2 - 1}{x^2} = 3 - \frac{1}{x^2}$$

Alternative method

$$f(x) = 3x + \frac{1}{x}$$

$$\Rightarrow \quad f(x+h) = 3(x+h) + \frac{1}{(x+h)}$$

$f(x+h) - f(x)$

$$= \left[3(x+h) + \frac{1}{(x+h)}\right] - \left[3x + \frac{1}{x}\right]$$

$$= [3(x+h) - 3x] + \left[\frac{1}{(x+h)} - \frac{1}{x}\right]$$

$$= 3h - \frac{h}{x(x+h)}$$

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{3h - \frac{h}{x(x+h)}}{h}\right)$$

$$= \lim_{h \to 0}\left(3 - \frac{1}{x(x+h)}\right) = \left(3 - \frac{1}{x^2}\right)$$

Example 8

Obtain from first principles, the derivative of $f(x) = \sqrt{x}$.

Solution

To 'differentiate from first principles' means to find $f'(x)$ by using the expression

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$f(x) = \sqrt{x} \qquad \Rightarrow \quad f(x+h) = \sqrt{x+h}$$

$f(x+h) - f(x)$

$$= \sqrt{x+h} - \sqrt{x}$$

$$= \left(\sqrt{x+h} - \sqrt{x}\right)\frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{x + h - x}{\sqrt{x+h} + \sqrt{x}} = \frac{h}{\sqrt{x+h} + \sqrt{x}}$$

$$\therefore \quad \frac{dy}{dx} = f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{h}{h(\sqrt{x+h}+\sqrt{x})}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{\sqrt{x+h}+\sqrt{x}}\right)$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Example 9

Find, from first principle, the derivative of $f(x) = (2x+3)^2$

Solution

$$f(x) = (2x+3)^2$$

$$f^1(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f^1(x) = \lim_{h\to 0}\frac{4(x+h)^2+12(x+h)+9-(4x^2+12x+9)}{h}$$

$$f^1(x) = \lim_{h\to 0}\frac{8xh+4h^2+12h}{h}$$

$$f^1(x) = \lim_{h\to 0}\frac{h(8x+4h+12)}{h}$$

$$f^1(x) = \lim_{h\to 0}(8x+4h+12)$$

As $h \to 0$, $f^1(x) = 8x+4(0)+12$

$$f^1(x) = 8x+12 = 4(2x+3)$$

Example 10

Find, from first principle the derivative of x^3+x

Solution

$$f^1(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Let $f(x) = x^3+x$

$$f^1(x) = \lim_{h\to 0}\frac{f[(x+h)^3+(x+h)]-(x^3+x)}{h}$$

$$= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3+x+h-x^3-x}{h}$$

$$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3+h}{h}$$

$$= \lim_{h\to 0}\frac{h(3x^2+3xh+h^2+1)}{h}$$

$$= \lim_{h\to 0}3x^2+3xh+h^2+1$$

As $h \to 0$ $f^1(x) = 3x^2+0+0+1$

$\therefore f^1(x) = 3x^2+1$

Example 11

Obtain from first principles, the derivative of the following

i) $y = x^2$ ii) $y = x^3$

iii) $y = x^2-2x+1$ iv) $y = 3x^2+x$

v) $y = x+\frac{1}{x}$ vi) $y = 5x^3+x^2$

vii) $y = \frac{1}{x^2}$ viii) $y = x^2+3x$

Solution

i) $\frac{dy}{dx} = 2x$ ii) $\frac{dy}{dx} = 3x^2$

iii) $\frac{dy}{dx} = 2x-2$ iv) $\frac{dy}{dx} = 6x+1$

v) $\frac{dy}{dx} = 1-\frac{1}{x^2}$ vi) $\frac{dy}{dx} = 15x^2+2x$

vii) $\frac{dy}{dx} = \frac{-2}{x^3}$ viii) $\frac{dy}{dx} = 2x+3$

[Work out the solution in each case.]

Notation

Differentiation is the process of finding the **derivative** of a function. The derivative of a function is also called its **derived function** and also its **differential coefficient**.

The derivative of y with respect to x is usually written as $\frac{dy}{dx}$. This is called the **Leibniz notation**.

The derivative of $f(x)$ with respect to x is usually written as $f'(x)$ or $\frac{d}{dx}[f(x)]$.

When differentiating, results can be expressed in a number of ways.

For example:

i) If $y = 3x^2$, then $\frac{dy}{dx} = 6x$;

ii) If $f(x) = 3x^2$, then $f'(x) = 6x$;

iii) The differential coefficient of $3x^2$ is $6x$;

iv) The derivative of $3x^2$ is $6x$;

v) The gradient function of $3x^2$ is $6x$;

vi) $\frac{d}{dx}[3x^2] = 6x$.

Rules of Differentiation

Differentiation of a constant

If $y = c$ where c is a constant, then $\frac{dy}{dx} = 0$.

i.e. $y = c$ can be written as $y = cx^0$ and $\frac{dy}{dx} = 0.cx^{-1} = 0$. For example, if $y = 3$ then $\frac{dy}{dx} = 0$. Therefore the derivative of any constant is zero.

Differentiation of *ax* where *a* is a constant

If $y = ax$ where a is a constant, then $\frac{dy}{dx} = a$

For example, if $y = 3x$, then $\frac{dy}{dx} = 3$ and also if $y = -5x$, then $\frac{dy}{dx} = -5$

Differentiation of x^n

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$, where n is any real number, positive or negative or fraction. Thus, the general rule for differentiating x^n is to multiply the term by the power (index) and decrease the power by one.

For example, if $y = x^3$, then $\frac{dy}{dx} = 3x^2$.

Also if $y = x^2$, then $\frac{dy}{dx} = 2x$.

Differentiation of ax^n

If $y = ax^n$, where a is a constant, then $\frac{dy}{dx} = nax^{n-1}$

For example, if $y = 4x^3$, then $\frac{dy}{dx} = 12x^2$.

Also if $y = 3x^{-2}$, then $\frac{dy}{dx} = -6x^{-3}$.

Illustrative example

Find $\frac{dy}{dx}$ if:

i) $y = \frac{2}{x^3}$ ii) $y = \frac{1}{\sqrt{x}}$ iii) $y = \frac{1}{x^4}$

iv) $y = 3\sqrt{x}$ v) $y = \frac{1}{\sqrt[3]{x^4}}$

Solution

i) $y = \frac{2}{x^3}$

$$y = 2x^{-3} \Rightarrow \frac{dy}{dx} = -6x^{-4} = -\frac{6}{x^4}$$

ii) $y = \frac{1}{\sqrt{x}}$

$$y = \frac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -\tfrac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2\sqrt{x^3}}$$

iii) $y = \dfrac{1}{x^4}$

$$y = x^{-4} \quad \Rightarrow \quad \frac{dy}{dx} = -4x^{-5} = -\frac{4}{x^5}$$

iv) $y = 3\sqrt{x}$

$$y = 3x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \tfrac{3}{2}x^{-\frac{1}{2}} = \tfrac{3}{2}.\tfrac{1}{\sqrt{x}} = \frac{3}{2\sqrt{x}}$$

v) $y = \dfrac{1}{\sqrt[3]{x^4}}$

$$y = \frac{1}{x^{\frac{4}{3}}} = x^{-\frac{4}{3}}$$

$$\frac{dy}{dx} = -\tfrac{4}{3}x^{-\frac{7}{3}} = -\frac{4}{3\sqrt[3]{x^7}}$$

Differentiation of a sum or difference

The differential coefficient of a sum or difference is the sum or difference of the differential coefficients of the separate terms. Thus differentiate the sum or difference term by term. i.e.

$$\frac{d}{dx}[f(x) + g(x) - h(x)]$$

$$= \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] - \frac{d}{dx}[h(x)]$$

For example, if $y = x^4 + \dfrac{1}{x} - 6x$, then

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(x^{-1}) - \frac{d}{dx}(6x)$$

$$= 4x^3 - \frac{1}{x^2} - 6$$

Illustrative examples

Example 1

Differentiate with respect to x

i) $3x^2 + 4x$ ii) $\dfrac{3}{x^2} + 5x + 6$

iii) $5x^2 - 4x + 8$ iv) $4x^3 - 3x^2 + \dfrac{5}{x^2}$

v) $\dfrac{1}{2}x^2 - \dfrac{1}{2x^2}$

Solution

i) Let $y = 3x^2 + 4x$

$$\frac{dy}{dx} = 6x + 4$$

ii) Let $y = \dfrac{3}{x^2} + 5x + 6$

$$\frac{dy}{dx} = -\frac{6}{x^3} + 5$$

iii) Let $y = 5x^2 - 4x + 8$

$$\frac{dy}{dx} = 10x - 4$$

iv) Let $y = 4x^3 - 3x^2 + \dfrac{5}{x^2}$

$$\frac{dy}{dx} = 12x^2 - 6x - \frac{10}{x^3}$$

v) Let $y = \dfrac{1}{2}x^2 - \dfrac{1}{2x^2}$

$$\frac{dy}{dx} = x + \frac{1}{x^3}$$

Example 2

Differentiate each of the following with respect to x.

i) $y = x^2 + 5\sqrt{x}$ ii) $y = x^{\frac{3}{2}} + x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

iii) $y = \sqrt{x} + \sqrt{x^3}$ iv) $y = \dfrac{1}{x^2} - \dfrac{1}{x^3}$

v) $y=\frac{1}{\sqrt{x}}-\frac{2}{x}$ vi) $y=\frac{4}{x^3}+\frac{x^3}{4}$

vii) $y=\frac{4}{x}-\frac{5}{x^2}-\frac{6}{x^3}$ viii) $y=3\sqrt{x}-3x$

ix) $y=x-\frac{2}{x}-\frac{3}{x^3}$

Solution

i) $\frac{dy}{dx}=2x+\frac{5}{2\sqrt{x}}$

ii) $\frac{dy}{dx}=\frac{3}{2}x^{\frac{1}{2}}+\frac{1}{2\sqrt{x}}-\frac{1}{2x^{\frac{3}{2}}}$

iii) $\frac{dy}{dx}=\frac{1}{2\sqrt{x}}+\frac{3\sqrt{x}}{2}$

iv) $\frac{dy}{dx}=-\frac{2}{x^3}+\frac{3}{x^4}$

v) $\frac{dy}{dx}=\frac{-1}{2\sqrt{x^3}}+\frac{2}{x^2}$

vi) $\frac{dy}{dx}=\frac{-12}{x^4}+\frac{3x^2}{4}$

vii) $\frac{dy}{dx}=-\frac{4}{x^2}-\frac{10}{x^3}+\frac{18}{x^4}$

viii) $\frac{dy}{dx}=\frac{3}{2\sqrt{x}}-3$

ix) $\frac{dy}{dx}=1+\frac{2}{x^2}+\frac{9}{x^4}$

The rules of differentiation can be applied to the differentiation of expressions containing products and fractions provided they can be multiplied out or divided into separate terms.

Example 3

Find $\frac{dy}{dx}$ if

i) $y=x^4-2x+\frac{1}{x^2}$

ii) $y=\frac{3x^4-x}{x^3}$

iii) $y=\frac{(x^5+x^2-2)}{x^4}$

Solution

i) $y=x^4-2x+\frac{1}{x^2}$

$$\frac{dy}{dx}=4x^3-2-\frac{2}{x^3}$$

ii) $y=\frac{3x^4-x}{x^3}=\frac{3x^4}{x^3}-\frac{x}{x^3}=3x-\frac{1}{x^2}$

$$\frac{dy}{dx}=3+\frac{2}{x^3}$$

iii) $y=\frac{(x^5+x^2-2)}{x^4}$

$$=\frac{x^5}{x^4}+\frac{x^2}{x^4}-\frac{2}{x^4}=x+\frac{1}{x^2}-\frac{2}{x^4}$$

$$\frac{dy}{dx}=1-\frac{2}{x^3}+\frac{8}{x^5}$$

Example 4

Given that $v=\frac{2u^3-u^2}{3u}$, find $\frac{dv}{du}$.

Solution

$$v=\frac{2u^3-u^2}{3u}=\frac{2u^3}{3u}-\frac{u^2}{3u}=\tfrac{2}{3}u^2-\tfrac{1}{3}u$$

$$\frac{dv}{du}=\tfrac{4}{3}u-\tfrac{1}{3}$$

Example 5

Differentiate the following with respect to x:

i) $\frac{1}{x}\left(3+4x+5x^2\right)$

ii) $\frac{2}{x}+\frac{1}{\sqrt{x}}$

Solution

i) Let $y=\frac{1}{x}\left(3+4x+5x^2\right)$

$$y=\left(\frac{3}{x}+4+5x\right) \Rightarrow \frac{dy}{dx}=-\frac{3}{x^2}+5$$

ii) Let $y=\frac{2}{x}+\frac{1}{\sqrt{x}}=\frac{2}{x}+x^{-\frac{1}{2}}$

$$\frac{dy}{dx}=-\frac{2}{x^2}-\frac{1}{2\sqrt{x^3}}$$

Example 6

If $y=(x-3)(x^2+7x-1)$, find $\frac{dy}{dx}$

Solution

$$y=(x-3)(x^2+7x-1)=x^3+4x^2-22x+3$$

$$\therefore \quad \frac{dy}{dx}=3x^2+8x-22$$

Example 7

If $y=\frac{6x^2+x-4}{2x}$, find $\frac{dy}{dx}$.

Solution

$$y=\frac{6x^2+x-4}{2x}=\frac{6x^2}{2x}+\frac{x}{2x}-\frac{4}{2x}$$

$$=3x+\tfrac{1}{2}-\frac{2}{x}=3x+\tfrac{1}{2}-2x^{-1}$$

$$\therefore \quad \frac{dy}{dx}=3+0-(-2x^{-2})=3+\frac{2}{x^2}$$

Example 8

Differentiate the following with respect to x

i) $y=(x-3)(2x+5)$

ii) $y=(x-4)^2$

iii) $y=\frac{(x^3+2)}{x}$

iv) $y=\frac{x^2-7x+4}{x^3}$

v) $y=\sqrt{x}(x-1)$

vi) $y=\frac{(x-1)}{\sqrt{x}}$

Solution

i) $\frac{dy}{dx}=4x-1$

ii) $\frac{dy}{dx}=2x-8$

iii) $\frac{dy}{dx}=2x-\frac{2}{x^2}$

iv) $\frac{dy}{dx}=-\frac{1}{x^2}+\frac{14}{x^3}-\frac{12}{x^4}$

v) $\frac{dy}{dx}=\frac{3}{2}\sqrt{x}-\frac{1}{2\sqrt{x}}$

vi) $\frac{dy}{dx}=\frac{1}{2x^{\frac{1}{2}}}+\frac{1}{2x^{\frac{3}{2}}}$

Note: If the expression is such that it cannot be easily multiplied out or divided into separate terms the **product** or **quotient rules** are used.

Differentiation of a Product

1. If $y=au$ where a is a constant and u is a function of x, then $\frac{dy}{dx}=a\frac{du}{dx}$

For example, if $y=5x^3$, then

$$\frac{dy}{dx}=5(3x^2)=15x^2$$

2. If $y = uv$ where u and v are both functions of x, then by the **product rule**:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

The derivatives of functions such as $(x-2)(x^2+1)$, $2x^2\cos x$, are obtained by the formula above.

Illustrative example

Find $\frac{dy}{dx}$, if

i) $y = (3x^2+1)(5x-8)$
ii) $y = (2x-1)(3x+2)$
iii) $y = (2x^2-1)(x^2+3)$

Solution

i) $(3x^2+1)(5x-8)$ is a product of two terms $(3x^2+1)$ and $(5x-8)$

Let $u = (3x^2+1)$ and $v = (5x-8)$

$$\frac{du}{dx} = 6x \text{ and } \frac{dv}{dx} = 5$$

Using the product rule: $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$$\frac{dy}{dx} = (5x-8)(6x) + (3x^2+1)5$$
$$= 30x^2 - 48x + 15x^2 + 5$$
$$= 45x^2 - 48x + 5$$

Note: Differentiate the first function whilst maintaining the second one and differentiate the second function whilst maintaining the first.

ii) $y = (2x-1)(3x+2)$ is a product of two terms $(2x-1)$ and $(3x+2)$

Let $u = (2x-1)$ and $v = (3x+2)$

$$\frac{du}{dx} = 2 \text{ and } \frac{dv}{dx} = 3$$

Using the product rule: $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$$\frac{dy}{dx} = (3x+2)2 + (2x-1)3$$
$$= 6x + 4 + 6x - 3 = 12x + 1$$

iii) $y = (2x^2-1)(x^2+3)$ is a product of two terms $(2x^2-1)$ and (x^2+3)

Let $u = (2x^2-1)$ and $v = (x^2+3)$

$$\frac{du}{dx} = 4x \text{ and } \frac{dv}{dx} = 2x$$

Using the product rule: $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$$\frac{dy}{dx} = (x^2+3).(4x) + (2x^2-1).2x$$
$$= 4x^3 + 12x + 4x^3 - 2x$$
$$= 8x^3 + 10x = 2x(4x^2+5)$$

Differentiation of a quotient

If $y = \frac{u}{v}$, where u and v are both functions of x, then by the **quotient rule**:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

The derivatives of functions such as $\frac{(x-2)^2}{(x^2+1)}$, $\frac{\sin x}{\cos x}$, $\frac{\cos x}{\sqrt{x}}$ etc. are obtained by the formula above.

Illustrative examples

Example 1

Differentiate with respect to x

i) $y = \frac{2x+3}{x-5}$ ii) $y = \frac{1+x^2}{x-1}$

Solution

i) $y = \frac{2x+3}{x-5}$ is a quotient.

Let $u = 2x+3$ and $v = x-5$

$$\frac{du}{dx} = 2 \text{ and } \frac{dv}{dx} = 1$$

Using the quotient rule,

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}=\frac{(x-5).2-(2x+3).1}{(x-5)^2}$$

$$=\frac{2x-10-2x-3}{(x-5)^2}$$

Hence $\frac{dy}{dx}=\frac{-13}{(x-5)^2}$

ii) $y=\frac{1+x^2}{x-1}$ is a quotient.

Let $u=1+x^2$ and $v=x-1$

$\frac{du}{dx}=2x$ and $\frac{dv}{dx}=1$

Using the quotient rule,

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}=\frac{(x-1).2x-(1+x^2).1}{(x-1)^2}$$

$$=\frac{2x^2-2x-1-x^2}{(x-1)^2}$$

Hence $\frac{dy}{dx}=\frac{x^2-2x-1}{(x-1)^2}$

Example 2

Find the derivative with respect to x, of:

i) $\frac{1-x^2}{1+x^2}$ ii) $\frac{1+x}{1-x^2}$ iii) $\frac{x^2-1}{2x^2+1}$

Solution

i) $\frac{1-x^2}{1+x^2}$ is a quotient.

Let $u=1-x^2$ and $v=1+x^2$

$\frac{du}{dx}=-2x$ and $\frac{dv}{dx}=2x$

Using the quotient rule,

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$=\frac{(1+x^2).(-2x)-(1-x^2).2x}{(1+x^2)^2}$$

$$=\frac{-2x-2x^3-2x+2x^3}{(1+x^2)^2}$$

Hence $\frac{dy}{dx}=\frac{-4x}{(1+x^2)^2}$

ii) $\frac{1+x}{1-x^2}$ is a quotient.

Let $u=1+x$ and $v=1-x^2$

$\frac{du}{dx}=1$ and $\frac{dv}{dx}=-2x$

Using the quotient rule,

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$=\frac{(1-x^2).1-(1+x).(-2x)}{(1-x^2)^2}$$

$$=\frac{1-x^2+2x+2x^2}{(1-x^2)^2}$$

Hence $\frac{dy}{dx}=\frac{x^2+2x+1}{(1-x^2)^2}$

$$=\frac{(1+x)^2}{(1-x^2)^2}=\left(\frac{1+x}{1-x^2}\right)^2$$

iii) $\frac{x^2-1}{2x^2+1}$ is a quotient.

Let $u=x^2-1$ and $v=2x^2+1$

$\frac{du}{dx}=2x$ and $\frac{dv}{dx}=4x$

Using the quotient rule,

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$=\frac{(2x^2+1).2x-(x^2-1).4x}{(2x^2+1)^2}$$

$$= \frac{4x^3 + 2x - 4x^3 + 4x}{(2x^2+1)^2}$$

Hence $\frac{dy}{dx} = \frac{6x}{(2x^2+1)^2}$

Use of partial fractions

Rational functions with two or more factors in the denominator may be differentiated by first expressing the function in partial fractions.

For example,

$$y = \frac{x}{(x-2)(x-3)} = \frac{-2}{x-2} + \frac{3}{x-3}$$

Therefore

$$\frac{dy}{dx} = \frac{d}{dx}[-2(x-2)^{-1}] + \frac{d}{dx}[3(x-3)^{-1}]$$

$$\frac{dy}{dx} = 2(x-2)^{-2} - 3(x-3)^{-2}$$

$$= \frac{2}{(x-2)^2} - \frac{3}{(x-3)^2}$$

Illustrative example

Express the following in partial fractions and hence differentiate them.

i) $y = \frac{1}{(x-1)(x+1)}$

ii) $y = \frac{x}{(x-2)(x+1)}$

iii) $y = \frac{2x}{(2x-1)(x-3)}$

iv) $y = \frac{3x-1}{(x+2)(2x-1)}$

Solution

i) $\frac{dy}{dx} = \frac{-2x}{(x-1)^2(x+1)^2}$

ii) $\frac{dy}{dx} = -\frac{2}{3(x-2)^2} - \frac{1}{3(x+1)^2}$

iii) $\frac{dy}{dx} = \frac{4}{5(2x-1)^2} - \frac{6}{5(x-3)^2}$

iv) $\frac{dy}{dx} = -\frac{7}{5(x+2)^2} - \frac{2}{5(2x-1)^2}$

[Work out the solution in each case]

Differentiation of function of a function – the chain rule

If y is a function of u and u is a function of x, then y is called a **function of a function** of x.

This can be differentiated using the **chain rule**:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Functions such as $(x+3)^5$ are difficult to differentiate since it leads to tedious multiplication.

The formula above allows us to deal with such functions. The rule is to differentiate the function as we do for x^n and then multiply the result by the derivative of the expression in the bracket.

Thus if $y = (ax+b)^n$, then

$$\frac{dy}{dx} = n(ax+b)^{n-1} \cdot \frac{d}{dx}[ax+b]$$

This fact is needed very often and is quotable.

Derivatives of functions such as $\sin^3 x$, $\cos(2x^2+1)$, $(x^2+1)^{\frac{1}{2}}$ etc are obtained using the chain rule.

Illustrative examples

Example 1

Differentiate with respect to x.

i) $y = (3x+4)^5$ ii) $y = (2x^2 - 5x)^2$
iii) $y = (2x+3)^5$ iv) $y = (3x+1)^7$
v) $y = (x^2+2)^3$ vi) $y = (3x^4 - 5)^7$

Solution

i) $y=(3x+4)^5$.

Let $u=(3x+4) \quad \Rightarrow \quad y=u^5$

$$\frac{dy}{du}=5u^4 \text{ and } \frac{du}{dx}=3$$

$$\text{Hence } \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=5u^4(3)=15u^4$$

$$=15(3x+4)^4$$

ii) $y=(2x^2-5x)^2$

Let $u=(2x^2-5x)$

$$y=u^2$$

$$\frac{dy}{du}=2u \text{ and } \frac{du}{dx}=4x-5$$

$$\text{Hence } \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=2u(4x-5)$$

$$=2(2x^2-5x)(4x-5)$$

$$=16x^3-60x^2+50x$$

iii) $y=(2x+3)^5$

$$\frac{dy}{dx}=(5).2\,(2x+3)^4=10(2x+3)^4$$

iv) $y=(3x+1)^7$

$$\frac{dy}{dx}=(7).3(3x+1)^6=21(3x+1)^6$$

v) $y=(x^2+2)^3$

$$\frac{dy}{dx}=(2x).3(x^2+2)^2=6x(x^2+2)^2$$

vi) $y=(3x^4-5)^7$

$$\frac{dy}{dx}=(7).12x^3(3x^4-5)^6=84x^3(3x^4-5)^6$$

Example 2

Use the chain rule to differentiate each function with respect to x.

i) $y=(3x+4)^2$ ii) $y=(x-3)^4$

iii) $y=(4x+5)^5$ iv) $y=(2x-5)^{-4}$

v) $y=(1-2x)^{-5}$ vi) $\dfrac{1}{(1-5x)^2}$

Solution

i) $\dfrac{dy}{dx}=6(3x+1)$

ii) $\dfrac{dy}{dx}=4(x-3)^3$

iii) $\dfrac{dy}{dx}=20(4x+5)^4$

iv) $\dfrac{dy}{dx}=-8(2x-5)^{-5}$

v) $\dfrac{dy}{dx}=10(1-2x)^{-6}$

vi) $\dfrac{dy}{dx}=-10(1-5x)^{-3}$

Work out the missing steps

Higher derivatives

We have seen that if y is a function of x, then $\dfrac{dy}{dx}$ will also be a function of x. The result of differentiating $\dfrac{dy}{dx}$ with respect to x is called the second derivative of y with respect to x and is denoted by $\dfrac{d^2y}{dx^2}$. If the second derivative is also differentiated with respect to x, the third derivative $\dfrac{d^3y}{dx^3}$ is obtained, and so on.

Note: $\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left[\dfrac{dy}{dx}\right]$.

Also if f is a function of x then $f'(x)$ is the first derivative;

$f''(x)$ is the second derivative;

$f'''(x)$ is the third derivative and so on.

Illustrative examples

Example 1

Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ when $y = 3x^4 + 2x^3 + \frac{2}{x^2}$.

Solution

$$y = 3x^4 + 2x^3 + \frac{2}{x^2}$$

$$\frac{dy}{dx} = 12x^3 + 6x^2 - \frac{4}{x^3}$$

$$\frac{d^2y}{dx^2} = 36x^2 + 12x + \frac{12}{x^4}$$

$$\frac{d^3y}{dx^3} = 72x + 12 - \frac{48}{x^5}$$

Example 2

Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ when $y = x^6 + 2x^3 - 3x^2 + 4x + 1$.

Solution

$$y = x^6 + 2x^3 - 3x^2 + 4x + 1$$

$$\frac{dy}{dx} = 6x^5 + 6x^2 - 6x + 4$$

$$\frac{d^2y}{dx^2} = 30x^4 + 12x - 6 \text{ and}$$

$$\frac{d^3y}{dx^3} = 120x^3 + 12$$

Example 3

Given that $y = x^4 - x^3 + 4x - 1$, find $\frac{d^2y}{dx^2}$ and state the set of values of x for which $\frac{d^2y}{dx^2}$ is zero.

Solution

$y = x^4 - x^3 + 4x - 1 \Rightarrow \frac{dy}{dx} = 4x^3 - 3x^2 + 4$ and $\frac{d^2y}{dx^2} = 12x^2 - 6x = 6x(2x - 1)$

Putting $\frac{d^2y}{dx^2} = 0 \Rightarrow 6x(2x - 1) = 0$

$$\{x : x = 0 \text{ or } x = \tfrac{1}{2}\}$$

Differentiation of implicit functions

An **implicit function** in two variables x and y say, is one in which neither variables can be easily expressed in terms of the other. For example, the equation $x^3 + y^3 = 3xy$ is an implicit function since y cannot be easily isolated on one side of the equation.

To differentiate such an implicit function:

1. Differentiate it term by term to give an equation in $\frac{dy}{dx}$, x and y;
2. Make $\frac{dy}{dx}$ the subject of the equation;
3. It is useful to remember that $\frac{d}{dx}[y] = \frac{dy}{dx}$; $\frac{d}{dx}[y^2] = 2y\frac{dy}{dx}$; $\frac{d}{dx}[2y^3] = 6y^2\frac{dy}{dx}$ or in general $\frac{d}{dx}[g(y)] = g'(y)\frac{dy}{dx}$, where $g'(y)$ is the derivative of $g(y)$ with respect to y.
4. Treat terms such as $3xy$, xy^2 etc. as products and use the product rule. Also treat terms such as $\frac{y}{x^2}$, $\frac{2y}{x}$ etc. as quotients and use the quotient rule.

Illustrative examples

Example 1

Find $\dfrac{dy}{dx}$, if

i) $x^2 + y^2 = 0$ ii) $x^3 + y^3 = 3xy$

iii) $x^2 - 3xy + 6y^2 = 15$.

Solution

i) $x^2 + y^2 = 0$.

Differentiating term by term gives;

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

ii) $x^3 + y^3 = 3xy$.

Differentiating term by term gives;

$$3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$$

Note: Treat $3xy$ as a product

$$3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 3x^2$$

$$(y^2 - x)\frac{dy}{dx} = y - x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

iii) $x^2 - 3xy + 6y^2 = 15$.

Differentiating term by term gives;

$$2x - 3y - 3x\frac{dy}{dx} + 12y\frac{dy}{dx} = 0$$

(**Note:** Treat $3xy$ as a product)

$$12y\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 2x$$

$$(12y - 3x)\frac{dy}{dx} = 3y - 2x$$

$$\frac{dy}{dx} = \frac{3y - 2x}{12y - 3x}$$

Example 2

If i) $x^2 + xy - y^2 + 1 = 0$

ii) $y + xy + y^2 = 2$

iii) $3x^2 - 7y^2 + 4xy - 8x = 0$,

find $\dfrac{dy}{dx}$.

Solution

i) $x^2 + xy - y^2 + 1 = 0$.

Differentiating term by term gives;

$$2x + y + x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

(**Note:** Treat xy as a product)

$$x\frac{dy}{dx} - 2y\frac{dy}{dx} = -2x - y$$

$$(x - 2y)\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x - y}{x - 2y} = \frac{2x + y}{2y - x}$$

ii) $y + xy + y^2 = 2$.

Differentiating term by term gives;

$$\frac{dy}{dx} + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

(**Note:** Treat xy as a product)

$$\frac{dy}{dx} + x\frac{dy}{dx} + 2y\frac{dy}{dx} = -y$$

$$(1 + x + 2y)\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{1 + x + 2y}$$

iii) $3x^2 - 7y^2 + 4xy - 8x = 0$.

Differentiating term by term gives;

$$6x - 14y\frac{dy}{dx} + 4y + 4x\frac{dy}{dx} - 8 = 0$$

(**Note:** Treat $4xy$ as a product)

$$-14y\frac{dy}{dx} + 4x\frac{dy}{dx} = -6x - 4y + 8$$

$$(-14y + 4x)\frac{dy}{dx} = -6x - 4y + 8$$

$$\frac{dy}{dx} = \frac{-6x - 4y + 8}{-14y + 4x} = \frac{4 - 3x - 2y}{2x - 7y}$$

Example 3

Given that $x^2 - 3xy + 2y^2 - 2x = 4$, find the value of $\frac{dy}{dx}$ at the point (1, –1).

Solution

$x^2 - 3xy + 2y^2 - 2x = 4$

Differentiating term by term gives

$$2x - 3y - 3x\frac{dy}{dx} + 4y\frac{dy}{dx} - 2 = 0$$

(**Note**: Treat $3xy$ as a product)

$$-3x\frac{dy}{dx} + 4y\frac{dy}{dx} = 3y - 2x + 2$$

$$(-3x + 4y)\frac{dy}{dx} = 3y - 2x + 2$$

$$\frac{dy}{dx} = \frac{3y - 2x + 2}{-3x + 4y}$$

At the point (1, –1), $\frac{dy}{dx} = \frac{3(-1) - 2(1) + 2}{-3(1) + 4(-1)}$

$$= \frac{-3}{-7} = \frac{3}{7}$$

Example 4

If $x^2 + y^2 = 2pxy$, where p is a constant, find $\frac{dy}{dx}$

Solution

$x^2 + y^2 = 2pxy.$

Differentiating term by term gives;

$$2x + 2y\frac{dy}{dx} = 2py + 2px\frac{dy}{dx}$$

(**Note**: Treat $2pxy$ as a product)

$$2y\frac{dy}{dx} - 2px\frac{dy}{dx} = 2py - 2x$$

$$(y - px)\frac{dy}{dx} = py - x$$

$$\frac{dy}{dx} = \frac{py - x}{y - px}$$

Example 5

If $y = \frac{5}{(1-x^2)^3}$, show that

$$(1 - x^2)\frac{dy}{dx} = 6xy$$

Solution

For such questions, first clear the fractions and differentiate the result implicitly.

i.e. $y = \frac{5}{(1-x^2)^3} \Rightarrow y(1-x^2)^3 = 5$

We now consider the LHS as a product and differentiate implicitly.

i.e. $(1-x^2)^3\frac{dy}{dx} + y(3)(-2x)(1-x^2)^2 = 0$

$$(1-x^2)^3\frac{dy}{dx} - 6xy(1-x^2)^2 = 0$$

$$(1-x^2)^3\frac{dy}{dx} = 6xy(1-x^2)^2$$

Dividing through by $(1-x^2)^2$, we have:

$(1-x^2)\frac{dy}{dx} = 6xy$ as required.

Note that in the differentiation of $(1-x^2)^3$, the chain rule was used.

Example 6

If $y = \frac{15}{\sqrt{(x^3+1)}}$, show that

$$2(x^3 + 1)\frac{dy}{dx} = -3x^2y$$

Solution

For such questions, first clear the fractions and differentiate the result implicitly.

i.e. $y=\dfrac{15}{\sqrt{(x^3+1}}$ $\quad\Rightarrow\quad$ $y(x^3+1)^{\frac{1}{2}}=15$

We now consider the LHS as a product and differentiate implicitly

i.e. $(x^3+1)^{\frac{1}{2}}\dfrac{dy}{dx}+y(\tfrac{1}{2})(3x^2)(x^3+1)^{-\frac{1}{2}}=0$

$$(x^3+1)^{\frac{1}{2}}\frac{dy}{dx}+\frac{3yx^2}{2(x^3+1)^{\frac{3}{2}}}=0$$

$$(x^3+1)^{\frac{1}{2}}\frac{dy}{dx}=-\frac{3yx^2}{2(x^3+1)^{\frac{3}{2}}}$$

Dividing through by $2(x^3+1)^{\frac{1}{2}}$, we have:

$2(x^3+1)\dfrac{dy}{dx}=-3x^2y$ as required.

Note that in the differentiation of $(x^3+1)^{\frac{1}{2}}$, the chain rule was used.

Example 7

If $y=\dfrac{ax+b}{x^2}$, show that

$$x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=0$$

Solution

For such questions, first clear the fractions and differentiate the result implicitly.

i.e. $y=\dfrac{ax+b}{x^2}$ $\quad\Rightarrow\quad$ $yx^2=ax+b$

We now consider the LHS as a product and differentiate implicitly

i.e. $x^2\dfrac{dy}{dx}+y(2x)=a$

$$x^2\frac{dy}{dx}+2xy=a$$

Differentiating again to get the second derivative, we have

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+2x\frac{dy}{dx}+2y=0$$

$x^2\dfrac{d^2y}{dx^2}+4x\dfrac{dy}{dx}+2y=0$ as required.

Example 8

If $(1+x^2)(1+y^2)-px^2=0$, where p is a constant, show that $xy(1+x^2)\dfrac{dy}{dx}=(1+y^2)$

Solution

$(1+x^2)(1+y^2)-px^2=0$ ....(1)

$$1+y^2+x^2+x^2y^2-px^2=0$$

$$2y\frac{dy}{dx}+2x+2xy^2+2x^2y\frac{dy}{dx}-2px=0$$

$$2y(1+x^2)\frac{dy}{dx}=2px-2x-2xy^2$$

$$y(1+x^2)\frac{dy}{dx}=px-x-xy^2$$

Multiplying through by x

$$xy(1+x^2)\frac{dy}{dx}=px^2-x^2(1+y^2)$$

From (1), $(1+x^2)(1+y^2)=px^2$

$$xy(1+x^2)\frac{dy}{dx}=(1+x^2)(1+y^2)-x^2(1+y^2)$$

$$xy(1+x^2)\frac{dy}{dx}=(1+y^2)(1+x^2-x^2)$$

$xy(1+x^2)\dfrac{dy}{dx}=(1+y^2)$ as required.

Try:

1. If $y=\dfrac{x}{x^2+5}$, show that

   $(x^2+5)\dfrac{d^2y}{dx^2}+4x\dfrac{dy}{dx}+2y=0$.

2. Given that $y=\dfrac{x}{x-1}$, show that

   $(x-1)\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}=0$

Differentiation of simple trigonometric functions

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. Also, if $y = \sin ax$, then $\frac{dy}{dx} = a\cos ax$
4. If $y = \cos ax$, then $\frac{dy}{dx} = -a\sin ax$

In general,

If $y = \sin f(x)$, then $\frac{dy}{dx} = f'(x)\cos f(x)$

For example,

$$\frac{d}{dx}[\sin(x^2 - 3x + 1)]$$
$$= (2x - 3)\cos(x^2 - 3x + 1)$$

If $y = \cos f(x)$, then $\frac{dy}{dx} = -f'(x)\sin f(x)$.

This is the chain rule.

For example,

$$\frac{d}{dx}[\sin(2x^3 + x^2 - 1)]$$
$$= -(6x^2 + 2x)\cos(2x^3 + x^2 - 1)$$

Illustrative example

Write down the derivatives of the following trigonometric functions

i) $\cos x + \sin x$

ii) $3 - \cos x$

iii) $3\cos(5x^2 + 2)$

Solution

i) $\frac{d}{dx}(\cos x + \sin x) = -\sin x + \cos x$

ii) $\frac{d}{dx}(3 - \cos x) = -(-\sin x) = \sin x$

iii) $\frac{d}{dx}[3\cos(5x^2 + 2)]$

$$= (3)(-10x)\sin(5x^2 + 2)$$
$$= -30x\sin(5x^2 + 2)$$

Using the chain rule

The function of function or chain rule can also be used to find the derivative of powers of simple trigonometric functions.

Note that by the chain rule:

$$\frac{d}{dx}\sin f(x) = f'(x)\cos f(x),$$

where $f'(x) = \frac{d}{dx}[f(x)]$

Eg. $\frac{d}{dx}\sin(3x^2 + 1) = 6x^2\cos(3x^2 + 1)$

Also, $\frac{d}{dx}\cos f(x) = -f'(x)\sin f(x)$,

Eg. $\frac{d}{dx}\cos(4x^3) = -12x^2\sin(4x^3)$

Illustrative example

If $y = \cos^3 x$, find $\frac{dy}{dx}$

Solution

$y = \cos^3 x = (\cos x)^3$

Let $u = \cos x \quad \Rightarrow \quad y = u^3$

$\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = -\sin x$

From the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 3u^2(-\sin x)$$
$$= -3\cos^2 x\sin x$$

Differentiation of powers of trigonometric functions

In general, if $y = \cos^n x$,

then $\frac{dy}{dx} = -n\cos x^{n-1} x \sin x$

Similarly, if $y = \sin^n x$,

then $\frac{dy}{dx} = n\sin^{n-1} x \cos x$

Illustrative examples

Example 1

If $y = 2\cos^2(x^2+1)$, find $\frac{dy}{dx}$

Solution

Let $u = \cos(x^2+1)$

$y = 2u^2$

$\frac{dy}{du} = 4u$ and $\frac{du}{dx} = -2x\sin(x^2+1)$

From the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 4u(-2x\sin(x^2+1)$$

$$= -8x\cos(x^2+1)\sin(x^2+1)$$

Example 2

Find $\frac{dy}{dx}$ if

i) $y = \sin^2 x$

ii) $y = \sin^2(x^2+1)$

Solution

i) $\frac{dy}{dx} = 2\sin x\cos x$

ii) $\frac{dy}{dx} = 4x\sin(x^2+1)\cos(x^2+1)$

[Work out the solution in each case]

The derivatives of other trigonometric functions

1. $\frac{d}{dx}(\tan x) = \sec^2 x$
2. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
3. $\frac{d}{dx}(\sec x) = \sec x\tan x$
4. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x\cot x$

Illustrative example

Find $\frac{dy}{dx}$ if i) $y = \tan^2 x$

ii) $y = -4\cos 2x$

iii) $y = 3\tan^4 x$

Solution

i) $\frac{dy}{dx} = 2\tan x\sec^2 x$

ii) $\frac{dy}{dx} = 8\sin 2x$

iii) $\frac{dy}{dx} = 36\tan^3 3x\sec^2 3x$

[Using the chain rule, work out the missing steps.]

Using the product and quotient rules

The product and quotient rules can also be used to differentiate trigonometric expressions such as:

$$y = x\sin 3x \text{ or } y = \frac{\sin x}{\sqrt{x}} \text{ or } y = x^3\cos 2x$$

For $y = x\sin 3x$; let $u = x \Rightarrow \frac{du}{dx} = 1$

and $v = \sin 3x \Rightarrow \frac{dv}{dx} = 3\cos 3x$

From the product rule,

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= \sin 3x(1) + x(3\cos 3x)$$
$$= \sin 3x + 3x\cos 3x$$

For $y = \frac{\sin x}{\sqrt{x}}$; let $u = \sin x$

$\frac{du}{dx} = \cos x$ and $v = \sqrt{x} = x^{\frac{1}{2}}$

$\frac{dv}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

From the quotient rule:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{\sqrt{x}\cos x - \sin x(\frac{1}{2}x^{-\frac{1}{2}})}{(\sqrt{x})^2}$$
$$= \frac{\sqrt{x}\cos x + \frac{1}{2}x^{-\frac{1}{2}}\sin x}{x}$$
$$= \frac{2x\cos x - \sin x}{2x\sqrt{x}}$$

Illustrative examples

Example 1

If i) $y = \cos x \sin^3 x$ ii) $y = 2x^3\cos 3x$, find $\frac{dy}{dx}$.

Solution

i) $y = \cos x \sin^3 x$;

Let $u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$ and

$v = \sin^3 x \Rightarrow \frac{dv}{dx} = 3\sin^2 x \cos x$

From the product rule:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= \sin^3 x(-\sin x) + \cos x(3\sin^2 x\cos x)$$
$$= -\sin^4 x + 3\sin^2 x\cos^2 x$$

ii) $y = 2x^3\cos 3x$;

Let $u = 2x^3 \Rightarrow \frac{du}{dx} = 6x^2$ and

$v = \cos 3x \Rightarrow \frac{dv}{dx} = -3\sin 3x$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= \cos 3x(6x^2) + 2x^3(-3\sin 3x)$$
$$= 6x^2\cos 3x - 6x^3\sin 3x$$
$$= 6x^2(\cos 3x - x\sin 3x)$$

Example 2

If $y = \cos x - \sin x$, find $\frac{d^2y}{dx^2}$

Solution

$y = \cos x - \sin x$

$\frac{dy}{dx} = -\sin x - \cos x$

$\frac{d^2y}{dx^2} = -\cos x + \sin x$

Note that $\frac{d}{dx}(\sin y) = \cos y\frac{dy}{dx}$ and $\frac{d}{dx}(\cos y) = -\sin y\frac{dy}{dx}$

Example 3

Differentiate the following

i) $\cos 3x$
ii) $\sin(2x-1)$
iii) $\sin^9 4x$
iv) $\cos(x^2 - x)$
v) $x^4\cos x$
vi) $2\sin^3 x$
vii) $\sin x \sin 2x$
viii) $\sin(2x^2 + 3)$
ix) $\frac{\cos 2x}{\sin x}$
x) $\frac{x^2}{\sin x}$
xi) $\frac{\tan x}{x}$
xii) $\sin x\cos 2x$

Solution

Work out the solution and compare answers.

Review exercise 19

1. Find

i) $\lim_{x \to -3} (x-1)$

ii) $\lim_{x \to -2} (3x^2 - 1)$

iii) $\lim_{x \to -1} (2x^2 - 3x + 1)$

iv) $\lim_{x \to 0} \left\{ \frac{\sqrt{x+h} - \sqrt{x}}{h} \right\}$

v) $\lim_{x \to -2} \left\{ \frac{x^2 - 4}{x+2} \right\}$

vi) $\lim_{x \to 3} \left\{ \frac{x^2 - 9}{x-3} \right\}$

vii) $\lim_{x \to 2} \left\{ \frac{x^2 - 3x + 2}{x^2 - 4} \right\}$

viii) $\lim_{x \to -1} \left\{ \frac{x^3 + 1}{x+1} \right\}$

ix) $\lim_{x \to 16} \left\{ \frac{16 - x}{4 - \sqrt{x}} \right\}$

x) $\lim_{x \to \infty} \left(\frac{3x - 1}{2x + 3} \right)$

xi) $\lim_{x \to \infty} \left(\frac{7x^2}{3x^2 - 4} \right)$

xii) $\lim_{x \to \infty} \left(\frac{x^3 + 2x + 3}{2x^3 + 2} \right)$

xiii) $\lim_{x \to \infty} \left(\frac{x - 1}{x + 1} \right)$

xiv) $\lim_{x \to \infty} \left(\frac{x^2 - 5x + 6}{2x^2 + 7x - 2} \right)$

xv) $\lim_{x \to \infty} \left(\frac{x^2 + 2}{x^2 + 1} \right)$

xvi) $\lim_{x \to 5} (2x^2 - 3x + 4)$

xvii) $\lim_{x \to -2} \left\{ \frac{x^3 + 2x^2 - 1}{5 - 3x} \right\}$

xviii) $\lim_{x \to 1} \left\{ \frac{x^2 - 1}{x - 1} \right\}$

xix) $\lim_{t \to 0} \left\{ \frac{\sqrt{t^2 + 9} - 3}{t^2} \right\}$

xx) $\lim_{x \to -3} \left\{ \frac{x^2 - x - 12}{x + 3} \right\}$

xxi) $\lim_{x \to 9} \left\{ \frac{x^2 - 81}{\sqrt{x} - 3} \right\}$

2. If $f(x) = 2x^3 + 1$, evaluate $\frac{f(x+h) - f(x)}{h}$ where $h \neq 0$. Use your result to find the derivative of $f(x)$ with respect to x.

3. Obtain from first principles, the derivative with respect to x of the function defined by

i) $y = x + \frac{1}{x}$ $x \neq 0$ ii) $y = x^2 - 3x + 2$

iii) $y = \frac{3}{x^2 - 4}$ iv) $y = 3x^2 + 5$

v) $y = x^2 + 5x - 3$ vi) $y = 4x^2$

vii) $y = 5x^3$ viii) $y = x^3 + \frac{1}{x^2}$

ix) $y = x^2 - 3x$

4. Determine from first principles the derivative of the following with respect to x

i) $y = 5x$

ii) $y = \frac{1}{x^2}$

iii) $y = 4x^2$

iv) $y = -2x^2 + 3x - 12$

v) $f(x) = 9x$

vi) $f(x)=\frac{2x}{3}$

vii) $f(x)=9x^2$

viii) $f(x)=7x^3$

ix) $f(x)=x^2+15x-4$

5. Find $\frac{dy}{dx}$ if

i) $y=\frac{2}{x^3}$ ii) $y=\frac{1}{\sqrt{x}}$

iii) $y=\frac{1}{x^4}$ iv) $y=3\sqrt{x}$

v) $y=\frac{5}{\sqrt[3]{x^4}}$ vi) $y=\frac{x^2+1}{x}$

6. Differentiate with respect to x

i) $3x^2+4x$

ii) $\frac{3}{x^2}+5x+6$

iii) $5x^2-4x+8$

iv) $4x^3-3x^2+\frac{5}{x^2}$

v) $\frac{1}{2}x^2-\frac{1}{2x^2}$

vi) $2\sin(3x-2)$

vii) $\sqrt{x}-\frac{1}{x}+\frac{1}{x^3}$

viii) $x^{\frac{3}{2}}-x^{\frac{2}{3}}+x^{-\frac{1}{3}}$

ix) $\sqrt{x}+\frac{1}{\sqrt{x}}$

x) $\cos^2 x$

xi) $2\cos^2(x^2+1)$

xii) $\frac{1}{\sin x}$

7. Differentiate the following expressions

i) $(x-2)^2(x^2+1)$

ii) $(x+1)(x^3-2)$

iii) $x\sin x$

8. Differentiate the following expressions with respect to x

i) $\frac{(x-3)^2}{x^2+1}$ ii) $\frac{x}{1+x^2}$

iii) $\frac{3x-1}{\sqrt{x^2+1}}$ iv) $\frac{2\cos 3x}{x^3}$

v) $-5\cos 3x$ vi) $\frac{x-1}{x+1}$

vii) $3x^3(x^2+4)^2$ viii) $\sqrt{x}(x+3)^2$

9. Differentiate the following:

i) $y=\sin 5x$ ii) $y=\cos(3x+\frac{\pi}{2})$

iii) $y=x^2\sin 3x$ iv) $y=\sqrt{\frac{x}{x+1}}$

v) $y=\frac{3x^2}{4x^3+1}$ vi) $y=\frac{1}{x^2}\sqrt{(1+x^3)}$

vii) $y=\frac{x}{\sqrt{x-1}}$ viii) $\frac{\sqrt{x+1}}{x}$

10. Differentiate the following with respect to x

i) $y=(1-8x)^{\frac{1}{3}}$ ii) $y=\sqrt{(1-4x)}$

iii) $y=(2x+3)^{\frac{1}{3}}$ iv) $y=(1-2x)^{-5}$

v) $y=(4-2x)^5$ vi) $y=\frac{1}{(1-5x)^2}$

vii) $y=\frac{1}{\sqrt{(3-x)}}$ viii) $y=(3x+1)^{-1}$

ix) $y=\frac{1}{x^2}+\frac{2}{x^3}$ x) $y=\sqrt{\frac{x-1}{x+1}}$

xi) $y=\frac{x^2-1}{x^3+1}$ xii) $y=\frac{\sin x}{1+\cos x}$

11. Find an expression for $\frac{dy}{dx}$ for the equation of the curve
$x^2 + xy + y^2 - x - y = 0$

12. Differentiate the following with respect to x

i) $x^2 + xy^2 + y^3 = 2$

ii) $x^2 + y^2 = 4$

iii) $x^2 + xy + y^2 = 0$

iv) $\sin x + \sin y = 1$

v) $x^2 + 3xy - y^2 = 3$

vi) $y^2(x+3) = 20$

13. Differentiate the following with respect to x

i) $\cos 6x$ ii) $\tan 2x$

iii) $\operatorname{cosec} x$ iv) $\cot x$

v) $\cos(2x-1)$

14. Obtain from first principle, the derivate with respect to x of the function defined by
$y = x + \frac{1}{x},\ x \neq 0$

15. If $y = \frac{1+2x}{1-2x}$, find $\frac{d^2y}{dx^2}$ in its simplest form.

16. If $y = \frac{x-3}{x+1}$, show that

$$(x+1)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + = 0$$

17. If $x^2 - y^2 = a^2$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y.

18. Express $y = \frac{2x^2}{(2x-1)(x+1)}$ in partial fractions. Hence or otherwise, find the value of $\frac{d^2y}{dx^2}$ when $x = 1$.

19. If $y = \sqrt{(5x^2+3)}$, show that

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 5$$

20. If $x^2y = a\cos nx$, show that

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (n^2x^2 + 2)y = 0$$

21. If $y = x^2 + y^2 = 2y$, find $\frac{dy}{dx}$ in terms of x and y. Prove that $\frac{d^2y}{dx^2} = \frac{1}{(1-y)^3}$.

22. If $y = x^2\cos 2x$, find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = \frac{\pi}{2}$.

23. Differentiate the following with respect to x.

i) $y = x^2\cos\frac{2}{3}x$

ii) $y = 2x\cos\frac{x}{2}$

iii) $y = x^2\sin 2x$

iv) $y = \tan\left(\frac{\pi}{5} - 3x\right)$

v) $y = \sin x\cos 3x$

vi) $y = \frac{x}{2-\cos x}$

24. If $y = \sin 2x$, find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

25. If $y = \frac{x-2}{2x-1}$, show that

$$(2x-1)\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 0$$

26. If $y = \frac{1}{1+x}$, show that

$$(1+x)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

27. If $y = x\sin 2x$, find the value of $\frac{dy}{dx}$ when $x = \frac{\pi}{2}$

28. If $y = \tan 2x$, show that $\frac{dy}{dx} = 2(1 + y^2)$

CHAPTER 20

APPLICATIONS OF DIFFERENTIATION

1. The gradient function

If y or f is a function of x, then the first derivative $\frac{dy}{dx}$ or $f'(x)$ is called the **gradient function.** The gradient of a curve at any point $P(x_1, y_1)$ is obtained by substituting the values of x_1 and y_1 into the expression for $\frac{dy}{dx}$. This is the same as the gradient of the tangent at that point.

Illustrative examples

Example 1

If $f(x) = (x^2 + 3)^2$, find the gradient of $f(x)$ at $x = \frac{1}{2}$.

Solution

If $f(x) = (x^2 + 3)^2$, then the gradient is given by; $f'(x) = (2).2x\,(x^2 + 3) = 4x(x^2 + 3)$
$= 4x^3 + 12x$

Therefore the gradient at $x = \frac{1}{2}$ is the value of $f'(x)$ at $x = \frac{1}{2}$

i.e. $f'(\frac{1}{2}) = 4(\frac{1}{2})^3 + 12(\frac{1}{2}) = \frac{1}{2} + 6 = 6\frac{1}{2}$.

Example 2

Find the gradient of the curve $y = (x-3)(x^2+2)$ at the point where $x = 1$.

Solution

$y = (x-3)(x^2+2) = x^3 - 3x^2 + 2x - 6$

Therefore $\frac{dy}{dx} = 3x^2 - 6x + 2$, when $x = 1$,

$\frac{dy}{dx} = 3(1)^2 - 6(1) + 2 = 3 - 6 + 2 = -1$

Therefore the gradient of $y = (x-3)(x^2+2)$ at the point where $x = 1$ is -1.

Example 3

Find the gradient of $x^2 + y^2 = 9$ at the point where $x = 1$.

Solution

$x^2 + y^2 = 9$.

Differentiating term by term gives;

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -\frac{x}{y}.$$

This is the gradient function.

At $x = 1$, $1^2 + y^2 = 9$

$y^2 = 8 \Rightarrow y = \pm 2\sqrt{2}$

Therefore the gradient at $x = 1$ is:

$$-\frac{1}{\pm 2\sqrt{2}} = \pm\tfrac{1}{4}\sqrt{2}.$$

Example 4

Find the gradient of the given curve at the given point on the curve

i) $y = x^2 - 3$ at the point where $x = 1$

ii) $y = 3x^2 - 2$ at the point where $x = 3$

iii) $y = \sqrt{x}$ at the point where $x = 2$

iv) $y = (x-3)(x+4)$ at the point where $x = -1$

v) $y = \frac{1}{x}$ at the point where $x = 3$

vi) $y=(2x-3)(x+1)$ at the point where $x=0$

vii) $y=1-\frac{1}{x}$ at the point where $x=3$

viii) $y=x^3+7x-4$ at the point where $x=-3$

ix) $y=\frac{3x-1}{x}$ at the point where $x=\frac{1}{2}$

x) $y=x^2+\frac{3}{x}$ at the point where $x=-1$

Solution

i) 2 ii) 18 iii) $\frac{\sqrt{2}}{4}$ iv) -1

v) $-\frac{1}{9}$ vi) -1 vii) $\frac{1}{9}$ viii) 34

ix) 4 x) -5

[Work out the solution in each case]

Example 5

Find the coordinates of the point on the curve $y=\frac{2}{x^2}$ at which its gradient is $\frac{1}{2}$.

Solution

$$y=\frac{2}{x^2}=2x^{-2} \quad\Rightarrow\quad \frac{dy}{dx}=-4x^{-3}$$

$$\text{When } \frac{dy}{dx}=\tfrac{1}{2} \quad\Rightarrow\quad -4x^{-3}=\tfrac{1}{2}$$

$$\frac{-4}{x^3}=\tfrac{1}{2} \quad\Rightarrow\quad x^3=-8$$

$$x=-2$$

When $x=-2$, $y=\frac{2}{(-2)^2}=\frac{2}{4}=\frac{1}{2}$

Therefore the gradient of $y=\frac{2}{x^2}$ is $\frac{1}{2}$ at the point $(-2, \frac{1}{2})$.

Example 6

Find the values of x for which the gradient function of the curve $y=\frac{1}{4}x^4+\frac{2}{3}x^3-\frac{1}{2}x^2-2x+1$ is zero.

Solution

Gradient function $=0 \Rightarrow \frac{dy}{dx}=0$

$$y=\tfrac{1}{4}x^4+\tfrac{2}{3}x^3-\tfrac{1}{2}x^2-2x+1$$

$$\frac{dy}{dx}=x^3+2x^2-x-2=0$$

When $x=-1$, $\frac{dy}{dx}=0 \Rightarrow (x+1)$ is a factor of $x^3+2x^2-x-2=0$. Using the long division method to find the other factors, we have:

$$\begin{array}{r} x^2+x-2 \\ x+1\,\overline{)\,x^3+2x^2-x-2} \\ \underline{x^3+x^2} \\ x^2-x-2 \\ \underline{x^2+x} \\ -2x-2 \\ \underline{-2x-2} \\ 0 \end{array}$$

$$(x+1)(x^2+x-2)=0$$

But $x^2+x-2=(x-1)(x+2)$

$$(x+1)(x-1)(x+2)=0$$

$$x=-1 \text{ or } x=1 \text{ or } x=-2$$

$\therefore$ The values of x for which the gradient function of the curve is zero are $x=-2$, $x=-1$ and $x=1$

Example 7

Find the coordinates of the point(s) on the given curve at which its gradient has the given value.

i) $y=x^2-x+3$, gradient $=1$

ii) $y=5+3x-2x^2$, gradient $=-3$

iii) $y=(x-1)(x+1)$, gradient $=2$

iv) $y=(x+3)(x-5)$, gradient $=0$

v) $y=x^3+x^2$, gradient $=1$

vi) $y=\frac{1}{3}x^3+3x^2+7x+1$, gradient $=-1$

vii) $y=x+\dfrac{1}{x}$, gradient $=2$

viii) $y=\sqrt{x}$, gradient $=2$

ix) $y=1-\dfrac{1}{x}$, gradient $=4$

x) $y=\dfrac{1}{x^2}$, gradient $=\frac{1}{4}$

Solution

i) $(1, 3)$ ii) $(\frac{3}{2}, 5)$

iii) $(1, 0)$ iv) $(1, -16)$

v) $(-1, 0)$, $(\frac{1}{3}, \frac{4}{27})$ vi) $(-4, -\frac{1}{3})$, $(-2, -3\frac{2}{3})$

vii) no points viii) $(\frac{1}{16}, \frac{1}{4})$

ix) $(\frac{1}{2}, -1)$, $(-\frac{1}{2}, 3)$ x) $(-2, \frac{1}{4})$

[Work out the solution in each case]

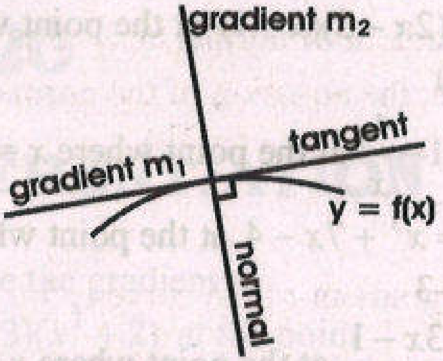
2. Tangent and normal to a curve

1. If y is a function of x, then the gradient of the **tangent** to the curve at any point $P(x_1, y_1)$ is the value of $\dfrac{dy}{dx}$ at that point.

The gradient of a tangent to the curve at any point $P(x_1, y_1)$ is obtained by substituting the values of x_1 and y_1 into the expression for $\dfrac{dy}{dx}$. Hence the **equation of the tangent** is given by: $y-y_1=m(x-x_1)$,

Where the gradient, $m=\dfrac{dy}{dx}$ at $P(x_1, y_1)$.

2. The **normal** to a curve at any point $P(x_1, y_1)$ is the straight line that is perpendicular to the tangent at that point. Therefore the gradient of the normal is $-\frac{1}{m}$. Hence the **equation of the normal** is given by: $y-y_1=-\frac{1}{m}(x-x_1)$.



Illustrative examples

Example 1

Given that $y=x^3-4x^2+5x-2$, find $\dfrac{dy}{dx}$.

P is the point on the curve where $x=3$.

i) Calculate the y-coordinate of P.

ii) Calculate the gradient at P.

iii) Find the equation of the tangent at P.

iv) Find the equation of the normal at P.

v) Find the values of x for which the curve has gradient of 5.

Solution

If $y=x^3-4x^2+5x-2$ then

$$\frac{dy}{dx}=3x^2-8x+5$$

i) At P, $x=3$

$$y=3^3-4(3)^2+5(3)-2=4.$$

Therefore the y-coordinate of P is 4.

ii) The gradient at P is given by $\dfrac{dy}{dx}$ at $x=3$ i.e. $\dfrac{dy}{dx}=3(3)^2-8(3)+5$

$$=27-24+5=8$$

iii) Gradient of tangent at P is 8, from (ii) above. Therefore, the equation of the tangent at $P(3, 4)$ is $y-4=8(x-3)$

$$y=8x-20.$$

iv) Gradient of normal at P is the negative reciprocal of gradient of tangent at P.

Hence gradient of normal at $x = 3$ is $-\frac{1}{8}$.
Therefore, the equation of the normal at $P(3, 4)$ is $y - 4 = -\frac{1}{8}(x - 3)$

$$8y + x = 35.$$

v) The gradient of the curve is 5

$$\frac{dy}{dx} = 5 \quad \Rightarrow \quad 3x^2 - 8x + 5 = 5$$

$$3x^2 - 8x = 0 \quad \Rightarrow \quad x(3x - 8) = 0$$

$$x = 0 \text{ and } 3x - 8 = 0$$

$$x = 0 \text{ or } x = \frac{8}{3}$$

Example 2

Find the equation of the tangent and the normal to the curve $y = x^3 - 3x + 2$ at the point $(2, 4)$.

Solution

If $y = x^3 - 3x + 2$ then $\frac{dy}{dx} = 3x^2 - 3$.

Note that $\frac{dy}{dx}$ gives the gradient of the tangent at any point on the curve (i.e. the gradient function).
At $(2, 4)$, $x = 2$, hence the gradient of the tangent is $3(2)^2 - 3 = 12 - 3 = 9$.
Therefore, the equation of the tangent with gradient 9, passing through $(2, 4)$ is;
$y - 4 = 9(x - 2) \quad \Rightarrow \quad y = 9x - 14$.
Since the gradient of the tangent at $(2, 4)$ is 9, the gradient of the normal is $-\frac{1}{9}$.
Therefore the equation of the normal to the curve at $(2, 4)$ is given by;

$$y - 4 = -\frac{1}{9}(x - 2)$$

$$9y + x = 38$$

Example 3

Find the coordinates of the points on the curve $y = x^3 - 4x^2 - 3x - 2$ where the tangent is parallel to the x-axis.

Solution

A line parallel to the x-axis has zero gradient. Therefore if the tangent is parallel to the x-axis then, $\frac{dy}{dx}$ must be zero at the point of contact.

$$y = x^3 - 4x^2 - 3x - 2$$

$$\frac{dy}{dx} = 3x^2 - 8x - 3 = 0$$

$$(3x + 1)(x - 3) = 0 \quad x = -\frac{1}{3} \text{ or } 3$$

Therefore at $x = -\frac{1}{3}$ and 3, the tangent is parallel to the x-axis i.e. the gradient of the curve is zero. To get the corresponding y values, substitute these values into the expression for y
i.e. at $x = 3$, $y = (3)^3 - 4(3)^2 - 3(3) - 2 = -20$
and at $x = -\frac{1}{3}$,

$$y = (-\tfrac{1}{3})^3 - 4(-\tfrac{1}{3})^2 - 3(-\tfrac{1}{3}) - 2 = -\frac{64}{27}.$$

$\therefore$ The required points are $(3, -20)$ and $(-\frac{1}{3}, -\frac{64}{27})$.

Example 4

The gradient of a curve at any point $P(x, y)$ is given by $4x^2 - 3x + 1$.
If the curve passes through the point $A(1, 0)$, find the equation of the tangent to the curve at the point A.

Solution

The gradient of a curve at any point $P(x, y)$ is given by $\frac{dy}{dx} = 4x^2 - 3x + 1$
At $A(1, 0)$, $x = 1$, hence the gradient of the tangent is $4(1)^2 - 3(1) + 1 = 4 - 3 + 1 = 2$
Therefore, the equation of the tangent with gradient 2, passing through $A(1, 0)$ is;

$$y - 0 = 2(x - 1)$$

$$y = 2x - 2.$$

Example 5
Given that $y = (x-2)(2x^2-3x-5)$, find $\frac{dy}{dx}$. Find the gradient of the tangents to the curve, at where it cuts the x – axis.

Solution
$y = (x-2)(2x^2-3x-5)$ is a product of $(x-2)$ and $(2x^2-3x-5)$
Let $u = (x-2)$ and $v = (2x^2-3x-5)$

$$\frac{du}{dx} = 1 \text{ and } \frac{dv}{dx} = 4x-3$$

Using the product rule: $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$$\frac{dy}{dx} = (2x^2-3x-5).1 + (x-2).(4x-3)$$
$$= (2x^2-3x-5) + (4x^2-3x-8x+6)$$
$$= 6x^2-14x+1$$

Note that $\frac{dy}{dx}$ gives the gradient of the tangent at any point on the curve.
At the point where it cuts the x – axis, $y = 0$

$$(x-2)(2x^2-3x-5) = 0$$
$$(x-2)(x+1)(2x-5) = 0$$
$$x = -1, 2 \text{ or } \tfrac{5}{2}$$

At $x = -1$, $\frac{dy}{dx} = 6(-1)^2 - 14(-1) + 1 = 21$

At $x = 2$, $\frac{dy}{dx} = 6(2)^2 - 14(2) + 1 = -3$

At $x = \frac{5}{2}$, $\frac{dy}{dx} = 6(\frac{5}{2})^2 - 14(\frac{5}{2}) + 1 = \frac{7}{2}$

Hence the gradients of the tangents to the curve, at where it cuts the x–axis are -3, $\frac{5}{2}$ and 21.

Example 6
Find the gradient and the equations of the tangent and normal to $y = 5x^3 - 7x^2 + 3x + 2$ at (1, 3).

Solution
If $y = 5x^3 - 7x^2 + 3x + 2$
then $\frac{dy}{dx} = 15x^2 - 14x + 3$

Note that $\frac{dy}{dx}$ gives the gradient of the tangent at any point on the curve.
At (1, 3), $x = 1$, hence the gradient of the tangent is $15(1)^2 - 14(1) + 3 = 4$.
Therefore, the equation of the tangent with gradient 4, passing through (1, 3) is;
$y - 3 = 4(x-1) \Rightarrow y = 4x - 1$.
Since the gradient of the tangent at (1, 3) is 4, the gradient of the normal is $-\frac{1}{4}$.
Therefore, the equation of the normal to the curve at (1, 3) is given by;
$y - 3 = -\frac{1}{4}(x-1) \Rightarrow 4y + x = 13$

Example 7
Find the gradient and the equations of the tangent and normal to $y = 4x^2 - 12x + 3$ at (2, 1).

Solution
If $y = 4x^2 - 12x + 3$ then $\frac{dy}{dx} = 8x - 12$

Note that $\frac{dy}{dx}$ gives the gradient of the tangent at any point on the curve.
At (2, 1), $x = 2$, hence the gradient of the tangent is $8(2) - 12 = 4$.
Therefore, the equation of the tangent with gradient 4, passing through (2, 1) is;
$y - 1 = 4(x-2) \Rightarrow y = 4x - 7$.
Since the gradient of the tangent at (2, 1) is 4, the gradient of the normal is $-\frac{1}{4}$.
Therefore, the equation of the normal to the curve at (2, 1) is given by;
$y - 1 = -\frac{1}{4}(x-2)$
$4y + x = 6$

Example 8

Find the equation of the tangent to the curve $y = x^3 - 2x + 1$ at $(-1, 2)$.

Solution

If $y = x^3 - 2x + 1$, then $\frac{dy}{dx} = 3x^2 - 2$.

Note that $\frac{dy}{dx}$ gives the gradient of the tangent at any point on the curve.

At $(-1, 2)$, $x = -1$, hence the gradient of the tangent is $3(-1)^2 - 2 = 1$.

Therefore, the equation of the tangent with gradient 1, passing through $(-1, 2)$ is;

$$y - 2 = 1[x - (-1)]$$

$$y - 2 = x + 1 \Rightarrow \quad y = x + 3$$

Example 9

Find the equation of the tangent to the curve $y = 6 - x - x^4$ at the point where $x = 2$.

Solution

If $y = 6 - x - x^4$, then $\frac{dy}{dx} = -1 - 4x^3$

Note that $\frac{dy}{dx}$ gives the gradient of the tangent at any point on the curve.

At $x = 2$, the gradient of the tangent is: $-1 - 4(2)^3 = -33$.

When $x = 2$, $y = 6 - (2) - (2)^4 = -12$

i.e. finding the y coordinate of the point

Therefore, the equation of the tangent with gradient -33, passing through $(2, -12)$ is;

$$y - (-12) = -33(x - 2)$$

$$y + 12 = -33x + 66$$

$$y + 33x = 54$$

Example 10

Given that $y = x^3 - 4x^2 + 5x - 2$ and P is the point on the curve where $x = 3$, calculate the

i) gradient at P;

ii) equation of the tangent at P;

iii) equation of the normal at P;

iv) values of x for which the gradient of the tangent to the curve is 5.

Solution

i) If $y = x^3 - 4x^2 + 5x - 2$

then $\frac{dy}{dx} = 3x^2 - 8x + 5$

When $x = 3$

$$\frac{dy}{dx} = 27 - 24 + 5 = 8$$

$\therefore$ Gradient at $P = 8$

ii) $x = 3 \Rightarrow y = 27 - 36 + 15 - 2 = 4$

$\therefore \; P = (3, 4)$

Gradient of the tangent = 8

Therefore equation of tangent at P is

$$y - 4 = 8(x - 3)$$

$$y - 8x + 20 = 0$$

iii) Gradient of normal $= -\frac{1}{8}$

Therefore equation of normal at P is

$$y - 4 = -\tfrac{1}{8}(x - 3)$$

$$8y + x - 35 = 0$$

iv) Gradient $= \frac{dy}{dx} = 3x^2 - 8x + 5 = 5$

$$x(3x - 8) = 0$$

$$x = 0 \text{ or } 2\tfrac{2}{3}$$

Example 11

The tangent to the curve $y = x^3 + bx$, where b is a constant, at $x = 1$ passes through the points $A(-1, 6)$ and $B(2, -15)$. Find the value of b.

Solution

$$y = x^3 + bx \quad \Rightarrow \quad \frac{dy}{dx} = 3x^2 + b$$

At $x = 1$, $\frac{dy}{dx} = 3 + b =$ Gradient of AB

Again, gradient of $AB = \dfrac{6+15}{-1-2} = -7$

$\therefore\ 3 + b = -7 \Rightarrow b = -10$

Example 12

a) Find the gradient of $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$ at the point $(3, \frac{8}{5})$.

b) The equation of a curve is $x^2y + 3xy + y + 3 = 0$

   i) Determine the gradient of the curve at point $P(-2, 3)$.

   ii) Find the equation of the normal to the curve at P.

Solution

a) $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1 \Rightarrow \dfrac{2x}{25} + \dfrac{2y}{4}\dfrac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{4x}{25y}$$

At the point $(3, \frac{8}{5})$

$$\frac{dy}{dx} = -\left(\frac{4}{25} \bullet \frac{3}{8}\right) \times 5 = -\frac{3}{10}$$

$\therefore$ Gradient at point $(3, \frac{8}{5})$ is $-\frac{3}{8}$

b) i) $x^2y + 3xy + y + 3 = 0$

$$2xy + x^2\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y + \frac{dy}{dx} = 0$$

$$(x^2 + 3x + 1)\frac{dy}{dx} = -2xy - 3y$$

$$\frac{dy}{dx} = -\frac{2xy + 3y}{x^2 + 3x + 1} = -\frac{y(2x+3)}{x^2 + 3x + 1}$$

ii) At $P(-2, 3)$

$$\frac{dy}{dx} = -\frac{3(-4+3)}{4 + (-6) + 1} = -3$$

Gradient of tangent $= \dfrac{dy}{dx} = -3$

Gradient of normal $= \frac{1}{3}$

Equation of the normal to the curve at P is

$y - 3 = \frac{1}{3}(x + 2) \quad \Rightarrow \quad 3y - x - 11 = 0$

Example 13

Find the equations of the tangent and normal to the given curve at the given point on the curve.

i) $y = x^2 - 2$ at the point where $x = 1$

ii) $y = x^2 + 3x - 1$ at the point where $x = 0$

iii) $y = \dfrac{1}{x}$ at the point where $x = -1$

iv) $y = (x-2)(x^2+1)$ at the point where $x = -1$

v) $y = x^2 - 5x + 2$ at the point where $x = 3$

vi) $y = x^2 - 2$ at the point where $x = 0$

vii) $y = x^2 - 5x + 7$ at the point where $x = 2$

viii) $y = (1 - 2x)^4$ at the point where $x = 1$

Solution

i) tangent: $y - 2x + 3 = 0$
normal: $x + 2y + 1 = 0$

ii) tangent: $3x - y - 1 = 0$
normal: $x + 3y + 3 = 0$

iii) tangent: $x + y + 2 = 0$
normal: $x - y = 0$

iv) tangent: $8x - y + 1 = 0$
normal: $x + 8y + 49 = 0$

v) tangent: $x - y - 7 = 0$
normal: $x + y + 1 = 0$

vi) tangent: $y = 2$
normal: $x = 0$

vii) tangent: $x + y - 3 = 0$
normal: $x - y - 1 = 0$

viii) tangent: $8x - y - 7 = 0$
normal: $x + 8y - 9 = 0$

[Work out the solution in each case]

3. Rates of change

Suppose we have some quantity y which depends on and varies with a quantity x, then the **rate of change** of y with respect to x is given by $\frac{dy}{dx}$. Thus, for example, the rate of change of pressure p with respect to height, h is $\frac{dp}{dh}$.

Rates of change are often, but not always with respect to time. However, by convention "rate of change of a quantity S" means "the rate of change of S with respect to time" i.e. $\frac{dS}{dt}$. For a sphere of volume V, and radius r:

$\frac{dV}{dt}$ is the rate of change of volume V with respect to time.

$\frac{dV}{dr}$ is the rate of change of volume V with respect to radius.

$\frac{dr}{dt}$ is the rate of change of radius r with respect to time.

If the quantity y is **increasing** with respect to x then $\frac{dy}{dx}$ is **positive** and

If y is **decreasing** with respect to x then $\frac{dy}{dx}$ is **negative**.

Rates of change can be related by the **chain rule:** $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Thus the chain rule enables us to find the rate of change of y with respect to x, if y is a function of u and the rate of change of u with respect to x is known.

Illustrative examples

Example 1

A circular ink blot has a radius of 2 cm. At what rate is the area increasing with respect to the radius.

Solution

Let r cm be the radius of the ink blot and A cm^2, the area, then

$A = \pi r^2$ since ink blot is circular.

The rate of increase of A with respect to r is $\frac{dA}{dr} = 2\pi r$. If $r = 2$ cm then

$\frac{dA}{dr} = 2\pi(2) = 4\pi$ cm i.e. the rate at which the area is increasing with respect to the radius is 4π cm.

Example 2

A spherical balloon is being inflated and the volume is increasing at the rate of 15cm^3/sec. At what rate is the radius increasing when it is 10 cm?

Solution

The volume of the spherical balloon,

$V = \frac{4}{3}\pi r^3$, where r is the radius

$$\frac{dV}{dr} = 4\pi r^2$$

The rate of increase of volume, $\frac{dV}{dt} = 15$ cm^3/sec, since the volume is *increasing*.

The rate of increase of the radius is $\frac{dr}{dt}$

Using the chain rule,

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{1}{4\pi r^2} \times 15$$

$$\frac{dr}{dt} = \frac{15}{4\pi r^2}$$

Note that $\frac{dr}{dV} = \frac{1}{4\pi r^2}$ [since $\frac{dV}{dr} = 4\pi r^2$]

Now when $r = 10$ cm,

$$\frac{dr}{dt} = \frac{15}{4\pi(10)^2} = \frac{15}{400\pi} = 0.012 \text{ cm/s}.$$

Example 3

A spherical balloon is blown up so that its volume increases at the rate of 2 cm^3/sec. Find the rate of increase of the radius when the volume of the balloon is 50 cm^3?

Solution

The volume of the spherical balloon,
$V = \frac{4}{3}\pi r^3$, where r is the radius

$$\frac{dV}{dr} = 4\pi r^2$$

We wish to find the rate of increase of radius, $\frac{dr}{dt}$,

We are give the rate of change of volume, $\frac{dV}{dt} = 2$ cm^3/sec, since the volume is *increasing* and we know that $\frac{dV}{dr} = 4\pi r^2$

Using the chain rule,

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{1}{4\pi r^2} \times 2 = \frac{2}{2\pi r^2}$$

Note that $\frac{dr}{dV} = \frac{1}{4\pi r^2}$ [since $\frac{dV}{dr} = 4\pi r^2$]

Now from $V = \frac{4}{3}\pi r^3$, when $V = 50$ cm^3, $r = 2.29$ cm,

$$\frac{dr}{dt} = \frac{1}{2\pi(2.29)^2} = 0.03 \text{ cm/s}.$$

So the radius is increasing at 0.03 cm/s when the volume is 50 cm^3

Example 4

A spherical bubble is decreasing in volume at a rate of 2 cm^3/s. Find the rate at which the surface area is diminishing when the radius is 3 cm.

Solution

The volume of the spherical bubble,
$V = \frac{4}{3}\pi r^3$, where r is the radius

$$\frac{dV}{dr} = 4\pi r^2$$

The surface area, $S = 4\pi r^2$

$$\frac{dS}{dr} = 8\pi r$$

Using the chain rule, $\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$

But $\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{1}{4\pi r^2} \times (-2) = -\frac{1}{2\pi r^2}$

Note: $\frac{dV}{dt} = -2$ cm^3/s, since the volume is *decreasing*.

$$\therefore \quad \frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt} = 8\pi r \times \left(-\frac{1}{2\pi r^2}\right) = -\frac{4}{r}$$

Now when $r = 3$ cm,

$$\frac{dS}{dt} = -\frac{4}{3} \text{ cm}^2\text{/s}.$$

Therefore the surface area is diminishing at a rate $\frac{4}{3}$ cm^2/s.

Alternative method

Using the chain rule, $\frac{dS}{dt} = \frac{dS}{dV} \times \frac{dV}{dt}$

$dS = 8\pi r dr$ and $dV = 4\pi r^2 dr$

$$\frac{dS}{dV} = \frac{8\pi r dr}{4\pi r^2 dr} = \frac{2}{r}$$

$$\therefore \quad \frac{dS}{dt} = \frac{dS}{dV} \times \frac{dV}{dt}$$

$$\frac{dS}{dt} = \frac{2}{r} \times (-2) = \frac{-4}{r}$$

When $r = 3$ cm,

$$\frac{dS}{dt} = -\frac{4}{3} \text{ cm}^2\text{/s}.$$

Example 5

A spherical balloon is inflated by a gas being pumped in at the constant rate of 200 cm^3/s. What is the rate of increase of the surface area of the balloon when the radius is 100 cm?

Solution

The volume of the spherical balloon,

$V = \frac{4}{3}\pi r^3$, where r is the radius

$$\frac{dV}{dr} = 4\pi r^2$$

The surface area, $S = 4\pi r^2$

$$\frac{dS}{dr} = 8\pi r$$

Using the chain rule, $\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$

But $\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{1}{4\pi r^2} \times (200) = \frac{50}{\pi r^2}$

(Note that $\frac{dV}{dt}$ = 200 cm^3/s).

$$\therefore \quad \frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt} = 8\pi r \times \left(\frac{50}{\pi r^2}\right) = \frac{400}{r}$$

Now when r = 100 cm,

$$\frac{dS}{dt} = \frac{400}{100} = 4\,\text{cm}^2/\text{s}.$$

Therefore, the surface area is increasing at a rate 4 cm^2/s

Alternative method

Using the chain rule, $\frac{dS}{dt} = \frac{dS}{dV} \times \frac{dV}{dt}$

$dS = 8\pi r dr$ and $dV = 4\pi r^2 dr$

$$\frac{dS}{dV} = \frac{8\pi r dr}{4\pi r^2 dr} = \frac{2}{r}$$

$$\therefore \quad \frac{dS}{dt} = \frac{dS}{dV} \times \frac{dV}{dt}$$

$$\frac{dS}{dt} = \frac{2}{r} \times (200) = \frac{400}{r}$$

When r = 100 cm, $\frac{dS}{dt} = \frac{400}{100} = 4$ cm^2/s.

Example 6

An elastic spherical balloon is being blown up so that the radius is increasing at a rate of 1 cm per second. Calculate the rate at which the volume of the balloon is increasing when the radius is 5 cm.

Solution

The volume of the spherical balloon,

$V = \frac{4}{3}\pi r^3$, where r is the radius

$$\frac{dV}{dr} = 4\pi r^2$$

The rate of increase of radius,

$\frac{dr}{dt} = 1$ cm/sec, since the radius is *increasing*.

The rate of increase of the volume is $\frac{dV}{dt}$

Using the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times 1 = 4\pi r^2$$

Now when r = cm,

$$\frac{dV}{dt} = 4\pi(5)^2 = 100\pi\ \text{cm}^3/\text{s}.$$

Example 7

The side of a cube is increasing at the rate of 6 cm/s. Find the rate of increase of the volume when the length of the side is 9 cm.

Solution

The volume of a cube, $V = x^3$, where x is the side $\Rightarrow \quad \frac{dV}{dx} = 3x^2$

The rate of increase of side, $\frac{dx}{dt} = 6$ cm/s, since the side is *increasing*.

The rate of increase of the volume is $\frac{dV}{dt}$.

Using the chain rule, $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$$= 3x^2 \times 6 = 18x^2$$

Now when $x = 9$ cm,

$$\frac{dV}{dt} = 18(9)^2 = 1458\text{ cm}^3/\text{s}.$$

Example 8

The volume of a cube is increasing at the rate of $2\frac{1}{2}\text{ cm}^3\text{s}^{-1}$. Find the rate of change of the side of the base when its length is 2 cm.

Solution

Let x = side of the cube

Volume, $V = x^3 \Rightarrow \frac{dV}{dx} = 3x^2$

$$\frac{dl}{dV} = \frac{1}{3x^2}$$

$$\frac{dx}{dt} = \frac{dV}{dt} \times \frac{dx}{dV} = 2\tfrac{1}{2} \times \frac{1}{3x^2}$$

When $x = 2$ cm

$$\frac{dx}{dt} = \frac{5}{2} \times \frac{1}{3 \times 2^2} = \frac{5}{24}\text{ cms}^{-1}\text{ or } 0.21\text{ cms}^{-1}$$

Example 9

If $y = (x^2 + 3x)^2$ and $\frac{dx}{dt} = 2$,

find $\frac{dy}{dt}$, when $x = 2$.

Solution

$$y = (x^2 + 3x)^2$$

$$\frac{dy}{dx} = 2(x^2 + 3x)(2x + 3)$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= 2(2x + 3)(x^2 + 3x) \times 2$$

$$= 4(2x + 3)(x^2 + 3x)$$

At $x = 2$,

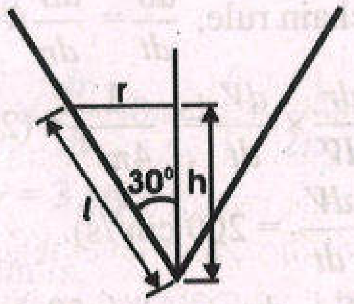
$$\frac{dy}{dt} = 4(4 + 3)(4 + 6)$$

$$= 4(7)(10) = 280$$

Example 10

A vessel containing water is in the form of an inverted cone with semi-vertical angle of 30°. There is a small hole at the vertex of the cone and the water is running out at a rate of 3 cm³/s. Find the rate at which the surface area in contact with water is changing when there are 81π cm³ of water remaining in the cone.

Solution



At any time the volume of water in the vessel is, $V = \frac{1}{3}\pi r^2 h$

r and h are variables so we have to find an expression for one in terms of the other.

From the diagram, $\tan 30° = \frac{r}{h}$

$$r = h\tan 30° = \frac{h}{\sqrt{3}}$$

Therefore, $V = \frac{\pi h^3}{9} \Rightarrow \frac{dV}{dh} = \frac{\pi h^2}{3}$

The wet surface area, $S = \pi r l$ and from the diagram $\sin 30° = \frac{r}{l} \Rightarrow l = \frac{h}{\sin 30°} = 2h$

Therefore, $S = \pi r l = \pi\left(\frac{h}{\sqrt{3}}\right)2h = \frac{2\pi h^2}{\sqrt{3}}$

$$\frac{dS}{dh} = \frac{4\pi h}{\sqrt{3}}$$

Now $\frac{dV}{dt} = -3$ and $\frac{dS}{dt}$ is required.

Using the chain rule, $\frac{dS}{dt} = \frac{dS}{dh} \times \frac{dh}{dV} \times \frac{dV}{dt}$

$$\frac{dS}{dt} = \left(\frac{4\pi h}{\sqrt{3}}\right)\left(\frac{3}{\pi h^3}\right)(-3) = -\frac{12\sqrt{3}}{h^2}$$

Now when $V = 81\pi$ cm^3,

$$\frac{\pi h^3}{9} = 81\pi \quad \Rightarrow \quad h = 9$$

Therefore $\frac{dS}{dt} = -\frac{12\sqrt{3}}{9^2} = -\frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$

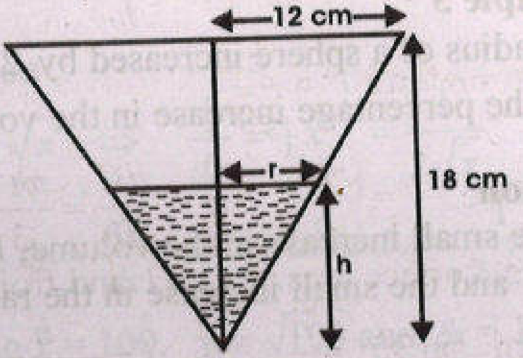
Hence the surface area is decreasing at $\frac{4\sqrt{3}}{3}$ cm^2/s.

Example 11

A hollow right circular cone of height 18 cm and radius 12 cm is held beneath a tap that is leaking at the rate of 2 cm^{3}s^{-1}. Find the rate of rise of the water level when the depth is 6 cm. Leave your answer in terms of π.

Solution

At any time the volume of water in the vessel is $V = \frac{1}{3}\pi r^2 h$. Since r and h are variables, we have to find an expression for one in terms of the other.



From the diagram, using similar triangles

$$\frac{r}{h} = \frac{12}{18} = \frac{2}{3} \quad \Rightarrow \quad r = \tfrac{2}{3}h$$

Therefore, $V = \frac{1}{3}\pi(\frac{2}{3}h)^2 h = \frac{4}{27}\pi h^3$

$$\frac{dV}{dh} = \tfrac{4}{9}\pi h^2$$

Now $\frac{dV}{dt} = 2$ and $\frac{dh}{dt}$ is required.

Using the chain rule, $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$$\frac{dh}{dt} = \left(\frac{9}{4\pi h^2}\right)(2) = \frac{9}{2\pi h^2}$$

Now when the depth, $h = 6$ cm,

$$\frac{dh}{dt} = \frac{9}{2\pi(6)^2} = \frac{9}{72\pi} = \frac{1}{8\pi}$$

Therefore the rate of rise of the water level when the depth is 6 cm is $\frac{1}{8\pi}$ cms^{-1}.

4. Small changes

Since $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$, then $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$ when δx is small.

Therefore, $\delta y \approx \frac{dy}{dx} \times \delta x$

This approximation can be used to estimate the **small change** δy in y, if $\frac{dy}{dx}$ can be found and the small change δx in x is given. It can also be used to estimate **percentage changes.**

In general, if x is increased by p %, then

$$\delta x = \frac{p}{100} \times x \quad \text{and}$$

The approximate percentage increase in y is $\frac{\delta y}{y} \times 100$.

Illustrative examples

Example 1

The surface area of a sphere is $4\pi r^2$. If the radius of the sphere is increased from 10 cm to 10.1 cm, what is the approximate increase in the surface area?

Solution

Let the small increase in the surface area, A be δA and the small increase in radius, r be δr.

The surface area, $A = 4\pi r^2 \Rightarrow \dfrac{dA}{dr} = 8\pi r$

Using $\delta A \approx \dfrac{dA}{dr} \times \delta r \approx 8\pi r.\delta r$

When $r = 10$ cm and $\delta r = 10.1 - 10 = 0.1$ cm, we have; $\delta A \approx 8\pi(10)(0.1) = 25.14\text{ cm}^2$

Therefore the required increase in area is 25.14 cm^2.

Example 2

The radius r of a circle is 5 cm. Find the increase in the area A of the circle when the radius expands by 0.01 cm.

Solution

Let the small increase in the area, A be δA and the small increase in the radius, r be δr.

The surface area, $A = \pi r^2 \Rightarrow \dfrac{dA}{dr} = 2\pi r$

Using $\delta A \approx \dfrac{dA}{dr} \times \delta r \approx 2\pi r.\delta r$

When $r = 5$ cm and $\delta r = 0.01$ cm, we have; $\delta A \approx 2\pi(5)(0.01) = 0.3142\text{ cm}^2$

Therefore the required increase in area is 0.3142 cm^2.

Example 3

The side of a cube increases by 5 %. Find the corresponding percentage increase in the volume.

Solution

Let the small increase in the volume, V be δV and the small increase in the side, x be δx. The volume of the cube, $V = x^3$

$$\frac{dV}{dx} = 3x^2$$

Using $\delta V \approx \dfrac{dV}{dx} \times \delta x \approx 3x^2\delta x$

When $\delta x = 5\ \%$ of $x = 0.05x$, we have;

$\delta V \approx 3x^2(0.05x) = 0.15x^3$

The approximate percentage change in volume $= \dfrac{\delta V}{V} \times 100 = \dfrac{0.15x^3}{x^3} \times 100 = 15\%$

Example 4

Find the approximate value for the square root of 16.01.

Solution

Let $x = 16$ and $\delta x = 0.01$. Let $y = \sqrt{x}$.

Let the small increase in y be δy and the small increase in x be δx.

If $y = \sqrt{x} \quad \Rightarrow \quad \dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

Using $\delta y \approx \dfrac{dy}{dx} \times \delta x \approx \dfrac{1}{2\sqrt{x}}\delta x$

When $x = 16$ and $\delta x = 16.01 - 16 = 0.01$, we have;

$$\delta y \approx \frac{1}{2\sqrt{16}}(0.01) = \frac{0.01}{8} = 0.00125$$

Therefore the square root of 16.01 is $(4 + 0.00125) = 4.00125$.

Example 5

The radius of a sphere increased by $2\frac{1}{2}\%$. Find the percentage increase in the volume.

Solution

Let the small increase in the volume, V be δV and the small increase in the radius, r be δr.

The volume, $V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \dfrac{dV}{dr} = 4\pi r^2$

Using $\delta V \approx \dfrac{dV}{dr} \times \delta r \approx 4\pi r^2.\delta r$

When $\delta r = 2\frac{1}{2}\%$ of $r = 0.025r$, we have;

$\delta V \approx 4\pi r^2(0.025r) = 0.1\pi r^3$

The approximate percentage change in

volume $= \dfrac{\delta V}{V} \times 100 = \dfrac{0.1\pi r^3}{\frac{4}{3}\pi r^3} \times 100 = 7.5\%$.

Example 6

If $y = \sqrt{x}$, find the approximate increase in y if x is increased from 4 to 4.01.

Solution

Let the small increase in y be δy and the small increase in x be δx.

If $y = \sqrt{x} \quad \Rightarrow \quad \dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

Using $\delta y \approx \dfrac{dy}{dx} \times \delta x$ we have $\delta y \approx \dfrac{1}{2\sqrt{x}}\delta x$.

When $x = 4$ and $\delta x = 4.01 - 4 = 0.01$, we

have; $\delta y \approx \dfrac{1}{2\sqrt{4}}(0.01) = \dfrac{0.01}{4} = 0.0025$

Therefore the required increase in y is 0.0025.

Example 7

Using $y = \sqrt{x}$, find the approximate value of $\sqrt{101}$.

Solution

If $y = \sqrt{x} \quad \Rightarrow \quad \dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

Using $\delta y \approx \dfrac{dy}{dx} \times \delta x$ we have $\delta y \approx \dfrac{1}{2\sqrt{x}}\delta x$.

Taking $x = 100$, $y = \sqrt{100}$ and $\delta x = 1$ gives:

$\delta y \approx \dfrac{1}{2\sqrt{100}}(1) = \dfrac{1}{20} = 0.05$

Therefore

$\sqrt{101} \approx y + \delta y \approx \sqrt{100} + 0.05 \approx 10.05$

5. Stationary points

1. Condition

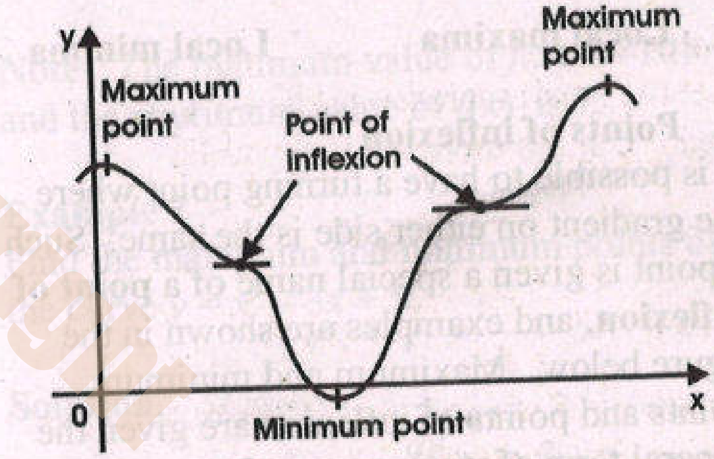
A point on a curve at which $\dfrac{dy}{dx} = 0$ is called a **stationary point** and the value of the function or y at this point is called the **stationary value.**

At the *stationary points*, the tangent to the curve is *parallel* to the x–axis.

To find the stationary points, put $\dfrac{dy}{dx} = 0$ and solve the resulting equation.

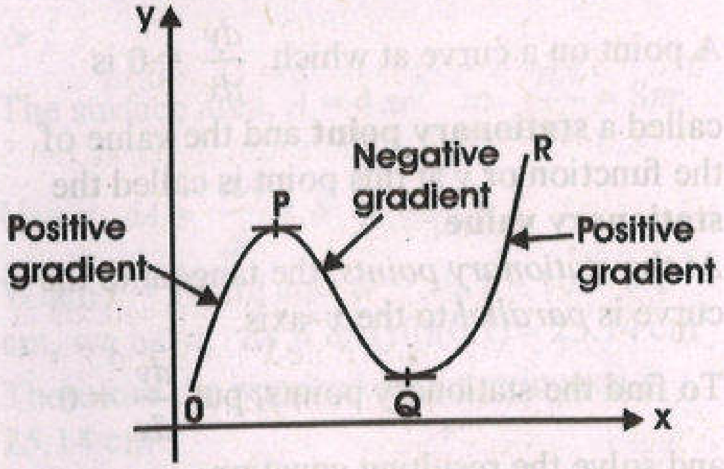
The stationary point is also called the **turning point.**

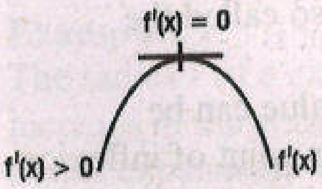
The stationary point or value can be maximum or minimum or point of inflexion.



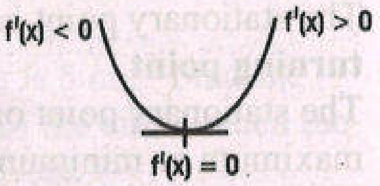
2. Local maxima and minima

As shown in the figure below, the gradient of a curve changes from positive between O and P to negative between P and Q, and then positive between Q and R. At point P, the gradient is zero and, as x increases, the gradient of the curve changes from positive just before P to negative just after. Such a point is called a **maximum point** and appears as the 'crest of a wave'. At point Q, the gradient is zero and, as x increases, the gradient of the curve changes from negative just before Q to positive just after. Such a point is called a **minimum point** and appears as the 'bottom of a valley'. Points such as P and Q are called **turning points**.





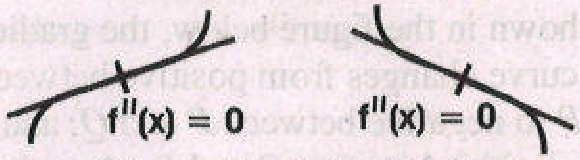
Local maxima



Local minima

3. Points of inflexion

It is possible to have a turning point where the gradient on either side is the same. Such a point is given a special name of a **point of inflexion**, and examples are shown in the figure below. Maximum and minimum points and points of inflexion are given the general term of **stationary points**.



General points of inflexion

Procedure for testing and distinguishing between stationary points

1. Given $y = f(x)$, determine $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ (i.e. $f'(x)$ and $f''(x)$) .
2. Put $\frac{dy}{dx} = 0$ and solve the resulting equation for values of x.
3. Substitute the values of x into the original equation, $y = f(x)$, to find the corresponding y–ordinate values. This establishes the coordinates of the stationary point(s).
4. To determine the nature of the stationary points:

***Method 1*: Using the second derivative**
Substitute the values of x found in step (2) into $\frac{d^2y}{dx^2}$

i) If $\frac{d^2y}{dx^2} < 0$ (i.e. **negative**) – the point is a **maximum** one.

ii) If $\frac{d^2y}{dx^2} > 0$ (i.e. **positive**) – the point is a **minimum** one.

iii) If $\frac{d^2y}{dx^2} = 0$ (i.e. **zero**) – the point is a point of inflexion.

Note that at the point of inflexion, $\frac{dy}{dx}$ need not be zero. Therefore to test for points of inflexion, first put $\frac{d^2y}{dx^2} = 0$ and solve for the values of x.

***Method 2*: Using the signs of the gradient, $\frac{dy}{dx}$, on either side of the turning points**

Sign to left	*Sign to right*	*Type of point*
+	–	Maximum
–	+	Minimum
– +	– +	Point of inflexion

Illustrative examples

Example 1

A curve is defined by the function $f(x) = x^3 - 6x^2 - 15x - 1$, find:

i) the derivative of $f(x)$ with respect to x;

ii) the gradient of the curve at the point where $x = 1$;

iii) the maximum and the minimum points.

Solution

i) If $f(x) = x^3 - 6x^2 - 15x - 1$, then the first derivative of $f(x)$ with respect to x is given by; $\dfrac{dy}{dx} = f'(x) = 3x^2 - 12x - 15.$

ii) At the point where $x = 1$, the gradient of the curve is:
$f'(1) = 3(1)^2 - 12(1) - 15 = -24.$
[Thus put $x = 1$ into $f'(x)$]

iii) From (i) above, $\dfrac{dy}{dx} = 3x^2 - 12x - 15$

At the turning point, $\dfrac{dy}{dx} = 0$

$3x^2 - 12x - 15 = 0$
$x^2 - 4x - 5 = 0$
$(x - 5)(x + 1) = 0$
$x = -1$ or $x = 5$

Therefore the turning points occur at $x = -1$ or $x = 5$.

Now we have to investigate whether the function is maximum or minimum at these points.

The second derivative, $\dfrac{d^2y}{dx^2} = 6x - 12.$

To test for maximum or minimum, we substitute $x = -1$ and $x = 5$ into $\dfrac{d^2y}{dx^2}$.

At $x = -1$, $\dfrac{d^2y}{dx^2} = 6(-1) - 12 = -18 < 0.$

Therefore, $f(x)$ has maximum point at $x = -1$ since $\dfrac{d^2y}{dx^2}$ is negative.

At $x = 5$, $\dfrac{d^2y}{dx^2} = 6(5) - 12 = 18 > 0.$

Therefore, $f(x)$ has a minimum point at $x = 5$ since $\dfrac{d^2y}{dx^2}$ is positive.

To find the corresponding y or $f(x)$ values at $x = -1$ and $x = 5$, substitute $x = -1$ and $x = 5$ into the expression for y or $f(x)$.

When $x = -1$,
$f(-1) = (-1)^3 - 6(-1)^2 - 15(-1) - 1 = 7.$
Hence the maximum point is $(-1, 7)$.
When $x = 5$,
$f(5) = (5)^3 - 6(5)^2 - 15(5) - 1 = -101.$
Hence the minimum point is $(5, -101)$.

Note: The minimum value of $f(x)$ is -101 and the maximum value of $f(x)$ is 7.

Example 2

Find the maximum and minimum points of the curve $y = x^3 - 3x + 5$.

Solution

$y = x^3 - 3x + 5 \Rightarrow \dfrac{dy}{dx} = 3x^2 - 3$

At the turning point, $\dfrac{dy}{dx} = 0$

$3x^2 - 3 = 0 \qquad \Rightarrow x^2 - 1 = 0$
$(x - 1)(x + 1)\ 0$
$x = -1$ or $x = 1$

Therefore the turning points occur at $x = -1$ or $x = 1$.

Now we have to investigate whether the function is maximum or minimum at these points.

The second derivative, $\dfrac{d^2y}{dx^2} = 6x$

To test for maximum or minimum, we substitute $x = -1$ or $x = 1$ into $\frac{d^2y}{dx^2}$.

At $x = -1$, $\frac{d^2y}{dx^2} = 6(-1) = -6 < 0$.

Therefore, y has maximum point at $x = -1$, since $\frac{d^2y}{dx^2}$ is negative.

At $x = 1$, $\frac{d^2y}{dx^2} = 6(1) = 6 > 0$.

Therefore, y has a minimum point at $x = 1$, since $\frac{d^2y}{dx^2}$ is positive.

To find the corresponding y values at $x = -1$ and $x = 1$, substitute $x = -1$ and $x = 1$ into the expression for y.
When $x = -1$, $y = (-1)^3 - 3(-1) + 5 = 7$.
Hence the maximum point is $(-1, 7)$.
When $x = 1$, $y = (1)^3 - 3(1) + 5 = 3$.
Hence the minimum point is $(1, 3)$

Example 3
Find the stationary points of $y = \frac{1}{3}x^3 - 2x^2 + 3x$ and identify their nature.

Solution
$y = \frac{1}{3}x^3 - 2x^2 + 3x$

$\frac{dy}{dx} = x^2 - 4x + 3$ and $\frac{d^2y}{dx^2} = 2x - 4$

At the turning point, $\frac{dy}{dx} = 0$

$x^2 - 4x + 3 = 0$
$(x - 3)(x - 1) = 0 \Rightarrow x = 1$ or $x = 3$
Therefore the turning points occur at $x = 1$ or $x = 3$.
Now we have to investigate whether the function is maximum or minimum at these points.

The second derivative, $\frac{d^2y}{dx^2} = 2x - 4$

To test for maximum or minimum, we substitute $x = 1$ or $x = 3$ into $\frac{d^2y}{dx^2}$.

At $x = 1$, $\frac{d^2y}{dx^2} = 2(1) - 4 = -2 < 0$.

Therefore, y has maximum point at $x = 1$, since $\frac{d^2y}{dx^2}$ is negative.

At $x = 3$, $\frac{d^2y}{dx^2} = 2(3) - 4 = 2 > 0$.

Therefore, y has a minimum point at $x = 3$, since $\frac{d^2y}{dx^2}$ is positive.

To find the corresponding y values at $x = 1$ and $x = 3$, substitute $x = 1$ and $x = 3$ into the expression for y.
When $x = 1$, $y = \frac{1}{3}(1)^3 - 2(1)^2 + 3(1) = \frac{4}{3}$
Hence the maximum point is $(1, \frac{4}{3})$.
When $x = 3$, $y = \frac{1}{3}(3)^3 - 2(3)^2 + 3(3) = 0$.
Hence the minimum point is $(3, 0)$.

Example 4
The curve $y = x^2 - ax + b$ has turning point at $(1, 3)$. Find a and b.

Solution
$y = x^2 - ax + b \Rightarrow \frac{dy}{dx} = 2x - a$

At the turning point, $\frac{dy}{dx} = 0$

$2x - a = 0$.
This occurs at $(1, 3)$ i.e. at $x = 1$
$2(1) - a = 0 \Rightarrow a = 2$
Since the point $(1, 3)$ is on the curve it means $(1, 3)$ satisfies the equation of the curve.

i.e. $3 = (1)^2 - 2(1) + b$

$3 = -1 + b \quad \Rightarrow \quad b = 4$

Example 5

The function $f(x) = ax^2 + bx + c$ has gradient function $4x + 2$ and stationary value of 1. Find the values of a, b and c.

Solution

$f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

But we know that $f'(x) = 4x + 2$

$2ax + b \equiv 4x + 2$

Therefore comparing corresponding terms,

$2a = 4 \quad \Rightarrow \quad a = 2$ and $b = 2$

The stationary value of $f(x)$ occurs when $f'(x) = 0$ i.e. when $4x + 2 = 0$

$x = -\frac{1}{2}$

The stationary value is

$f(x) = 2(-\frac{1}{2})^2 + 2(-\frac{1}{2}) + c = -\frac{1}{2} + c$

But the stationary value has been given as 1

$-\frac{1}{2} + c = 1 \quad \Rightarrow \quad c = \frac{3}{2}$

Example 6

Find the stationary values of the following functions and investigate their nature.

i) $y = x^3 - 3x$

ii) $y = x + \dfrac{4}{x}$

iii) $y = x - \dfrac{4}{x^2}$

iv) $y = x^2(3 - x)$

v) $y = (x - 3)(2x + 1)$

vi) $y = x^2(3 + 2x - 3x^2)$

Solution

i) -2 minimum, 2 maximum

ii) 4 minimum, -4 maximum

iii) -3 maximum

iv) 0 minimum, 4 maximum

v) $-\frac{49}{8}$ minimum

vi) 0 minimum, $\frac{5}{16}$ maximum, 2 maximum

[Work out the solution in each case]

Example 7

Locate the stationary points on the curve and determine the nature of each one.

i) $y = 4x^3 + 3x^2 - 6x - 1$

ii) $y = x^4 + 4x^3 - 6$

iii) $y = 2x^3 + x^2 - 4x + 1$

Solution

i) $(\frac{1}{2} - 2\frac{3}{4})$ minimum point,

$(-1, 4)$ maximum point

ii) $(-3, -33)$ minimum,

$(0, -4)$ point of inflexion

iii) $(\frac{2}{3} - \frac{17}{27})$ minimum point,

$(-1, 4)$ maximum point

[Work out the solution in each case.]

Practical problems involving maximum and minimum values

There are many practical problems involving maximum and minimum values which occur in science and engineering. Usually, an equation has to be determined from the given data, and rearranged where necessary, so that it contains only one variable.

The general procedure is:

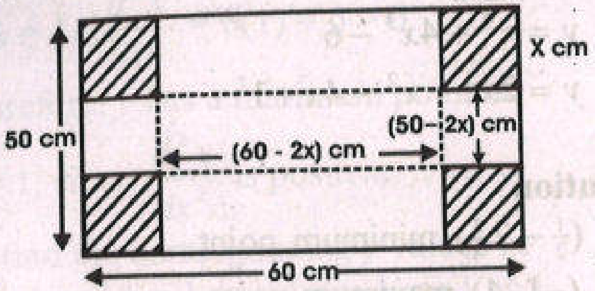
1. Write an expression for the required quantity;
2. Use the given conditions to rewrite it in terms of a single variable;
3. Find the stationary point(s) and their nature.

Illustrative examples

Example 1

A rectangular sheet of metal 60 cm by 50 cm has squares each of side x cm cut off from its corners. The remainder is then formed into a cuboid of volume V cm^3. Find the value of x for which the volume of the box will be a maximum.

Solution



From the diagram above, the sides of the box are $(60 - 2x)$, $(50 - 2x)$ and x cm.

The volume V cm^3 of the box is therefore given by: $V = (60 - 2x)(50 - 2x)x$

$$= 4x^3 - 220x^2 + 3000x.$$

[Note: Volume = length × breadth × height]

$$\frac{dV}{dx} = 12x^2 - 440x + 3000.$$

At the stationary value, $\frac{dV}{dx} = 0$

$$12x^2 - 440x + 3000 = 0$$
$$3x^2 - 110x + 750 = 0$$
$$x = 27.61 \text{ or } x = 9.05$$

(Note: Use the formula method to solve the quadratic equation)

Now we have to investigate whether the function is maximum or minimum at these points.

To test for the maximum or minimum value, we substitute $x = 9.05$ and $x = 27.61$ into $\frac{d^2V}{dx^2}$. Now $\frac{d^2V}{dx^2} = 24x - 440$.

When $x = 9.05$, $\frac{d^2V}{dx^2} = 24(9.05) - 440 < 0$

maximum value.

When $x = 27.61$,

$$\frac{d^2V}{dx^2} = 24(27.61) - 440 = 226.64 > 0$$

minimum value.

Hence the value of x for which the volume of the box will be a maximum is $x = 9.05$.

Note: The maximum volume $V_{\max}$ of the box is given by substituting $x = 9.05$ into the expression for V. i.e.

$$V_{\max} = [(60 - 2(9.05)][(50 - 2(9.05)]9.05$$
$$= (41.9)(31.9)9.05 = 12096.32 \text{ cm}^3.$$

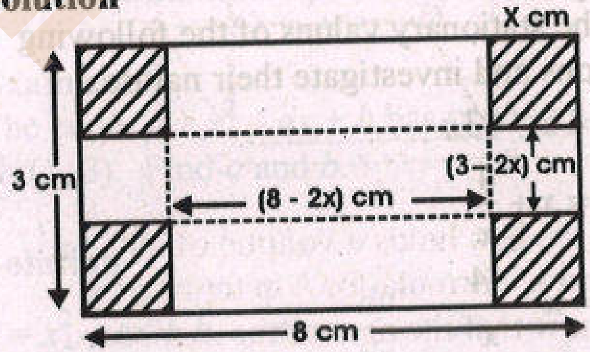
Example 2

The lengths of the sides of a rectangular sheet of metal are 8 cm and 3 cm. A square of side x cm is cut from each corner of the sheet and the remaining piece is folded to make an open box.

i) Show that the volume V of the box is given by $V = 4x^3 - 22x^2 + 24x$ cm^3

ii) Find the value of x for which the volume of the box is a maximum. Calculate the maximum volume.

Solution



i) The volume of the box,

$$V = (8 - 2x)(3 - 2x)x$$
$$= 4x^3 - 22x^2 + 24x \text{ cm}^3$$

ii) Differentiating the expression for V with respect to x, $\frac{dV}{dx} = 12x^2 - 44x + 24$

For a maximum (or minimum) value of V,

$\frac{dV}{dx} = 0 \Rightarrow 12x^2 - 44x + 24 = 0$

$(3x-2)(x-3) = 0 \Rightarrow x = \frac{2}{3}$ or 3

Clearly x cannot be 3 cm since the initial width of the sheet is only 3 cm. So $x = \frac{2}{3}$.

Differentiating again gives $\frac{d^2V}{dx^2} = 24x - 44$

When $x = \frac{2}{3}$, $\frac{d^2V}{dx^2} = 24(\frac{2}{3}) - 44 < 0$

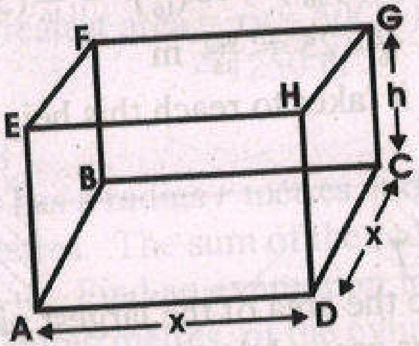
i.e. V is a maximum.

The maximum volume, $V_{\max}$ of the box is given by substituting $x = \frac{2}{3}$ into the expression for V.

i.e. $V_{\max} = 4(\frac{2}{3})^3 - 22(\frac{2}{3})^2 + 24(\frac{2}{3}) = \frac{200}{27}$ cm^3

Example 3

A rectangular box with a square base x cm and height of h cm is to be made from a cardboard.



i) If the box holds a volume of 200 cm^3, find a formula for h in terms of x.

ii) Show that the area of the cardboard needed is given by $2x^2 + \frac{800}{x}$.

iii) Hence find the least area of cardboard needed to make the box (correct to one decimal place)

Solution

i) The volume of the box, $V = x^2h$

$xh = \frac{V}{x}$

The surface area of the box,

$A = 2x^2 + 4xh = 2x^2 + \frac{4V}{x} = 2x^2 + \frac{800}{x}$,

since $V = 200$ cm^3

ii) Differentiating the expression for A with respect to x, $\frac{dA}{dx} = 4x - \frac{800}{x^2}$

For the least value of A, $\frac{dA}{dx} = 0$

i.e. $4x - \frac{800}{x^2} = 0 \Rightarrow 4x^3 = 800$

$x^3 = 200 \Rightarrow x = 5.848$

The least surface area, $A_{\min}$ of the box is given by substituting $x = 5.848$ into the expression for A. i.e.

$A_{\min} = 2(5.848)^2 + \frac{800}{5.848}$

$= 205.2$ cm^2, correct to one decimal place

Example 4

A rectangle has perimeter 28 m.
What is its maximum area?

Solution

Let the sides of the rectangle be x m and y m.

Its perimeter $P = 2x + 2y = 28$

$2y = 28 - 2x$

$y = \frac{1}{2}(28 - 2x) = 14 - x$

Area of rectangle, $A = xy$. But $y = 14 - x$

$A = x(14 - x) = 14x - x^2$

$\frac{dA}{dx} = 14 - 2x$.

For the greatest area, $\frac{dA}{dx} = 0$

$14 - 2x = 0 \quad \Rightarrow \quad x = 7$ m

When $x = 7$, $y = 14 - 7 = 7$ cm.

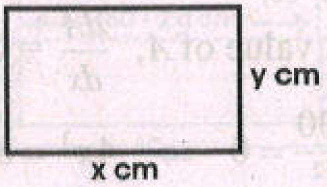
Now substitute these values into the expression for the area to get the greatest area.

$\therefore$ The greatest area $= xy = 7(7) = 49$ m^2.

Example 5

100 cm of fencing is to be used to make a rectangular enclosure. Find the greatest possible area of the enclosure.

Solution



Let the sides of the fence be x cm and y cm (see figure above).

Area of fence, $A = xy$ ……… (1)

Perimeter of fence,

$P = 2x + 2y = 100$ ……..(2)

$\therefore$ Making y the subject in equation (2)

$y = \frac{1}{2}(100 - 2x) = 50 - x$

Substituting $y = 50 - x$ into equation (1), we have

$A = x(50 - x) = 50x - x^2$

$$\frac{dA}{dx} = 50 - 2x$$

For the greatest area, $\frac{dA}{dx} = 0$

$50 - 2x = 0 \quad \Rightarrow \quad x = 25$ cm

To find the corresponding y value, substitute $x = 25$ into the expression for y.

When $x = 25$, $y = 50 - 25 = 25$ cm.

Now substitute these values into the expression for the area to get the greatest area.

$\therefore$ The greatest area $= xy = 25(25)$
$= 625$ cm^2.

Alternatively, we can substitute $x = 25$ into $A = 50 - x^2$, i.e.

$A = 50(25) - 25^2 = 625$ cm^2.

Example 6

A ball is thrown vertically upwards from the ground level and its height after t seconds is $(50t - 16t^2)$ m, find;

a) the greatest height it reaches; and
b) the time it takes to reach there.

Solution

Let the height be $h = 50t - 16t^2$

$$\frac{dh}{dt} = 50 - 32t$$

At the greatest height, $\frac{dh}{dt} = 0$

$50 - 32t = 0 \quad \Rightarrow \quad t = \frac{25}{16}$ s

For the greatest height, substitute $t = \frac{25}{16}$ into the expression for h.

$\therefore$ The greatest height,

$$h_{\max} = 50\left(\tfrac{25}{16}\right) - 16\left(\tfrac{25}{16}\right)^2 = \tfrac{25}{16}(50 - 25)$$

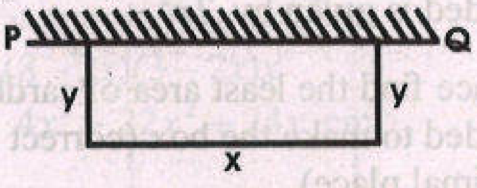
$$= \tfrac{25}{16} \times 25 = \tfrac{625}{16} \text{ m}$$

The time it takes to reach this height is $t = \frac{25}{16}$ s.

Example 7

Determine the area of the largest piece of rectangular ground that can be enclosed by 100 m of fencing, if part of an existing straight wall is used as one side.

Solution



Let the dimensions of the rectangle be x and y as shown, where PQ represents the straight wall.

Perimeter of fence,
$P = x + 2y = 100$ ....(1)
(Note that $P = 100$ m).
Area of fence, $A = xy$ ....(2)

Since the maximum area is required, a formula for area is needed in terms of one variable only. From equation (1)

$x = 100 - 2y$

Substituting $x = 100 - 2y$ into equation (2), we have;

$A = xy = (100 - 2y)y = 100y - 2y^2$

$$\frac{dA}{dy} = 100 - 4y$$

For the greatest area, $\frac{dA}{dy} = 0$

$$100 - 4y = 0 \quad \Rightarrow y = 25 \text{ m}$$

To find the corresponding y value, substitute $y = 25$ into the expression for x.

When $y = 25$, $x = 100 - 2(25) = 50$ m.

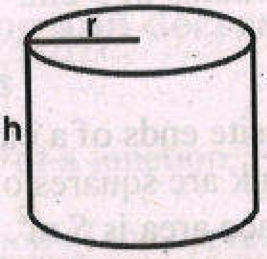
Now substitute these values into the expression for the area to get the greatest area.

$$\therefore \quad \text{The greatest area} = xy = 50(25) = 1250 \text{ m}^2.$$

Example 8

A cylinder has a radius r metres and a height h metres. The sum of the radius and height is 2 m. Find an expression for the volume, V cubic metres, of the cylinder in terms of r only. Hence find the maximum volume.

Solution



$V = \pi r^2 h$ and $r + h = 2$ or $h = 2 - r$

$$\therefore \quad V = \pi r^2(2 - r) = \pi(2r^2 - r^3)$$

Now for maximum volume, $\frac{dV}{dr} = 0$

i.e. $\frac{dV}{dr} = \pi(4r - 3r^2) = 0$

$$\pi r(4 - 3r) = 0$$

$$r = 0 \text{ and } r = \tfrac{4}{3}$$

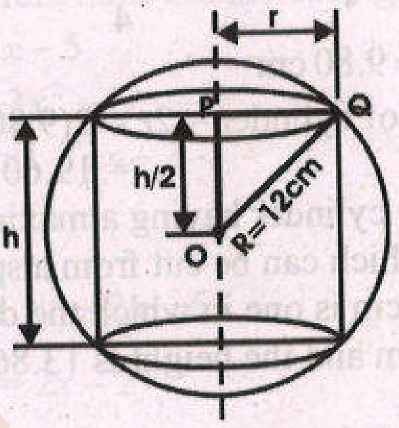
The maximum value of V occurs when $r = \tfrac{4}{3}$ and this is given by:

$$V = \pi(\tfrac{3}{4})^2(2 - \tfrac{4}{3}) = \tfrac{32\pi}{27} \text{ m}^3$$

Example 9

Find the diameter and height of a cylinder of maximum volume which can be cut from a sphere of radius 12 cm.

Solution



The diagram shows a cylinder of radius r and height h enclosed in a sphere of radius $R = 12$ cm.

Volume of cylinder, $V = \pi r^2 h$ ....(1)

Using the right angled triangle OPQ shown in the diagram

$$r^2 + \left(\tfrac{h}{2}\right)^2 = R^2$$

$$r^2 + \tfrac{h^2}{4} = 144 \text{ ....(2)}$$

Since the maximum volume is required, a formula for the volume V is required in terms of one variable only.

From equation (2), $\quad r^2 = 144 - \tfrac{h^2}{4}$

Substituting into (1) gives:

$$V = \pi(144 - \tfrac{h^2}{4})h = 144\pi h - \frac{\pi h^3}{4}$$

$$\frac{dV}{dh} = 144\pi - \frac{3\pi h^2}{4} = 0 \text{ for maximum or}$$
minimum value

$$144\pi = \frac{3\pi h^2}{4}$$

$$h = \sqrt{\frac{144 \times 4}{3}} = 13.86\,\text{cm}$$

The second derivative, $\frac{d^2V}{dh^2} = -\frac{6\pi h}{4}$

When $h = 13.86$, $\frac{d^2V}{dh^2}$ is negative, giving a maximum value.

From equation (2),

$$r^2 = 144 - \tfrac{h^2}{4} = 144 - \frac{(13.86)^2}{4}$$

$$r = 9.80\,\text{cm}$$

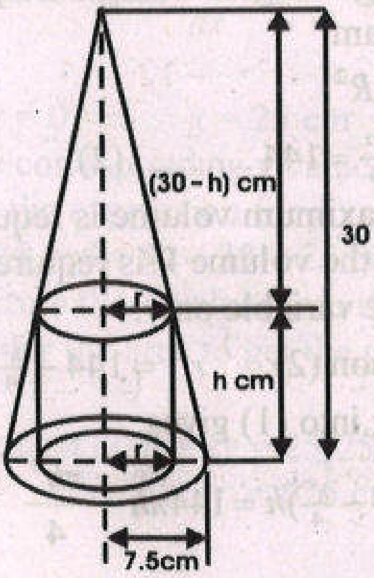
Diameter of cylinder $= 2r = 2(9.80)$
$= 19.60$ cm

Hence the cylinder having a maximum volume which can be cut from a sphere of radius 12 cm is one in which the diameter is 19.60 cm and the height is 13.86 cm.

Example 10

Find the height and radius of a cylinder of maximum volume which can be cut from a cone of height 30 cm and base radius 7.5 cm

Solution



The diagram shows a cylinder of radius r and height h enclosed in a cone of height 30 cm and base radius 7.5 cm

Volume of cylinder, $V = \pi r^2 h$ ……….….(1)

By similar triangles, $\frac{30-h}{r} = \frac{30}{7.5}$ …...(2)

Since the maximum volume is required, a formula for the volume V must be obtained in terms of one variable only.

From equation (2), $\frac{30-h}{r} = 4$

$$30 - h = 4r$$

$$h = 30 - 4r \text{ ……….(3)}$$

Substituting $h = 30 - 4r$ into (1) gives:

$$V = \pi r^2(30 - 4r) = 30\pi r^2 - 4\pi r^3$$

$$\frac{dV}{dr} = 60\pi r - 12\pi r^2 = 0 \text{ for maximum}$$
or minimum value.

$$12\pi r(5 - r) = 0 \quad \Rightarrow \quad r = 0 \text{ or } r = 5 \text{ cm.}$$

$\frac{d^2V}{dr^2} = 60\pi - 24\pi r$. When $r = 0$, $\frac{d^2V}{dh^2}$ is positive, giving a minimum value.

When $r = 5$, $\frac{d^2V}{dh^2}$ is negative, giving a maximum value.

From equation (3),

height $h = 30 - 4r = 30 - 4(5) = 10\,\text{cm}$

Hence the cylinder having a maximum volume which can be cut from a cone of height 30 cm and radius 7.5 cm, has a height of 10 cm and radius of 5 cm.

Example 11

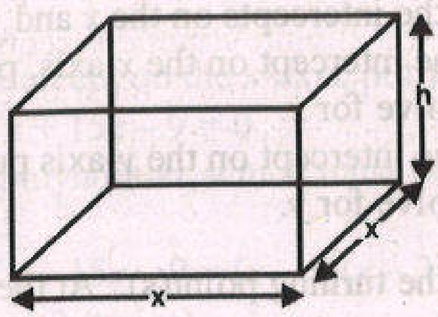
The two opposite ends of a closed rectangular tank are squares of side x cm. The total surface area is S m^2.

i) Show that the volume of the tank is $\frac{1}{4}x(S - 2x^2)\,\text{m}^3$.

ii) If the value of S is 2,400, find the value of x for which the volume is maximum.

Solution

i)



The surface area of the tank $= 2x^2 + 4xh$

$$S - 2x^2 = 4xh \quad \Rightarrow \quad h = \frac{S - 2x^2}{4x}$$

Volume of the tank $V = x^2 h$

$$V = x^2\left(\frac{S - 2x^2}{4x}\right)$$

$$V = \tfrac{1}{4}x(S - 2x^2)\,\text{m}^3$$

ii) $V = \frac{1}{4}xS - \frac{1}{2}x^3 \quad \Rightarrow \quad \frac{dV}{dx} = \frac{1}{4}S - \frac{3}{2}x^2$

For maximum volume, $\frac{dV}{dx} = 0$

$\frac{1}{4}S - \frac{3}{2}x^2 = 0$; But $S = 2400$

$\frac{1}{4} \times 2400 - \frac{3}{2}x^2 = 0 \quad \Rightarrow \quad 3x^2 = 1200$

$x = \sqrt{400} = 20$ cm

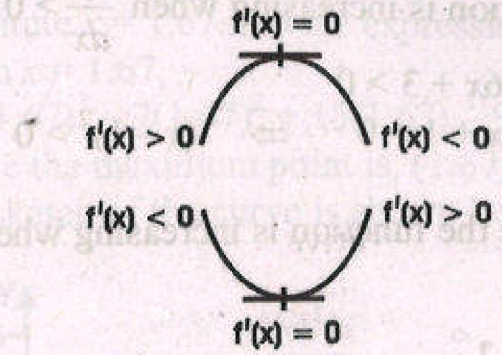
$\therefore$ The value of x for which the volume is maximum is 20 cm.

6. Increasing and decreasing functions

The derivative of a function, $\frac{dy}{dx}$, gives the *gradient function* of the curve.

Where the derivative is positive (i.e. $\frac{dy}{dx} > 0$), the function is **increasing**.

Where the derivative is negative (i.e. $\frac{dy}{dx} < 0$), the function is **decreasing** as shown in the figure below.



Illustrative examples

Example 1

Find the range of values of x for which the following functions are increasing.

i) $x^2 - 2x - 5$

ii) $2x^3 + 3x^2 - 12x + 5$

iii) $x^3 - 3x^2 + 3x + 2$

Solution

i) Let $y = x^2 - 2x - 5$

$$\frac{dy}{dx} = 2x - 2$$

The function is increasing when $\frac{dy}{dx} > 0$

i.e. $2x - 2 > 0 \quad \Rightarrow \quad x > 1$

Therefore the function is increasing when $x > 1$.

ii) Let $y = 2x^3 + 3x^2 - 12x + 5$

$$\frac{dy}{dx} = 6x^2 + 6x - 12$$

The function is increasing when $\frac{dy}{dx} > 0$

i.e. $6x^2 + 6x - 12 > 0$

$x^2 + x - 2 > 0 \quad \Rightarrow \quad (x + 2)(x - 1) > 0$

$x < -2$ or $x > 1$

Therefore the function is increasing when $x < -2$ or $x > 1$.

iii) Let $y = x^3 - 3x^2 + 3x + 2$

$$\frac{dy}{dx} = 3x^2 - 6x + 3$$

The function is increasing when $\frac{dy}{dx} > 0$

i.e. $3x^2 - 6x + 3 > 0$

$$x^2 - 2x + 1 > 0 \quad \Rightarrow \quad (x-1)^2 > 0$$

$$x > 1$$

Therefore the function is increasing when $x > 1$.

Example 2

If $f(x) = x^3 + x^2 - 5x - 5$, find the interval on which f is increasing and the interval on which f is decreasing.

Solution

Let $y = x^3 + x^2 - 5x - 5$

$$\frac{dy}{dx} = 3x^2 + 2x - 5$$

The function is increasing when $\frac{dy}{dx} > 0$

i.e. $3x^2 + 2x - 5 > 0$

$$(3x+5)(x-1) > 0$$

$$x < -\tfrac{5}{3} \text{ or } x > 1$$

Therefore the function is increasing when $x < -\frac{5}{3}$ or $x > 1$.

The function is decreasing when $\frac{dy}{dx} < 0$

i.e. $3x^2 + 2x - 5 < 0$

$$(3x+5)(x-1) < 0 \quad \Rightarrow \quad -\tfrac{5}{3} < x < 1$$

Therefore the function is decreasing when $-\frac{5}{3} < x < 1$.

Example 3

Find the range of values of x for which the function $f(x) = x^3 - 12x + 5$ is

i) increasing ii) decreasing

Solution

Work out the solution.

7. Curve sketching

The following steps are used to sketch any curve when the equation of the curve is given.

Steps

1. Find the intercepts on the x and y axes. For the intercept on the x-axis, put $y = 0$ and solve for x.
   For the intercept on the y-axis put $x = 0$ and solve for y.
2. Find the turning point(s). At the turning points, put $\frac{dy}{dx} = 0$ and solve the resulting equation for x. Substitute the values of x into the original equation, $y = f(x)$, to find the corresponding y–ordinate values. This establishes the coordinates of the stationary point(s).
3. Test whether the turning points are maximum or minimum. Use the following conditions:
   i) If $\frac{d^2y}{dx^2} > 0$ (i.e. positive) the turning point is minimum.
   ii) If $\frac{d^2y}{dx^2} < 0$ (i.e. negative) the turning point is maximum.

Note: First find $\frac{d^2y}{dx^2}$ and put the values of x obtained in step 2 into the expression for $\frac{d^2y}{dx^2}$.

Illustrative examples

Example 1

Sketch the curve $y = x^3 - 7x^2 + 15x - 9$ indicating clearly its points of intersection with the axes and its turning points.

Solution

The equation of the curve is
$y = x^3 - 7x^2 + 15x - 9$

For the intercept on the y-axis, put $x = 0$

$y = -9$

$\therefore$ The point $(0, -9)$ is the intercept on the y-axis

For the intercept on the x-axis, put $y = 0$

$x^3 - 7x + 15x - 9 = 0$

This is a polynomial, which must be factorized.

$\therefore$ $x^3 - 7x + 15x - 9 = (x - 1)(x - 3)^2 = 0$

$x = 1$ or 3 (twice)

$\therefore$ The intercepts on the x-axis are $(1, 0)$ and $(3, 0)$.

$y = x^3 - 7x^2 + 15x - 9$

$$\frac{dy}{dx} = 3x^2 - 14x + 15.$$

At the turning points, $\frac{dy}{dx} = 0$.

$3x^2 - 14x + 15 = 0 \quad \Rightarrow \quad x = 1.67$ or 3

Next we have to investigate which of these points give maximum or minimum points.

Now, $\frac{dy}{dx} = 3x^2 - 14x + 15$

$$\frac{d^2y}{dx^2} = 6x - 14.$$

At $x = 3$, $\frac{d^2y}{dx^2} = 6(3) - 14 = 4 > 0$

Therefore a minimum turning point occurs at $x = 3$, since $\frac{d^2y}{dx^2}$ is positive.

To find the corresponding y–value substitute $x = 3$ into the expression for y.

When $x = 3$, $y = 0$. Hence $(3, 0)$ is the minimum turning point.

At $x = 1.67$, $\frac{d^2y}{dx^2} = 6(1.67) - 14 = -3.98 < 0$

Therefore a maximum turning point occurs at $x = 1.67$, since $\frac{d^2y}{dx^2}$ is negative.

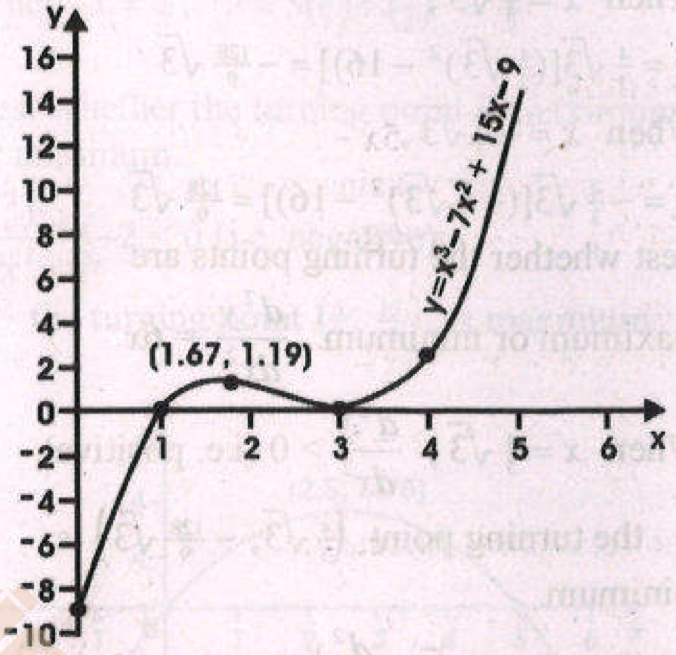
To find the corresponding y-values substitute $x = 1.67$ into the expression for y.

When $x = 1.67$,

$y = (1.67)^3 - 7(1.67)^2 + 15(1.67) - 9 = 1.19$

Hence the maximum point is $(1.67, 1.19)$.

The sketch of the curve is shown below.



Example 2

Sketch the curves

i) $y = x(x^2 - 16)$ ii) $y = x^2 - 4x - 5$

iii) $y = x^2 - 6x$ iv) $y = 5x - x^2$

Solution

i) $y = x(x^2 - 16)$

For the intercept on the x-axis, put $y = 0$ and solve for x. i.e. $x(x^2 - 16) = 0$

$x(x - 4)(x + 4) = 0 \Rightarrow x = -4, 0, 4$

$\therefore$ The points $(0, 0)$, $(-4, 0)$ and $(4, 0)$ are the intercepts on the x-axis

For the intercept on the y-axis put $x = 0$ and solve for y i.e. $y = 0$

$\therefore$ The point $(0, 0)$ is the intercept on the y-axis.

At the turning points, put $\frac{dy}{dx} = 0$ and solve the resulting equation for x.

$\frac{dy}{dx} = 3x^2 - 16 = 0 \quad \Rightarrow \quad x = \pm\tfrac{4}{3}\sqrt{3}$

Substitute the values of x into the original equation, $y = f(x)$, to find the corresponding y-ordinate values. This establishes the coordinates of the stationary point(s)

When $x = \tfrac{4}{3}\sqrt{3}$,

$y = \tfrac{4}{3}\sqrt{3}[(\tfrac{4}{3}\sqrt{3})^2 - 16)] = -\tfrac{128}{9}\sqrt{3}$

When $x = -\tfrac{4}{3}\sqrt{3}$,

$y = -\tfrac{4}{3}\sqrt{3}[(-\tfrac{4}{3}\sqrt{3})^2 - 16)] = \tfrac{128}{9}\sqrt{3}$

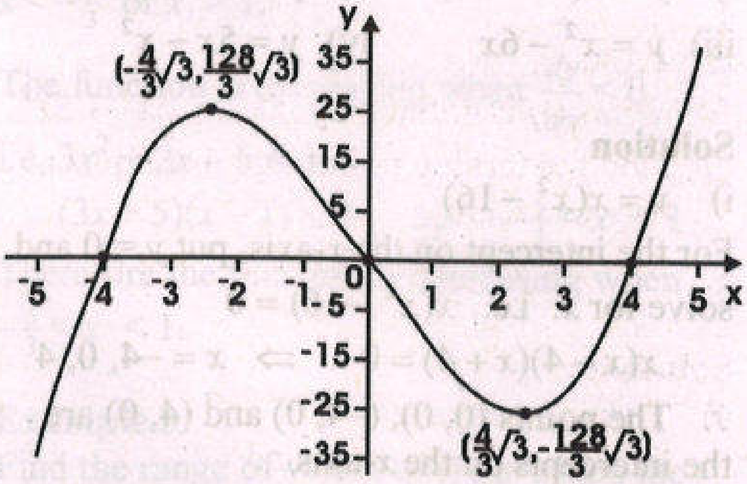
Test whether the turning points are maximum or minimum. $\frac{d^2y}{dx^2} = 6x$

When $x = \tfrac{4}{3}\sqrt{3}$, $\frac{d^2y}{dx^2} > 0$ (i.e. positive)

the turning point $\left(\tfrac{4}{3}\sqrt{3}, -\tfrac{128}{9}\sqrt{3}\right)$ is minimum.

When $x = -\tfrac{4}{3}\sqrt{3}$, $\frac{d^2y}{dx^2} < 0$ (i.e. negative)

the turning point $\left(-\tfrac{4}{3}\sqrt{3}, \tfrac{128}{9}\sqrt{3}\right)$ is maximum.



ii) $y = x^2 - 4x - 5$

For the intercept on the x-axis, put $y = 0$ and solve for x. i.e. $x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0 \quad \Rightarrow \quad x = -1 \text{ or } 5$

$\therefore$ The points $(-1, 0)$ and $(5, 0)$ are the intercepts on the x-axis

For the intercept on the y-axis put $x = 0$ and solve for y i.e. $y = -5$

$\therefore$ The point $(0, -5)$ is the intercept on the y-axis

At the turning points, put $\frac{dy}{dx} = 0$ and solve the resulting equation for x.

$\frac{dy}{dx} = 2x - 4 = 0 \quad \Rightarrow \quad x = 2$

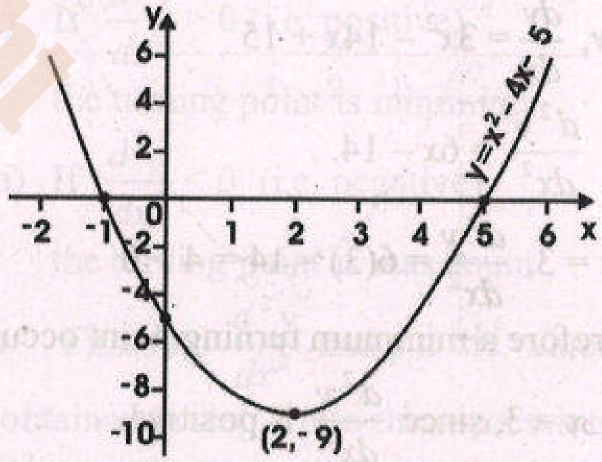
Substitute the value of x into the original equation, $y = f(x)$, to find the corresponding y–ordinate value. This establishes the coordinates of the stationary point(s)

When $x = 2$, $y = (2)^2 - 4(2) - 5 = -9$

Test whether the turning point is maximum or minimum.

$\frac{d^2y}{dx^2} = 2 > 0$ (i.e. positive)

the turning point $(2, -9)$, is minimum.



iii) $y = x^2 - 6x$

For the intercept on the x-axis, put $y = 0$ and solve for x. i.e. $x^2 - 6x$

$x(x-6) = 0 \quad \Rightarrow \quad x = 0 \text{ or } 6$

$\therefore$ The points $(0, 0)$ and $(6, 0)$ are the intercepts on the x-axis.

For the intercept on the y-axis put $x = 0$ and solve for y i.e. $y = 0$

$\therefore$ The point (0, 0) is the intercept on the y-axis.

At the turning points, put $\frac{dy}{dx} = 0$ and solve the resulting equation for x.

$$\frac{dy}{dx} = 2x - 6 = 0 \quad \Rightarrow \quad x = 3$$

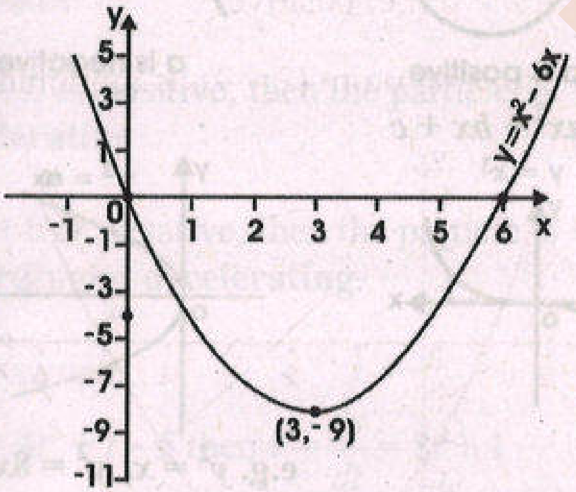
Substitute the value of x into the original equation, $y = f(x)$, to find the corresponding y–ordinate value. This establishes the coordinates of the stationary point(s).

When $x = 3$, $y = (3)^2 - 6(3) = -9$

Test whether the turning point is maximum or minimum.

$$\frac{d^2y}{dx^2} = 2 > 0 \text{ (i.e. positive)}$$

the turning point (3, –9), is minimum.



iv) $y = 5x - x^2$

For the intercept on the x-axis, put $y = 0$ and solve for x. i.e. $5x - x^2 = 0$

$$x(5 - x) = 0 \quad \Rightarrow \quad x = 0 \text{ or } 5$$

$\therefore$ The points (0, 0) and (5, 0) are the intercepts on the x-axis.

For the intercept on the y-axis put $x = 0$ and solve for y i.e. $y = 0$

$\therefore$ The point (0, 0) is the intercept on the y-axis.

At the turning points, put $\frac{dy}{dx} = 0$ and solve the resulting equation for x.

$$\frac{dy}{dx} = 5 - 2x = 0 \quad \Rightarrow \quad x = \tfrac{5}{2}$$

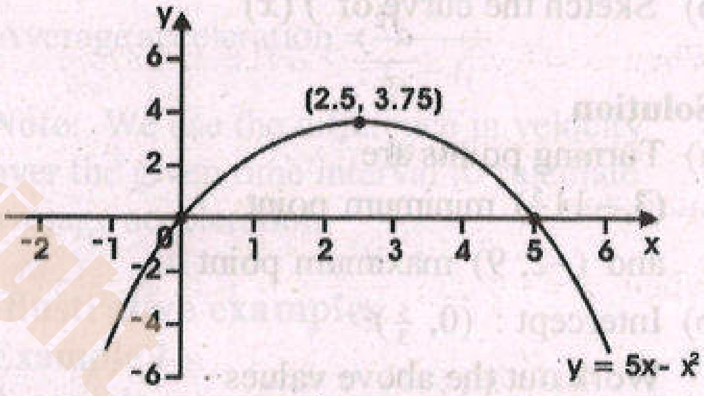
Substitute the value of x into the original equation, $y = f(x)$, to find the corresponding y–ordinate value. This establishes the coordinates of the stationary point(s)

When $x = \frac{5}{2}$, $y = 5\left(\frac{5}{2}\right) - \left(\frac{5}{2}\right)^2 = \frac{15}{4}$

Test whether the turning point is maximum or minimum.

$$\frac{d^2y}{dx^2} = -2 < 0 \text{ (i.e. negative)}$$

the turning point $\left(\frac{5}{2}, \frac{15}{4}\right)$, is maximum.



Example 3

A function, f is defined by $f(x) = 2x^3 + 3x^2$

a) Find the turning points and distinguish between them

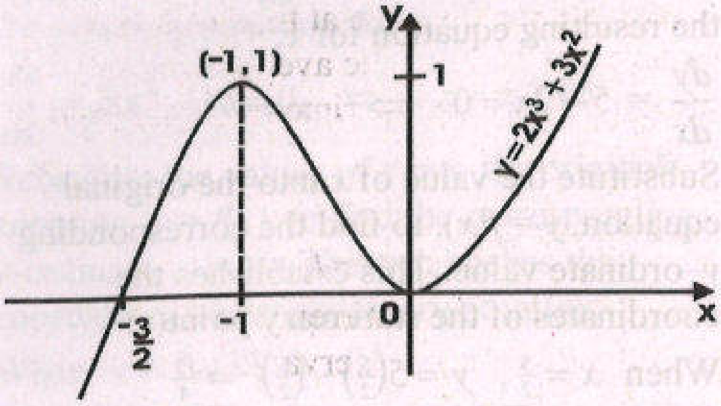
b) Sketch the curve, $f(x) = 2x^3 + 3x^2$

Solution

a) Turning points are
(0, 0) minimum point and
(–1, 1) maximum point

b) Intercepts : (0, 0), (–1, 0)
[Work out the above values]

The sketch of $f(x) = 2x^3 + 3x^2$ is shown below.



Example 4

A function, f is defined by

$$f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 6x + \frac{5}{3}$$

a) Find the turning points and distinguish between them

b) Sketch the curve of $f(x)$

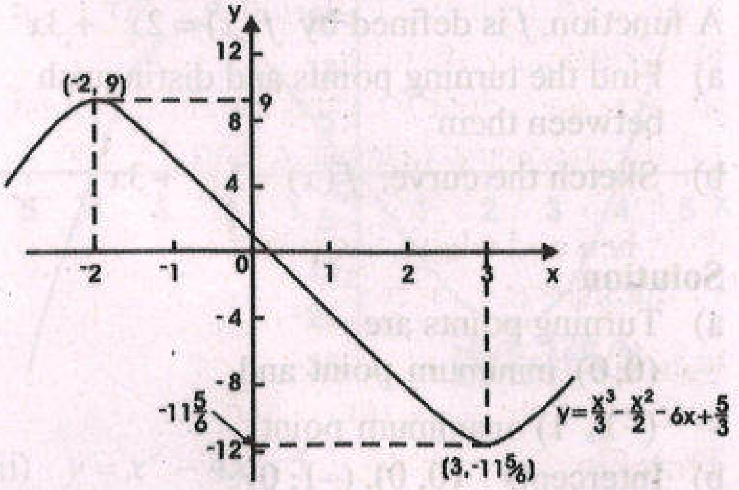
Solution

a) Turning points are $(3, -11\frac{5}{6})$ minimum point and $(-2, 9)$ maximum point

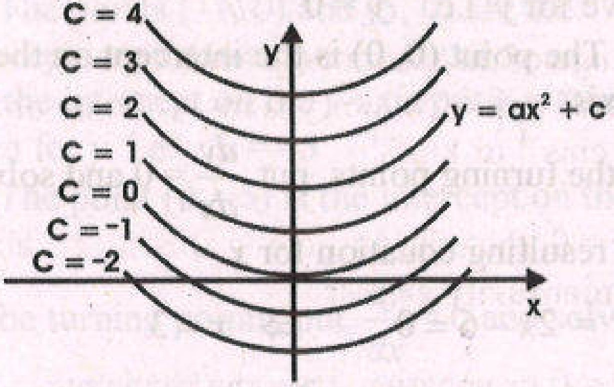
b) Intercept : $(0, \frac{5}{3})$

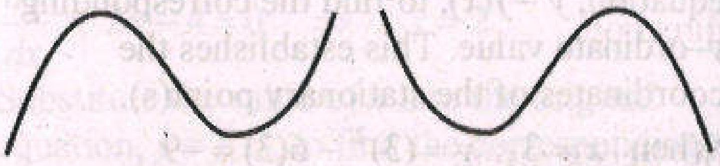
Work out the above values

The sketch is shown below.



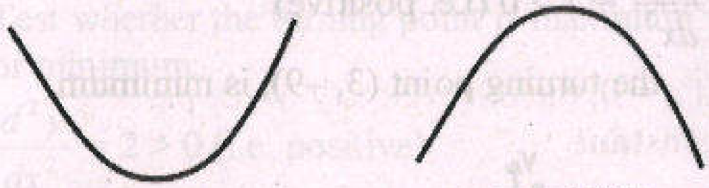
Sketch of some standard curves





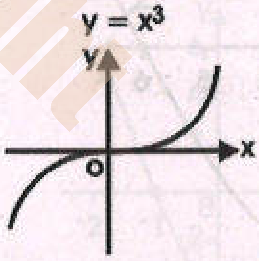
a is positive *a* is negative

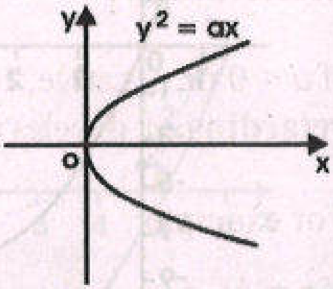
$y = ax^3 + bx^2 + cx + d$



a is positive *a* is negative

$y = ax^2 + bx + c$





e.g. $y^2 = x$, $y^2 = 8x$ etc.

8. Linear Kinematics

Kinematics is the study of displacement, velocity and acceleration.

Consider a particle which is moving in a straight line such that its displacement from a fixed point is s metres after time t seconds then:

i) **displacement,** $s = f(t)$;

The SI unit of displacement is m.

Other units are cm or km.

ii) **velocity,** $v = f'(t) = \dfrac{ds}{dt}$ i.e. the rate of change of displacement s with time t

The SI unit of velocity is ms^{-1}. Other units are cms^{-1} or kms^{-1}.

If $v = 0$, then the particle is at **rest** or **momentarily at rest**

If $v < 0$ or negative, then the particle is moving in **opposite direction** to the initial direction.

iii) **acceleration,** $a = \dfrac{dv}{dt}$ or $a = f''(t) = \dfrac{d^2s}{dt^2}$

i.e. the rate of change of velocity v with time t. The SI unit of acceleration is ms^{-2}. Other units are cms^{-2} or kms^{-2}.

If $a = 0$, then the velocity of the particle is **constant**

If $a > 0$ or positive, then the particle is **accelerating**

If $a < 0$ or negative, then the particle is **retarding** or **decelerating**.

For example,

if $s = 4t^2 + t - 5$ then $v = \dfrac{ds}{dt} = 8t + 1$

and $a = \dfrac{dv}{dt} = 8$

Therefore given an expression for distance s in terms of t, we can find the velocity, v by differentiating the expression for s with respect to t. Also knowing the velocity v, we can find the acceleration, a by differentiating the expression for v with respect to t.

Average velocity

If the distance travelled by a particle at time t_1 is s_1 and the distance at time t_2 is s_2 (where $t_2 > t_1$), then the average velocity of the particle within the time interval t_1 and t_2 is given by:

$$\text{Average velocity} = \frac{s_2 - s_1}{t_2 - t_1}$$

Note: We use the difference in distance over the given time interval to calculate average velocity.

Average acceleration

If the velocity of a particle at time t_1 is v_1 and the velocity at time t_2 is v_2 (where $t_2 > t_1$) then the average acceleration of the particle within the time interval t_1 and t_2 is given by:

$$\text{Average acceleration} = \frac{v_2 - v_1}{t_2 - t_1}$$

Note: We use the difference in velocity over the given time interval to calculate average acceleration.

Illustrative examples

Example 1

A particle moves in a straight line in such a way that its distance s metres from a fixed point O after time t seconds is given by $s = t^3 - 2t^2 + t + 10$. Calculate:

a) its velocity after 2s;
b) average velocity during the third second;
c) the average acceleration from $t = 1$ to $t = 3$;
d) when and where the particle is momentarily at rest.

Solution

a) $s = t^3 - 2t^2 + t + 10$,

Therefore the velocity

$v = \dfrac{ds}{dt} = 3t^2 - 4t + 1$

When $t = 2$,
$v = 3(2)^2 - 4(2) + 1 = 12 - 8 + 1$
$= 5$ m/s
Therefore the velocity after 2 s is 5 m/s.

b) The third second is between $t_1 = 2$s and $t_2 = 3$s (Note this carefully)
From $s = t^3 - 2t^2 + t + 10$
When $t_1 = 2$ s,
$s_1 = (2)^3 - 2(2)^2 + 2 + 10 = 12$ m
When $t_2 = 3$ s,
$s_2 = (3)^3 - 2(3)^2 + 3 + 10 = 22$ m

$$\text{Average velocity} = \frac{s_2 - s_1}{t_2 - t_1}$$
$$= \frac{22-12}{3-2} = 10 \text{ ms}^{-1}$$

Hence the average velocity during the third second is 10 ms^{-1}.

c) From (a), $v = 3t^2 - 4t + 1$
When $t_1 = 1$, $v_1 = 3(1)^2 - 4(1) + 1 = 0$ m/s
and
When $t_2 = 3$, $v_2 = 3(3)^2 - 4(3) + 1 = 16$ m/s

$$\text{Average acceleration} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{16-0}{3-1}$$
$$= \tfrac{16}{2} = 8 \text{ ms}^{-2}$$

Hence the average acceleration within the time interval $t = 1$ and $t = 3$ is 8 ms^{-2}.

d) When the particle is momentarily at rest the velocity, $v = 0$
But $v = 3t^2 - 4t + 1 \quad \Rightarrow \quad 3t^2 - 4t + 1 = 0$
$t = 1$ s or $t = 0.33$ s

Hence at $t = 0.33$ s and $t = 1$ s, the particle is momentarily at rest.
$s = t^3 - 2t^2 + t + 10$
When $t = 1$, $s = 1^3 - 2(1)^2 + 1 + 10 = 10$ m
When $t = 0.33$, $s = (0.33)^3 - 2(0.33)^2 + 10$
$= 10.15$ m

Hence at $s = 10$ m and 10.15 m, the particle is momentarily at rest.

Example 2

An object is thrown into the air such that its height, h metres, after time t seconds, is $h = 36t - 4t^2$. Find

a) the velocity at the $t = 2$
b) the time taken to reach the maximum height.

Solution

a) $h = 36t - 4t^2$

$$\text{the velocity, } v = \frac{dh}{dt} = 36 - 8t$$

When $t = 2$, $v = 36 - 16 = 20$ m/s
$\therefore$ The velocity at $t = 2$ is 20 m/s.

b) When the object reaches the maximum height, the velocity, $v = 0$
i.e. $36 - 8t = 0$
$$t = \frac{36}{8} = 4.5 \text{ s}$$
The time taken to reach the maximum height is 4.5 s.

Example 3

An object projected vertically upwards satisfies the relation $h = 27t - 3t^2$, where h m is the height after t seconds

a) Find the time it takes to reach the highest point.
b) How high does it go?

Solution

a) $h = 27t - 3t^2$

$$\text{the velocity, } v = \frac{dh}{dt} = 27 - 6t$$

When the object reaches the highest point, the velocity, $v = 0$.
i.e. $27 - 6t = 0$
$t = \tfrac{27}{6} = 4.5$ s
The time taken to reach the highest point is 4.5 s.

b) To find the highest point, substitute $t = 4.5$ s into the expression for h;
i.e. the highest point,
$h_{\max} = 27(4.5) - 3(4.5)^2 = 60.75$ m.

Example 4

A ball is thrown vertically upwards. Its height (h m) above the ground at time t seconds is given by $h = 32t - 8t^2$. Find:

a) the velocity of the ball at time $t = 1$ s;
b) the maximum height reached.

Solution

a) $h = 32t - 8t^2$

the velocity, $v = \dfrac{dh}{dt} = 32 - 16t$

when $t = 1$ s, $v = 32 - 16(1) = 32 - 16$
$= 16 \text{ ms}^{-1}$

b) When the object reaches the maximum height, the velocity, $v = 0$
i.e. $32 - 16t = 0 \quad \Rightarrow \quad t = 32/16 = 2$ s
The time taken to reach the maximum height is 2 s.
To find the maximum height, substitute $t = 2$ s into the expression for h;
i.e. the maximum height,
$h_{\max} = 32(2) - 8(2)^2 = 32$ m.

Example 5

A particle P moves in a straight line so that its distance from a fixed point O after t seconds is s metres where $s = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t$. Show that the particle is at rest at two different times and find these times. Find the acceleration of the particle at these times and interpret your results.

Solution

$s = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t$,

$\therefore$ velocity, $v = \dfrac{ds}{dt} = t^2 - 3t + 2$

When the particle is at rest, $v = 0$
i.e. $t^2 - 3t + 2 = 0 \quad \Rightarrow \quad (t-1)(t-2) = 0$
$t = 1$ s and $t = 2$ s. Therefore the particle is at rest at two different times.

Acceleration, $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} = 2t - 3$

When $t = 1$ s, $a = 2(1) - 3 = -1 \text{ ms}^{-2}$
i.e. the particle is being retarded (or slowing down).
When $t = 2$ s, $a = 2(2) - 3 = 1 \text{ ms}^{-2}$
i.e. the particle is being accelerated (or speeding up).

Example 6

A particle moves in a straight line so that its distance, s, from a fixed point O after t s where $t \geq 0$, is given by $s = 9t^2 - 2t^3$. What is the velocity and acceleration of P when $t = 3$ s? Find also the distance of P from O when $t = 4$ s and show that it is moving towards O.

Solution

$s = 9t^2 - 2t^3$,

$\therefore$ velocity $v = \dfrac{ds}{dt} = 18t - 6t^2$

At time $t = 3$ s, the velocity of P is .
$v = 18(3) - 6(3)^2 = 0 \text{ ms}^{-1}$
i.e. the particle P is stationary.

Acceleration, $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} = 18 - 12t$

At time $t = 3$ s, the acceleration of P is
$a = 18 - 12(3) = -18 \text{ ms}^{-2}$
When $t = 4$ s, $s = 9(4)^2 - 2(4)^3 = 16$ m and
$v = 18(4) - 6(4)^2 = -24 \text{ ms}^{-1}$
Hence at $t = 4$ s, P is 16 m from O and is moving towards O.

Example 7

A particle projected from a fixed point with velocity 10 ms^{-1}, moves in a straight line such that its velocity after time t seconds is

$v = bt^2 + 7t + c$, where b and c are constants. One second after projection, its acceleration is $1\ \text{ms}^{-2}$. Find the values of b and c.

Solution

$v = bt^2 + 7t + c$

The initial velocity of the particle is $10\ \text{ms}^{-1}$ (i.e. at $t = 0$, $v = 10\ \text{ms}^{-1}$)

Substituting $t = 0$, $v = 10\ \text{ms}^{-1}$ into the expression for velocity, we have $c = 10$.

Acceleration, $a = \dfrac{dv}{dt} = 2bt + 7$

When $t = 1$ s, acceleration, $a = 1$

$$1 = 2b(1) + 7 \quad \Rightarrow \quad b = -3$$

Example 8

A particle moves along a straight line such that its distance x from a fixed point after t s is $x = -\frac{1}{3}t^3 + t^2 - \frac{5}{7}$. Find:

i) the time when the particle is momentarily at rest;

ii) the average acceleration of the particle between $t = 0$ and $t = 3$.

Solution

$x = -\frac{1}{3}t^3 + t^2 - \frac{5}{7}$,

$\therefore$ velocity $v = \dfrac{dx}{dt} = -t^2 + 2t$

i) When the particle is momentarily at rest, velocity $v = 0$ i.e. $-t^2 + 2t = 0$

$$t(-t + 2) = 0 \quad \Rightarrow \quad t = 0 \text{ s and } t = 2 \text{ s}$$

The particle is momentarily at rest when $t = 2$ s. Note that $t = 0$ is the starting point of the motion.

ii) When $t = 0$, $v = -(0)^2 + 2(0) = 0\ \text{ms}^{-1}$

and when $t = 3$, $v = -(3)^2 + 2(3) = -3\ \text{ms}^{-1}$

Average acceleration $= \dfrac{-3 - 0}{3 - 0} = -1\ \text{ms}^{-2}$

Example 9

A particle P moves along the x-axis, Ox, such that its distance s from a fixed point O after t seconds is given by $s = t^3 - 6t^2 + 9t$. Calculate the:

i) times at which the particle is stationary;

ii) distance of P from O at these times;

iii) acceleration of P at these times;

iv) velocity of P when its acceleration is zero.

Solution

$s = t^3 - 6t^2 + 9t$,

$\therefore$ velocity $v = \dfrac{ds}{dt} = 3t^2 - 12t + 9$

i) When the particle is stationary, $v = 0$

i.e. $3t^2 - 12t + 9 = 0$

$$(t - 1)(t - 3) = 0 \quad \Rightarrow \quad t = 1 \text{ s and } t = 3 \text{ s}$$

Hence the times at which the particle is stationary are $t = 1$ s and $t = 3$ s.

ii) $s = t^3 - 6t^2 + 9t$. Substitute the two times obtained in (i) above into s.

When $t = 1$ s, $s = (1)^3 - 6(1)^2 + 9(1) = 4$ m

When $t = 3$ s, $s = (3)^3 - 6(3)^2 + 9(3) = 0$ m

i.e. the particle is at starting point O again.

iii) Acceleration, $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} = 6t - 12$.

Substitute the two times obtained in (i) above into a.

When $t = 1$ s, $a = 6(1) - 12 = -6\ \text{ms}^{-2}$

i.e. the particle is being retarded (or slowing down).

When $t = 3$ s, $a = 6(3) - 12 = 6\ \text{ms}^{-2}$

i.e. the particle is being accelerated (or speeding up).

iv) When the particle's acceleration is zero,

$$a = 6t - 12 = 0 \quad \Rightarrow \quad t = 2 \text{ s}$$

At time $t = 2$, $v = 3(2)^2 - 12(2) + 9 = -3\ \text{ms}^{-1}$

i.e. particle is moving towards O.

Review exercise 20

1. If the equation of a curve is $y = 2x^2 - 3x - 2$, find
   i) the gradient at the point where $x = 0$
   ii) the coordinates of the points of intersection where the curve cuts the x-axis.
   iii) the gradient at each of the points found in (ii) .

2. Find the gradient of the curve $y = 2x^3 - 3x^2 + 5x - 1$ at the point
   i) $(0, -1)$
   ii) $(1, 3)$
   iii) $(-1, -11)$

3. Find the equation of the normal to the curve $y = x^2 + 4x - 3$ at the point where the curve cuts the y-axis.

4. Find the equation of the tangent to the curve $y = x^2 - 3x - 4$ at the point where this curve cuts the line $x = 5$.

5. Find the equation of the tangent to the curve $y = (2x - 3)(x - 1)$ at each of the points where this curve cuts the x-axis. Find the point of intersection of these tangents.

6. Find the equation of the normal to the curve $y = x^2 - 6x + 5$ at each of the points where the curve cuts the x-axis.

7. Find the equation of the tangent to the curve $y = 3x^2 + 5x - 1$ at each of the points of intersection of the curve and the line $y = x - 1$.

8. Find the equations of the tangent to the curve $y = x^2 + 5x - 3$ at the points where the line $y = x + 2$ crosses the curve.

9. Find the coordinates of the point on the curve $y = x^2$ at which the gradient is 8. Hence find the equation of the tangent to $y = x^2$ whose gradient is 8.

10. Find the coordinates of the point to the curve $y = 3x^2 - 1$ at which the gradient is 3.

11. Find the equation of the tangent to the curve $y = 4x^2 + 3x$ which has a gradient of -1.

12. Find the equation of the normal to the curve $y = 2x^2 - 2x + 1$ which has a gradient of $\frac{1}{2}$.

13. Find the value of k for which $y = 2x + k$ is a tangent to the curve $y = 2x^2 - 3$.

14. Find the equation of the tangent to the curve $y = (x - 5)(2x + 1)$ which is parallel to the x-axis.

15. Find the coordinates of the point(s) on the curve $y = x^2 - 5x + 3$ where the gradient of the normal is $\frac{1}{3}$.

16. A curve has the equation $y = x^3 - px + q$. The tangent to this curve at the point $(2, -8)$ is parallel to the x-axis. Find the values of p and q. Find also the coordinates of the other point where the tangent is parallel to the x-axis.

17. A is a fixed point on a straight line AB. The displacement from A of a particle P moving on this line, is given by $s = t^3 - 6t^2 + 9t$ at any time t. Find an expression for the velocity of P at time t and hence, using metres and seconds, find

a) the initial velocity,

b) the velocity when $t = 2$,

c) the speed when $t = 2$,

d) the times when P is momentarily at rest.

18. A particle starts from rest and travels in a straight line so that its acceleration is $(t + 3)\,\text{ms}^{-2}$ at any time t. Find the distance travelled in the interval of time from $t = 1$ to $t = 3$.

19. i) A particle P moves in a straight line so that its displacement, s metres, from a fixed point A on the line, is given by $s = (3t^2 + 4)$ at time t seconds.

a) Find the velocity and the acceleration of P when $t = 2$.

b) What was the initial distance of P from A.

ii) A particle P is moving on a straight line and O is a fixed point on the line. The displacement of P from O at time t seconds is s metres, where $s = 6 + 10t - t^2$. Find

a) the distance travelled from $t = 1$ to $t = 4$,

b) the velocity when $t = 6$,

c) the speed when $t = 6$.

20. The displacement, x metres, from O of an object P moving on a straight line Ox, is given by $x = 42t - 3t^2$ at time t seconds.

a) Find the velocity when $t = 6$, when $t = 7$ and when $t = 8$. Explain what happens during this interval of time.

b) Find the maximum distance of P from O during this time.

21. Find the equation of the tangent to the curve $y = 3x^3 - x$ at the point where $x = \frac{2}{3}$. Find also the equation of the tangent at the point where $x = \frac{1}{3}$. Find the equations of the normals at these points.

22. a) Differentiate $(3x + \sqrt{x})^2$ with respect to x.

b) Find the x coordinates of the points on the curve $y = \dfrac{x^2 - 1}{x}$ at which the gradient of the curve is 5.

23. Using implicit differentiation, or otherwise, find an expression for the slope of the tangent at the point (x_0, y_0) on the curve $x^2 + xy + y^2 - x - y = 0$. Find:

a) the equation of the tangent to the curve at the point $(\frac{2}{3}, \frac{2}{3})$,

b) the coordinates of the points at which the tangents to the curve are parallel to the x-axis,

c) the maximum and minimum values of y, distinguishing between them,

d) the coordinates of the points at which the tangents to the curve are parallel to the y-axis.

24. A hemispherical bowl of radius 6 cm contains water which is flowing into it at a constant rate. When the height of the water is h cm the volume V of water in the bowl is given by $V = \pi(6h^2 - \frac{1}{3}h^3)\text{ cm}^3$. Find the rate at which the water level is rising when $h = 3$, given that the time taken to fill the bowl is 1 minute.

25. An elastic sphere filled with water has a small hole from which the water leaks. When the water leaks away at the rate of $16\pi\text{ cm}^3/\text{s}$, the radius is decreasing at the rate of $\frac{1}{4}$ cm/s. Assuming that the spherical shape is preserved at the water leaks away, calculate, at the instant given by the above data,
a) the radius,
b) the rate at which the surface area is decreasing.

26. The radius of a circular oil slick is increasing at 1.5 m/s. Taking π to be 3.14, find, to 2 significant figures, the rate at which the area of the slick is increasing when its radius is 300 m.

27. a) The radius of a circular disc is increasing at a constant rate of 0.003 cm/s. Find the rate at which the area is increasing when the radius is 20 cm.
b) The area of another circular disc increases from 100π to $101\pi\text{ cm}^2$. Use calculus to find the corresponding increase in the radius.

28. The curve for which
$$\frac{dy}{dx} = a(x - p)(x - q),$$
where a, p and q are constants, has turning-points at (2, 0) and (1, 1).
i) State the value of p and q.
ii) Using these values, determine the value of a.

29. The curve $y = x^2 + ax + b$ has a turning point at (1, 3). Find a and b.

30. An open rectangular box is made of very thin sheet metal. Its volume is 128 cm^3, its width is x cm, and its length is $4x$ cm. Obtain an expression for its depth in terms of x. Show that the total surface area of its base, its ends and its sides is equal to $\left(4x^2 + \frac{320}{x}\right)\text{cm}^2$.

Calculate the dimensions of the box for which the surface area is minimum; explain why your answer gives a minimum rather than a maximum.

31. Differentiate the function $\dfrac{x+3}{\sqrt{(1+x^2)}}$ with respect to x. Find the value of x at which the function has a stationary value and determine the nature of the stationary value

32. A ball is thrown vertically upwards. Its height (h m) at time t (sec) is given by $h = 5 + 30t - 5t^2$. Find:
i) The velocity of the ball at time $t = 2$ s
ii) The maximum height

33. A particle moves in a straight line in such a way that its distance is metres from a fixed point O after time t seconds is given by
$s = 5t^3 - \frac{21}{2}t^2 + 6t - 4$.
Calculate
a) its initial velocity and velocity after 3 s.
b) the value of t when the body came to rest
c) its acceleration after 2 s

d) the value of t when the acceleration is 24 m/s^2

e) the average velocity during the third second

34. A missile fired from ground level rises x metres vertically upwards in t seconds and $x = 100t - \frac{25}{2}t^2$. Find

a) the initial velocity of the missile

b) the time when the height of the missile is maximum

c) the maximum height reached

d) the velocity with which the missile strikes the ground

35. Find the maximum and minimum points of the curve $y = x^3 - x^2 - 8x + 1$ and sketch the curve.

36. Find the coordinates of the stationary points on the curve $y = x^4 - 4x^3$ and identify their nature.

37. From each of the four corners of a rectangular sheet of metal 50 cm by 80 cm, a square of sides r cm is removed. The remainder is then formed into a tray, cuboid in shape. Find the value of r for which the volume contained by the tray is maximum. Find also, its maximum volume.

38. A rectangular sheet of metal 20 cm by 12 cm has squares each of side x cm cut off from its corners. The remainder is then formed into an open box of volume V cm^3. Find the value of x for which the volume of the box will be a maximum. Find this maximum volume.

39. Resistance to motion, F, of a moving vehicle is given by $F = \dfrac{5}{x} + 100x$. Determine the minimum value of resistance

40. A lidless box with square ends is to be made from a thin sheet of metal. Determine the least area of the metal for which the volume is 3.5 m^3.

41. Determine the maximum area of a rectangular piece of land that can be enclosed by 1200 m of fencing.

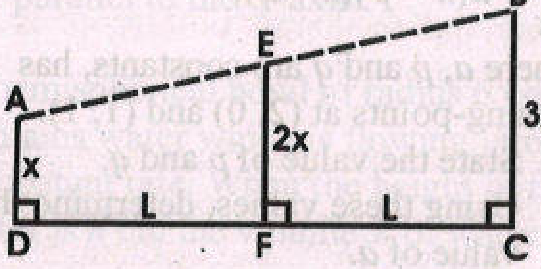
42. A rectangular area is formed having a perimeter of 40 cm. Determine the length and breadth of the rectangle if it is to enclose the maximum possible area.

43. Determine the height and radius of a cylinder of volume 200 cm^3 which has the least surface area.

44. The volume of a sphere is given by $\frac{4}{3}\pi r^3$. An elastic spherical balloon is being blown up so that the radius is increasing at a rate of 2 cm per second. Calculate the rate at which the volume is increasing when the radius is 10 cm.

45. If $k = 3\sqrt{x}$, find the approximate increase in the value of k if x increases from 4 to 4.01.

46. A farmer wishes to enclose two adjacent fields, each the shape of a trapezium as shown in the diagram below.



The side of *AB* is along the bank of a stream and requires no fencing. The farmer has 600 m of fencing material available for the remaining sides (including *EF*).

i) Express in terms of x
  a) the length L
  b) the total area A of the two fields.
ii) Find, in hectares, the maximum value of $A(1 \text{ ha} = 10^4 \text{m}^2)$.

47. The distance s m of a particle moving in a straight line from a fixed point O at a time t s is given by $s = 24 + 10t - 20t^2$. Find

i) the velocity and acceleration of the particle at time t.
ii) the time the particle is momentarily at rest and the distance from O at this instant.

48. A particle moves in a straight line in such a way that its distance s m from a fixed point O after t s is given by $s = t^3 - 2t^2 + t + 10$. Calculate

i) its velocity after 2s.
ii) its average velocity during the third seconds.
iii) the average acceleration form $t = 1$ to $t = 3$
iv) when and where the particle is momentarily at rest.

49. A hollow circular cone is held vertically downward beneath a water tap leaking at the rate of 2 cm^3/s. Find the rate of rise of water level when the depth is 6 cm given that the height of the cone is 1.8 cm and its radius is 12 cm.

50. Determine the turning points on the curve $y = 4 \sin x - 3 \cos x$ in the range $x = 0$ to $x = 360°$, and distinguish between them. Sketch the curve over one cycle.

51. Determine the height and radius of a cylinder of maximum volume which can be cut from a cone of height 10 cm and base radius 5 cm.

52. Find the height and radius of a closed cylinder of volume 125 cm^3 which has the least area.

53. A closed cylindrical container has a surface area of 400 cm^2. Determine the dimensions for maximum volume.

54. Calculate the height of a cylinder of maximum volume which can be cut from a cone of height 20 cm and base radius 80 cm.

55. A hollow circular cone is held vertically downward beneath a water tap leaking at the rate of 2 cm^3/s. Find the rate of rise of water level when the depth is 6 cm given that the height of the cone is 1.8 cm and its radius is 12 cm.

56. The curve $y = ax^2 + bx + c$ crosses the y-axis at the point (0, 3) and has stationary point at (1, 2). Find the values of a, b and c.

57. The gradient of the tangent to the curve $y = px^2 - qx - r$ at the point (1, –2) is 1. If the curve crosses the x- axis where $x = 2$, find the values of p, q and r. Find the point of intersection with the x-axis and sketch the curve.

58. Sketch the following curves:

i) $y = (3 - x)(1 + x)$
ii) $y = x^3 - 7x^2 + 15x - 9$
iii) $y = 3x^2 + 2$
iv) $y = x^3 - x$
v) $y = 2x^3 + x^2 - 4x + 1$
vi) $y = x^2 + 3$
vii) $y = x^3 + 3$
viii) $y = 3 - x + x^2$
ix) $y = x^2 - 4$

59. A closed cylindrical can has height h and base radius r. The volume is 0.01 cm^3. Show that $h = \dfrac{1}{100\pi r^2}$. Show further that S, the surface area, is given by $S = 2\pi r^2 + \dfrac{1}{50r}$. Hence find the value of r for which S is minimum.

60. An open rectangular box is made from a square sheet of cardboard by removing a square from each corner and joining the cut edges. If the cardboard is of edge 0.5 m, find the maximum volume of the box.

61. Find the equation of the tangent to the curve $y = 3\cos x - 2\sin x$ at $x = \frac{\pi}{4}$.

62. A cylindrical tin has a volume of 128π cm^3. Find the dimensions for the tin in order to have the minimum possible surface area, and find the common possible surface area.

63. Locate the stationary points on the curve $y = 2x^3 - 3x^2 - 12x + 18$ and determine the nature of each one. Hence sketch the curve.

64. Find the value of x when the maximum point occurs on the curve $y = x - 2\cos x$, $0 \le x \le 2\pi$

65. A curve whose equation is of the form $y = ax^2 + bx + 3$ has a turning point at P with coordinates (1, 2).

i) Calculate the values of a and b.
ii) Determine whether P is a maximum or minimum point.

66. Find the value of x when the maximum point on the curve $y = x - 2\sin x$, $0 \le x \le 2\pi$, occurs

67. The cubic polynomial defined by
$f(x) = ax^3 + bx^2 + cx + d$
has a minimum value –4 when $x = 1$ and a maximum value 4 when $x = -1$. Find $f(x)$.

68. If the volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$, write an expression for the approximate change in volume corresponding to a change δr in the radius. When the radius is 5 cm it is increased by 2%. Find the approximate percentage increase in the volume.

CHAPTER 21

INTEGRATION

Indefinite integrals – the process of integration

The process of **integration** reverses the process of differentiation. The process of finding a function from its derived function is called integration. In differentiation, if $y = 2x^2$ then $\frac{dy}{dx} = 4x$. Thus the integral of $4x$ is $2x^2$, i.e. integration is the process of moving from $\frac{dy}{dx}$ to y. Integration is also a process of summation or adding parts together and an elongated *S*, shown as $\int$, is used to replace the words 'the integral of'. In integration, the variable of integration is shown by adding *d*(the variable) after the function to be integrated. Thus $\int 4x\,dx$ means 'the integral of $4x$ with respect to x'.

To allow for the possible presence of a constant, whenever the process of integration is performed, a constant '*c*' is added to the result. Thus $\int 4x\,dx = 2x^2 + c$.

The function to be integrated is called the **integrand** and *c* is called the **arbitrary constant of integration**. The process of integration where the constant of integration is not known is called **indefinite integral**.

$\int \ldots dx$ means the integral of …with respect to x.

The general solution of integrals of the form ax^n

The general solution of integrals of the form $\int ax^n\,dx$, where *a* and *n* are constants is given by:

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$$

This rule is true when *n* is fractional, zero or a positive or negative integer, with the exception of $n = -1$.

The rule is:

To integrate a power or index of *x*, *add one* to the power or index and divide by the *new index*. Using this rule gives:

i) $\int x^5\,dx = \frac{x^{5+1}}{5+1} + c = \frac{1}{6}x^6 + c$

ii) $\int 3x^2\,dx = \frac{3x^{2+1}}{2+1} + c = x^3 + c$

iii) $\int 2x^4\,dx = \frac{2x^{4+1}}{4+1} + c = \frac{2}{5}x^5 + c$

iv) $\int \frac{2}{x^2}\,dx = \int 2x^{-2}\,dx = \frac{2x^{-2+1}}{-2+1} + c$

$= \frac{2x^{-1}}{-1} + c = -\frac{2}{x} + c$

v) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{2}{3}x^{\frac{3}{2}} + c$

vi) $\int x^{-4}\,dx = \frac{x^{-4+1}}{-4+1} + c = -\frac{1}{3}x^{-3} + c$

Note: *c* is called the constant of integration. This must be added to the result in the case of indefinite integrals.

Integrating a constant, *k*

The integral of a constant *k* is $kx + c$

i.e. $\int k\,dx = kx + c$.

For example, $\int 7\,dx = 7x + c$.

Integrating a sum or difference of functions

The integral of a function consisting of the sum or difference of a number of terms is given by the integral of each term. Thus

$$\int\{f(x)+g(x)-h(x)\}dx$$

$$=\int f(x)dx+\int g(x)dx-\int h(x)dx+c$$

This result can be generalised to cover any finite number of functions.

Illustrative examples

Example 1

i) $\int(3x+2x^2-5)\,dx=\int 3x dx+\int 2x^2 dx-\int 5dx$

$$=\frac{3x^2}{2}+\frac{2x^3}{3}-5x+c$$

ii) $\int\left(x^3+\frac{1}{x^2}+\sqrt{x}\right)dx=\int\left(x^3+x^{-2}+x^{\frac{1}{2}}\right)dx$

$$=\tfrac{1}{4}x^4-x^{-1}+\tfrac{2}{3}x^{\frac{3}{2}}+c$$

Example 2

Find the integrals of the following:

i) $\int\left(\frac{t^3+3t^2-1}{t^2}\right)dt$ ii) $\int\left(\frac{2x^3-3x}{4x}\right)dx$

iii) $\int(1-t)^2dt$ iv) $\int\left(\frac{(1+x)^2}{\sqrt{x}}\right)dx$

Solution

i) $\int\left(\frac{t^3+3t^2-1}{t^2}\right)dt=\int\left(\frac{t^3}{t^2}+\frac{3t^2}{t^2}-\frac{1}{t^2}\right)dt$

$$=\int(t+3-t^{-2})\,dt=\frac{t^2}{2}+3t-\frac{t^{-1}}{-1}+c$$

$$=\tfrac{1}{2}t^2+3t+\frac{1}{t}+c$$

ii) $\int\left(\frac{2x^3-3x}{4x}\right)dx=\int\left(\frac{2x^3}{4x}-\frac{3x}{4x}\right)dx$

$$=\int\left(\tfrac{1}{2}x^2-\tfrac{3}{4}\right)dx=\left(\tfrac{1}{2}\right)\frac{x^{2+1}}{2+1}-\tfrac{3}{4}x+c$$

$$=\tfrac{1}{6}x^3-\tfrac{3}{4}x+c$$

iii) $\int(1-t)^2dt=\int\left(1-2t+t^2\right)dt$

$$=t-\frac{2t^{1+1}}{1+1}+\frac{t^{2+1}}{2+1}+c=t-t^2+\tfrac{1}{3}t^3+c$$

iv) $\int\left(\frac{(1+x)^2}{\sqrt{x}}\right)dx=\int\frac{(1+2x+x^2)}{x^{\frac{1}{2}}}dx$

$$=\int\left(\frac{1}{x^{\frac{1}{2}}}+\frac{2x}{x^{\frac{1}{2}}}+\frac{x^2}{x^{\frac{1}{2}}}\right)dx$$

$$=\int\left(x^{-\frac{1}{2}}+2x^{\frac{1}{2}}+x^{\frac{3}{2}}\right)dx$$

$$=2x^{\frac{1}{2}}+\tfrac{4}{3}x^{\frac{3}{2}}+\tfrac{2}{5}x^{\frac{5}{2}}+c$$

This problem shows that functions often have to be arranged into the standard form $\int ax^n\,dx$ before it is possible to integrate them.

Example 3

Find the integrals of the following with respect to x:

i) $\int\left(2x^2-\frac{1}{x^2}+x\right)dx$

ii) $\int\left(\sqrt{x}+\frac{1}{\sqrt[3]{x}}\right)dx$

iii) $\int\left(3x^3+\frac{1}{x^3}+3\right)dx$

iv) $\int\left(\frac{4}{x^3}-\frac{1}{x^2}-x^2\right)dx$

Solution

i) $\int\left(2x^2-\frac{1}{x^2}+x\right)dx=\frac{2}{3}x^3+\frac{1}{x}+\frac{x^2}{2}+c$

ii) $\int\left(\sqrt{x}+\frac{1}{\sqrt[3]{x}}\right)dx=\frac{2}{3}x^{\frac{3}{2}}+\frac{3}{2}x^{\frac{2}{3}}+c$

iii) $\int\left(3x^3+\frac{1}{x^3}+3\right)dx$

$=\frac{3}{4}x^4-\frac{1}{2}x^{-2}+3x+c$

iv) $\int\left(\frac{4}{x^3}-\frac{1}{x^2}-x^2\right)dx=-\frac{2}{x^2}+\frac{1}{x}-\frac{x^3}{3}$

[Work out the missing steps]

Definite integrals

Integrals containing an arbitrary constant c in their results are called **indefinite integrals** since their precise value cannot be determined without further information.

Definite integrals are those in which limits are applied. The definite integral from a to b of $f(x)$ is given by:

$$\int_a^b f(x)dx=[F(x)]_a^b=F(b)-F(a)$$

where $F(x)=\int f(x)dx$

This means the value of the integral of $f(x)$ with respect to x when $x=b$ minus the value of this integral when $x=a$. The symbols a and b are called the *lower* and *upper* limits of integration respectively. Such integrals are called definite integrals because their exact values can be found. Note that the constant c always cancels out when the limits are applied and need not be shown with definite integrals.

Illustrative examples

Example 1

Evaluate i) $\int_1^3(3x^2+2x+4)dx$

ii) $\int_1^2\frac{(x+1)(x^2-2x+2)}{x^2}dx$

Solution

i) $\int_1^3(3x^2+2x+4)dx=\left[x^3+x^2+4x\right]_1^3$

$=\left(3^3+3^2+4(3)\right)-\left(1^3+1^2+4(1)\right)$

$=(48)-(6)=42$

ii) For this type of integral, first expand to remove the brackets and then divide before pursuing the integration.

$$\int_1^2\frac{(x+1)(x^2-2x+2)}{x^2}dx$$

$$=\int_1^2\left(\frac{x^3-2x^2+2x+x^2-2x+2}{x^2}\right)dx$$

$$=\int_1^2\left(\frac{x^3-x^2+2}{x^2}\right)dx$$

$$=\int_1^2\left(x-1+\frac{2}{x^2}\right)dx$$

$$=\int_1^2\left(x-1+2x^{-2}\right)dx=\left[\frac{x^2}{2}-x-\frac{2}{x}\right]_1^2$$

$$=\left(\frac{4}{2}-2-\frac{2}{2}\right)-\left(\frac{1}{2}-1-\frac{2}{1}\right)=(-1)-(-\tfrac{5}{2})=\tfrac{3}{2}$$

Example 2

Evaluate i) $\int_{-2}^2\left(\frac{x^3-x^2+1)}{x^2}\right)dx$

ii) $\int_{-2}^2\left(x^2-2\right)(2x-1))dx$

Solution

i) $\int_{-2}^2\left(\frac{x^3-x^2+1)}{x^2}\right)dx$

$$=\int_{-2}^2\left(\frac{x^3}{x^2}-\frac{x^2}{x^2}+\frac{1}{x^2}\right)dx$$

$$=\int_{-2}^2\left(x-1+x^{-2}\right)dx=\left[\frac{x^2}{2}-x-\frac{1}{x}\right]_{-2}^2$$

$$=\left(\frac{4}{2}-2-\frac{1}{2}\right)-\left(\frac{4}{2}+2+\frac{1}{2}\right)$$

$$=(-\tfrac{1}{2})-(\tfrac{9}{2})=-5$$

ii) $\int_{-2}^{2}\left(x^2-2\right)\left(2x-1\right)dx$

$$=\int_{-2}^{2}\left(2x^3-x^2-4x+2\right)dx$$

$$=\left[\frac{x^4}{2}-\frac{x^3}{3}-2x^2+2x\right]_{-2}^{2}$$

$$=\left(\frac{16}{2}-\frac{8}{3}-2(2)^2+2(2)\right)-\left(\frac{16}{2}+\frac{8}{3}-2(-2)^2+2(-2)\right)$$

$$=(\tfrac{4}{3})-(-\tfrac{4}{3})=\tfrac{8}{3}$$

Example 3

$\int_{-1}^{a}2(x+2)dx=15$ where $a>0$,

find the value of a.

Solution

$$\int_{-1}^{a}2(x+2)dx=15$$

$$\int_{-1}^{a}(2x+4)dx=15$$

$$\left[x^2+4x\right]_{-1}^{a}=15$$

$$(a^2+4a)-[(-1)^2+4(-1)]=15$$

$$a^2+4a+3=15$$

$$a^2+4a-12=0$$

$$(a-2)(a+6)=0$$

$$a=2 \text{ or } a=-6$$

But $a>0 \therefore a=2$

Integration by substitution

Functions which require integrating are not always in the 'standard' forms shown above. However, it is often possible to change a function into a form which can be integrated by using an algebraic **substitution**. With algebraic substitutions, the substitution usually made is to let u be equal to $f(x)$ such that $f(u)du$ is a standard integral. It is found that integrals of the forms:

$$k\int f'(x)[f(x)]^n\,dx \quad \text{and} \quad k\int\frac{f'(x)}{[f(x)]^n}dx$$

can both be integrated by substituting u for $f(x)$

Illustrative examples

Example 1

Find $\int x\sqrt{1-x^2}\,dx$

Solution

Let $u=1-x^2$, then $\frac{du}{dx}=-2x$

$$dx=\frac{du}{-2x} \text{ (i.e. rearranging)}$$

Hence

$$\int x\sqrt{1-x^2}\,dx=\int x\sqrt{u}\,\frac{du}{-2x}=-\frac{1}{2}\int u^{\frac{1}{2}}du$$

$$=-\frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right)+c=-\frac{1}{3}u^{\frac{3}{2}}+c$$

$$=-\tfrac{1}{3}\sqrt{u^3}+c$$

Rewriting $u=1-x^2$ gives:

$$\int x\sqrt{1-x^2}\,dx=-\tfrac{1}{3}\sqrt{(1-x^2)^3}+c$$

Example 2

Integrate the following integrals with respect to x

i) $\int\cos(3x+7)\,dx$ ii) $\int(2x-5)^7\,dx$

iii) $\int 3x(4x^2+3)^5\,dx$ iv) $\int\frac{2x}{\sqrt{(4x^2-1)}}dx$

v) $\int\cos x\sin^3 x\,dx$

Solution

i) Let $u = 3x + 7$, then $\frac{du}{dx} = 3$

$$dx = \frac{du}{3} \text{ (i.e. rearranging)}$$

Hence $\int \cos(3x + 7)\,dx = \int (\cos u)\frac{du}{3}$

$$= \frac{1}{3}\int \cos u \; du = \tfrac{1}{3}\sin u + c$$

Rewriting $u = 3x + 7$ gives:

$\int \cos(3x + 7)\,dx = \tfrac{1}{3}\sin(3x + 7) + c$

ii) Let $u = 2x - 5$, then $\frac{du}{dx} = 2$

$$dx = \frac{du}{2} \text{ (i.e. rearranging)}$$

Hence $\int (2x - 5)^7\,dx = \int u^7 \frac{du}{2} = \frac{1}{2}\int u^7\,du$

Rewriting $u = 2x - 5$ gives:

$\int (2x - 5)^7\,dx = \tfrac{1}{16}(2x - 5)^8 + c$

iii) Let $u = 4x^2 + 3$, then $\frac{du}{dx} = 8x$

$$dx = \frac{du}{8x} \text{ (i.e. rearranging)}$$

Hence

$$\int 3x(4x^2 + 3)^5\,dx = \int 3x\,u^5 \frac{du}{8x} = \frac{3}{8}\int u^5\,du$$

$$= \frac{3}{8}\left(\frac{u^6}{6}\right) + c = \frac{1}{16}u^6 + c$$

Rewriting $u = 4x^2 + 3$ gives:

$$\int 3x(4x^2 + 3)^5\,dx = \frac{1}{16}(4x^2 + 3)^6 + c$$

iv) Let $u = 4x^2 - 1$, then $\frac{du}{dx} = 8x$

$$dx = \frac{du}{8x} \text{ (i.e. rearranging)}$$

Hence

$$\int \frac{2x}{\sqrt{(4x^2 - 1)}}\,dx = \int \frac{2x}{\sqrt{u}}\frac{du}{8x} = \frac{1}{4}\int \frac{1}{\sqrt{u}du}$$

$$= \frac{1}{4}\int u^{-\frac{1}{2}}\,du = \frac{1}{4}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right] + c = \tfrac{1}{2}\sqrt{u} + c$$

Rewriting $u = 4x^2 - 1$ gives:

$$\int \frac{2x}{\sqrt{(4x^2 - 1)}}\,dx = \tfrac{1}{2}\sqrt{(4x^2 - 1)} + c$$

v) Let $u = \sin x$, then $\frac{du}{dx} = \cos x$

$$dx = \frac{du}{\cos x} \text{ (i.e. rearranging)}$$

Hence

$$\int \cos x \sin^3 x\,dx = \int \cos x\, u^3 \frac{du}{\cos x}$$

$$= \int u^3\,du = \tfrac{1}{4}u^4 + c$$

Rewriting $u = \sin x$ gives:

$\int \cos x \sin^4 x\,dx = \tfrac{1}{4}\sin^4 x + c$

This result shows that in general:

$\int \cos x \sin^n x\,dx = \tfrac{1}{n+1}\sin^{n+1} x + c$ and also

$\int \sin x \cos^n x\,dx = \tfrac{-1}{n+1}\cos^{n+1} x + c$

Example 3

Find the following integrals by making the substitution suggested.

i) $\int x(x^2 - 3)^4\,dx$; $u = x^2 - 3$

ii) $\int x^2(1 - x^3)\,dx$; $u = 1 - x^3$

iii) $\int \cos x \sin^4 x dx$; $u = \sin x$

iv) $\int (2 + 7x)^7\,dx$; $u = 2 + 7x$

v) $\int \frac{1}{(4x + 5)^3}\,dx$; $u = 4x + 5$

vi) $\int \frac{1}{\sqrt{(1 - 2x)}}\,dx$; $u = 1 - 2x$

Solution

i) $\int x(x^2-3)^4\,dx = \frac{1}{10}((x^2-3)^5+c$

ii) $\int x^2(1-x^3)\,dx = \frac{1}{6}(1-x^3)^2+c$

iii) $\int \cos x\sin^4 x dx = \frac{1}{5}\sin^5 x + c$

iv) $\int (2+7x)^7\,dx = \frac{1}{56}(2+7x)^8+c$

v) $\int \frac{1}{(4x+5)^3}dx = -\frac{1}{8(4x+5)^2}+c$

vi) $\int \frac{1}{\sqrt{(1-2x)}}dx = -\frac{1}{2}\sqrt{(1-2x)}+c$

[Work out the solution in each case.]

Summary

1. $\int (ax+b)^n\,dx = \frac{1}{(a)(n+1)}(ax+b)^{n+1}$

   or make the substitution $u = ax + b$

   For example,

   $$\int (2x+1)^3\,dx = \frac{1}{(2)(3+1)}(2x+1)^{3+1}+c$$
   $$= \tfrac{1}{8}(2x+1)^4$$

2. If $\int f'(x)[f(x)]^n\,dx$ and $\int \frac{f'(x)}{[f(x)]^n}dx$ make the substitution $u = f(x)$.

Change of limits

When evaluating definite integrals involving substitutions, it is sometimes convenient to **change the limits** of the integrals as shown in the examples below. This allows direct calculation of the required integral instead of using the old limits which will require you to change to the original variables.

Illustrative examples

Example 1

Evaluate the following taking positive square roots only.

i) $\int_0^2 \frac{3x}{\sqrt{2x^2+1}}dx$ ii) $\int_1^3 5x\sqrt{2x^2+7}\,dx$

Solution

i) Let $u = 2x^2+1$, then $\frac{du}{dx} = 4x$

$$dx = \frac{du}{4x} \text{ (i.e. rearranging)}$$

Hence

$$\int_0^2 \frac{3x}{\sqrt{2x^2+1}}dx = \int_0^2 \frac{3x}{\sqrt{u}}\,\frac{du}{4x} = \frac{3}{4}\int_0^2 u^{-\frac{1}{2}}du$$

It is possible to change the limits of integration.

Since $u = 2x^2+1$, when

$x=2,\ u=2(2)^2+1=9$ and when

$x=0,\ u=2(0)^2+1=1$ and

Thus $\frac{3}{4}\int_0^2 u^{-\frac{1}{2}}du = \frac{3}{4}\int_1^9 u^{-\frac{1}{2}}du$

i.e. the limits have been changed

$$\frac{3}{4}\int_1^9 u^{-\frac{1}{2}}du = \frac{3}{4}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^9 = \tfrac{3}{2}(\sqrt{9}-\sqrt{1}) = 3$$

ii) Let $u = 2x^2+7$, then

$$\frac{du}{dx} = 4x \quad \Rightarrow dx = \frac{du}{4x}$$

(i.e. rearranging) It is possible to change the limits of integration.

Since $u = 2x^2+1$,when

$x=3,\ u=2(3)^2+7=25$ and when

$x=1,\ u=2(1)^2+7=9$ and hence

$$\int_1^3 5x\sqrt{2x^2+7}\,dx = \int_9^{25} 5x\sqrt{u}\,\frac{du}{4x}$$
$$= \frac{5}{4}\int_9^{25}\sqrt{u}\,du = \frac{5}{4}\int_9^{25} u^{\frac{1}{2}}du$$

Thus the limits have changed, and it is unnecessary to change the integral back in terms of x. Thus

$$\int_1^3 5x\sqrt{2x^2+7}\,dx$$

$$=\frac{5}{4}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_9^{25}=\frac{5}{6}\left[\sqrt{u^3}\right]_9^{25}$$

$$=\frac{5}{6}\left[\sqrt{25^3}-\sqrt{9^3}\right]=\tfrac{5}{6}(125-27)=81\tfrac{2}{3}$$

Example 2

Using the substitution $u = x + 1$, evaluate

$$\int_1^3 \frac{x-1}{(x+1)^3}\,dx.$$

Solution

$$\int_1^3 \frac{x-1}{(x+1)^2}dx \;; \quad u=x+1 \;\Rightarrow\; du=dx$$

When $x = 3$, $u = 3 + 1 = 4$

When $x = 1$, $u = 1 + 1 = 2$

$$\int_2^4 \frac{u-2}{u^3}du = \int_2^4 (u^{-2}-2u^{-3})\,du$$

$$=\left[-\frac{1}{u}+\frac{1}{u^2}\right]_2^4=\left[\frac{1}{u^2}-\frac{1}{u}\right]_2^4$$

$$=\left[\left(\frac{1}{16}-\frac{1}{4}\right)-\left(\frac{1}{4}-\frac{1}{2}\right)\right]$$

$$=\frac{1}{16}=0.0625$$

Example 3

Evaluate the following by making the appropriate substitution.

i) $\int_0^1 x^2\sqrt{x^3+1}\,dx$

ii) $\int_0^2 x(x^2+1)^3\,dx$

iii) $\int_0^{\frac{\pi}{2}} \sin^4 x\cos x\,dx$

iv) $\int_5^6 (6-x)^4\,dx$

v) $\int_{-1}^{12} \frac{3}{\sqrt[3]{(2y+3)}}dy$

vi) $\int_2^{10} \frac{1}{\sqrt{(2y+5)}}\,dy$

Solution

i) $\tfrac{2}{9}\left(2\sqrt{2}-1\right)$ ii) 78 iii) $\frac{1}{5}$

iv) $\frac{1}{5}$ v) 8 vi) 2

[Work out the solution in each case]

Integration of simple trigonometric functions

Referring to the derivatives of trigonometric functions in *chapter 19*, we can write the following standard integrals:

1. $\int\cos x\,dx=\sin x+c$
2. $\int\sin x\,dx=-\cos x+c$
3. $\int\sec^2 x\,dx=\tan x+c$
4. $\int\operatorname{cosec}^2 x\,dx=-\cot x+c$
5. $\int\sec x\tan x\,dx=\sec x+c$
6. $\int\operatorname{cosec}x\cot x\,dx=-\operatorname{cosec}x+c$

In general,

$$\int\cos(ax+b)\,dx=\frac{1}{a}\sin(ax+b)+c \quad \text{and}$$

$$\int\sin(ax+b)\,dx=-\frac{1}{a}\cos(ax+b)+c$$

Similar results apply for the remaining trigonometric integrals.

Illustrative example

Integrate the following integrals with respect to x

i) $\int 3\sec^2 x\,dx$

ii) $\int\sin 4x\,dx$

iii) $\int(\sec^2 2x-\operatorname{cosec}^2 4x)dx$

iv) $\int 3\cos(4x-\pi/2)\,dx$

v) $\int(\cos 3x+3\sin x)dx$

vi) $\int 2\sin(3x+\alpha)\,dx$

Solution

i) $\int 3\sec^2 x\,dx = 3\tan x + c$

ii) $\int \sin 4x\,dx = -\frac{1}{4}\cos 4x + c$

iii) $\int (\sec^2 2x - \operatorname{cosec}^2 4x)dx$

$= \frac{1}{2}\tan 2x + \frac{1}{4}\cot 4x + c$

iv) $\int 3\cos(4x - \pi/2)\,dx$

$= \frac{3}{4}\sin(4x - \pi/2) + c$

v) $\int (\cos 3x + 3\sin x)dx$

$= \frac{1}{3}\sin 3x - 3\cos x + c$

vi) $\int 2\sin(3x+\alpha)\,dx = -\frac{2}{3}\sin(3x+\alpha)$

Integration of powers of some trigonometric functions

Even powers of sin *x* and cos *x*

Use the double angle trigonometric identities:

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$ and

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

For example, to find $\int \cos^2 x\,dx$, we use $\cos^2 x = \frac{1}{2}(1+\cos 2x)$. Thus

$\int \cos^2 x\,dx = \int \frac{1}{2}(1+\cos 2x)dx$

$= \int (\frac{1}{2} + \frac{1}{2}\cos 2x)dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$

Also to find $\int \sin^2 x\,dx$, we use $\sin^2 x = \frac{1}{2}(1-\cos 2x)$. Thus

$\int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)dx$

$= \int (\frac{1}{2} - \frac{1}{2}\cos 2x)dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

Odd powers of sin *x* and cos *x*

Use the trigonometric identity:

$\cos^2 x + \sin^2 x = 1$

For example, to find $\int \sin^3 x\,dx$, we use $\sin^2 x = 1 - \cos^2 x$. Thus

$\int \sin^3 x\,dx = \int (\sin x)(\sin^2 x)dx$

$= \int \sin x(1-\cos^2 x)dx = \int (\sin x - \sin x\cos^2 x)dx$

$= -\cos x - \frac{1}{3}\cos^3 x + c$

Note: Use substitution or $\int \sin x\cos^n x\,dx = \frac{-1}{n+1}\cos^{n+1} x + c$ to find $\int \sin x\cos^2 x\,dx$.

Powers of tan *x*

The identity $\tan^2 x + 1 = \sec^2 x$ or $\tan^2 x = \sec^2 x - 1$ is helpful in integrating any power of tan *x*. Thus

$\int \tan^2 x\,dx = \int (\sec^2 x - 1)dx = \tan x - x + c$

Review exercise 21

1. Determine the following indefinite integrals:

i) $\int \frac{2}{5}x^2 dx$ ii) $\int \frac{5}{6}x^3 dx$

iii) $\int \sqrt{x^3}\,dx$ iv) $\int \frac{4}{3x^2}dx$

v) $\int \frac{3}{4x^4}dx$ vi) $\int \frac{1}{\sqrt{x}}dx$

vii) $\int (2+3x-4x^2)dx$ viii) $\int (x-4x^2)dx$

ix) $\int (x-1)^2 dx$ x) $\int \frac{3x^2-5x}{x}dx$

xi) $\int \frac{x^3-x+1}{x^3}dx$ xii) $\int (t-1)(t+2)dt$

xiii) $\int (x^3 - 3x + \frac{1}{2x^2})dx$

xiv) $\int \frac{x+1}{x^3}dx$

2. Evaluate the following definite integrals:

i) $\int_1^2 (x^2+3)dx$

ii) $\int_0^1 (x^4-5)dx$

iii) $\int_{-1}^1 (x^2+2x+2)dx$

iv) $\int_{-1}^3 (x+\frac{2}{x^2})dx$

v) $\int_1^3 2x(x+3)dx$

vi) $\int_{-2}^0 x(x-2)dx$

vii) $\int_1^4 (x^2+1)^2 dx$

viii) $\int_1^2 \left(\frac{x^2+1}{x^2}\right)dx$

ix) $\int_1^2 \left(\frac{x^4-1}{x^3}\right)dx$

x) $\int_0^1 x^2(1-x)^2 dx$

xi) $\int_0^1 x(x-2)(x+1)dx$

xii) $\int_{-1}^2 (x+1)(2x-3)dx$

xiii) $\int_0^2 x^3 dx$

xiv) $\int_1^2 (2x-1)dx$

xv) $\int_{1/2}^1 \frac{dx}{x^2}$

xvi) $\int_0^{\frac{1}{3}} x^2 dx$

xvii) $\int_{-2}^{-1} (x^2+x-1)dx$

xviii) $\int_1^0 x(x-1)dx$

xix) $\int_0^2 (x+1)^2 dx$

xx) $\int_1^4 \left(\frac{1}{(x+3)^2}\right)dx$

xxi) $\int_0^{\frac{\pi}{2}} \cos x\, dx$

xxii) $\int_0^1 x^2\sqrt{(x^3+1)}\, dx$

xxiii) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 2x\, dx$

xxiv) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^4 x \cos x\, dx$

3. Find a if

i) $\int_1^a (3x^2-2x-4)dx = 10$, $a>1$

ii) $\int_2^a (2x+3)dx = 8$, $a>0$

iii) $\int_{-1}^1 (ax^2-3x+4)dx = \frac{28}{3}$

4. Evaluate

i) $\int_0^2 (2x+3)^3 dx$

ii) $\int_{-2}^2 (x-2)(2x+1)dx$

5. Calculate the value of this integral

$$\int_4^9 \left(\frac{3}{\sqrt{x}} - \frac{1}{x\sqrt{x}}\right)dx$$

6. Evaluate

i) $\int_1^2 \left(\frac{x^4-1}{x^3}\right)dx$

ii) $\int_0^2 x(x-2)dx$

iii) $\int_1^2 \frac{(x-1)(x^2-2x-2)}{x^2}dx$

7. Determine the following indefinite integrals:

i) $\int (2x-1)^2 dx$

ii) $\int (2+7x)^3 dx$

iii) $\int 4\sqrt{(1-x)}\, dx$

iv) $\int 3\sqrt{(2-5x)}\,dx$

v) $\int \frac{1}{(4x+5)^3}\,dx$

vi) $\int \frac{1}{\sqrt{1-2x}}\,dx$

vii) $\int \frac{1}{\sqrt[3]{(3-7x)}}\,dx$

viii) $\int \left(\frac{3}{(2x+1)^3}+\sqrt{(1-2x)}\right)dx$

8. Determine the following indefinite integrals

i) $\int \sin 2x dx$

ii) $\int 3\cos(4x-\pi/2)dx$

iii) $\int \sec^2(\pi/3-2x)\,dx$

iv) $\int \operatorname{cosec}^2(3x+\pi/6)\,dx$

v) $\int (\cos 3x+3\sin x)dx$

vi) $\int (\sec^2 2x-\operatorname{cosec}^2 4x)dx$

9. Evaluate the following definite integrals:

i) $\int_2^3 (x^2+2x-1)dx$

ii) $\int_2^7 \sqrt{(x+2)}dx$

iii) $\int_0^{\frac{\pi}{2}} \sin 2x dx$

iv) $\int_0^{\frac{\pi}{6}} \sec x^2\, 2x dx$

v) $\int_0^{\frac{\pi}{2}} \cos x\sqrt{\sin x}\ dx$

vi) $\int_0^1 x^2(x^3+1)^4\,dx$

vii) $\int_1^3 \left(x^2+\frac{1}{x^2}\right)dx$

viii) $\int_0^{\frac{\pi}{6}} (\cos x\sqrt{1-2\sin x}\,)\,dx$

ix) $\int_0^2 (x+1)\sqrt{(x^2+2x)}dx$

x) $\int_0^{\pi} \left(\frac{\sin x}{\cos^4 x}\right)dx$

xi) $\int_0^{\frac{\pi}{6}} \sin^2 x dx$

xii) $\int_0^2 x(x^2+1)dx$

xiii) $\int_{-2}^5 (x^2-3x-10)dx$

xiv) $\int_{-1}^0 (x-1)^3\,dx$

xv) $\int_0^{\pi} \pi x^2\,dx$

xvi) $\int_{-1}^2 \frac{3}{\sqrt{x+2}}\,dx$

10. Given that

i) $\int_{-2}^{a} x(x+1)dx=\frac{16}{3}$

ii) $\int_{-2}^{3} (ax-1)dx=0$

iii) $\int_1^a \left(\frac{x^4-1}{x^3}\right)dx=0,$

find the value of a in each case.

11. Evaluate $\int_0^{\frac{\pi}{6}} \sin 3x dx$

12. Determine the following:

i) $\int \sin^4 x dx$

ii) $\int \cos^3 x dx$

iii) $\int \sin^3 x dx$

iv) $\int_2^4 \left(\frac{x-1}{x^3}\right)dx$

13. If $\int_{-2}^{3} (kt-1)dt=0$, find the value of k

CHAPTER 22

APPLICATIONS OF INTEGRATION

1. Finding the equation of a curve when given the gradient function

When the gradient function of a curve, $\frac{dy}{dx}$ is given, we can find the equation of the curve using integration.

Thus, given $\frac{dy}{dx} = f(x) \quad \Rightarrow \quad y = \int f(x)\,dx$,

where $f(x)$ is a known function of x.
Further information will be given to enable you determine the constant of integration.

Illustrative examples

Example 1
A curve passes through the point (1, 0) and its gradient at any point $P(x, y) = 3x^2 - 1$. Find the equation of the curve.

Solution
The gradient of the curve at any point is

$3x^2 - 1$ i.e. $\frac{dy}{dx} = 3x^2 - 1$

$\therefore \ y = \int (3x^2 - 1)\,dx = x^3 - x + c$, where c is an arbitrary constant which must be determined.
Now the curve passes through the point (1, 0) $\Rightarrow$ when $x = 1, y = 0$
Substituting these values in $y = x^3 - x + c$

$$0 = 1^3 - 1 + c \quad \Rightarrow \quad c = 0$$

Therefore the equation of the curve is $y = x^3 - x$.

Example 2
The gradient of a curve at any point $P(x, y)$ on it is given by $4x^2 - 3x + 1$, if the curve passes through the point $A(1, 0)$, find the equation of the curve.

Solution
The gradient of the curve at any point is

$4x^2 - 3x + 1$ i.e. $\frac{dy}{dx} = 4x^2 - 3x + 1$

$\therefore \ y = \int (4x^2 - 3x + 1)\,dx = \frac{4}{3}x^3 - \frac{3}{2}x^2 + x + c$

where c is an arbitrary constant which must be determined. Now the curve passes through the point $A(1, 0)$

when $x = 1, y = 0$

Substituting these values in

$$y = \tfrac{4}{3}x^3 - \tfrac{3}{2}x^2 + x + c$$

$$0 = \tfrac{4}{3} - \tfrac{3}{2} + 1 + c \quad \Rightarrow c = -\tfrac{5}{6}$$

Therefore the equation of the curve is

$$y = \tfrac{4}{3}x^3 - \tfrac{3}{2}x^2 + x - \tfrac{5}{6}$$

$$6y = 8x^3 - 9x^2 + 6x - 5$$

Example 3
The gradient of a curve at any point is given by $2x - 3x^2$. If the curve passes through the point (1, 2), find the equation of the curve.

Solution
The gradient of the curve at any point is

$2x - 3x^2$ i.e. $\frac{dy}{dx} = 2x - 3x^2$

$\therefore \ y = \int (2x - 3x^2)\,dx = x^2 - x^3 + c$ where c is an arbitrary constant which must be determined. Now the curve passes through the point (1, 2) $\Rightarrow$ when $x = 1, y = 2$
Substituting these values in $y = x^2 - x^3 + c$

$$2 = 1 - 1 + c \quad \Rightarrow \quad c = 2$$

Therefore the equation of the curve is $y = x^2 - x^3 + 2$.

Example 4

A curve passes through the point $(1, -11)$ and its gradient at any point is $ax^2 + b$, where a and b are constants. The tangent to the curve at the point $(2, -16)$ is parallel to the x-axis. Find

i) the values of a and b

ii) the equation of the curve

Solution

i) $a = 3$ and $b = -12$ ii) $y = x^3 - 12x$

2. Finding a quantity when the rate of change of the quantity is given

In some questions the rate of change of a quantity, say S, with respect to time (i.e. $\frac{dS}{dt}$) will be given and you will be asked to find the quantity. If the quantity y is **increasing** with respect to x then $\frac{dy}{dx}$ is **positive** and if y is **decreasing** with respect to x then $\frac{dy}{dx}$ is **negative** as in the examples below.

Illustrative examples

Example 1

Air is blown into a balloon such that the volume V cm^3 increased at the rate of $2(100 - t)$ cm^3/s after t seconds. If the volume V is 5 cm^3 at time $t = 0$;

i) Find the volume of the balloon at t seconds.

ii) Calculate the volume of the balloon at 3 seconds.

Solution

i) The rate of increase of volume V cm^3 is $2(100 - t)$ cm^3/s

i.e. $\frac{dV}{dt} = 2(100 - t) = 200 - 2t$

$\therefore\ V = \int (200 - 2t)\,dt = 200t - t^2 + c$

where c is an arbitrary constant which must be determined. When $t = 0$, $V = 5$.

Substituting these values in

$V = 200t - t^2 + c \Rightarrow \quad c = 5$

The volume of the balloon is

$V = 200t - t^2 + 5$

ii) When $t = 3$,

$V = 200(3) - (3)^2 + 5 = 600 - 9 + 5 = 596$ cm^3

Example 2

The volume V litres of petrol in a car's tank is decreasing at a rate of $(\frac{1}{5} - \frac{x}{2000})$ litres/km after travelling x kilometres.

If the volume V is 40 litres when $x = 0$;

i) Find the volume of the tank V in terms of x..

ii) After how many kilometres is the tank empty?

Solution

i) The rate of decrease of volume V litres is $(\frac{1}{5} - \frac{x}{2000})$ litres/km i.e. $\frac{dV}{dx} = -(\frac{1}{5} - \frac{x}{2000})$

Note that since V is **decreasing** with respect to x, $\frac{dV}{dx}$ is negative.

$\therefore\ V = \int (\frac{x}{2000} - \frac{1}{5})\,dx = \frac{1}{4000}x^2 - \frac{1}{5}x + c$

where c is an arbitrary constant which must be determined.

When $x = 0$, $V = 40$. Substituting these values in $V = \frac{1}{4000}x^2 - \frac{1}{5}x + c \Rightarrow \quad c = 40$

The volume of the tank is

$V = \frac{1}{4000}x^2 - \frac{1}{5}x + 40$

ii) When the tank is empty means $V = 0$,

i.e. $0 = \frac{1}{4000}x^2 - \frac{1}{5}x + 40$

$(x - 400)^2 = 0$

$x = 400$ km

3. Kinematics

Kinematics is the study of displacement, velocity and acceleration.

Consider a particle which is moving in a straight line such that its acceleration a is given as a function of t:

i) **acceleration**, $a = f(t)$;

ii) **velocity**, $v = \int a\,dt$

If $v = 0$, then the particle is at **rest**.

If $v < 0$ or negative, then the particle is moving in **opposite direction** to the initial direction.

iii) **distance**, $s = \int v\,dt$

Therefore given the acceleration function, a, we can find the velocity v by integrating the expression for a in terms of t and given the velocity function, v, we can find the distance s by integrating the expression for velocity v in terms of t.

Note the *Difference between kinematics under differentiation and kinematics under integration*

Distance → velocity → acceleration
(Differentiation)

Acceleration → velocity → distance
(Integration)

Average velocity

If the distance travelled by a particle at time t_1 is s_1 and the distance at time t_2 is s_2 (where $t_2 > t_1$) then the average velocity of the particle within the time interval t_1 and t_2 is given by:

$$\text{Average velocity} = \frac{s_2 - s_1}{t_2 - t_1}$$

Note: We use the difference in distance over the given time interval to calculate average velocity.

Average acceleration

If the velocity of a particle at time t_1 is v_1 and the velocity at time t_2 is v_2 (where $t_2 > t_1$) then the average acceleration of the particle within the time interval t_1 and t_2 is given by:

$$\text{Average acceleration} = \frac{v_2 - v_1}{t_2 - t_1}$$

Note: We use the difference in velocity over the given time interval to calculate average acceleration. .

Illustrative examples

Example 1

A particle moves in a straight line from a point O with an initial velocity of 2 ms^{-1}. Its acceleration t sec later is $(2t - 3)$ ms^{-2}. Calculate:

a) its velocity after 3 s.

b) its distances from O when it is momentarily at rest.

Solution

a) Since the acceleration function has been given and we want to find velocity, it implies we have to integrate. i.e.

$$a = (2t - 3)$$
$$v = \int a\,dt = \int (2t - 3)\,dt$$
$$v = t^2 - 3t + c$$

But when $t = 0$, $v = 2$ ms^{-1}, since the initial velocity of the particle is 2 ms^{-1}.

Therefore substituting $t = 0$ and $v = 2$ in the expression for v $\Rightarrow$ $2 = c$

$\therefore v = t^2 - 3t + 2$

When $t = 3$ s, $v = (3)^2 - 3(3) + 2 = 2\ \text{ms}^{-1}$
Hence the velocity of the particle after 3 s is $2\ \text{ms}^{-1}$.

b) When the particle is momentarily at rest the velocity, $v = 0$

$$t^2 - 3t + 2 = 0 \quad \Rightarrow \quad (t-1)(t-2)$$
$$t = 1 \text{ or } t = 2.$$

Hence at the instances $t = 1$ s and $t = 2$ s, the particle is momentarily at rest.

Now distance, $s = \int v\,dt$

$$s = \int (t^2 - 3t + 2)\,dt$$
$$s = \tfrac{1}{3}t^3 - \tfrac{3}{2}t^2 + 2t + c$$

At the point O i.e. the starting point, $t = 0$ and $s = 0$. Therefore, substituting $t = 0$ and $s = 0$ in the expression for $s \Rightarrow c = 0$

$\therefore\ s = \tfrac{1}{3}t^3 - \tfrac{3}{2}t^2 + 2t$. Hence its distances from O when the particle is momentarily at rest are at

$t = 1$ s and $t = 2$ s.

When $t = 1$, $s = \tfrac{1}{3}(1)^3 - \tfrac{3}{2}(1)^2 + 2(1) = \tfrac{5}{6}$ m.

When $t = 2$, $s = \tfrac{1}{3}(2)^3 - \tfrac{3}{2}(2)^2 + 2(2) = \tfrac{2}{3}$ m.

Hence the distances from O when it is momentarily at rest are $\tfrac{5}{6}$ m and $\tfrac{2}{3}$ m.

Example 2

A particle Q moving in a straight line passes through a point O with a speed of $3\ \text{ms}^{-1}$. The acceleration at time t sec. of Q passing through O is $(6t + 8)\ \text{ms}^{-2}$. Calculate

i) the velocity of Q when $t = 3$;
ii) the distance covered by Q between instances when $t = 2$ and $t = 5$.

Solution

i) Given the acceleration, $a = (6t + 8)\ \text{ms}^{-2}$

$$v = \int a\,dt = \int (6t + 8)\,dt$$
$$v = 3t^2 + 8t + c$$

When $t = 0$, $v = 3\ \text{ms}^{-1}$, since the initial velocity of the particle is 3 m. Therefore, substituting $t = 0$ and $v = 3$ in the expression for $v \quad \Rightarrow \quad c = 3$

$\therefore\ v = 3t^2 + 8t + 3$

When $t = 3$, $v = 3(3)^2 + 8(3) + 3 = 54\ \text{ms}^{-1}$.

ii) The distance, $s = \int v\,dt$

$$s = \int (3t^2 + 8t + 3)\,dt$$
$$s = t^3 + 4t^2 + 3t + c$$

At the point O i.e. the starting point, $t = 0$ and $s = 0$. Therefore substituting $t = 0$ and $s = 0$ in the expression for $s \Rightarrow c = 0$

$\therefore\ s = t^3 + 4t^2 + 3t$

When $t = 2$, $s = (2)^3 + 4(2)^2 + 3(2) = 30$ m

When $t = 5$, $s = (5)^3 + 4(5)^2 + 3(5) = 240$ m

The distance covered by Q between the instances when $t = 2$ and $t = 5$ is $(240 - 30)$ m $= 210$ m.

Example 3

A particle P is projected from a point O with a velocity $30\ \text{ms}^{-1}$ and it moves in a straight line in such a way that, after t s, the velocity v is given by $v = a + 7t + bt^2$, where a and b are constants. If 1 s after projection, the acceleration of the particle is $5\ \text{ms}^{-2}$. Find:

a) the values of a and b;
b) the time when the particle is momentarily at rest;
c) the distance travelled by that time.

Solution

a) The velocity, $v = a + 7t + bt^2$

The initial velocity of the particle is $30\ \text{ms}^{-1}$ i.e. when $t = 0$, $v = 30\ \text{ms}^{-1}$.

Therefore, substituting these values in the expression for v;

$$30 = a + 7(0) + b(0)^2$$
$$a = 30$$

$\therefore\ v = 30 + 7t + bt^2$

The acceleration, $a_1 = \dfrac{dv}{dt} = 7 + 2bt.$

But 1 s after projection, the acceleration of the particle is 5 ms^{-2} i.e. when $t = 1,\ a_1 = 5$

$$7 + 2b(1) = 5 \quad \Rightarrow \quad 2b = 5 - 7$$
$$b = -1$$

Hence $a = 30$ and $b = -1$.

Note: Here, we have denoted acceleration by a_1 to differentiate it from the constant 'a' in the expression for v.

b) Now $v = 30 + 7t - t^2$.

When the particle is momentarily at rest, $v = 0$

$$30 + 7t - t^2 = 0$$
$$t^2 - 7t - 30 = 0$$
$$(t - 10)(t + 3) = 0 \quad \Rightarrow \quad t = -3 \text{ or } 10.$$

Taking only the positive value of t since we do not have negative time.

$t = 10$ s is the time when the particle is momentarily at rest.

c) Distance, $s = \int v\,dt$

$$s = \int (30 + 7t - t^2)\,dt$$
$$s = 30t + \tfrac{7}{2}t^2 - \tfrac{1}{3}t^3 + c$$

At the point O i.e. the starting point, $t = 0$ and $s = 0$. Therefore, substituting $t = 0$ and $s = 0$ in the expression for $s \Rightarrow c = 0$

$$\therefore \quad s = 30t + \tfrac{7}{2}t^2 - \tfrac{1}{3}t^3$$

Putting $t = 10$ into s

$$s = 30(10) + \tfrac{7}{2}(10)^2 - \tfrac{1}{3}(10)^3$$
$$= 316.67 \text{ m, to 2 decimal places.}$$

Example 4

The speed v ms^{-1} of a body moving in a straight line from a point O in the time t seconds is given by $v = 24t - 3t^2$. Calculate:

a) the acceleration;

b) the distance from O at the end of 3 seconds.

Solution

a) $v = 24t - 3t^2$

acceleration, $a = \dfrac{dv}{dt} = 24 - 6t$

b) Distance, $s = \int v\,dt$

$$s = \int (24t - 3t^2)\,dt$$
$$s = 12t^2 - t^3 + c$$

At the point O, i.e. the starting point, $t = 0$ and $s = 0$

Therefore substituting $t = 0$ and $s = 0$ in the expression for s

$$c = 0 \qquad \therefore \quad s = 12t^2 - t^3$$

When $t = 3$ s, $s = 12(3)^2 - (3)^3 = 81$ m

Example 5

A particle is projected in a straight line from a point O with a speed 5 ms^{-1}. It moves in a line with an acceleration of $2(3 - t)$, t seconds after leaving O. Calculate:

i) the maximum speed of the particle;

ii) the average acceleration during the fourth second;

iii) the distance covered in 3 seconds.

Solution

i) acceleration, $a = 2(3 - t) = 6 - 2t$

Note: When the particle reaches the maximum speed, the acceleration is zero.

$$6 - 2t = 0 \quad \Rightarrow \quad t = 3 \text{ s}$$

Velocity, $v = \int a\,dt = \int (6 - 2t)\,dt$

$$v = 6t - t^2 + c$$

When $t = 0$, $v = 5$ ms^{-1}, since the initial velocity of the particle is 5 ms^{-1}.

Therefore, substituting $t = 0$ and $v = 5$ in the expression for $v \Rightarrow c = 5$

$\therefore \quad v = 6t - t^2 + 5$ When $t = 3$ s,

$$v = 6(3) - (3)^2 + 5 = 14 \text{ ms}^{-1}$$

The maximum speed of the particle is 14 ms^{-1}.

ii) The 4th second is between $t = 3$ s and $t = 4$s

At $t = 3$, $v_1 = 6(3) - (3)^2 + 5 = 14\text{ ms}^{-1}$
At $t = 4$, $v_2 = 6(4) - (4)^2 + 5 = 13\text{ ms}^{-1}$

$$\text{Average acceleration} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{13-14}{4-3}$$
$$= -1\text{ ms}^{-1}$$

(i.e. retardation)

iii) Distance, $s = \int v\,dt$

$$s = \int (6t - t^2 + 5)\,dt$$
$$s = 3t^2 - \tfrac{1}{3}t^3 + 5t + c$$

At the point O i.e. the starting point, $t = 0$ and $s = 0$.
Therefore, substituting $t = 0$ and $s = 0$ in the expression for $s \Rightarrow c = 0$

$$\therefore\quad s = 3t^2 - \tfrac{1}{3}t^3 + 5t$$

When $t = 3$ s,
$$s = 3(3)^2 - \tfrac{1}{3}(3)^3 + 5(3) = 27 - 9 + 15 = 33\text{ m}.$$

Example 6

The acceleration of a particle at time t seconds was $6(2t - t^2)$. When t was 1 s, the velocity was 2 ms^{-1} and its displacement was $3\frac{1}{2}$ m. Calculate

i) the velocity at time $t = 2$ s
ii) the displacement at time $t = 3$ s.

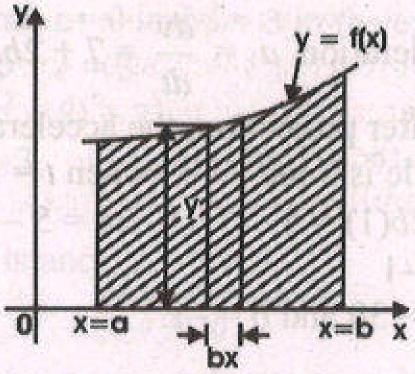
Solution

i) 6 ms^{-1} ii) $11\frac{1}{2}$ m
[Work out the solution]

4. Areas under and between curves

1. *Area between a curve and the x-axis*

Suppose $y = f(x)$ is the equation of a curve and that it is required to find the area enclosed by the curve, the x-axis and the lines $x = a$ and $x = b$ as shown in the following figure.

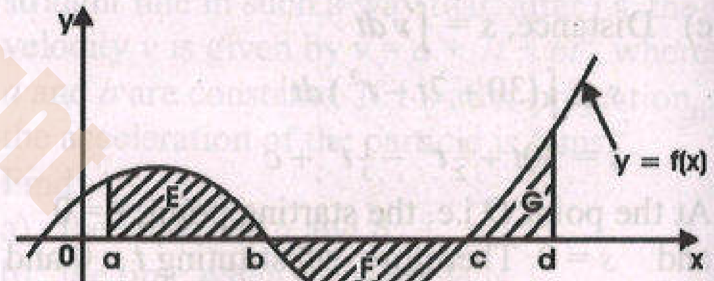


The area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by:

$$A = \int_a^b y\,dx\ .$$

Thus determining the area under a curve by integration merely involves evaluating a definite integral. The area is **positive** if the area is **above** the x–axis and **negative** if it is **below** the x-axis.

If the curve cuts the x-axis between the given limits as shown in the figure below:



1. find areas above and below the x-axis separately and

2. add their numerical values.

For the curve shown below, the total shaded area is given by (area E + area F + area G).

By integration, the **total shaded area**

$$= \int_a^b y\,dx + \int_b^c y\,dx + \int_c^d y\,dx$$

Note: It is always advisable to sketch the curve before using integration to find the required area because part of the curve may lie below the x-axis.

Note carefully that the total area **is not** $\int_a^d ydx$.

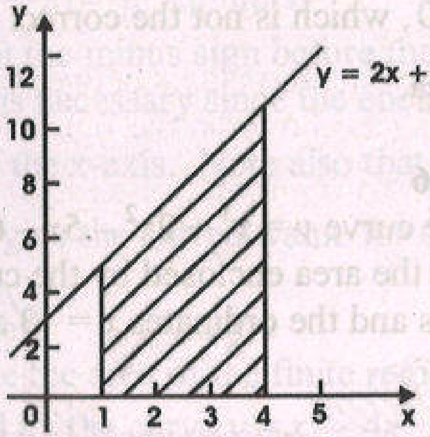
Illustrative examples

Example 1

Determine the area enclosed by $y = 2x + 3$, the x-axis and the ordinates $x = 1$ and $x = 4$.

Solution

Note: It is always advisable to sketch the curve before using integration to find the required area. $y = 2x + 3$ is a straight line graph as shown below.



The required area is shown shaded.

By integration, the required shaded area

$= \int_1^4 ydx = \int_1^4 (2x+3)dx = \left[x^2+3x\right]_1^4$

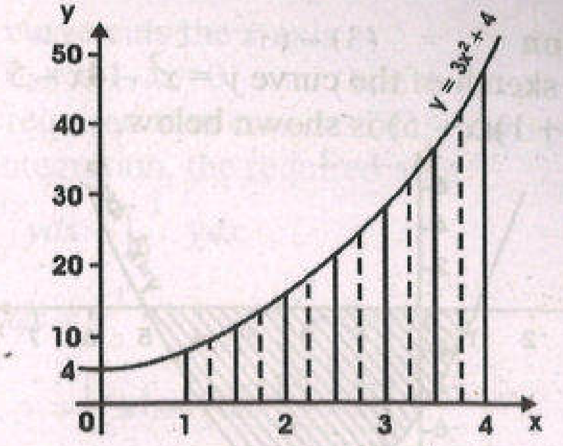
$= [16 + 12) - (1 + 3) = 24$ square units.

Example 2

Determine the area enclosed by $y = 3x^2 + 4$, the x-axis and the ordinates $x = 1$ and $x = 4$.

Solution

A sketch of the curve $y = 3x^2 + 4$ is shown in the following diagram.



By integration, the required area

$= \int_1^4 ydx = \int_1^4 (3x^2+4)dx = \left[x^3+4x\right]_1^4$

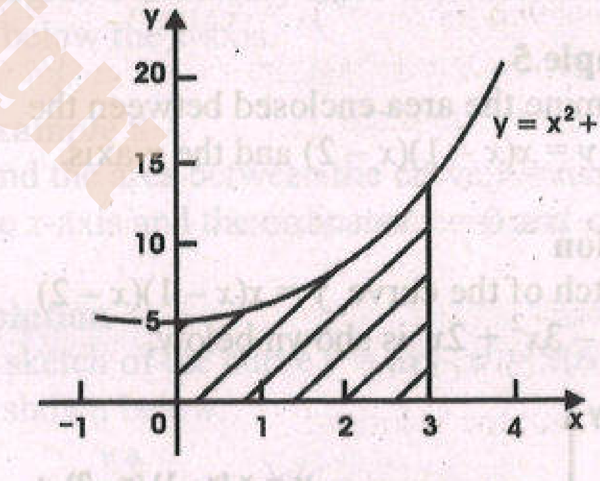
$= [64 + 16) - (1 + 4) = 75$ square units.

Example 3

Determine the area enclosed by $y = x^2 + 5$, the x-axis and the ordinates $x = 0$ and $x = 3$.

Solution

A sketch of the curve $y = x^2 + 5$ is shown below.



By integration, the required area

$= \int_0^3 ydx = \int_0^3 (x^2+5)dx = \left[\frac{1}{3}x^3+5x\right]_0^3$

$= [9 + 15) - (0 + 0) = 24$ square units.

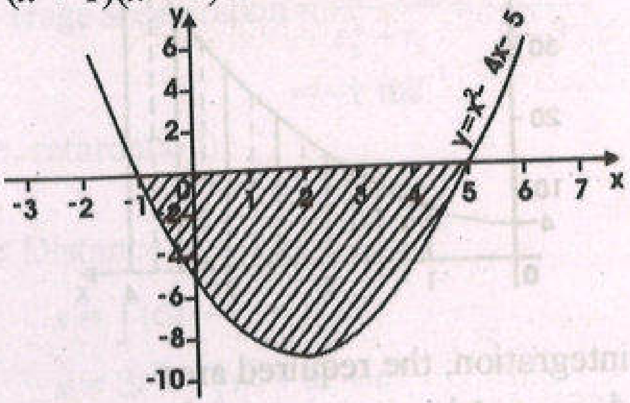
Example 4

i) Sketch the curve $y = x^2 - 4x - 5$.

ii) Find the area of the finite region bounded by the curve and the x-axis.

Solution

i) A sketch of the curve $y = x^2 - 4x - 5$ or $y = (x+1)(x-5)$ is shown below.



The curve cuts the x-axis at $x = -1$ and $x = 5$

ii) By integration, the required area

$$= \int_{-1}^{5} ydx = \int_{-1}^{5} (x^2 - 4x - 5)dx$$

$$= \left[\tfrac{1}{3}x^3 - 2x^2 - 5x\right]_{-1}^{5}$$

$$= \left(\tfrac{125}{3} - 50 - 25\right) - \left(-\tfrac{1}{3} - 2 + 5\right)$$

$$= -36 \text{ square units.}$$

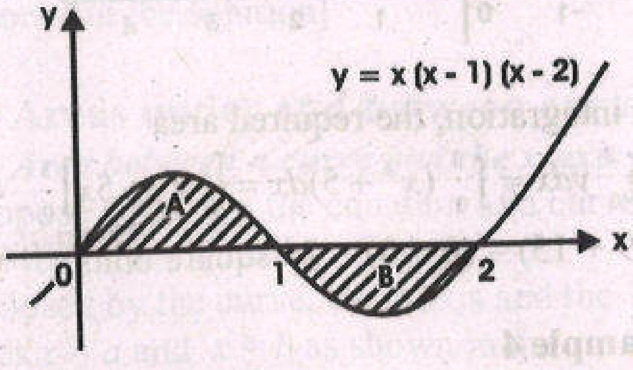
The negative value of the area means that it lies below the x-axis. Hence the required area is 36 square units.

Example 5

Determine the area enclosed between the curve $y = x(x-1)(x-2)$ and the x-axis.

Solution

A sketch of the curve $y = x(x-1)(x-2)$ $y = x^3 - 3x^2 + 2x$ is shown below.



The curve cuts the x-axis at $x = 0, x = 1, x = 2$

The required area consists of the two parts.

By integration, the required area

$$= \int_0^1 ydx - \int_1^2 ydx$$

$$\int_0^1 ydx = \int_0^1 (x^3 - 3x^2 + 2x)dx$$

$$= \left[\tfrac{1}{4}x^4 - x^3 + x^2\right]_0^1 = \tfrac{1}{4}$$

$$\int_1^2 ydx = \int_1^2 (x^3 - 3x^2 + 2x)dx$$

$$= \left[\tfrac{1}{4}x^4 - x^3 + x^2\right]_1^2 = -\tfrac{1}{4}$$

Hence the shaded area

$$= \int_0^1 ydx - \int_1^2 ydx$$

$$= \tfrac{1}{4} - \left(-\tfrac{1}{4}\right) = \tfrac{1}{2} \text{ square unit.}$$

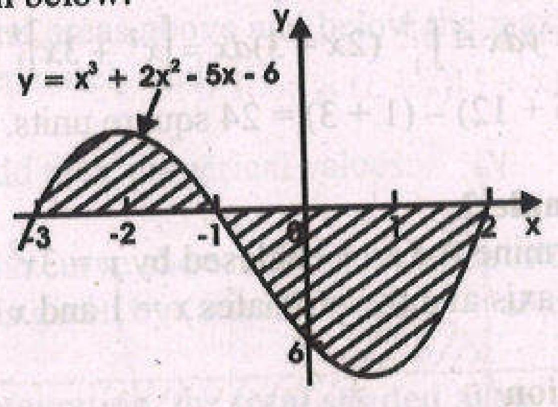
Note that the minus sign before the second integral is necessary since the enclosed area is below the x-axis. Note also that, $\int_0^2 ydx = 0$, which is not the correct value for the area.

Example 6

Sketch the curve $y = x^3 + 2x^2 - 5x - 6$ and determine the area enclosed by the curve, the x-axis and the ordinates $x = -3$ and $x = 2$.

Solution

A sketch of the curve $y = x^3 + 2x^2 - 5x - 6$ is shown below.



The curve cuts the x-axis at $x = 0, x = 1, x = 3$.

The required area consists of the two parts.

By integration, the required area

$$= \int_{-3}^{-1} ydx - \int_{-1}^{2} ydx$$

$$\int_{-3}^{-1} ydx = \int_{-3}^{-1} (x^3 + 2x^2 - 5x - 6)dx$$

$$= \left[\tfrac{1}{4}x^4 + \tfrac{2}{3}x^3 - \tfrac{5}{2}x^2 - 6x\right]_{-3}^{-1}$$

$$= \left(\tfrac{1}{4} - \tfrac{2}{3} - \tfrac{5}{2} + 6\right) - \left(\tfrac{81}{4} - 18 - \tfrac{45}{2} + 18\right)$$

$$= \left(\tfrac{37}{12}\right) - \left(-\tfrac{9}{4}\right) = \tfrac{16}{3}$$

$$\int_{-1}^{2} ydx = \int_{-1}^{2} (x^3 + 2x^2 - 5x - 6)dx$$

$$= \left[\tfrac{1}{4}x^4 + \tfrac{2}{3}x^3 - \tfrac{5}{2}x^2 - 6x\right]_{-1}^{2}$$

$$= \left(4 + \tfrac{16}{3} - 10 - 12\right) - \left(\tfrac{37}{12}\right)$$

$$= \left(-\tfrac{38}{3}\right) - \left(\tfrac{37}{12}\right) = -\tfrac{63}{4}$$

Hence the shaded area

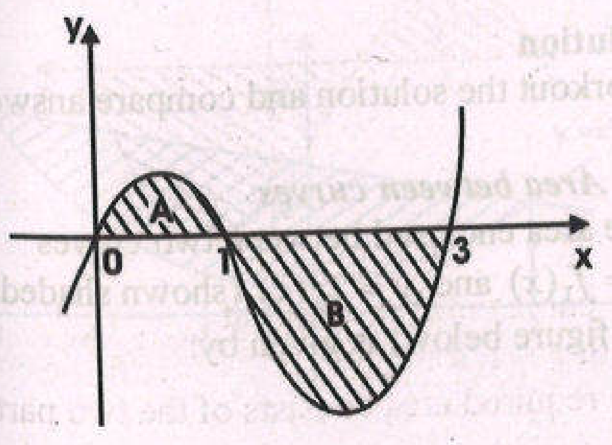
$$= \int_{-3}^{-1} ydx - \int_{-1}^{2} ydx$$

$$= \left(\tfrac{16}{3}\right) - \left(-\tfrac{63}{4}\right) = \tfrac{253}{12} \text{ sq. units}$$

Note that the minus sign before the second integral is necessary since the enclosed area is below the x-axis. Note also that $\int_{-3}^{2} ydx$ will not give the correct value for the area.

Example 7
Calculate the area of the finite region enclosed by the curve $y = x^3 - 4x^2 + 3x$ and the axis between $x = 0$ and $x = 3$

Solution
A sketch of the curve $y = x^3 - 4x^2 + 3x$ or $y = x(x-1)(x-3)$ is shown below.



The curve cuts the x-axis at $x = 0, x = 1, x = 3$.
The required area consists of the two parts.
By integration, the required area

$$= \int_{0}^{1} ydx - \int_{1}^{3} ydx$$

$$\int_{0}^{1} ydx = \int_{0}^{1} (x^3 - 4x^2 + 3x)dx$$

$$= \left[\tfrac{1}{4}x^4 - \tfrac{4}{3}x^3 + \tfrac{3}{2}x^2\right]_{0}^{1}$$

$$= \left(\tfrac{1}{4} - \tfrac{4}{3} + \tfrac{3}{2}\right) - (0) = \tfrac{5}{12}$$

$$\int_{1}^{3} ydx = \int_{1}^{3} (x^3 - 4x^2 + 3x)dx$$

$$= \left[\tfrac{1}{4}x^4 - \tfrac{4}{3}x^3 + \tfrac{3}{2}x^2\right]_{1}^{3}$$

$$= \left(\tfrac{81}{4} - \tfrac{108}{3} + \tfrac{27}{2}\right) - \left(\tfrac{5}{12}\right)$$

$$= \left(-\tfrac{27}{12}\right) - \left(\tfrac{5}{12}\right) = -\tfrac{32}{12}$$

Hence shaded area

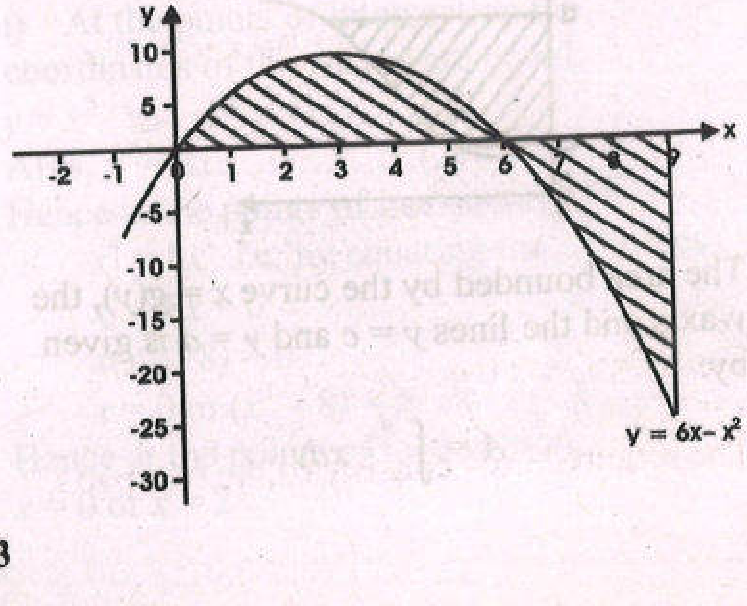
$$= \int_{0}^{1} ydx - \int_{1}^{3} ydx$$

$$= \left(\tfrac{5}{12}\right) - \left(-\tfrac{32}{12}\right) = \tfrac{37}{12} \text{ sq. units.}$$

Note that the minus sign before the second integral is necessary since the enclosed area is below the x-axis.

Example 8
Find the area between the curve $y = 6x - x^2$, the x-axis and the ordinates $x = 0$ and $x = 9$.

Solution
A sketch of the curve $y = 6x - x^2 = x(6 - x)$ is shown below.



The curve cuts the x-axis at $x = 0$ and $x = 6$. The required area consists of the two parts.

By integration, the required area

$= \int_0^6 ydx - \int_6^9 ydx$

$$\int_0^6 ydx = \int_0^6 (6x - x^2)dx = \left[3x^2 - \tfrac{1}{3}x^3\right]_0^6$$

$$= (108 - 72) - (0) = 36$$

$$\int_6^9 ydx = \int_6^9 (6x - x^2)dx = \left[3x^2 - \tfrac{1}{3}x^3\right]_6^9$$

$$= (243 - 243) - (36) = -36$$

Hence shaded area

$= \int_0^6 ydx - \int_6^9 ydx$

$= (36) - (-36) = 72$ sq. units

Example 9

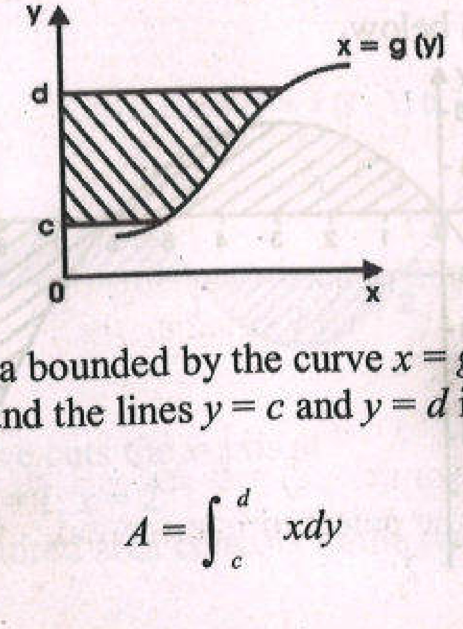
Find the area under each of the given curves between the given limits.

i) $y = 4x - x^2$; $x = 1, x = 2$ and the x-axis
ii) $y = x^2 + 3$; $x = -1, x = 2$ and the x-axis
iii) $y = x^2 + 1$; $x = 0, x = 1$ and the x-axis
iv) $y = x(x - 2)$; $x = 0, x = 4$ and the x-axis
v) $y = x^2(3 - x)$; $x = 4, x = 5$ and the x-axis
vi) $2y = 12 - x$; $x = 0, x = 12$ and the x-axis
vii) $y = x(x - 1)(x - 3)$ and the x-axis

Solution

Work out the solution and compare answers.

2. *Area between a curve and the y-axis*



The area bounded by the curve $x = g(y)$, the y-axis and the lines $y = c$ and $y = d$ is given by:

$$A = \int_c^d xdy$$

Illustrative examples

Example 1

Find the area between the curve $x = 4 - y^2$ and the y-axis.

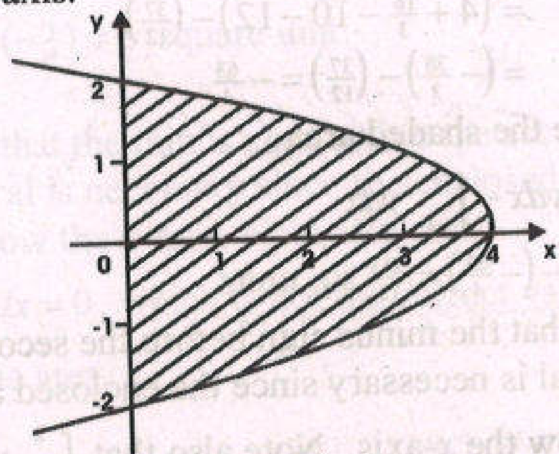
Solution

A sketch of the curve $x = 4 - y^2$ is shown below.

The curve cuts the y-axis at $y = -2$ and $y = 2$.

i.e. for intercepts on y-axis put $x = 0$.

The required area is bounded by curve and the y-axis.



By integration, the required area

$$= \int_{-2}^{2} xdy = \int_{-2}^{2} (4 - y^2)dy = \left[4y - \tfrac{1}{3}y^3\right]_{-2}^{2}$$

$$= \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right) = 10\tfrac{2}{3} \text{ square units.}$$

Example 2

Find the area enclosed by the y-axis and the following curves and straight lines.

i) $y = x^3$; $y = 1, y = 8$
ii) $x = y^2$; $y = 3$
iii) $y = x^2$; $y = 1, y = 4$ and x is positive

Solution

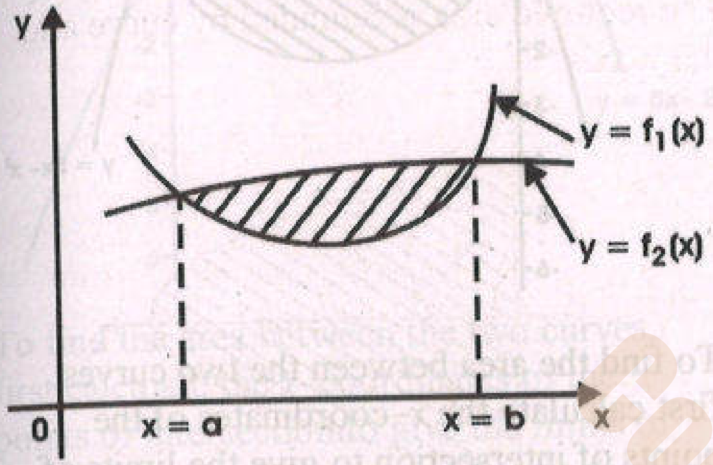
Workout the solution and compare answers.

3. *Area between curves*

The area enclosed between two curves $y = f_1(x)$ and $y = f_2(x)$ (shown shaded in the figure below) is given by:

$$\text{Shaded area} = \int_a^b f_2(x)dx - \int_a^b f_1(x)dx$$
$$= \int_a^b [f_2(x) - f_1(x)]dx$$

A sketch must always be made to show the relative position of the curves. To find the area between two curves, first calculate the x-coordinates of the points of intersection to give the limits of integration a and b.



Illustrative examples

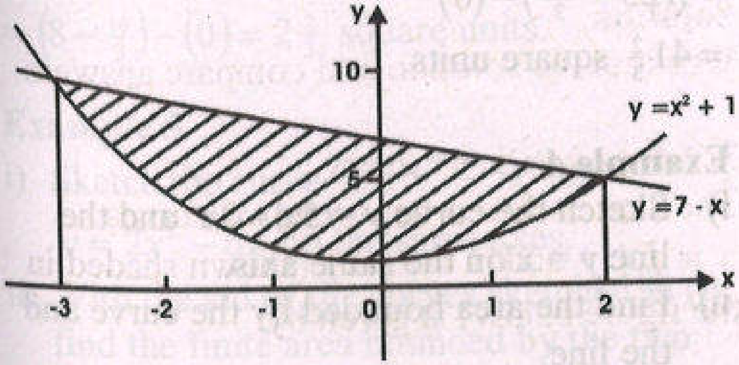
Example 1

i) Sketch the curves $y = x^2 + 1$ and $y = 7 - x$ on the same axes.

ii) Find the area enclosed between the curves.

Solution

A sketch must always be made to show the relative position of the curves. A sketch of the curves is shown below (Note: Go through the steps of sketching a curve).



To find the area between the two curves, first calculate the x–coordinates of the points of intersection to give the limits of integration. At the points of intersection, the curves are equal. Thus, equating the y-values of each curve gives:

$$x^2 + 1 = 7 - x$$
$$x^2 + x - 6 = 0$$

Factorising gives: $(x - 2)(x + 3) = 0$,
From which $x = 2$ and $x = -3$
Therefore shaded area

$$= \int_{-3}^{2} (7 - x)dx - \int_{-3}^{2} (x^2 + 1)dx$$
$$= \int_{-3}^{2} [(7 - x) - (x^2 + 1)]dx$$
$$= \int_{-3}^{2} (6 - x - x^2)dx$$
$$= \left[6x - \tfrac{x^2}{2} - \tfrac{x^3}{3}\right]_{-3}^{2}$$
$$= \left(12 - 2 - \tfrac{8}{3}\right) - \left(-18 - \tfrac{9}{2} + 9\right)$$
$$= 20\tfrac{5}{6} \text{ square units.}$$

Example 2

i) Determine the coordinates of the points of intersection of the curves $y = x^2$ and $y^2 = 8x$.

ii) Sketch the curves $y = x^2$ and $y^2 = 8x$ on the same axes.

iii) Calculate the area enclosed by the two curves.

Solution

i) At the points of intersection the coordinates of the curves are equal.

$y = x^2 \Rightarrow \quad y^2 = x^4 \ldots\ldots(1)$

Also, $y^2 = 8x \ldots\ldots\ldots\ldots\ldots(2)$

Hence at the points of intersection,

$x^4 = 8x$ i.e. by equating the y^2 values

$x^4 - 8x = 0$

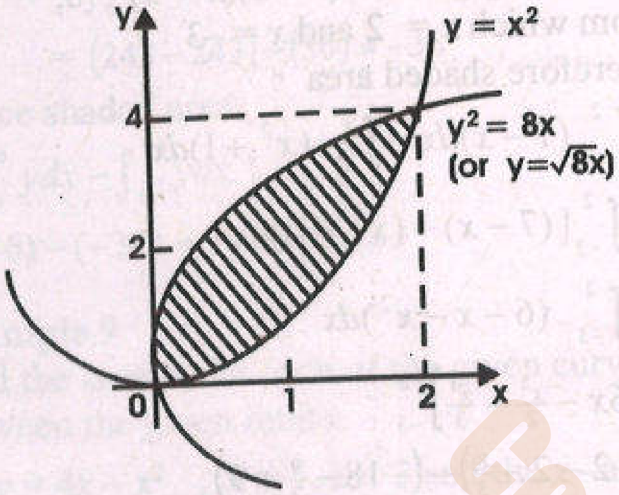
$x(x^3 - 8) = 0$

$x = 0$ or $(x^3 - 8) = 0 \Rightarrow \quad x = 2$

Hence at the points of intersection, $x = 0$ or $x = 2$

When $x = 0$, $y = 0$ and
When $x = 2$, $y = 2^2 = 4$
Hence the points of intersection of the curves $y = x^2$ and $y^2 = 8x$ are (0, 0) and (2, 4).

ii) A sketch of $y = x^2$ and $y^2 = 8x$ is shown in the figure below.



iii) Shaded area $= \int_0^2 (\sqrt{8x})dx - \int_0^2 (x^2)dx$

$$= \int_0^2 (\sqrt{8x} - x^2)dx$$

$$= \int_0^2 [(\sqrt{8})x^{\frac{1}{2}} - x^2]dx$$

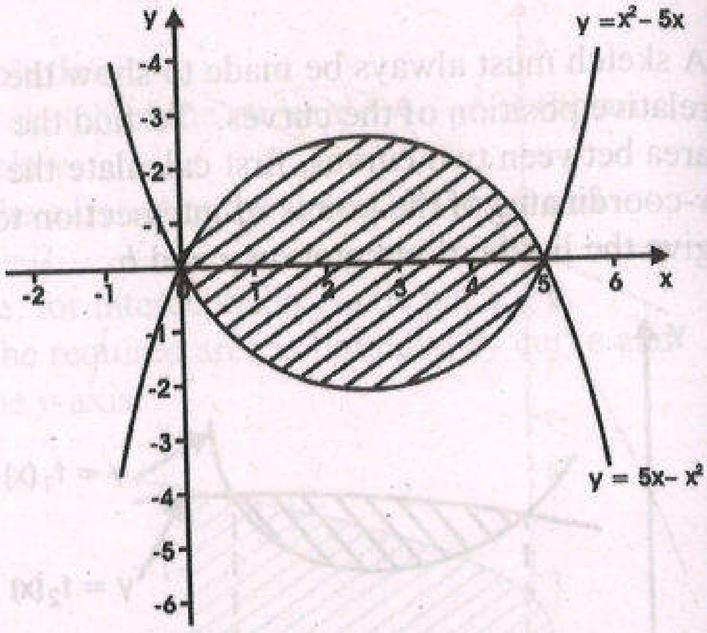
$$= \left[\sqrt{8}\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\right]_0^2$$

$$\text{Shaded area} = \left(\frac{\sqrt{8}\sqrt{8}}{\frac{3}{2}} - \frac{8}{3}\right) - (0)$$

$$= \frac{16}{3} - \frac{8}{3} = 2\tfrac{2}{3} \text{ square units.}$$

Example 3
On the same axes sketch and label clearly the curves $y = x^2 - 5x$ and $y = 5x - x^2$. Find the area of the finite region enclosed by the two curves.

Solution
A sketch of the curves is shown below (Go through the steps of sketching a curve).



To find the area between the two curves, first calculate the x–coordinates of the points of intersection to give the limits of integration. At the points of intersection, the curves are equal. Thus, equating the y-values of each curve gives:
$x^2 - 5x = 5x - x^2 \Rightarrow 2x^2 - 10x = 0$
Factorising gives $2x(x - 5) = 0$, from which $x = 0$ and $x = 5$.
Required area

$$= \int_0^5 (5x - x^2)dx - \int_0^5 (x^2 - 5x)dx$$

$$= \int_0^5 [(5x - x^2) - (x^2 - 5x)]dx$$

$$= \int_0^5 (10x - 2x^2)dx = \left[5x^2 - \tfrac{2x^3}{3}\right]_0^5$$

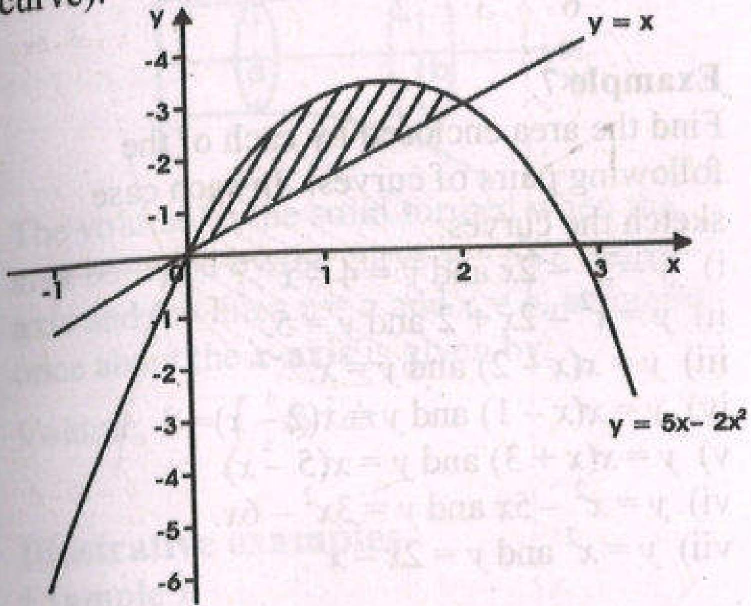
$$= \left(125 - \tfrac{250}{3}\right) - (0)$$

$$= 41\tfrac{2}{3} \text{ square units.}$$

Example 4
i) Sketch the curve $y = 5x - 2x^2$ and the line $y = x$ on the same axis.
ii) Find the area bounded by the curve and the line.

Solution

A sketch of the curves is shown in the figure below. (Go through the steps of sketching a curve).



To find the area between the two curves, first calculate the x–coordinates of the points of intersection to give the limits of integration. At the points of intersection, the curve and the line are equal. Thus, equating the y-values of each curve gives:

$$5x-2x^2=x$$

$$2x^2-4x=0$$

Factorising gives: $2x(x-2)=0$

From which $x=0$ and $x=2$.

Required area

$$=\int_0^2(5x-2x^2)dx-\int_0^2 xdx$$

$$=\int_0^2[(5x-2x^2)-(x)]dx$$

$$=\int_0^2(4x-2x^2)dx=\left[2x^2-\tfrac{2x^3}{3}\right]_0^2$$

$$=\left(8-\tfrac{16}{3}\right)-(0)=2\tfrac{2}{3}\text{ square units.}$$

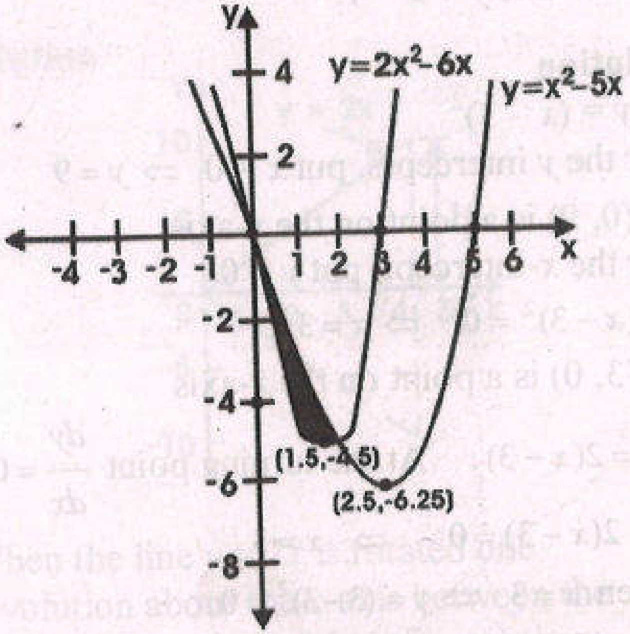
Example 5

i) Sketch the curve $y=x^2-5x$ and $y=2x^2-6x$ on the same axes.

ii) With the aid of the curves sketched in (i), find the finite area bounded by the two curves $y=2x^2-6x$ and x^2-5x

Solution

i) A sketch of $y=x^2-5x$ and $y=2x^2-6x$ is shown in the figure below (Go through the steps of sketching a curve).



ii) To find the area between the two curves, first calculate the x–coordinates of the points of intersection to give the limits of integration.

At the points of intersection, the two are equal. Thus, equating the y-values of each curve gives:

$$x^2-5x=2x^2-6x$$

$$x^2-x=0$$

$$x(x-1)=0$$

$$x=0\text{ or }1$$

These give the limits of integration

The finite region is

$$x^2-5x-(2x^2-6x)=x-x^2$$

$\therefore$ Area of the finite region

$$\int_0^1(x-x^2)dx=\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1$$

$$=\left[\frac{1}{2}-\frac{1}{3}\right]-0=\frac{1}{6}\text{ sq. units}$$

Example 6

i) Sketch the curve $y=(x-3)^2$ and the line $y=x-1$ on the same axis.

ii) Calculate the area of the finite region enclosed by the curve $y=(x-3)^2$ and the line $y=x-1$.

Solution

i) $y=(x-3)^2$

For the y intercepts, put $x=0 \Rightarrow y=9$

$\therefore$ (0, 9) is a point on the y-axis

For the x-intercept, put $y=0$

$$(x-3)^2=0 \quad \Rightarrow x=3$$

$\therefore$ (3, 0) is a point on the x-axis

$\dfrac{dy}{dx}=2(x-3)$. At the turning point $\dfrac{dy}{dx}=0$

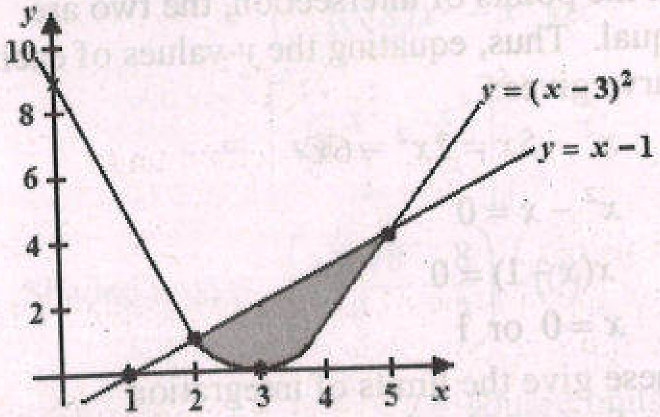
$$2(x-3)=0 \qquad \Rightarrow \quad x=3$$

When $x=3 \quad \Rightarrow y=(3-3)^2=0$

$\dfrac{d^2y}{dx^2}=2>0$ (Min. point)

$\therefore$ (3, 0) is a minimum turning point

A sketch of the curve and line is shown below.



ii) At the points of intersection

$$x^2-6x+9=x-1 \;\Rightarrow\; x^2-7x+10=0$$

$$(x-2)(x-5)=0 \quad\Rightarrow\quad x=2 \text{ or } 5$$

These give the limits of integration.

Required Area $=\int_2^5(7x-x^2-10)dx$

$$=\left[\frac{7x^2}{2}-\frac{x^3}{3}-10x\right]_2^5$$

$$=\left[\left(\frac{175}{2}-\frac{125}{3}-50\right)-\left(14-\frac{8}{3}-20\right)\right]$$

$$=-\frac{25}{6}+\frac{26}{3}=\frac{9}{2}=4.5\text{ sq. units}$$

Example 7

Find the area enclosed by each of the following pairs of curves. In each case sketch the curves.

i) $y=x^2-2x$ and $y=4-x^2$.

ii) $y=x^2-2x+2$ and $y=5$.

iii) $y=x(x-2)$ and $y=x$.

iv) $y=x(x-1)$ and $y=x(2-x)$

v) $y=x(x+3)$ and $y=x(5-x)$

vi) $y=x^2-5x$ and $y=3x^2-6x$.

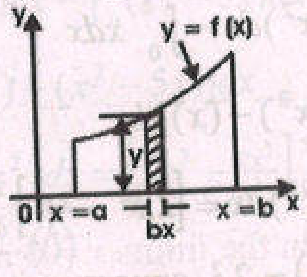
vii) $y=x^2$ and $y=2x-x^2$.

Solution

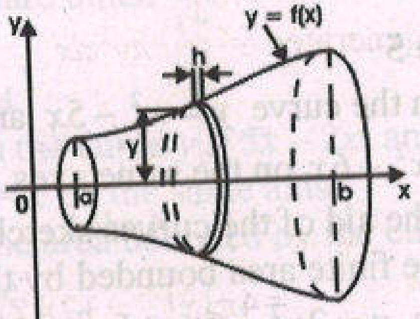
Workout the solution and compare answers

5. Volumes of revolution

If the area bounded by the curve $y=f(x)$, (shown in figure (a) below) between two ordinates $x=a$ and $x=b$, is rotated **360°** or **four right angles** about the x-axis, then a volume known as **solid of revolution** is formed as shown in figure (b) below. The volume of such a solid may be determined precisely using integration.

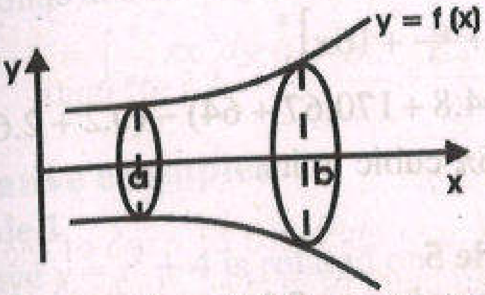


(a)



(b)

1. *Rotation about the x-axis*



The volume of the solid formed when the area bounded by the curve $y = f(x)$, the **x-axis** and the lines $x = a$ and $x = b$, is rotated once about the **x-axis** is given by:

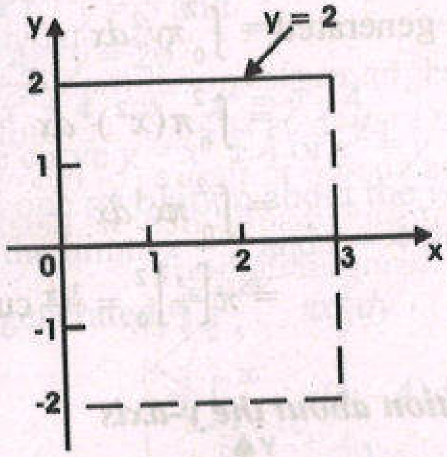
Volume, $V = \int_a^b \pi y^2\,dx$.

Illustrative examples

Example 1

Determine the volume of the solid of revolution formed when the line $y = 2$ is rotated 360° about the x-axis between the limits $x = 0$ and $x = 3$.

Solution



When the line $y = 2$ is rotated 360° about the x-axis between the limits $x = 0$ and $x = 3$ (see figure above) then:

$$\text{Volume generated} = \int_0^3 \pi y^2\,dx$$

$$= \int_0^3 \pi(2)^2\,dx$$

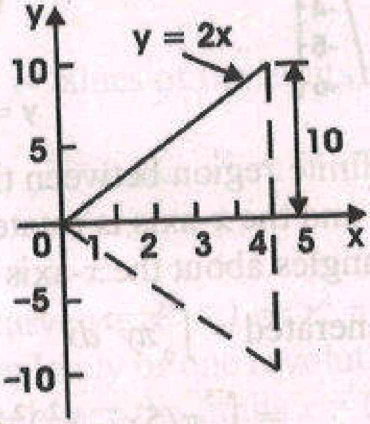
$$= \int_0^3 4\pi dx$$

$$= 4\pi[x]_0^3 = 12\pi \text{ cubic units}$$

Example 2

Find the volume of the solid of revolution formed when the line $y = 2x$ is rotated one revolution about the x-axis between the limits $x = 0$ and $x = 5$.

Solution



When the line $y = 2x$ is rotated one revolution about the x-axis between the limits $x = 0$ and $x = 5$ (see figure) then:

$$\text{Volume generated} = \int_0^5 \pi y^2\,dx$$

$$= \int_0^5 \pi(2x)^2\,dx$$

$$= \int_0^5 4\pi x^2\,dx$$

$$= 4\pi\left[\tfrac{x^3}{3}\right]_0^5$$

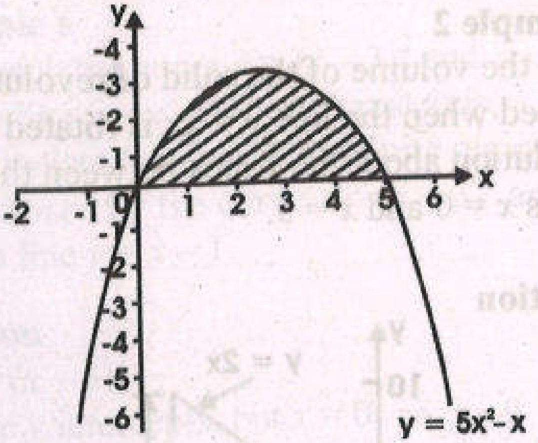
$$= \tfrac{500\pi}{3} \text{ cubic units}$$

Example 3

Find the volume of the solid generated when the finite region between the curve $y = 5x - x^2$ and the x-axis is rotated through four right angles about the x-axis, giving your answer in terms of π.

Solution

The curve cuts the x-axis at $x = 0$, $x = 5$. A sketch of the curve $y = 5x - x^2 = x(5 - x)$ is shown in the next page.



When the finite region between the curve $y = 5x - x^2$ and the x-axis is rotated through four right angles about the x-axis then:

$$\text{Volume generated} = \int_0^5 \pi y^2 dx$$

$$= \int_0^5 \pi(5x - x^2)^2 \, dx$$

$$= \int_0^5 \pi(25x^2 - 10x^3 + x^4)dx$$

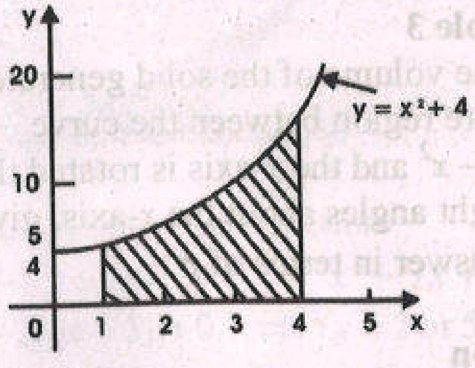
$$= \pi\left[\tfrac{25x^3}{3} - \tfrac{10x^4}{4} + \tfrac{x^5}{5}\right]_0^5$$

$$= 104.17\pi \text{ cubic units.}$$

Example 4

The curve $y = x^2 + 4$ is rotated one revolution about the x-axis between the limits $x = 1$ and $x = 4$. Determine the volume of the solid of revolution produced.

Solution



Revolving the shaded area shown in the figure above about the x-axis produces a solid of revolution given by:

Volume generated

$$= \int_1^4 \pi y^2 dx = \int_1^4 \pi(x^2 + 4)^2 dx$$

$$= \int_1^4 \pi(x^4 + 8x^2 + 16)dx$$

$$= \pi\left[\tfrac{x^5}{5} + \tfrac{8x^3}{3} + 16x\right]_1^4$$

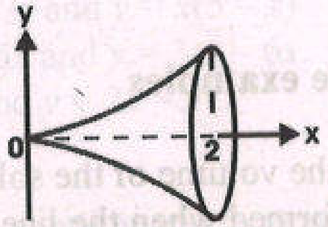
$$= \pi(204.8 + 170.67 + 64) - (0.2 + 2.67 + 16)$$

$$= 420.6\pi \text{ cubic units.}$$

Example 5

Find the volume of the solid of revolution formed when the curve $y = x^2$ is rotated one revolution about the x-axis between the limits $x = 0$ and $x = 2$.

Solution



When the curve $y = x^2$ is rotated one revolution about the x-axis between the limits $x = 0$ and $x = 2$(see figure above) then

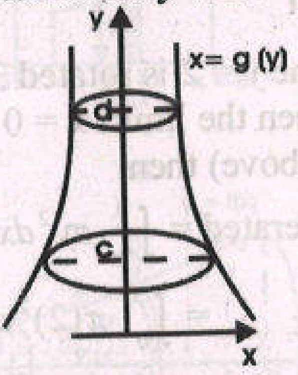
$$\text{Volume generated} = \int_0^2 \pi y^2 dx$$

$$= \int_0^2 \pi(x^2)^2 \, dx$$

$$= \int_0^2 \pi x^4 dx$$

$$= \pi\left[\tfrac{x^5}{5}\right]_0^2 = \tfrac{32\pi}{5} \text{ cubic units}$$

2. *Rotation about the y-axis*



The volume of the solid formed when the area bounded by the curve $x = g(y)$, the

y-axis and the lines $y = c$ and $y = d$, is rotated once about the **y-axis** is given by:

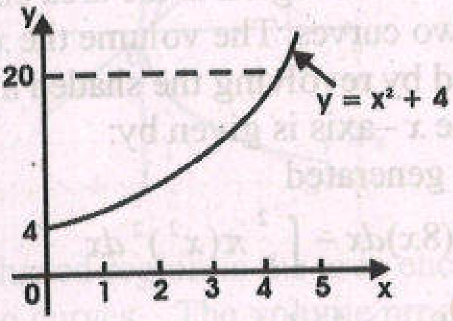
Volume, $V = \int_a^b \pi x^2 dy$.

Illustrative examples

Example 1

The curve $y = x^2 + 4$ is rotated one revolution about the y-axis between the limits $x = 1$ and $x = 4$. Determine the volume of the solid of revolution produced.

Solution



Finding the y-values of the limits of integration:

When $x = 1$, $y = 5$ and
when $x = 4$, $y = 20$

Also $y = x^2 + 4 \Rightarrow x^2 = y - 4$

When the curve $y = x^2 + 4$ or $x^2 = y - 4$ is rotated one revolution about the y-axis between the limits $x = 2$ and $x = 4$ then:

$$\text{Volume generated} = \int_5^{20} \pi x^2 dy$$

$$= \int_5^{20} \pi(y-4)dy$$

$$= \pi\left[\tfrac{y^2}{2} - 4y\right]_5^{20}$$

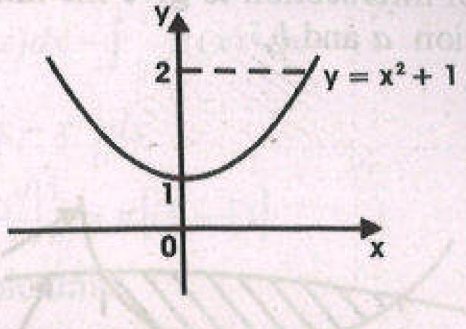
$$= \pi[(120) - (-\tfrac{15}{2})]$$

$$= 127\tfrac{1}{2}\pi \text{ cubic units.}$$

Example 2

The curve $y = x^2 + 1$ is rotated one revolution about the y-axis between the limits $x = 0$ and $x = 1$. Determine the volume of the solid of revolution produced.

Solution



Finding the y-values of the limits of integration:

When $x = 0$, $y = 1$ and when $x = 2$, $y = 2$

Also $y = x^2 + 1 \Rightarrow x^2 = y - 1$

When the curve $y = x^2 + 1$ or $x^2 = y - 1$ is rotated completely or one revolution about the y-axis between the limits $x = 0$ and $x = 1$ then:

$$\text{Volume generated} = \int_1^2 \pi x^2 dy$$

$$= \int_1^2 \pi(y-1)dy$$

$$= \pi\left[\tfrac{y^2}{2} - y\right]_1^2$$

$$= \pi[(2-2) - (\tfrac{1}{2} - 1)]$$

$$= \tfrac{1}{2}\pi \text{ cubic units.}$$

3. *Volume generated by rotation of the area between two curves*

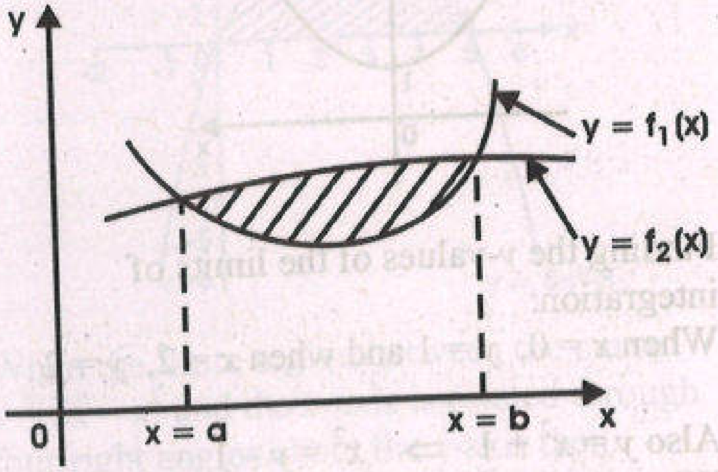
The volume generated by rotating the area enclosed between two curves $y_1 = f(x)$ and $y_2 = g(x)$ (shown shaded in the figure below), and the lines $x = a$ and $x = b$ about the x-axis is given by

$$\text{Volume} = \int_a^b \pi y_2^2 dx - \int_a^b \pi y_1^2 dx$$

$$= \int_a^b \pi[y_2^2 - y_1^2]dx$$

A sketch must always be made to show the relative position of the curves.

To find the volume between two curves, first calculate the x–coordinates of the points of intersection to give the limits of integration a and b.



Illustrative examples

Example 1

i) Determine the coordinates of the points of intersection of the curves $y = x^2$ and $y^2 = 8x$.

ii) Sketch the curves $y = x^2$ and $y^2 = 8x$ on the same axes .

iii) Calculate the volume of the solid formed when the area enclosed between the two curves is rotated 360° about the x-axis.

Solution

i) At the points of intersection the coordinates of the curves are equal.

$y = x^2 \;\Rightarrow\; y^2 = x^4$ ....(1)

Also, $y^2 = 8x$ ....(2)

Hence at the points of intersection, $x^4 = 8x$ i.e. by equating the y^2 values.

$$x^4 - 8x = 0$$

$$x(x^3 - 8) = 0$$

$$x = 0 \text{ or } (x^3 - 8) = 0 \;\Rightarrow\; x = 2$$

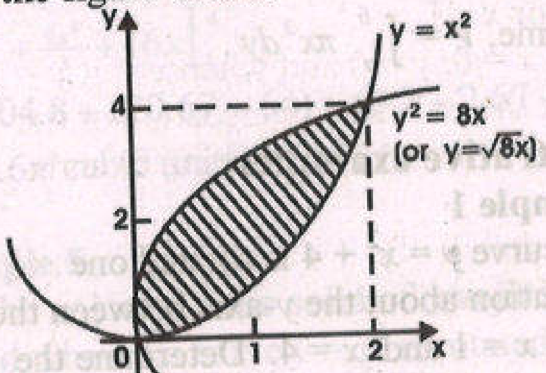
Hence at the points of intersection, $x = 0$ or $x = 2$

When $x = 0$, $y = 0$ and

When $x = 2$, $y = 2^2 = 4$

Hence the points of intersection of the curves $y = x^2$ and $y^2 = 8x$ are (0, 0) and (2, 4)

ii) A sketch of $y = x^2$ and $y^2 = 8x$ is shown in the figure below.



iii) The shaded region is the area enclosed by the two curves. The volume the x-axis produced by revolving the shaded area 360° about the x –axis is given by:

Volume generated

$$= \int_0^2 \pi(8x)dx - \int_0^2 \pi(x^2)^2\,dx$$

$$= \int_0^2 \pi(8x - x^4)dx$$

$$= \pi\left[4x^2 - \tfrac{x^5}{5}\right]_0^2$$

$$= \pi\left[(16 - \tfrac{32}{5}) - (0)\right]$$

$$= 9.6\pi \text{ cubic units}$$

Example 2

i) Determine the coordinates of the points of intersection of the curves $y = x$ and $y^2 = x$.

ii) Sketch the curves $y = x$ and $y^2 = x$ on the same axes.

iii) Calculate the volume of the solid formed when the area enclosed between the two curves is rotated 360° about the x-axis.

Solution

i) At the points of intersection the coordinates of the curves are equal.

$$y = x \;\Rightarrow\; y^2 = x^2$$

Hence the points of intersection, $x^2 = x$ i.e. by equating the y^2 values.

$$x^2 - x = 0 \;\Rightarrow\; x(x - 1) = 0$$

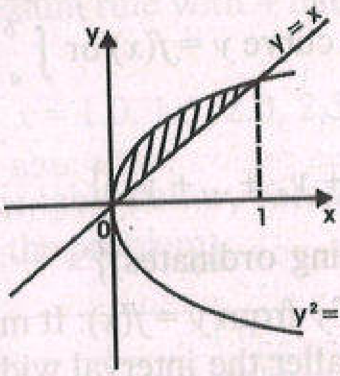
$x = 0$ or $(x - 1) = 0$

Hence at the points of intersection,
$x = 0$ or $x = 1$

When $x = 0$, $y = 0$ and when $x = 1$, $y = 1$

Hence the points of intersection of the curves $y = x$ and $y^2 = x$ are (0, 0) and (1, 1)

ii) A sketch of $y = x$ and $y^2 = x$ is shown in the figure below.



iii) The shaded region is the area enclosed by the two curves. The volume produced by revolving the shaded area 360° about the x–axis is given by:

Volume generated

$$= \int_0^1 \pi(x)dx - \int_0^1 \pi(x)^2\,dx$$

$$= \int_0^1 \pi(x - x^2)dx$$

$$= \pi\left[\tfrac{x^2}{2} - \tfrac{x^3}{3}\right]_0^1 = \pi\left[\left(\tfrac{1}{2} - \tfrac{1}{3}\right)\right]$$

$= \frac{1}{6}\pi$ cubic units

Example 3

i) Calculate the area of the finite region enclosed by the curve $y = x^2 + 2$ and the line $y = 4x + 2$.

ii) Calculate the volume of the solid generated when the finite region in (i) is rotated completely about the x-axis.

Solution

i) $\frac{32}{3}$ square units ii) 179.2π cubic units

Work out the missing steps.

---

6. Numerical integration – The trapezium rule

Even with more advanced methods of integration, there are many mathematical functions which cannot be integrated by analytical methods and therefore approximate methods may be used. Approximate methods of definite integral may be determined by what is termed **numerical integration**. Determining the value of definite integrals is, in fact, finding the area between a curve, the x-axis and the specified ordinates.

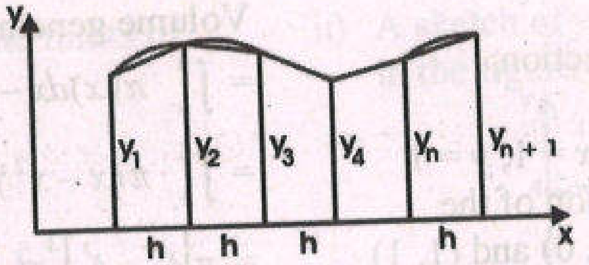
The **trapezium rule** gives an approximate method of finding areas under curves or evaluating definite integrals. There are other methods such as *Simpson's rule* and the *mid-ordinate rule* but they are beyond the scope of this book.

The trapezium rule

1. Finding definite integrals or area under a curve

Let the required definite integral be denoted by $\int_a^b y\,dx$ and be represented by the area under the graph of $y = f(x)$ between the limits $x = a$ and $x = b$ as shown in the figure on the next page. Let the range of integration be divided into n equal intervals each of width h, such that

$h = \frac{b-a}{n}$. The ordinates are labelled $y_1, y_2, y_3, \ldots y_{n+1}$ as shown in the following diagram.



Note that the number of ordinates = number of intervals + 1 (i.e $n + 1$).
An approximation to the area under the curve may be determined by joining the tops of the ordinates by straight lines. Each interval is thus a trapezium. The sum of the areas of these trapezia will give an approximate value for the area under the curve $y = f(x)$ or $\int_a^b y\,dx$.

The trapezium rule states that:

Area under the curve $= \int_a^b y\,dx \approx \frac{1}{2}h\{(y_1 + y_{n+1}) + 2(y_2 + y_3 + y_4 + \ldots + y_n)\}$

$\approx \frac{1}{2}$(width of interval){(first + last ordinate) + 2(sum of remaining ordinates)}

Note that tables are used where necessary to find the values of y from $y = f(x)$. It must be also noted that the greater the number of intervals used (i.e. the smaller the interval width) the more accurate will be the value of the definite integral. The exact value is found when the number of intervals is infinite.

2. *Finding the volume generated by rotating the area through 360° about the x-axis*

The volume generated by rotating the area under the graph of $y = f(x)$ through 360° or four right angles about the x-axis can also be determined using the trapezium rule. The required volume is the sum of the volumes generated by each trapezium (Note that the volume generated is either a circular disc or a cylinder). For a more accurate result the average of the ordinates (y-values) of each interval is taken as the radius of each cylinder or circular disc generated.

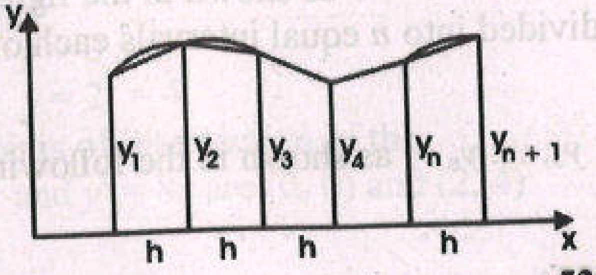
Volume generated ≈ sum of volume generated by each trapezium.

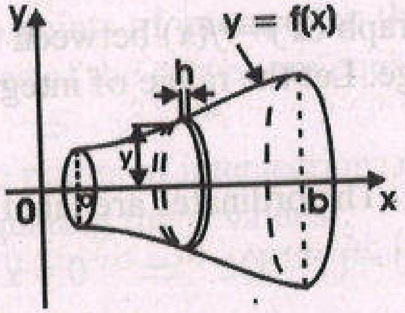
$$\text{Volume generated} \approx \pi h\left\{\left(\frac{y_1 + y_2}{2}\right)^2 + \left(\frac{y_2 + y_3}{2}\right)^2 + \ldots + \left(\frac{y_n + y_{n+1}}{2}\right)^2\right\}$$

$$\approx \tfrac{1}{4}\pi h\left\{(y_1 + y_2)^2 + (y_2 + y_3)^2 + \ldots + (y_n + y_{n+1})^2\right\}$$

Alternative method

$$\text{Volume generated} \approx \int_a^b \pi y^2\,dx \approx \tfrac{1}{2}h\{(y_1^2 + y_{n+1}^2) + 2(y_2^2 + y_3^2 + y_4^2 + \ldots + y_n^2)\}$$





Illustrative examples

Example 1

Use the trapezium rule with 4 intervals to evaluate the integral $\int_1^3 \frac{2}{\sqrt{x}}\,dx$, correct to 3 decimal places.

Solution

The range of integration is the difference between the upper and lower limits, i.e. $3 - 1 = 2$

Using trapezium rule with 4 intervals gives an interval width of $h = \frac{3-1}{4} = 0.5$ and ordinates situated at: $x = 1.0, 1.5, 2.0, 2.5$ and 3. Let $y = \frac{2}{\sqrt{x}}$. The corresponding values of $\frac{2}{\sqrt{x}}$ are shown in the table below, correct to 4 decimal places (which is one more decimal place than required in the problem).

x	1.0	1.5	2.0	2.5	3.0
$y = \frac{2}{\sqrt{x}}$	2.0000 y_1 (first)	1.6330 y_2	1.4142 y_3	1.2649 y_4	1.1547 y_5 (last)

Using the trapezium rule,

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx \tfrac{1}{2}(0.5)\{(2.0000 + 1.1547) + 2(1.6330 + 1.4142 + 1.2649)\} = 0.25\{3.1547 + 2(4.3121)\}$$

≈ 2.945 correct to 3 decimal places.

Example 2

Use the trapezium rule, with seven ordinates $x = 1, 1.5, 2, 2.5, 3, 3.5$ and 4 to calculate, correct to two decimal places, an approximate value for the integral $\int_1^4 \frac{dx}{x+3}$.

Solution

First find the corresponding values of $y = \frac{1}{x+3}$ at the given ordinates as shown in the table below, correct to 3 decimal places (which is one more decimal place than required in the problem). Note that the interval width $h = 0.5$.

x	1.0	1.5	2.0	2.5	3.0	3.5	4
$y = \frac{1}{x+3}$	0.250 y_1 (first)	0.222 y_2	0.200 y_3	0.182 y_4	0.167 y_5	0.154 y_6	0.143 y_7 (last)

Using the trapezium rule,

$$\int_1^4 \frac{dx}{x+3} \approx \tfrac{1}{2}(0.5)\{(0.250+0.143)+2(0.222+0.200+0.182+0.167+0.154)\} = 0.25\{0.393+2(0.925)\}$$

≈ 0.56 correct to 2 decimal places.

Example 3

Using the trapezoidal rule with values of x at intervals 0.5, calculate $\int_0^3 \frac{dx}{x+1}$

Solution

Since the interval is 0.5, and the range of integration is from 0 to 3, the ordinates are at $x = 0$, 0.5, 1, 1.5, 2, 2.5, and 3

The corresponding values of $y = \frac{1}{x+1}$ at the given ordinates are shown in the table below,

x	0	0.5	1.0	1.5	2.0	2.5	3
$y = \frac{1}{x+1}$	1.000 y_1 (first)	0.667 y_2	0.500 y_3	0.400 y_4	0.333 y_5	0.286 y_6	0.250 y_7 (last)

Using the trapezium rule,

$$\int_0^3 \frac{dx}{x+1} \approx \tfrac{1}{2}(0.5)\{(1.000 + 0.250) + 2(0.667 + 0.500 + 0.400 + 0.333 + 0.286)\}$$

$$\approx 0.25\{1.250 + 2(2.186)\}$$

$$\approx 1.41 \text{ correct to 2 decimal places}$$

Example 4

Using the trapezium rule with 7 ordinates, find an approximates value for $\int_1^4 \frac{2}{x+2}\,dx$.

Solution

The range of integration is the difference between the upper and lower limits, i.e. $4 - 1 = 3$

Using the trapezium rule with 7 ordinates (i.e. 6 intervals) gives an interval width of $h = \frac{4-1}{6} = 0.5$ and ordinates situated at: $x = 1.0$, 1.5, 2.0, 2.5, 3, 3.5 and 4.

Corresponding values of $y = \frac{2}{x+2}$ are shown in the table below.

Note carefully: Number of intervals = number of ordinates – 1

x	1.0	1.5	2.0	2.5	3.0	3.5	4
$y = \frac{2}{x+2}$	0.667 y_1 (first)	0.571 y_2	0.500 y_3	0.444 y_4	0.400 y_5	0.364 y_6	0.333 y_7 (last)

Using the trapezium rule,

$$\int_1^4 \frac{2}{x+2}\,dx \approx \tfrac{1}{2}(0.5)\{(0.667 + 0.333) + 2(0.571 + 0.500 + 0.444 + 0.400 + 0.364)\}$$

$$\approx 0.25\{1.000 + 2(2.279) \approx 1.39 \text{ correct to 2 decimal places.}$$

Example 5

Copy and complete the table below for the values of $\frac{4}{x+5}$ at the ordinates 0, 0.5, 1.0, 1.5, 2.0 and 2.5.

x	0	0.5	1.0	1.5	2.0	2.5
$\frac{4}{x+5}$	0.80		0.67		0.57	0.53

Use the trapezium rule to find the value of $\int_0^{2.5} \frac{4}{x+5}\,dx$.

Solution

The completed table is shown below. Note that the interval width $h = 0.5$

x	0	0.5	1.0	1.5	2.0	2.5
$\frac{4}{x+5}$	0.80 y_1 (first)	0.73 y_2	0.67 y_3	0.62 y_4	0.57 y_5	0.53 y_6 (last)

Using the trapezium rule,

$$\int_0^{2.5} \frac{4}{x+5}\,dx \approx \tfrac{1}{2}(0.5)\{(0.80+0.53)+2(0.73+0.67+0.62+0.57)\} = 0.25\{1.33+2(2.59)\}$$

$$\approx 1.63 \text{ correct to 2 decimal places.}$$

Example 6

Use the trapezium rule, with five ordinates, to estimate, correct to two decimal places;

a) the area bounded by the curve $y = \frac{2}{1+x^2}$, the x-axis and the lines $x = 0$ and $x = 1$.

b) the volume of the solid of revolution by rotating the area in (a) through four right angles about the x-axis (take $\pi = 3.142$).

Solution

The area bounded by the curve $y = \frac{2}{1+x^2}$, the x-axis and the lines $x = 0$ and $x = 1$ is given by the definite integral $\int_0^1 \frac{2}{1+x^2}\,dx$. The range of integration is the difference between the upper and lower limits, i.e. $1 - 0 = 1$. Using the trapezium rule with 5 ordinates (i.e. 4 intervals) gives an interval width of $h = \frac{1-0}{4} = 0.25$ and ordinates situated at: $x = 0, 0.25, 0.50, 0.75$ and 1. Corresponding values of $y = \frac{2}{1+x^2}$ are shown in the following table, correct to 3 decimal places (which is one more decimal place than required in the problem).

Note carefully: Number of intervals = number of ordinates – 1

x	0	0.25	0.50	0.75	1.0
$1+x^2$ $y=\frac{2}{1+x^2}$	1 2.000 y_1 (first)	1.0625 1.8824 y_2	1.25 1.600 y_3	1.5625 1.280 y_4	2.0000 1.000 y_5 (last)

a) Using the trapezium rule, the required area

$$\int_0^1 \frac{2}{1+x^2}\,dx \approx \tfrac{1}{2}(0.25)\{(2.000+1.000)+2(1.8824+1.600+1.280)\}=0.125\{3.000+2(4.762)\}$$

$\approx$ 1.57 square units, correct to 2 decimal places

b) The volume of the solid of revolution by rotating the area in (a) through four right angles about the x-axis is given by:

Volume generated $\approx$ sum of volume generated by each interval

$$\approx \pi h\left\{\left(\frac{y_1+y_2}{2}\right)^2+\left(\frac{y_2+y_3}{2}\right)^2+\ldots+\left(\frac{y_n+y_{n+1}}{2}\right)^2\right\}$$

$$\approx \tfrac{1}{4}\pi h\{(y_1+y_2)^2+(y_2+y_3)^2+\ldots+(y_n+y_{n+1})^2\}$$

Volume generated

$\approx \tfrac{1}{4}(3.142)(0.25)\{(2+1.8824)^2+(1.8824+1.6)^2+(1.6+1.28)^2+(1.28+1.0)^2\}$

$\approx (0.1964)\{15.073+12.127+8.294+5.198\} \approx 0.1964(40.692)$

$\approx$ 7.99 cubic units, correct to 2 decimal places

Note that the required volume by analytical method (i.e. integration) is 8.078 cubic units.

Alternative method

Volume generated $\approx \int_a^b \pi y^2\,dx \approx \tfrac{1}{2}h\{(y_1^2+y_{n+1}^2)+2(y_2^2+y_3^2+y_4^2+\ldots+y_n^2)\}$

$\approx \tfrac{1}{2}(0.25)\{(4.0000+1.0000)+2(3.5434+2.5600+1.6384)\}$

$\approx 0.125\{3.000+2(4.762)\}$

$\approx$ 8.04 cubic units, correct to 2 decimal places

Example 7

A river is 30 m wide. The depth D m, at a distance X m from one bank is given by the table below.

X (m)	0	3	6	9	12	15	18	21	24	27	30
D (m)	0	1.5	2.2	3.0	4.2	4.8	4.1	3.2	3.0	2.0	1.1

Using the trapezium rule, find approximately, the area of the cross-section of the river, correct to one decimal place.

Solution

Using the trapezium rule, the required area

$$\approx \int_0^{30} DdX \approx \tfrac{1}{2}h\{(y_1 + y_{n+1}) + 2(y_2 + y_3 + y_4 + \ldots + y_n)\}$$

$$\approx \tfrac{1}{2}(3)\{(0 + 1.1) + 2(1.5 + 2.2 + 3.0 + 4.2 + 4.8 + 4.1 + 3.2 + 3.0 + 2.0)\} \approx 1.5\{1.1 + 2(28)\}$$

≈ 85.7 square units, correct to 1 decimal place.

Note that the interval width $h = 3$.

Example 8

a) Copy and complete the table of values for $\frac{x+1}{x+2}$ at the ordinates 0, 1, 2, 3, 4 and 5.

x	0	1	2	3	4	5
$\frac{x+1}{x+2}$	0.5000		0.7500	0.8000		0.8571

b) Using the trapezium rule, evaluate $\int_0^5 \frac{x+1}{x+2}dx$, correct to three decimal places.

Solution

a) Let $y = \frac{x+1}{x+2}$; When $x = 1 \Rightarrow y = \frac{1+1}{1+2} = \frac{2}{3} = 0.6667$

When $x = 4 \Rightarrow y = \frac{4+1}{4+2} = \frac{5}{6} = 0.8333$

The completed table is shown below.

x	0	1	2	3	4	5
$\frac{x+1}{x+2}$	0.5000	**0.6667**	0.7500	0.8000	**0.8333**	0.8571

b) Using the trapezium rule

$$\int_0^5 \frac{x+1}{x+2}dx = \frac{h}{2}[(y_1 + y_6) + 2(y_2 + y_3 + y_4 + y_5)], \text{ where } h = 1$$

$$\int_0^5 \frac{x+1}{x+2}dx = \tfrac{1}{2}[0.5 + 0.8571 + 2(0.6667 + 0.75 + 0.8 + 0.8333)]$$

$$= 0.5[1.3571 + 6.1] = 3.729 \text{ (3d.p)}$$

Example 9

Using the trapezium rule with seven ordinates, evaluate $\int_0^6 \frac{2}{\sqrt{(x^2+1)}}dx$, correct to three decimal places.

Solution

Form the table of values as shown below.

x	0	1	2	3	4	5	6
$\frac{2}{\sqrt{(x^2+1)}}$	2.0000	1.4142	0.8944	0.6325	0.4851	0.3922	0.3288

$$\int_0^6 \frac{2}{\sqrt{(x^2+1)}}\,dx = \frac{1}{2}[(2+0.3288)+2(1.4142+0.8944+0.6325+0.4851+0.3922)]$$

$$=\frac{1}{2}(2.3288+7.6368)=\frac{1}{2}(9.9656)=4.983 \text{ correct to three decimal places.}$$

Review exercise 22

1. The gradient of a curve at any point (x, y) is $6x^2-6x+11$. If the curve passes through the point (1, 3), find its equation.

2. The gradient of a curve at any point $P(x, y)$ on it is given by $6x + 2$. If the curve passes through the point $A(1, 0)$, find the equation of the curve.

3. A particle is projected in a straight line from a point O with a speed 6 ms^{-1}. At time t s later its acceleration is $(1 + 2t)$ ms^{-1}. For the time when $t = 4$, calculate for the particle:
   i) its velocity;
   ii) its distance from O.

4. A particle moves in a straight line in such a way that its acceleration a ms^{-2} at time t s after starting from rest is given by $a = t(t + 2)$. Calculate:
   i) the velocity of the particle after 3 s;
   ii) the distance travelled in the first 3 s.

5. The velocity of a particle, v, is given by $v = 2 + 3t$. Given that $s = 3$ when $t = 0$, find an expression for s in terms of t.

6. The velocity of a particle, v, is given by $v = 15 - 3t$. When does it come to rest?

7. Sketch the curves $y^2 = x$ and $y = x - 5$. Find
   i) the coordinates of the point of intersection of the curves.
   ii) the area of the finite region enclosed between them.

8. Sketch the curve $y = x^3 - x$, indicating all turning points and the points of intersection with the axes. Calculate the finite area bounded by the curve and the x-axis.

9. a) Sketch the graph of the function defined by $f(x) = x^2 - 6x + 11$
   b) Calculate the area of the region S bounded by the curve in (a) above and the lines $x = 0$, $x = 3$, and $y = 0$.
   c) Calculate the volume of the solid formed when the region S is rotated completed about the line $y = 0$ (You may leave your answer in terms of π).

10. i) Calculate the area of the finite region enclosed between the curve $y = x^2$ and the line $y = 4x + 2$.

ii) Calculate also the volume of the solid generated when the finite region in (i) is rotated completely about the x-axis. (You may leave your answer in terms of π).

11. A curve is defined in the $x - y$ plane by $y = (x - 1)(ax^2 + bx + c)$ where a, b and c are constants. The curve touches the x-axis at the point where $x = 1$ and the line $y = 3x + 6$ at the point $(-2, 0)$.

a) Find the values of a, b and c.

b) Sketch the curve, and on the same axes, draw the line $y = 3x + 6$, indicating clearly the points of the intersection.

c) Calculate the area of the finite region bounded by the curve and the line $y = 3x + 6$.

12. A curve is defined in the $x - y$ plane by $y = ax^2 + bx + c$ where a, b and c are constants. The curve passes through the point $(0, -2)$ and its gradient at the point $(-1, 0)$ is -5.

a) Find the values of a, b and c.

b) Sketch the curve and find the area of the finite region bounded by the curve and the x-axis

13. a) Sketch the curve $y = 3x^2 + 1$. Determine the area enclosed by the curve, the x-axis and the ordinates $x = -1$ and $x = 3$.

b) Sketch the curves $y = x^2 + 3$ and $y = 7 - 3x$. Determine the area enclosed by them.

14. Sketch the curve $y = 3x^2 + 2$. Determine the area enclosed by the curve, the x-axis and the ordinates $x = 1$ and $x = 3$.

15. Determine the volume of the solid of revolution formed by rotating the areas enclosed by the given curves, the x-axis and the given ordinates through one revolution about the x-axis.

i) $y = 5x$; $x = 1, x = 4$

ii) $y = 2x^2 + 3$; $x = 0, x = 2$

iii) $y = x^2$; $x = -2, x = 3$.

16. If part of the curve $y = x^2$ from $x = 1$ to $x = 2$ is rotated completely about the y-axis, find the volume of the solid formed.

17. i) Determine the coordinates of the points of intersection of the curves $y = x^2$ and $y^2 = x$.

ii) Sketch the curves $y = x^2$ and $y^2 = x$ on the same axes.

iii) Find the area enclosed between the two curves

iv) Calculate the volume of the solid formed when the area enclosed between the two curves is rotated 360° about the x-axis.

18. Sketch the curve $y = x^2 - x$. The part of the curve below the x-axis is rotated about the x-axis to form a solid of revolution. Find its volume.

19. Sketch the curve $y^2 = x - 1$. The area enclosed by the curve, the y-axis, and the lines $y = \pm 2$ is completely rotated about the y-axis. Find the volume of the solid formed.

20. Using the trapezium rule, with seven ordinates at $x = 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}$ and 4, calculate to three decimal places, the approximate value of $I = \int_1^4 \frac{dx}{1+x}$.

If the correct value of I is 0.9163, calculate correct to two decimal places the percentage error.

Note : Percentage error

$$= \frac{\text{Difference between the two values}}{\text{Correct value}} \times 100$$

21. Using the trapezium rule, with ordinates at $x = 1, 2, 3$ and 4 estimate, correct to two decimal places, the value of $\int_1^4 \sqrt{(x^2+16)}\, dx$.

22. Use the trapezium rule, with ordinates at $x = 1, 2, 3, 4$, and 5 to evaluate, correct to two decimal places, an approximate for $\int_1^5 \sqrt{(2x + \frac{8}{x^2})}\, dx$.

23. Using a trapezium rule with interval of 0.5, calculate, the approximate value of $\int_{1\frac{1}{2}}^{4\frac{1}{2}} \sqrt{(x^2+1)}\; dx$.

24. Use the trapezium rule to find $\int_0^1 \frac{dx}{\sqrt{1+x^2}}$ at intervals of 0.2 correct to three decimal place

25. Use the trapezium rule to find $\int_0^{2.5} \frac{dx}{\sqrt{8-x^2}}$ at intervals of 0.25 correct to three decimal places.

26. Using the trapezium rule, with ordinates at $x = 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}$ and 4, estimate correct to two decimal places the value of $I = \int_1^3 \frac{x+1}{x}\, dx$.

27. Use the trapezium rule, with ordinates at $x = 1, 1.5, 2, 2.5$ and 3 to calculate, correct to two decimal places, an approximate value for the integral $\int_1^3 \frac{\sqrt{x+1}}{x}\, dx$

28. Use the trapezium rule, with ordinates at $x = 0, 0.1, 0.2, 0.3, 0.4$ and 0.5 to evaluate, correct to two decimal places, an approximate for $\int_0^{0.5} \frac{4}{\sqrt{1-x^2}}\, dx$.

29. Use the trapezium rule to find $\int_1^2 \sqrt{1+\frac{1}{x}}\; dx$ at intervals of 0.2 correct to two decimal places

30. Use the trapezium rule to find $\int_0^2 \frac{1}{9-x^2}\, dx$ at intervals of 0.5 correct to two decimal places

31. Copy and complete the table below for $y = \sqrt{(x-2)(1-x)}$.

x	1	1.2	1.4	1.6	1.8	2
$\sqrt{(x-2)(1-x)}$	0		0.490			0

Using the trapezium rule:

a) find the area bounded by the curve $y = \sqrt{(x-2)(1-x)}$, the x-axis and the lines $x = 0$ and $x = 1$

b) the volume of the solid of revolution by rotating the area in (a) through four right angles about the x-axis (Take $\pi = 3.142$).

32. Given the curve $y = (3 - x)(1 + x)$

a) Find

i) the coordinates of the points of intersection with the x and y-axis.

ii) the gradients of the curve at these points.

iii) the coordinates of its turning points.

b) Sketch the curve

c) Calculate the area of the finite region bounded by the curve and the x-axis.

33. a) On the same axis, sketch and label clearly the curves $y = x^2 - 5x$ and $y = 5x - x^2$.

b) Find the volume of the solid generated when the finite region between the curve $y = 5x - x^2$ and the x-axis is rotated through four right angles about the x-axis, giving the answer in terms of π.

34. Sketch the curve $y = x^3 - 7x^2 + 15x - 9$, indicating clearly its points of intersection with the axes and its turning points. Calculate, to one decimal place, the area of the finite region enclosed by the curve and the lines $y = 0$, $x = 1$ and $x = 3$

35. The velocity of a particle, which starts from rest at a point O and travels in a straight line, is v ms^{-1} at any time t seconds. If v is given by $v = 2\sqrt{t}$, find

a) the acceleration when $t = 4$,

b) the displacement of the particle from O when $t = 9$.

36. The velocity after t seconds of a particle moving in a straight line is v ms^{-1} where $v = 6t^2 + 1$. Find how far the particle travels in the interval from $t = 1$ to $t = 4$.

37. A particle moving in a straight line starts from rest at a point O on the line, and t seconds later, has an acceleration $(t - 6)$ ms^{-2}. Find expressions for the velocity and the displacement of the particle O after t seconds. Find also the velocity and the displacement from O after 6 seconds.

38. The velocity of a particle moving in a straight line is $3t^2$ ms^{-1} at anytime, t seconds. What is the acceleration of the particle at time t? If the particle passes through a fixed point A on the line, with velocity 12 ms^{-1}, find the displacement from A, 6 seconds later.

39. The acceleration of a particle travelling in a straight line is given by $15\sqrt{(t-1)}$ at any time t seconds. When $t = 2$ the particle passes through a fixed O on the line with velocity 2 ms^{-1}. Find the expressions for the velocity of the particle and for its displacement from O at time t.

40. The velocity at time, t, of a particle P travelling in a straight line, is v ms^{-1} where $v = 2 + \frac{1}{t^2}$. What is the acceleration of P when $t = 1$? Find the distance travelled by P during the time interval between $t = 2$ and $t = 6$.

41. A particle moves in a straight line with velocity v ms^{-1} where, after t seconds,

$$v = 2 + \frac{1}{(t+1)^2}.$$ When $t = 0$ the particle

passes through a fixed point 0 on the line.

a) Find an expression for the displacement of the particle from O after t seconds.

b) Show that the velocity has a limiting value (i.e. v approaches a fixed value as t approaches infinity), and state this value.

42. a) Find the area between the curve $y = 4x - x^2$ and the x-axis.

b) Determine the area enclosed by the curve and $y = 5x^2 + 2$, the x-axis and the ordinates $x = 0$ and $x = 3$. Find also the area enclosed by the curve and the y-axis between the same limits.

43. Calculate the area enclosed between $y = x^3 - 4x^2 - 5x$ and the x-axis using an approximate method and compare your result with the true area obtained by integration.

44. Sketch the curves $y = x^2 + 2$ and $y - 12 = 3x$ from $x = -3$ to $x = 6$.

a) Determine the co-ordinates of the points of intersection of the two curves

b) Determine the area enclosed by the two curves.

c) If the enclosed area is rotated 360° about the x-axis, calculate the volume of the solid produced

45. A particle moves in a straight line so that its velocity, v m/s, is given by $v = 12t - t^2$, where t is the time, in seconds, measured from the start of the motion. Find:

i) the acceleration when $t = 20$,

ii) the values of t at which the particle is stationary,

iii) the value of t when the particle is again at its starting point.

46. A particle P moves in a straight line so that its velocity v m/s at time t seconds, where $t \geq 0$, is given $v = 12t - 5t^2$. Calculate:

a) the value of t when the acceleration of P is zero;

b) the distance covered by P between the instants when $t = 0$ and $t = 2$.

47. i) Sketch the curve $y = (x-1)(x-2)(x-3)$, indicating clearly the coordinates of the points where it cuts the axes.

ii) Find the area, above the x-axis, bounded by the curve and x-axis.

48. a) On the same axes sketch the graphs of $y = x(x^2 - 12)$ and $y = -3x$.

i) Find the finite area enclosed between the two graphs

ii) Find the equation of the tangent to the curve $y = x(x^2 - 12)$ at the point where $x = 0$

49. Find the area enclosed by $y^2 = 2x$, the y-axis and the line $y = 4$.

50. Find the area enclosed by $y = x^2 - x - 2$ and the x-axis.

51. i) Find the area enclosed by the curve $y = x^2 - 3x - 4$ and the line $y = 0$.

ii) The finite area enclosed between the curve $y = kx - x^2$ and the x-axis is 288 square units. Find the value of the constant k.

52. Find the area of the finite region enclosed by the curve $y = 2x^2 - x - 1$ and the x-axis.

53. A particle is projected from a point O, with a speed of 5 m s^{-1}. It moves in a straight line with an acceleration of $(6 - 2t)$ ms^{-2}, t seconds after leaving O. Calculate

i) the speed after 2 seconds.

ii) Calculate the maximum speed of the particle.

54. The volume V litres of petrol in a car's tank is decreasing at a rate of $(\frac{2}{5} - \frac{x}{500})$ litres/km after travelling x kilometres. If the volume V is 40 litres when $x = 0$;

i) Find the volume of the tank V in terms of x..

ii) After how many kilometres is the tank empty?

55. A curve passes through the point (2, 2) and its gradient at a point is $ax^2 + b$, where a and b are constants. The tangent to the curve at the point (1, –2) is parallel to the x-axis. Find

i) the values of a and b

ii) the equation of the curve

56. A curve passes through the point (1, 0) and its gradient at a point is $ax + b$, where a and b are constants. The tangent to the curve at the point (2, 4) is $y = 5x - 6$. Find

i) the values of a and b

ii) the equation of the curve

57. The gradient at any point $P(x, y)$ on a curve is given by $-2x$. The curve passes through the point $A(0, 1)$.

i) Find the equation of the curve and sketch the curve showing clearly the points where it cuts the x-axis.

ii) Calculate the area of the finite region enclosed by the curve and the x-axis.

iii) Calculate the volume of the solid generated when the finite region in (ii) is rotated completely about the x-axis.

CHAPTER 23

VECTORS IN A PLANE

Definition

A **vector quantity** is any quantity which has *both magnitude* and *direction.* Examples are: displacement, acceleration, momentum, velocity, force, weight etc.

A **scalar quantity** has magnitude only. Examples are: time, speed, distance, mass, area, pressure etc.

Notation

Free vectors are denoted by a pair of block letters and an arrow. For example, $\overrightarrow{AB}$ denotes the directed line segment representing a movement from A to another point B.

The magnitude of $\overrightarrow{AB}$ is the length of line AB. Also the direction of the vector $\overrightarrow{AB}$ is that of the line AB.

Position vectors are denoted by small letters printed in bold type, for example, **a, b, c, p, q, r** etc. In hand writing, we underline the small letters to represent position vectors.

Generally, for the vector $\overrightarrow{AB} = \begin{pmatrix} a \\ b \end{pmatrix}$, a and b are called the components of the vector. The horizontal component, a, is measured along the x-axis and the vertical component, b, is measured along the y-axis from A to B.

Let a particle be moved 3 units to the right and 4 units upwards from a point A to another point B. The vector $\overrightarrow{AB}$ can be represented in the column form by $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. To draw the vector $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, we start from the point A, move 3 units to the right and 4 units upwards. If the x-component is negative, move to the left and if positive move to the right. Also if the y-component is negative, move downwards and if positive move upwards.

Representation of vectors in standard basis form

We know that vectors may be represented in **2-dimensions** by:

1. the **column** or **components** form i.e. $\begin{pmatrix} a \\ b \end{pmatrix}$. For example, $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

2. the **magnitude and bearing** form (direction from the geographic north). For example,

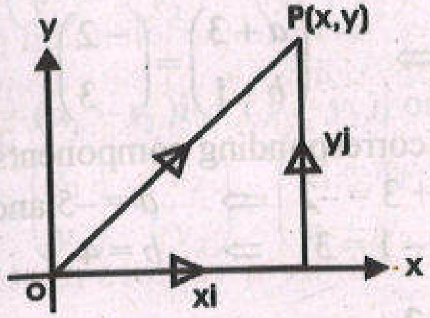
$$\overrightarrow{OB} = (3 \text{ km}, 060^\circ) \text{ or } \overrightarrow{OP} = (5 \text{ m}, 330^\circ)$$

Let us now learn how to represent vectors in the standard basis form.

Vectors can be related to the basic coordinate systems which we use by the introduction of what we call "**unit vectors**." A unit vector is one which has a magnitude of 1 and is often indicated by putting a hat (or circumflex) on top of the vector symbol, for example unit vector of a vector **a** is denoted by $\hat{\mathbf{a}}$.

Let us consider the two-dimensional or x-y Cartesian coordinate system. We can define a unit vector in the x-direction by **i**, sometimes denoted by $\hat{\mathbf{i}}$.

Similarly, we can define a unit vector in the y-direction by **j**, sometimes denoted by $\hat{\mathbf{j}}$. Any two-dimensional vector can therefore be represented by employing multiples of the unit vectors, **i** and **j**, as illustrated by the diagram below.



Thus any vector can be represented algebraically by $x\mathbf{i} + y\mathbf{j}$, where **i** and **j** are the unit vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ along the x and y axes respectively and x and y are the components of the vector in the **i** and **j** directions respectively.

For example, $\begin{pmatrix}2\\-3\end{pmatrix}$ can be written in the standard basis form as $2\mathbf{i} - 3\mathbf{j}$. Also, the vector $\begin{pmatrix}-3\\5\end{pmatrix}$ can be written in the standard basis form as $-3\mathbf{i} + 5\mathbf{j}$.

Note: Vector quantities are represented by *arrowed lines* in diagrams. The length of the line represents the magnitude and the arrow shows the direction.

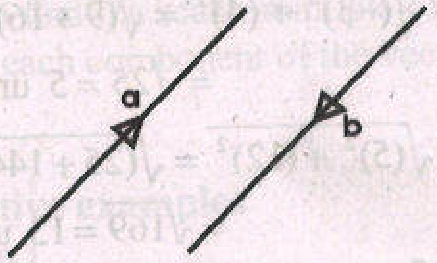
Zero vector

It is a vector with **zero** components i.e. $\mathbf{o} = \begin{pmatrix}0\\0\end{pmatrix} = 0\mathbf{i} + 0\mathbf{j}$. It has zero magnitude.

The negative or inverse of a vector

If $\overrightarrow{AB} = \begin{pmatrix}x\\y\end{pmatrix}$, then $\overrightarrow{BA} = -\overrightarrow{AB} = \begin{pmatrix}-x\\-y\end{pmatrix}$, is called the **inverse** or **negative vector** of $\overrightarrow{AB}$.

A vector and its inverse have the same magnitude but opposite directions.



$\mathbf{a} = -\mathbf{b}$

Illustrative examples

Example 1

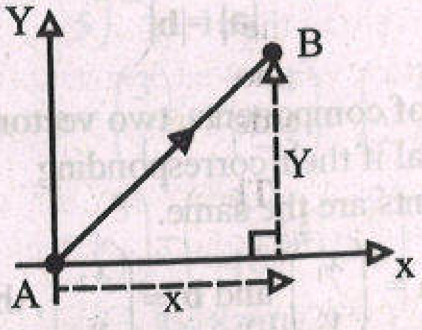
If $\overrightarrow{PQ} = \begin{pmatrix}-3\\-4\end{pmatrix}$ then $\overrightarrow{QP} = -\overrightarrow{PQ} = \begin{pmatrix}3\\4\end{pmatrix}$

Example 2

If $\overrightarrow{AB} = \begin{pmatrix}1\\-3\end{pmatrix}$ then $\overrightarrow{BA} = -\overrightarrow{AB} = \begin{pmatrix}-1\\3\end{pmatrix}$

Magnitude or length of a vector

If $\overrightarrow{AB} = \begin{pmatrix}x\\y\end{pmatrix} = x\mathbf{i} + y\mathbf{j}$, then the **magnitude** of $\overrightarrow{AB}$ denoted by $|\overrightarrow{AB}| = AB = \sqrt{x^2 + y^2}$



Illustrative examples

Example 1

If $\overrightarrow{CD} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$, find $|\overrightarrow{CD}|$ and $|\overrightarrow{AB}|$.

Solution

$$|\overrightarrow{CD}| = \sqrt{(-3)^2 + (4)^2} = \sqrt{(9+16)}$$
$$= \sqrt{25} = 5 \text{ units}$$
$$|\overrightarrow{AB}| = \sqrt{(5)^2 + (12)^2} = \sqrt{(25+144)}$$
$$= \sqrt{169} = 13 \text{ units}$$

Example 2

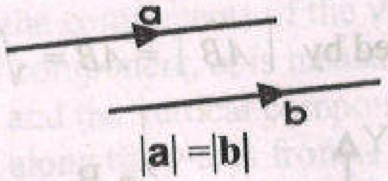
Find the length of the vector $\overrightarrow{AB} = -5\mathbf{i} + 10\mathbf{j}$.

Solution

$$|\overrightarrow{AB}| = \sqrt{(-5)^2 + (10)^2} = \sqrt{(25+100)}.$$
$$= \sqrt{125} = 5\sqrt{5} \text{ units}$$

Equality of vectors

Two vectors are said to be **equal** or **equivalent** if they have the same magnitude and direction.
In particular, if $\mathbf{a} = \mathbf{b}$ then $|\mathbf{a}| = |\mathbf{b}|$ and $\mathbf{a}$ and $\mathbf{b}$ have the same direction. i.e. they are parallel.



In terms of components, two vectors are said to be equal if their corresponding components are the same.

Thus if $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then $\mathbf{a} = \mathbf{b}$

$x_1 = x_2$ and $y_1 = y_2$

Illustrative examples

Example 1

If $\mathbf{p} = \begin{pmatrix} a+3 \\ b-1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, find a and b, if $\mathbf{p} = \mathbf{q}$

Solution

$$\mathbf{p} = \mathbf{q} \quad \Rightarrow \quad \begin{pmatrix} a+3 \\ b-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

Equating corresponding components;

$$a + 3 = -2 \quad \Rightarrow \quad a = -5 \text{ and}$$
$$b - 1 = 3 \quad \Rightarrow \quad b = 4$$

Example 2

If $\mathbf{a} = \begin{pmatrix} 3x+1 \\ 2y+3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -8 \\ 5 \end{pmatrix}$, find x and y, if $\mathbf{a} = \mathbf{b}$

Solution

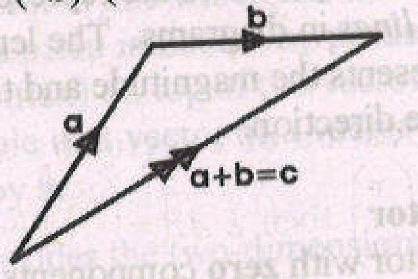
$$\mathbf{a} = \mathbf{b} \quad \Rightarrow \quad \begin{pmatrix} 3x+1 \\ 2y+3 \end{pmatrix} = \begin{pmatrix} -8 \\ 5 \end{pmatrix}$$

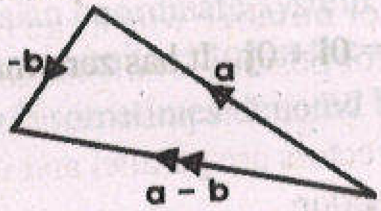
Equating corresponding components;

$$3x + 1 = -8 \quad \Rightarrow \quad 3x = -8 - 1 = -9$$
$$x = -3 \text{ and } 2y + 3 = 5$$
$$2y = 5 - 3 = 2 \quad \Rightarrow \quad y = 1$$

Addition and subtraction of vectors

The triangle law is used to add or subtract vectors. As shown from the diagrams below,
$\mathbf{a} + \mathbf{b} = \mathbf{c}$ (addition) and
$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$ (subtraction).





In general, if $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j}$ or $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then

i) $\mathbf{a} + \mathbf{b} = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j}$ or

$$= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

ii) $\mathbf{a} - \mathbf{b} = (x_1 - x_2)\mathbf{i} + (y_1 - y_2)\mathbf{j}$ or

$$= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}$$

Illustrative examples

Example 1

If $\mathbf{r} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then;

i) $\mathbf{r} + \mathbf{s} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3+1 \\ 4+2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

[Add the corresponding components].

ii) $\mathbf{r} - \mathbf{s} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3-1 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

[Subtract the corresponding components].

Example 2

If $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} - 5\mathbf{j}$, then;

i) $\mathbf{a} + \mathbf{b} = (3\mathbf{i} + \mathbf{j}) + (4\mathbf{i} - 5\mathbf{j}) = 7\mathbf{i} - 4\mathbf{j}$

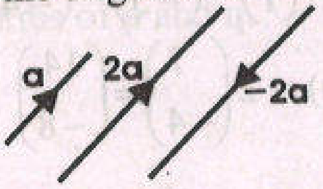
[Add the corresponding components].

ii) $\mathbf{a} - \mathbf{b} = (3\mathbf{i} + \mathbf{j}) - (4\mathbf{i} - 5\mathbf{j}) = -\mathbf{i} + 6\mathbf{j}$

[Subtract the corresponding components].

Multiplication of a vector by a scalar

If $\overrightarrow{AB} = \begin{pmatrix} x \\ y \end{pmatrix}$ then $k\,\overrightarrow{AB} = \begin{pmatrix} kx \\ ky \end{pmatrix}$, where k is **a scalar** or **number** which can be a negative or positive whole number or fraction. When k is positive, it implies that the vectors are parallel and in the same direction. When k is negative, it implies that the vectors are parallel but in opposite directions. The length of the new vector is $|k|$ times the length of the original vector.



Note: To find the scalar multiple of a vector, multiply each component of the vector by the scalar, k.

Illustrative examples

Example 1

If

$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ then $2\overrightarrow{AB} = \begin{pmatrix} 2 \times (-1) \\ 2 \times 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 10 \end{pmatrix}$

and $-3\overrightarrow{AB} = \begin{pmatrix} 3 \\ -15 \end{pmatrix}$.

Example 2

If $\mathbf{a} = -\mathbf{i} + 2\mathbf{j}$ then,

$-3\mathbf{a} = 3\mathbf{i} - 6\mathbf{j}$ and $2\mathbf{a} = -2\mathbf{i} + 4\mathbf{j}$

Example 3

If $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$, find

i) $2\mathbf{b}$ ii) $\mathbf{a} - 2\mathbf{b} + \mathbf{c}$ iii) $2(\mathbf{a} + \mathbf{b})$

Solution

i) $2\mathbf{b} = 2\begin{pmatrix} 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 8 \\ -10 \end{pmatrix}$

ii) $\mathbf{a} - 2\mathbf{b} + \mathbf{c} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 4 \\ -5 \end{pmatrix} + \begin{pmatrix} -2 \\ -6 \end{pmatrix}$

$$= \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 8 \\ -10 \end{pmatrix} + \begin{pmatrix} -2 \\ -6 \end{pmatrix}$$

$$= \begin{pmatrix} 3 - 8 + (-2) \\ 1 - (-10) + (-6) \end{pmatrix} = \begin{pmatrix} -7 \\ 5 \end{pmatrix}$$

iii) First, find the expression in the bracket i.e. $\mathbf{a} + \mathbf{b}$.

$$\mathbf{a}+\mathbf{b} = \begin{pmatrix}3\\1\end{pmatrix}+\begin{pmatrix}4\\-5\end{pmatrix}=\begin{pmatrix}7\\-4\end{pmatrix}$$

$$\therefore \quad 2(\mathbf{a}+\mathbf{b}) = 2\begin{pmatrix}7\\-4\end{pmatrix}=\begin{pmatrix}14\\-8\end{pmatrix}$$

Example 4

If $\mathbf{r} = 3\mathbf{i}+\mathbf{j}$ and $\mathbf{s} = -2\mathbf{i}+\mathbf{j}$, calculate $6(\mathbf{r}+2\mathbf{s})$.

Solution

First find the expression in the bracket i.e. $\mathbf{r}+2\mathbf{s}$

$$\begin{aligned}\mathbf{r}+2\mathbf{s} &= (3\mathbf{i}+\mathbf{j})+2(-2\mathbf{i}+\mathbf{j})\\ &= (3\mathbf{i}+\mathbf{j})+(-4\mathbf{i}+2\mathbf{j}) = -\mathbf{i}+3\mathbf{j}\end{aligned}$$

$$\therefore\ 6(\mathbf{r}+2\mathbf{s}) = 6(-\mathbf{i}+3\mathbf{j}) = -6\mathbf{i}+18\mathbf{j}$$

Note: If the vector is given in the Cartesian form you can write it in the column form before doing the calculations.

Example 5

If $\mathbf{p}=\begin{pmatrix}1\\2\end{pmatrix}$, $\mathbf{q}=\begin{pmatrix}-2\\3\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}1\\1\end{pmatrix}$, find $2\mathbf{p}-\mathbf{q}+\mathbf{r}$.

Solution

$$\begin{aligned}2\mathbf{p}-\mathbf{q}+\mathbf{r} &= 2\begin{pmatrix}1\\2\end{pmatrix}-\begin{pmatrix}-2\\3\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}\\ &= \begin{pmatrix}2\\4\end{pmatrix}-\begin{pmatrix}-2\\3\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}5\\2\end{pmatrix}\end{aligned}$$

Example 6

If $\mathbf{a} = 3\mathbf{i}+4\mathbf{j}$ and $\mathbf{b} = 10\mathbf{i}-2\mathbf{j}$, calculate $\mathbf{b}-2\mathbf{a}$.

Solution

$$\begin{aligned}\mathbf{b}-2\mathbf{a} &= (10\mathbf{i}-2\mathbf{j})-2(3\mathbf{i}+4\mathbf{j})\\ &= (10\mathbf{i}-2\mathbf{j})-(6\mathbf{i}+8\mathbf{j}) = 4\mathbf{i}-10\mathbf{j}\end{aligned}$$

Example 7

Two vectors are represented by $\mathbf{r} = 2\mathbf{a}$ and $\mathbf{s} = 3\mathbf{a}$, where $\mathbf{a} = 2\mathbf{i}-\mathbf{j}$. Find $|\mathbf{r}-\mathbf{s}|$.

Solution

$\mathbf{r} = 2\mathbf{a} = 2(2\mathbf{i}-\mathbf{j}) = 4\mathbf{i}-2\mathbf{j}$ and

$\mathbf{s} = 3\mathbf{a} = 3(2\mathbf{i}-\mathbf{j}) = 6\mathbf{i}-3\mathbf{j}$

$\mathbf{r}-\mathbf{s} = (4\mathbf{i}-2\mathbf{j})-(6\mathbf{i}-3\mathbf{j}) = -2\mathbf{i}+\mathbf{j}$

Alternatively

$\mathbf{r}-\mathbf{s} = 2\mathbf{a}-3\mathbf{a} = -\mathbf{a} = -(2\mathbf{i}-\mathbf{j}) = -2\mathbf{i}+\mathbf{j}$

$$\begin{aligned}|\mathbf{r}-\mathbf{s}| &= \sqrt{(-2)^2+(1)^2}\\ &= \sqrt{(4+1)} = \sqrt{5}\text{ units}\end{aligned}$$

Example 8

If $\mathbf{p}=\begin{pmatrix}2\\-2\end{pmatrix}$ and $\mathbf{q}=\begin{pmatrix}3\\4\end{pmatrix}$, find $\mathbf{r}$ such that

$$\tfrac{1}{2}\mathbf{p}-\mathbf{q}+\mathbf{r}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Solution

$$\tfrac{1}{2}\mathbf{p}-\mathbf{q}+\mathbf{r}=\begin{pmatrix}0\\0\end{pmatrix} \quad\Rightarrow\quad \mathbf{r}=\begin{pmatrix}0\\0\end{pmatrix}+\mathbf{q}-\tfrac{1}{2}\mathbf{p}$$

$$\begin{aligned}\mathbf{r} &= \begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}3\\4\end{pmatrix}-\tfrac{1}{2}\begin{pmatrix}2\\-2\end{pmatrix}\\ &= \begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}3\\4\end{pmatrix}-\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}2\\5\end{pmatrix}\end{aligned}$$

Example 9

If $\mathbf{p}=\begin{pmatrix}2\\1\end{pmatrix}$, $\mathbf{q}=\begin{pmatrix}1\\1\end{pmatrix}$, $\mathbf{r}=\begin{pmatrix}13\\7\end{pmatrix}$ and $k\mathbf{p}+m\mathbf{q}=\mathbf{r}$, where k and m are real numbers, find the values of k and m.

Solution

$k\mathbf{p}+m\mathbf{q}=\mathbf{r}$

$$k\begin{pmatrix}2\\1\end{pmatrix}+m\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}13\\7\end{pmatrix}$$

$$\begin{pmatrix}2k\\k\end{pmatrix}+\begin{pmatrix}m\\m\end{pmatrix}=\begin{pmatrix}13\\7\end{pmatrix}$$

$$\begin{pmatrix}2k+m\\k+m\end{pmatrix}=\begin{pmatrix}13\\7\end{pmatrix}$$

Equating corresponding components:

$2k+m=13$ ....(1) and

$k+m=7$ ....(2)

Solving equations (1) and (2) simultaneously, we have

Eqn1 − eqn2 $\Rightarrow$ $k=6$

Substitute $k=6$ into either eqn1 or eqn2

$m=1$

Therefore $k=6$ and $m=1$

Example 10

If $\mathbf{a}=\begin{pmatrix}1\\-3\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}2\\3\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}8\\3\end{pmatrix}$, find;

i) m and n such that $\mathbf{c}=m\mathbf{a}+n\mathbf{b}$, where m and n are scalars.

ii) $|\mathbf{d}|$, if $\mathbf{d}=\mathbf{c}-2\mathbf{a}$

Solution

$\mathbf{c}=m\mathbf{a}+n\mathbf{b}$

$$\begin{pmatrix}8\\3\end{pmatrix}=m\begin{pmatrix}1\\-3\end{pmatrix}+n\begin{pmatrix}2\\3\end{pmatrix}$$

$$\begin{pmatrix}8\\3\end{pmatrix}=\begin{pmatrix}m\\-3m\end{pmatrix}+\begin{pmatrix}2n\\3n\end{pmatrix}$$

$$\begin{pmatrix}8\\3\end{pmatrix}=\begin{pmatrix}m+2n\\-3m+3n\end{pmatrix}$$

Equating corresponding components:

$m+2n=8$ ....(1) and

$-3m+3n=3$ ....(2)

Solving equations (1) and (2) simultaneously, we have

$3\times$ Eqn1 + eqn2 $\Rightarrow$ $9n=27$ $\Rightarrow$ $n=3$

Substitute $n=3$ into either eqn1 or eqn2

$m=2$

Therefore $m=2$ and $n=3$

Example 11

Given that $\mathbf{p}=\begin{pmatrix}3\\5\end{pmatrix}$, $\mathbf{q}=\begin{pmatrix}2\\-1\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}5\\17\end{pmatrix}$, express **r** in terms of **p** and **q**.

Solution

$\mathbf{r}=a\mathbf{p}+b\mathbf{q}$

Substituting for **r**, **p** and **q**

$$\begin{pmatrix}5\\17\end{pmatrix}=a\begin{pmatrix}3\\5\end{pmatrix}+b\begin{pmatrix}2\\-1\end{pmatrix}$$

$3a+2b=5$..................(1)

$5a-b=17$..................(2)

$(1)-(2)\times 2$

$13a=39$ $\Rightarrow a=3$

Substituting $a=3$ into (2)

$b=5\times 3-17=-2$

$\therefore$ $\mathbf{r}=3\mathbf{p}-2\mathbf{q}$

Example 12

Two sides of a triangle are represented by the vectors $\mathbf{p}=\begin{pmatrix}2\\3\end{pmatrix}$ and $\mathbf{q}=\begin{pmatrix}3\\-2\end{pmatrix}$.

If $\mathbf{r}=\lambda\mathbf{p}+\mu\mathbf{q}$, where $\mathbf{r}=\begin{pmatrix}1\\-5\end{pmatrix}$, find the values of the scalars λ and μ.

Solution

$$\mathbf{r}=\lambda\mathbf{p}+\mu\mathbf{q}$$

$$\begin{pmatrix}1\\-5\end{pmatrix}=\lambda\begin{pmatrix}2\\3\end{pmatrix}+\mu\begin{pmatrix}3\\-2\end{pmatrix}$$

$$\begin{pmatrix}1\\-5\end{pmatrix}=\begin{pmatrix}2\lambda\\3\lambda\end{pmatrix}+\begin{pmatrix}3\mu\\-2\mu\end{pmatrix}$$

$$\begin{pmatrix}1\\-5\end{pmatrix}=\begin{pmatrix}2\lambda+3\mu\\3\lambda-2\mu\end{pmatrix}.$$

Equating corresponding components, we have

$1=2\lambda+3\mu$ ....(1)

$-5=3\lambda-2\mu$ ....(2)

Solving equations (1) and (2) simultaneously, we have $\lambda = -1$ and $\mu = 1$

Example 13

The position vectors of the points P, Q, R and S relative to a fixed point O are $\mathbf{p} = 2\mathbf{i} + 3\mathbf{j}$; $\mathbf{q} = 6\mathbf{i} + 23\mathbf{j}$; $\mathbf{r} = 2\mathbf{i} - 7\mathbf{j}$ and $\mathbf{s} = 7\mathbf{i} + 4\mathbf{j}$.

Find the scalars m and n such that

$\mathbf{s} = m\mathbf{p} + n\mathbf{q} + \mathbf{r}$

Solution

a) $\mathbf{p} = 2\mathbf{i} + 3\mathbf{j}$; $\quad \mathbf{q} = 6\mathbf{i} + 23\mathbf{j}$;
$\mathbf{r} = 2\mathbf{i} - 7\mathbf{j}$; $\quad \mathbf{s} = 7\mathbf{i} + 4\mathbf{j}$.

$7\mathbf{i} + 4\mathbf{j} = m(2\mathbf{i} + 3\mathbf{j}) + n(6\mathbf{i} + 23\mathbf{j}) + 2\mathbf{i} - 7\mathbf{j}$

$7\mathbf{i} + 4\mathbf{j} = (2m + 6n + 2)\mathbf{i} + (3m + 23n - 7)\mathbf{j}$

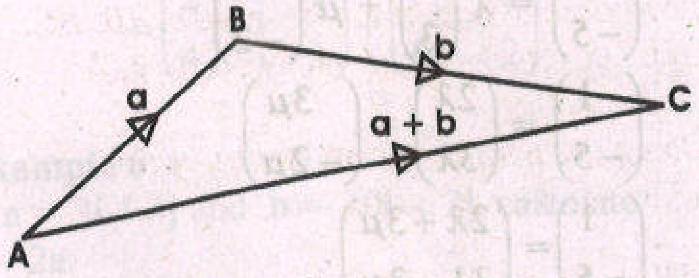
$2m + 6n = 5$ ....(1) and

$3m + 23n = 11$........(2)

Solving eqns (1) and (2) simultaneously, gives $m = \frac{7}{4}$ and $n = \frac{1}{4}$

Resultant of vectors

If $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are two vectors, then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. The vector $\overrightarrow{AC}$ is said to be the **resultant** of the two vectors. Note that the end point of the first vector should be the same as the starting point of the second vector. Thus, the resultant of the two vectors **a** and **b** is given by **a** + **b**.



Now if $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{b}\ x_2\mathbf{i} + y_2\mathbf{j}$

Then the resultant of **a** and **b** is

$\mathbf{a} + \mathbf{b} = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j}$

Illustrative examples

Example 1

Find the resultant of the vectors **a** and **b** if $\mathbf{a} = -\mathbf{i} - 5\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j}$.

Solution

$\mathbf{a} = -\mathbf{i} - 5\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j}$

The resultant of **a** and **b** is

$\mathbf{a} + \mathbf{b} = (-\mathbf{i} - 5\mathbf{j}) + (3\mathbf{i} + 2\mathbf{j}) = 2\mathbf{i} - 3\mathbf{j}$

Example 2

Let $\mathbf{a} = 2\mathbf{i} - 5\mathbf{j}$ and $\mathbf{b} = -\mathbf{i} + 3\mathbf{j}$.

Find the resultant of 2**a** and –**b**

Solution

$2\mathbf{a} = 2(2\mathbf{i} - 5\mathbf{j}) = 4\mathbf{i} - 10\mathbf{j}$ and

$-\mathbf{b} = -(-\mathbf{i} + 3\mathbf{j}) = \mathbf{i} - 3\mathbf{j}$

The resultant of 2**a** and –**b** is

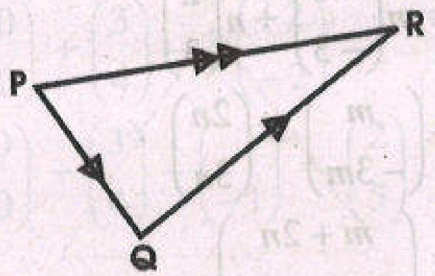
$2\mathbf{a} + (-\mathbf{b}) = (4\mathbf{i} - 10\mathbf{j}) + (\mathbf{i} - 3\mathbf{j}) = 5\mathbf{i} - 13\mathbf{j}$

Example 3

If $\overrightarrow{PQ} = -\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{RQ} = -5\mathbf{i} - 8\mathbf{j}$,

find $\overrightarrow{PR}$.

Solution



From the vectors given, $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$

$\overrightarrow{PQ}$ has been given but

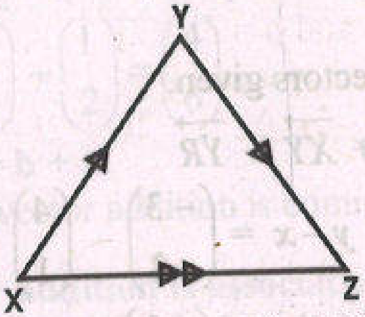
$\overrightarrow{QR} = -\overrightarrow{RQ} = 5\mathbf{i} + 8\mathbf{j}$ [Negative vectors]

$\therefore \quad \overrightarrow{PR} = (-\mathbf{i} + 3\mathbf{j}) + (5\mathbf{i} + 8\mathbf{j}) = 4\mathbf{i} + 11\mathbf{j}$

Example 4

If $\overrightarrow{XY} = -2\mathbf{i} + 5\mathbf{j}$ and $\overrightarrow{ZY} = 3\mathbf{i} + 2\mathbf{j}$, find $\overrightarrow{XZ}$.

Solution



From the vectors given, $\overrightarrow{XZ} = \overrightarrow{XY} + \overrightarrow{YZ}$

$\overrightarrow{XY}$ has been given but

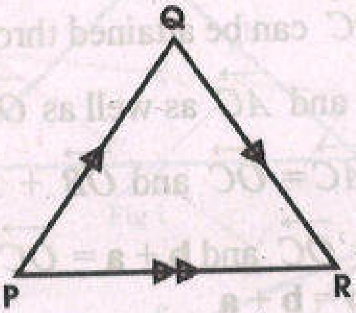
$\overrightarrow{YZ} = -\overrightarrow{ZY} = -3\mathbf{i} - 2\mathbf{j}$ [Negative vectors]

$\therefore \quad \overrightarrow{XZ} = (-2\mathbf{i} + 5\mathbf{j}) + (-3\mathbf{i} - 2\mathbf{j}) = -5\mathbf{i} + 3\mathbf{j}$

Example 5

If $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\overrightarrow{RQ} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$ find $\overrightarrow{PR}$.

Solution



From the vectors given, $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$

$\overrightarrow{PQ}$ has been given but

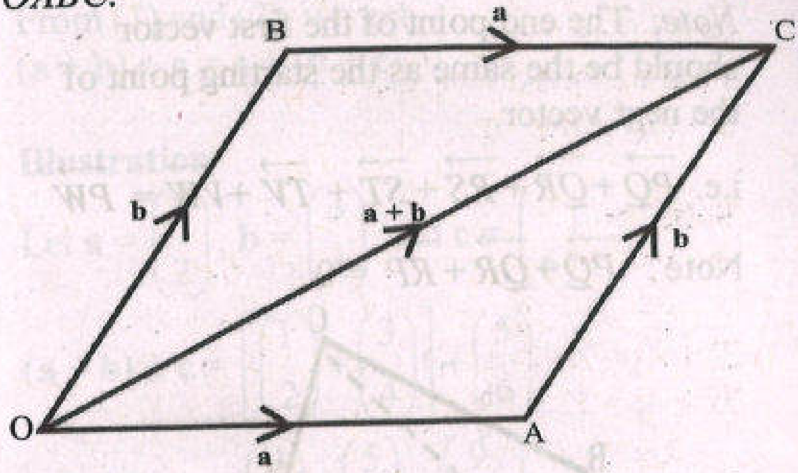
$\overrightarrow{QR} = -\overrightarrow{RQ} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ [Negative vectors].

$\therefore \quad \overrightarrow{PR} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$

Parallelogram and polygon laws of vector addition

Parallelogram law of vector addition

Consider two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in the figure below. Complete the parallelogram *OABC*.



Then $\overrightarrow{AC} = \overrightarrow{OB}$

$\therefore \quad \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$

Hence $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$ i.e. $\mathbf{a} + \mathbf{b} = \mathbf{c}$

This can be written in words as:

The sum of two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is represented in magnitude and direction by the diagonal of the parallelogram through *O*, of which *OA* and *OB* are adjacent.

Polygon law of vector addition

The concept of addition of two vectors can be extended to the addition of three or more vectors.

Consider the polygon whose sides *AB*, *BC*, *CD*, *DE* represent the vectors **a, b, c, d** respectively

Using the parallelogram law of vector addition, we can write

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = \mathbf{a} + \mathbf{b}$$

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD} = \mathbf{a} + \mathbf{b} + \mathbf{c}$$

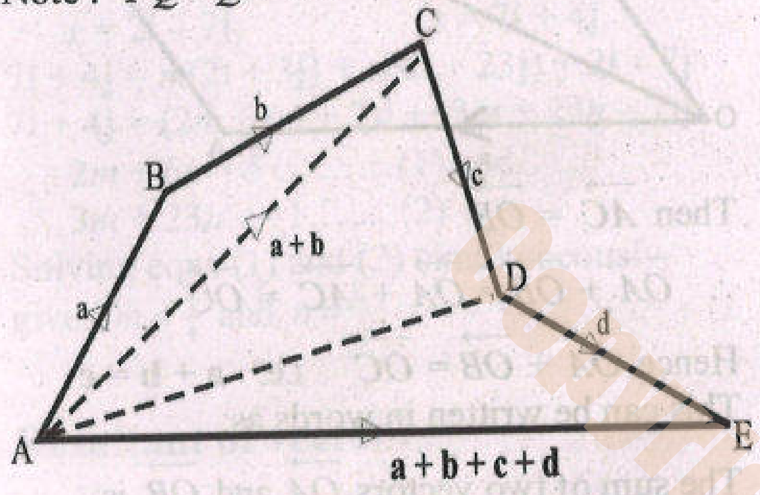
$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AE} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$$

i.e. if the vectors **a**, **b**, **c**, **d**, and so on are represented by the sides of an open polygon taken in order, then the line that closes the polygon represents the vector **a + b + c + d** +…. This is also referred to as **Polygon law of vectors**.

Note: The end point of the first vector should be the same as the starting point of the next vector.

i.e. $\overrightarrow{PQ}+\overrightarrow{QR}+\overrightarrow{RS}+\overrightarrow{ST}+\overrightarrow{TV}+\overrightarrow{VW}=\overrightarrow{PW}$

Note : $\overrightarrow{PQ}+\overrightarrow{QR}+\overrightarrow{RP}=0$



Illustrative examples

Example 1

Find the resultant of the following vectors

$3\overrightarrow{OA}$, $6\overrightarrow{BZ}$, $2\overrightarrow{AO}$, $\overrightarrow{AB}$ and $5\overrightarrow{OB}$

Solution

The resultant of the given vectors

$= 3\overrightarrow{OA} + 6\overrightarrow{BZ} + 2\overrightarrow{AO} + \overrightarrow{AB} + 5\overrightarrow{OB}$

$= (2\overrightarrow{OA} + 2\overrightarrow{AO}) + \overrightarrow{OA} + 6\overrightarrow{BZ} + \overrightarrow{AB} + 5\overrightarrow{OB}$

$= 0 + (\overrightarrow{OA} + \overrightarrow{AB}) + 6\overrightarrow{BZ} + 5\overrightarrow{OB}$

$= \overrightarrow{OB} + 6\overrightarrow{BZ} + 5\overrightarrow{OB}$

$= (\overrightarrow{OB} + 5\overrightarrow{OB}) + 6\overrightarrow{BZ}$

$= 6(\overrightarrow{OB} + \overrightarrow{BZ}) = 6\overrightarrow{OZ}$

Example 2

$\overrightarrow{XT}=\begin{pmatrix}-1\\2\end{pmatrix}$ and $\overrightarrow{YR}=\begin{pmatrix}-2\\-3\end{pmatrix}$. If $X(4, 1)$ and $Y(-3, 2)$, find $\overrightarrow{TR}$.

Solution

From the vectors given,

$\overrightarrow{TR}=\overrightarrow{TX}+\overrightarrow{XY}+\overrightarrow{YR}$

But $\overrightarrow{XY}= \mathbf{y}-\mathbf{x}=\begin{pmatrix}-3\\2\end{pmatrix}-\begin{pmatrix}4\\1\end{pmatrix}=\begin{pmatrix}-7\\1\end{pmatrix}$

and $\overrightarrow{TX}=-\overrightarrow{XT}=\begin{pmatrix}1\\-2\end{pmatrix}$

$\therefore\ \overrightarrow{TR}=\overrightarrow{TX}+\overrightarrow{XY}+\overrightarrow{YR}$

$=\begin{pmatrix}1\\-2\end{pmatrix}+\begin{pmatrix}-7\\1\end{pmatrix}+\begin{pmatrix}-2\\-3\end{pmatrix}=\begin{pmatrix}-8\\-4\end{pmatrix}$

Properties of addition of vectors

1. Vector addition is commutative

For any two vectors **a** and **b**

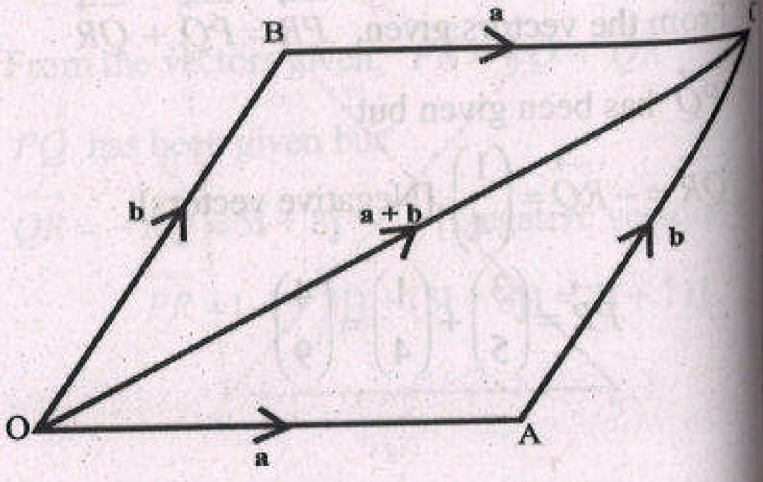
a + b = b + a

From the figure below, observe that the diagonal $\overrightarrow{OC}$ can be attained through two routes, $\overrightarrow{OA}$ and $\overrightarrow{AC}$ as well as $\overrightarrow{OB}$ and $\overrightarrow{BC}$.

i.e. $\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OC}$ and $\overrightarrow{OB}+\overrightarrow{BC}=\overrightarrow{OC}$

$\therefore$ **a + b** $=\overrightarrow{OC}$ and **b + a** $=\overrightarrow{OC}$

Hence **a + b = b + a**



Illustration

Let $\mathbf{a} = \begin{pmatrix}1\\2\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}3\\4\end{pmatrix}$

$$\mathbf{a}+\mathbf{b} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}4\\6\end{pmatrix}$$

$$\mathbf{b}+\mathbf{a} = \begin{pmatrix}3\\4\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}4\\6\end{pmatrix}$$

$$\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$$

Therefore vector addition is commutative.

2. Vector addition is associative

For any three vectors **a**, **b** and **c**,

$$(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$$

Let **a**, **b** and **c** be any three vectors represented by $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{RS}$ respectively as shown in the figures below.

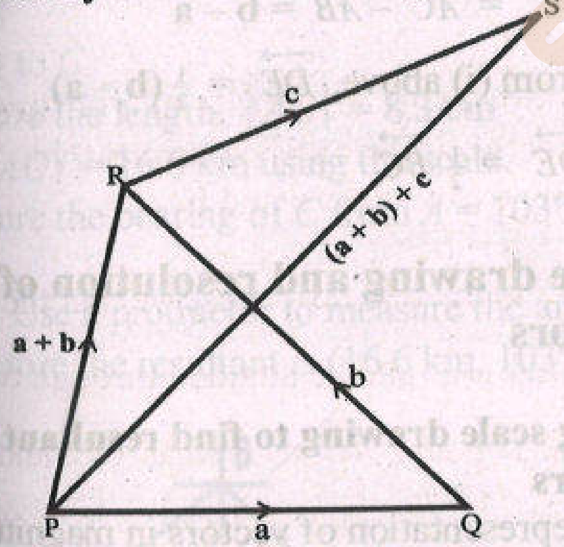


Fig i

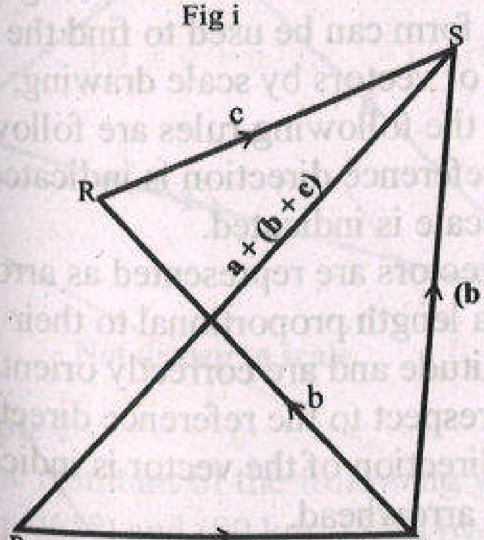


Fig ii

From Fig.(i), $\mathbf{a}+\mathbf{b} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

$\therefore\ (\mathbf{a}+\mathbf{b})+\mathbf{c} = \overrightarrow{PR} + \overrightarrow{RS} = \overrightarrow{PS}$ ……(1)

From Fig.(ii), $\mathbf{b}+\mathbf{c} = \overrightarrow{QR} + \overrightarrow{RS} = \overrightarrow{QS}$

$\therefore\ \mathbf{a}+(\mathbf{b}+\mathbf{c}) = \overrightarrow{PQ} + \overrightarrow{QS} = \overrightarrow{PS}$ ……(2)

From (1) and (2), we get

$(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$

Illustration

Let $\mathbf{a} = \begin{pmatrix}1\\2\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}3\\4\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}5\\6\end{pmatrix}$

$$(\mathbf{a}+\mathbf{b})+\mathbf{c} = \left[\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}3\\4\end{pmatrix}\right] + \begin{pmatrix}5\\6\end{pmatrix}$$

$$= \begin{pmatrix}4\\6\end{pmatrix} + \begin{pmatrix}5\\6\end{pmatrix} = \begin{pmatrix}9\\12\end{pmatrix}$$

$$\mathbf{a}+(\mathbf{b}+\mathbf{c}) = \begin{pmatrix}1\\2\end{pmatrix} + \left[\begin{pmatrix}3\\4\end{pmatrix} + \begin{pmatrix}5\\6\end{pmatrix}\right]$$

$$= \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}8\\10\end{pmatrix} = \begin{pmatrix}9\\12\end{pmatrix}$$

$$(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$$

Therefore vector addition is associative

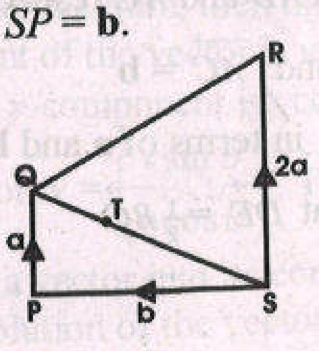
3. Distributive Law

If **a** and **b** are two vectors and k is a scalar, then : $k(\mathbf{a}+\mathbf{b}) = k\mathbf{a} + k\mathbf{b}$

Illustrative examples

Example 1

In the diagram, $ST = 2TQ$, $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{SR} = 2\mathbf{a}$ and $\overrightarrow{SP} = \mathbf{b}$.



a) Find the terms of **a** and **b**:

i) $\overrightarrow{SQ}$ ii) $\overrightarrow{TQ}$ iii) $\overrightarrow{RQ}$

iv) $\overrightarrow{PT}$ v) $\overrightarrow{TR}$

b) What do your answers to (iv) and (v) tell you about the points P, T, R?

Solution

a) i) $\overrightarrow{SQ} = \overrightarrow{SP} + \overrightarrow{PQ} = \mathbf{b} + \mathbf{a}$

ii) $\overrightarrow{TQ} = \frac{1}{3}\overrightarrow{SQ} = \frac{1}{3}(\mathbf{a} + \mathbf{b})$

iii) $\overrightarrow{RQ} = \overrightarrow{RS} + \overrightarrow{SQ} = -2\mathbf{a} + (\mathbf{a} + \mathbf{b})$

$= \mathbf{b} - \mathbf{a}$

iv) $\overrightarrow{PT} = \overrightarrow{PS} + \overrightarrow{ST} = -\mathbf{b} + \frac{2}{3}\overrightarrow{SQ}$

(since $\overrightarrow{ST} = \frac{2}{3}\overrightarrow{SQ}$, i.e $\overrightarrow{ST} = 2\overrightarrow{TQ}$)

$\overrightarrow{PT} = -\mathbf{b} + \frac{2}{3}(\mathbf{a} + \mathbf{b}) = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$

$= \frac{1}{3}(2\mathbf{a} - \mathbf{b})$

v) $\overrightarrow{TR} = \overrightarrow{TS} + \overrightarrow{SR}$

$= -\frac{2}{3}(\mathbf{a} + \mathbf{b}) + 2\mathbf{a} = \frac{4}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$

$= \frac{2}{3}(2\mathbf{a} - \mathbf{b})$

b) Since $\overrightarrow{PT} = \frac{1}{3}(2\mathbf{a} - \mathbf{b})$ and $\overrightarrow{TR} = \frac{2}{3}(2\mathbf{a} - \mathbf{b})$, $\overrightarrow{PT}$ and $\overrightarrow{TR}$ are both multiples of the same vector $(2\mathbf{a} - \mathbf{b})$. Hence PT and TR are parallel and T is common to both lines, so P, T, R lie on the same line, i.e. they are collinear.

Example 2

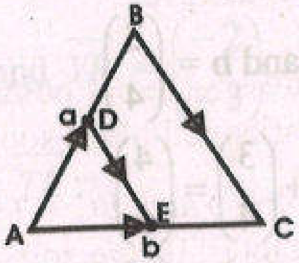
In a triangle ABC, D and E are the midpoints of AB and AC respectively.

If $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$

i) Find $\overrightarrow{DE}$ in terms of **a** and **b**

ii) Show that $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC}$

Solution



i) Since D and E are midpoints of AB and AC respectively,

$\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\mathbf{a}$ and $\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}\mathbf{b}$

From the figure, $\overrightarrow{AD} + \overrightarrow{DE} = \overrightarrow{AE}$

$\overrightarrow{DE} = \overrightarrow{AE} - \overrightarrow{AD}$

$= \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$

ii) Again, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

But from (i) above, $\overrightarrow{DE} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$

$\Rightarrow \overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC}$

Scale drawing and resolution of vectors

Using scale drawing to find resultant of vectors

The representation of vectors in magnitude-direction form can be used to find the resultant of vectors by scale drawing, provided the following rules are followed:

1. The reference direction is indicated.
2. The scale is indicated.
3. The vectors are represented as arrows with a length proportional to their magnitude and are correctly orientated with respect to the reference direction.
4. The direction of the vector is indicated by an arrowhead.
5. The arrows should be labelled to show which vectors they represent.

Illustrative examples

Example 1

Three poles are situated at points A, B and C in the same horizontal plane such that

$\overrightarrow{AB} = (8 \text{ km}, 060°)$ and $\overrightarrow{BC} = (12 \text{ km}, 130°)$.

Find the resultant of the vectors by scale drawing.

Solution

Step 1

Using a scale of 1 cm = 2 km, draw $\overrightarrow{AB}$ at an angle of 60° to the vertical (i.e. 30° above the horizontal line) and length 4 cm.

Step 2

From B, draw a line at an angle of 130° from the north (i.e. 40° below the horizontal line) and length 6 cm to get the point C.

Step 3

Join A to C

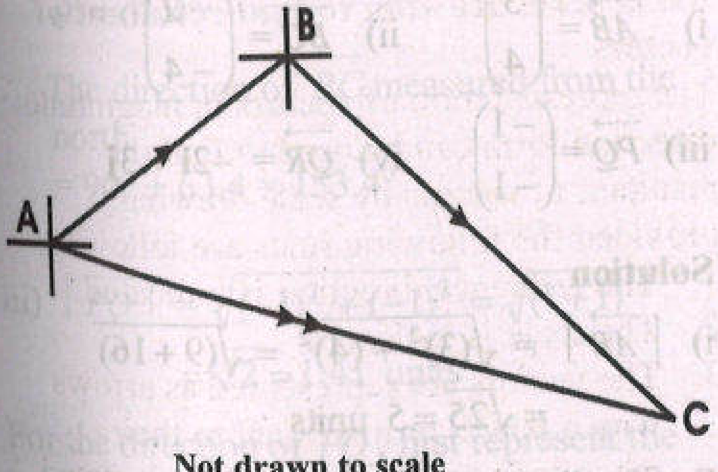
Measure the length, $|AC| = 8.3$ cm

$\therefore \quad |AC| = 16.6$ km using the scale.

Measure the bearing of C from $A = 103°$

Note: Use a protractor to measure the angles.

Therefore the resultant is (16.6 km, 103°)



Not drawn to scale

Example 2

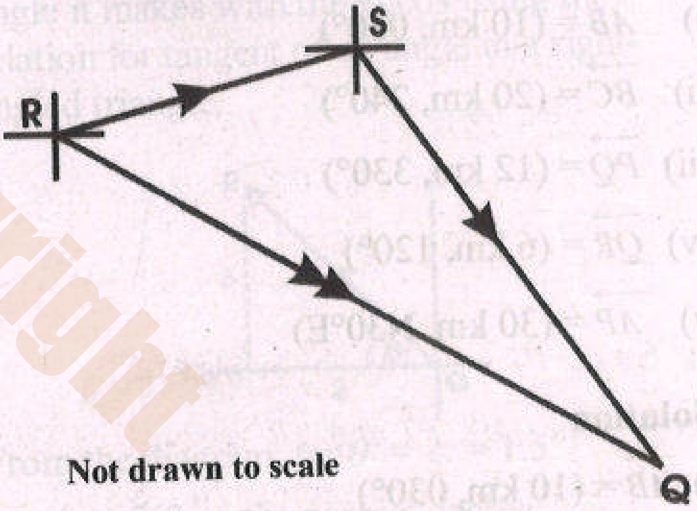
Find the resultant of the following vectors (54 km, 065°) and (80 km, 155°) by scale drawing.

Solution

Use a scale of 1 cm = 10 km,

1. Draw $\overrightarrow{RS}$ in the direction 65° from the vertical or 25° from the horizontal and length 5.4 cm.
2. From S, draw $\overrightarrow{SQ}$ in the direction 155° from the vertical or 65° below the horizontal and length 8.0 cm.
3. Measure $|RQ|$ and convert it to km using the scale. The distance between R and Q is 9.6 cm = 96 km using the scale.
4. Measure the bearing of Q from R. The bearing of Q from R is 121°.

Therefore the resultant is (97 km, 121°).



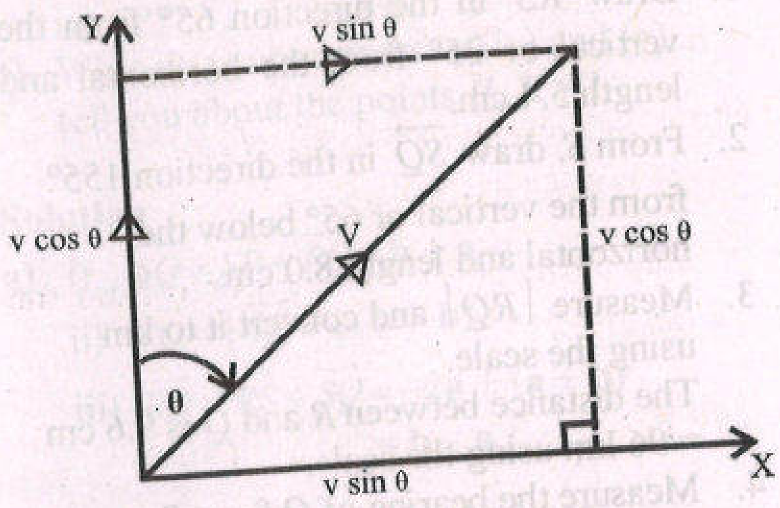
Not drawn to scale

Resolution of vectors

If the magnitude and bearing of any vector are given, its x and y components can easily be found. If v is the magnitude of the vector $\mathbf{v}$ and θ is its bearing [i.e. $\mathbf{v} = (v, \theta)$] as shown in the following diagram, then the horizontal or x-component of the vector is $v\sin\theta$ and the vertical or y-component is $v\cos\theta$.

Thus the vector, $\mathbf{v} = \begin{pmatrix} v\sin\theta \\ v\cos\theta \end{pmatrix}$. This process of expressing a vector into its components is known as resolution of the vector.

Note: The bearing or direction determines the sign of the components



Illustrative example

Express the following as column vectors

i) $\overrightarrow{AB} = (10 \text{ km}, 030°)$

ii) $\overrightarrow{BC} = (20 \text{ km}, 240°)$

iii) $\overrightarrow{PQ} = (12 \text{ km}, 330°)$

iv) $\overrightarrow{QR} = (6 \text{ km}, 120°)$

v) $\overrightarrow{AP} = (30 \text{ km}, \text{N}30°\text{E})$

Solution

i) $\overrightarrow{AB} = (10 \text{ km}, 030°)$

$$\overrightarrow{AB} = \begin{pmatrix} 10\sin 30° \\ 10\cos 30° \end{pmatrix}$$

$$= \begin{pmatrix} 10 \times 0.5 \\ 10 \times 0.866 \end{pmatrix} = \begin{pmatrix} 5 \\ 8.66 \end{pmatrix}$$

$$= 5\mathbf{i} + 8.66\mathbf{j}$$

ii) $\overrightarrow{BC} = (20 \text{ km}, 240°)$

$$\overrightarrow{BC} = \begin{pmatrix} 20\sin 240° \\ 20\cos 240° \end{pmatrix}$$

$$= \begin{pmatrix} -20 \times 0.866 \\ -20 \times 0.5 \end{pmatrix} = \begin{pmatrix} -17.32 \\ -10 \end{pmatrix}$$

$$= -17.32\mathbf{i} - 10\mathbf{j}$$

iii) $\overrightarrow{PQ} = (12 \text{ km}, 330°)$

$$\therefore \overrightarrow{PQ} = \begin{pmatrix} 12\sin 330° \\ 12\cos 330° \end{pmatrix}$$

$$= \begin{pmatrix} -12 \times 0.5 \\ 12 \times 0.866 \end{pmatrix} = \begin{pmatrix} -6 \\ 10.4 \end{pmatrix}$$

$$= -6\mathbf{i} + 10.4\mathbf{j}$$

iv) $\overrightarrow{QR} = (6\text{km}, 120°)$

$$\overrightarrow{QR} = \begin{pmatrix} 6\sin 120° \\ 6\cos 120° \end{pmatrix}$$

$$= \begin{pmatrix} 6 \times 0.866 \\ -6 \times 0.5 \end{pmatrix} = \begin{pmatrix} 5.2 \\ -3 \end{pmatrix}$$

$$= 5.2\mathbf{i} - 3\mathbf{j}$$

Finding the magnitude and direction (bearing) of a column vector

Illustrative examples

Example 1

Find the magnitudes and directions (bearings) of the following column vectors

i) $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ ii) $\overrightarrow{BC} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

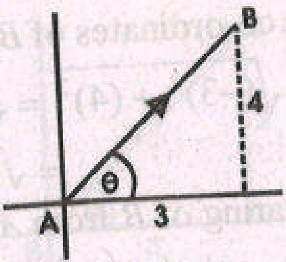
iii) $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ iv) $\overrightarrow{QR} = -2\mathbf{i} + 3\mathbf{j}$

Solution

i) $|\overrightarrow{AB}| = \sqrt{(3)^2 + (4)^2} = \sqrt{(9 + 16)}$

$= \sqrt{25} = 5$ units

For the direction of $\overrightarrow{AB}$, first represent the vector on a rough diagram, and find the acute angle it makes with the x-axis using the relation for tangent of an angle in a right-angled triangle.



From the diagram, $\tan\theta = \frac{4}{3} = 1.33$

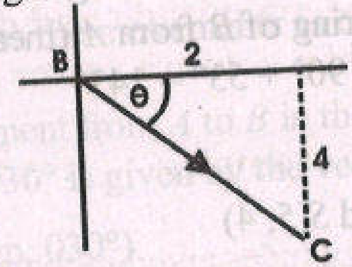
$\theta = 53°$

∴ The direction of $\overrightarrow{AB}$ measured from the north $= 90° - 53° = 37°$

ii) $|\overrightarrow{BC}| = \sqrt{(2)^2 + (-4)^2} = \sqrt{(4+16)}$

$= \sqrt{20} = 4.47$ units

For the direction of $\overrightarrow{BC}$, first represent the vector on a rough diagram, and find the acute angle it makes with the x-axis using the relation for tangent of an angle in a right-angled triangle



From the diagram, $\tan\theta = \frac{4}{2} = 2$

$\theta = 63.4°$

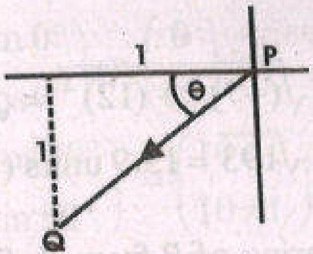
∴ The direction of $\overrightarrow{BC}$ measured from the north

$= 90° + 63.4 = 153.4°$

iii) $|\overrightarrow{PQ}| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{(1+1)}$

$= \sqrt{2} = 1.41$ units

For the direction of $\overrightarrow{PQ}$, first represent the vector on a rough diagram, and find the acute angle it makes with the x-axis using the relation for tangent of an angle in a right-angled triangle



From the diagram, $\tan\theta = \frac{1}{1} = 1$

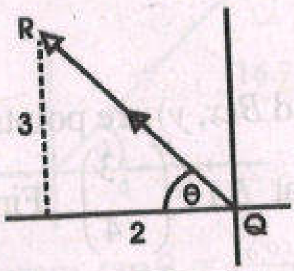
$\theta = 45°$

∴ The direction of $\overrightarrow{PQ}$ measured from the north $= 270° - 45° = 225°$

iv) $|\overrightarrow{QR}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{(4+9)}$

$= \sqrt{13} = 3.6$ units

For the direction of $\overrightarrow{QR}$, first represent the vector on a rough diagram, and find the acute angle it makes with the x-axis using the relation for tangent of an angle in a right-angled triangle.



From the diagram, $\tan\theta = \frac{3}{2} = 1.5$

$\theta = 56°$ to the nearest degree

∴ The direction of $\overrightarrow{QR}$ measured from the north $= 270° + 56° = 326°$

Example 2

Given the vector $\overrightarrow{AB} = \begin{pmatrix} -7 \\ 12 \end{pmatrix}$, calculate the:

i) length $|\overrightarrow{AB}|$, correct to three significant figures

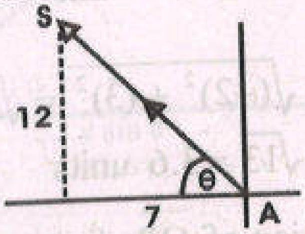
ii) bearing of B from A, correct to the nearest degree.

Solution

i) $|\overrightarrow{AB}| = \sqrt{(-7)^2 + (12)^2} = \sqrt{(49 + 144)}$

$= \sqrt{193} = 13.9$ units (to 3 s. f.)

ii) For the bearing of B from A, first represent the vector $\overrightarrow{AB}$ on a rough diagram, and find the acute angle it makes with the x-axis using the relation for tangent of an angle in a right-angled triangle.



From the diagram, $\tan\theta = \frac{12}{7} = 1.714$

$\theta = 60°$ to the nearest degree

$\therefore$ The bearing of B from A (measured from the north) $= 270° + 60° = 330°$

Example 3

If $A(-1, 2)$ and $B(x, y)$ are points in the Oxy plane such that $\overrightarrow{BA} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$. Find

i) the co-ordinate of B

ii) $|\overrightarrow{BA}|$

iii) the bearing of B from A.

Solution

i) $\overrightarrow{BA} = \mathbf{a} - \mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1-x \\ 2-y \end{pmatrix}$

But $\overrightarrow{BA} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1-x \\ 2-y \end{pmatrix}$

Comparing corresponding components

$-3 = -1 - x \quad \Rightarrow \quad x = 2$ and

$4 = 2 - y \quad \Rightarrow \quad y = -2$

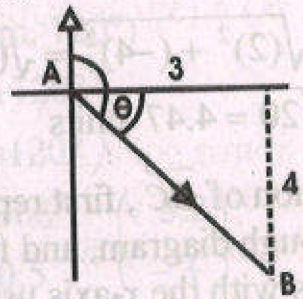
Therefore the co-ordinates of $B = (2, -2)$

ii) $|\overrightarrow{BA}| = \sqrt{(-3)^2 + (4)^2} = \sqrt{(9 + 16)}$

$= \sqrt{25} = 5$ units

iii) For the bearing of B from A, first find the vector $\overrightarrow{AB} = -\overrightarrow{BA} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

Represent it on a rough diagram and find the acute angle it makes with the x-axis using the relation for tangent of an angle in a right-angled triangle.



From the diagram, $\tan\theta = \frac{4}{3} = 1.333$

$\theta = 53°$ to the nearest degree

$\therefore$ The bearing of B from A (measured from the north) $= 90° + 53° = 143°$

Example 4

If $R(2, 2)$ and $S(5, 4)$

i) Calculate the magnitude of $\overrightarrow{RS}$.

ii) Express $\overrightarrow{RS}$ in the form (k, θ), where k is the magnitude and θ the bearing.

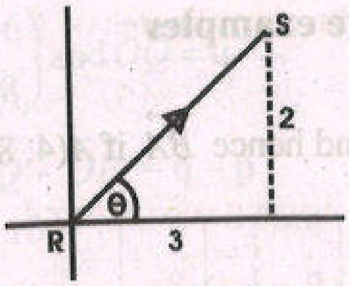
Solution

$\overrightarrow{RS} = \mathbf{s} - \mathbf{r} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

i) $|\overrightarrow{RS}| = \sqrt{(3)^2 + (2)^2} = \sqrt{(9 + 4)}$

$= \sqrt{13} = 3.6$ units

ii) For the bearing of $\overrightarrow{RS}$, first represent the vector $\overrightarrow{RS}$ on a rough diagram, and find the acute angle it makes with the x-axis using the relation for tangent of an angle in a right-angled triangle.



From the diagram, $\tan\theta = \frac{2}{3} = 0.667$

$\theta = 34°$ to the nearest degree

∴ The bearing of S from R (measured from the north) = 90° − 34° = 56°

∴ $\overrightarrow{RS} = (k, \theta) = (3.6 \text{ units}, 056°)$

Example 5

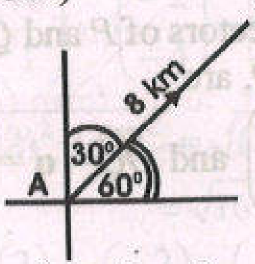
A boat starts from a point A, travels 8 km in the direction 030° to B and then 10 km north to C. Find the boat's distance and bearing from A.

Solution

First express the movements as column vectors.

The movement from A to B in the direction 030° is given by the vector

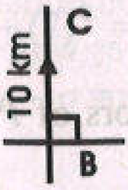
$\overrightarrow{AB} = (8 \text{ km}, 030°)$



$$\overrightarrow{AB} = \begin{pmatrix} 8\sin 30° \\ 8\cos 30° \end{pmatrix} = \begin{pmatrix} 4 \\ 6.93 \end{pmatrix} \text{ or}$$

$$\overrightarrow{AB} = \begin{pmatrix} 8\cos 60° \\ 8\sin 60° \end{pmatrix} = \begin{pmatrix} 8\times 0.500 \\ 8\times 0.867 \end{pmatrix} = \begin{pmatrix} 4 \\ 6.93 \end{pmatrix}$$

The movement from B to C due north is given by the vector $\overrightarrow{BC} = (10 \text{ km}, 000°)$



$$\overrightarrow{BC} = \begin{pmatrix} 10\sin 0° \\ 10\cos 0° \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \end{pmatrix} \text{ or}$$

$$\overrightarrow{BC} = \begin{pmatrix} 10\cos 90° \\ 10\sin 90° \end{pmatrix} = \begin{pmatrix} 10\times 0 \\ 10\times 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$$

i) The boats distance from A is given by $|\overrightarrow{AC}|$.

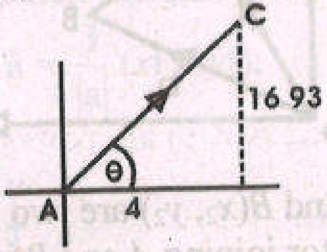
But $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \begin{pmatrix} 4 \\ 6.93 \end{pmatrix} + \begin{pmatrix} 0 \\ 10 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ 16.93 \end{pmatrix}$

$\therefore \quad |\overrightarrow{AC}| = \sqrt{(4)^2 + (16.93)^2}$

$= \sqrt{302.62} = 17.4 \text{ km}$

ii) To find the bearing of C from A, first represent the vector $\overrightarrow{AC}$ on a rough diagram and calculate the acute angle θ, it makes with the x-axis.



From the diagram, $\tan\theta = \frac{16.93}{4} = 4.93$

$\theta = 77°$

The actual direction measured from the north = 90° − 77° = 13°

∴ The bearing of C from A is 13°.

Hence the distance from A is 17.40 km and the direction is 013°

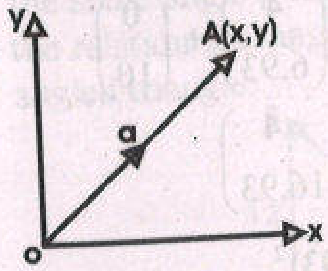
Position vectors

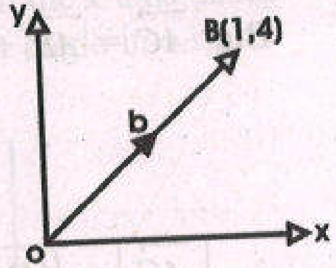
Let $A(x, y)$ be any point and O be an origin.

If the point $A(x, y)$ is referred to O then $\overrightarrow{OA}$ or **a** is the **position vector** of A relative to O, where $\overrightarrow{OA} = \mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$. For example, if $A(-2, 3)$, then the position vector of A

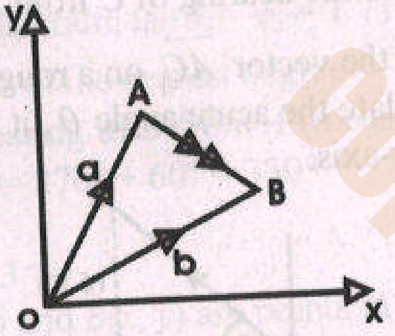
referred to an origin, *O*, is $\overrightarrow{OA} = \mathbf{a} = \begin{pmatrix}-2\\3\end{pmatrix}$.

Also if *B*(1, 4) then the position vector of *B* referred to an origin, *O*, is $\overrightarrow{OB} = \mathbf{b} = \begin{pmatrix}1\\4\end{pmatrix}$ etc.





Expressing two given points as a vector



If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two given points, then the vector joining *A* and *B* is given by

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \mathbf{b} - \mathbf{a} = \begin{pmatrix}x_2\\y_2\end{pmatrix} - \begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}x_2 - x_1\\y_2 - y_1\end{pmatrix}$$

Similarly, given the points *X* and *Y*, the vector $\overrightarrow{XY} = \overrightarrow{OY} - \overrightarrow{OX} = \mathbf{y} - \mathbf{x}$.

Also $\overrightarrow{CD} = \mathbf{d} - \mathbf{c}$, etc.
Note the order of subtraction.

That is the vector $\overrightarrow{AB}$
= Position vector of *B* – Position vector of *A*.

$\overrightarrow{CD}$ = Position vector of *D* – Position vector of *C*, etc.

Illustrative examples

Example 1

Find $\overrightarrow{AB}$ and hence $\overrightarrow{BA}$ if *A*(4, 8) and *B*(–1, 5).

Solution

The position vectors of *A* and *B* referred to the origin *O* are:

$\overrightarrow{OA} = \mathbf{a} = \begin{pmatrix}4\\8\end{pmatrix}$ and $\overrightarrow{OB} = \mathbf{b} = \begin{pmatrix}-1\\5\end{pmatrix}$

respectively.

$$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a} = \begin{pmatrix}-1\\5\end{pmatrix} - \begin{pmatrix}4\\8\end{pmatrix}$$

$$= \begin{pmatrix}-5\\-3\end{pmatrix}$$

$$\overrightarrow{BA} = -\overrightarrow{AB} = \begin{pmatrix}5\\3\end{pmatrix}$$

Example 2

If *P*(–1,3) and *Q*(2,5), find $\overrightarrow{PQ}$.

Solution

The position vectors of *P* and *Q* referred to the origin, *O*, are:

$\overrightarrow{OP} = \mathbf{p} = \begin{pmatrix}-1\\3\end{pmatrix}$ and $\overrightarrow{OQ} = \mathbf{q} = \begin{pmatrix}2\\5\end{pmatrix}$

respectively.

$$\therefore \quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \mathbf{q} - \mathbf{p} = \begin{pmatrix}2\\5\end{pmatrix} - \begin{pmatrix}-1\\3\end{pmatrix}$$

$$= \begin{pmatrix}3\\2\end{pmatrix}$$

Example 3

If *P*(–6,8) and *Q*(–10,6) find $\overrightarrow{PQ}$ and $|PQ|$.

Solution

The position vectors of *P* and *Q* referred to the origin *O* are:

$\overrightarrow{OP} = \mathbf{p} = \begin{pmatrix} -6 \\ 8 \end{pmatrix}$ and $\overrightarrow{OQ} = \mathbf{q}$

$$\therefore \quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \mathbf{q} - \mathbf{p}$$

$$= \begin{pmatrix} -10 \\ 6 \end{pmatrix} - \begin{pmatrix} -6 \\ 8 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

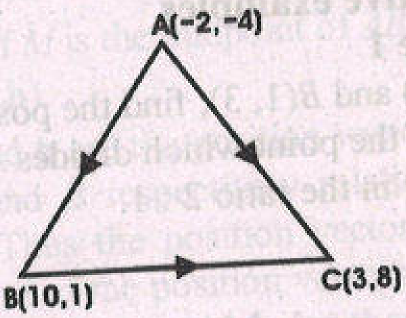
$$|PQ| = \sqrt{(-4)^2 + (-2)^2} = \sqrt{(16+4)}\,.$$

$$= \sqrt{20} = 2\sqrt{5} \text{ units}$$

Example 4

A triangle ABC has vertices $A(-2, -4)$, $B(10, 1)$ and $C(3, 8)$. Find the length of the sides and show that the triangle is isosceles.

Solution



$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 10 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

$$\therefore \quad |\overrightarrow{AB}| = \sqrt{(12)^2 + (5)^2} = \sqrt{(144+25)}$$

$$= \sqrt{169} = 13 \text{ units}$$

$$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 3 \\ 8 \end{pmatrix} - \begin{pmatrix} -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$$

$$\therefore \quad |\overrightarrow{AC}| = \sqrt{(5)^2 + (12)^2} = \sqrt{(25+144)}$$

$$= \sqrt{169} = 13 \text{ units}$$

$$\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = \begin{pmatrix} 3 \\ 8 \end{pmatrix} - \begin{pmatrix} 10 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ 7 \end{pmatrix}$$

$$\therefore \quad |\overrightarrow{BC}| = \sqrt{(-7)^2 + (7)^2} = \sqrt{(49+49)}$$

$$= \sqrt{98} = 7\sqrt{2} \text{ units}$$

Since two sides of the triangle are equal i.e. $|\overrightarrow{AB}| = |\overrightarrow{AC}| = 13$ units, it implies the triangle is isosceles.

Unit vectors

A **unit vector** is a vector whose magnitude is 1. The unit vector of a vector $\mathbf{a}$ is denoted by $\hat{\mathbf{a}}$

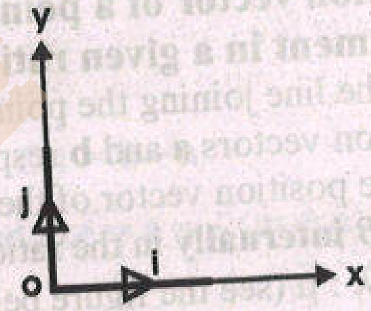
Thus if $\hat{\mathbf{a}}$ is the unit vector of $\mathbf{a}$, then $|\hat{\mathbf{a}}| = 1$. The unit vector of any vector is given by the formula:

$$\text{unit vector} = \frac{\text{vector}}{\text{magnitude}}$$

$$\text{i.e. } \hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} \ldots\ldots (1)$$

Hence any vector, $\mathbf{a}$, can be expressed as product of its *magnitude* and the *unit vector* in the same direction. i.e. $\mathbf{a} = |\mathbf{a}|\,\hat{\mathbf{a}}$.

If $\mathbf{a} = x\mathbf{i} + y\mathbf{j}$, then the unit vector parallel to $\mathbf{a}$ is given by $\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}(x\mathbf{i} + y\mathbf{j})$.



Note that the two basic unit vectors are $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the directions of the x and y axes respectively.

Illustrative examples

Example 1

Find the unit vector in the direction of the vector $\mathbf{a} = 5\mathbf{i} + 12\mathbf{j}$.

Solution

The magnitude of **a** is

$$|\mathbf{a}| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$$

∴ The unit vector of **a** is

$$\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|} \times \mathbf{a} = \frac{1}{13}(5\mathbf{i} + 12\mathbf{j}) = \tfrac{5}{13}\mathbf{i} + \tfrac{12}{13}\mathbf{j}$$

Example 2

Find the unit vector in the direction of the vector $\mathbf{u} = -3\mathbf{i} + 4\mathbf{j}$.

Solution

The magnitude of **u** is

$$|\mathbf{u}| = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$$

∴ The unit vector of **u** is

$$\hat{\mathbf{u}} = \frac{1}{|\mathbf{u}|} \times \mathbf{u}$$

$$= \frac{1}{5}(-3\mathbf{i} + 4\mathbf{j})$$

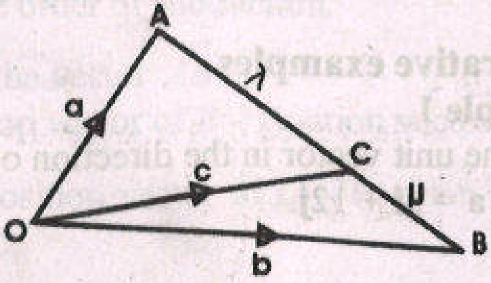
$$= -\tfrac{3}{5}\mathbf{i} + \tfrac{4}{5}\mathbf{j}$$

The position vector of a point dividing a line segment in a given ratio

Consider the line joining the points A and B with position vectors **a** and **b** respectively. Let **c** be the position vector of the point C, dividing AB **internally** in the ratio $AC : CB = \lambda : \mu$ (see the figure below). The position vector of C is given by:

$$\mathbf{c} = \frac{\lambda\mathbf{b} + \mu\mathbf{a}}{\lambda + \mu} = \frac{\lambda\overrightarrow{OB} + \mu\overrightarrow{OA}}{\lambda + \mu}$$

Proof of formula



From the given ratio,

$$\frac{AC}{CB} = \frac{\lambda}{\mu} \quad \Rightarrow \quad \lambda\overrightarrow{CB} = \mu\overrightarrow{AC}$$

$$\lambda(\mathbf{b} - \mathbf{c}) = \mu(\mathbf{c} - \mathbf{a})$$

$$\lambda\mathbf{b} - \lambda\mathbf{c} = \mu\mathbf{c} - \mu\mathbf{a}$$

$$(\lambda + \mu)\mathbf{c} = \mu\mathbf{a} + \lambda\mathbf{b}$$

$$\mathbf{c} = \frac{\lambda\mathbf{b} + \mu\mathbf{a}}{\lambda + \mu}$$

(Note this formula carefully)

For **external** division, the ratio λ : μ is taken to be negative (usually the smaller number in the ratio is taken as negative). Therefore the position vector of C is:

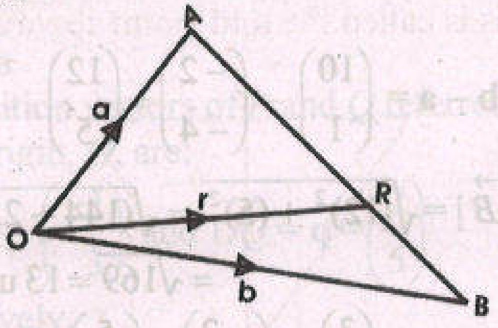
$$\mathbf{c} = \frac{\lambda\mathbf{b} - \mu\mathbf{a}}{\lambda - \mu} = \frac{\lambda\overrightarrow{OB} - \mu\overrightarrow{OA}}{\lambda - \mu}$$

Illustrative examples

Example 1

If $A(1, 6)$ and $B(1, 3)$, find the position vector of the point which divides AB internally in the ratio 2 : 1.

Solution



The position vectors of A and B are

$\mathbf{a} = \mathbf{i} + 6\mathbf{j}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{j}$

Let the required position vector be **r**, then

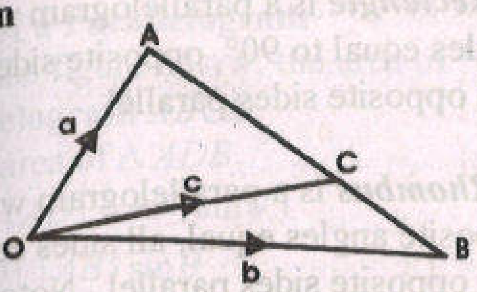
$$\mathbf{r} = \frac{2\mathbf{b} + \mathbf{a}}{2 + 1} = \tfrac{1}{3}(2\mathbf{b} + \mathbf{a}) = \tfrac{1}{3}(3\mathbf{i} + 12\mathbf{j}) = \mathbf{i} + 4\mathbf{j}$$

Since $2\mathbf{b} + \mathbf{a} = (2\mathbf{i} + 6\mathbf{i}) + (\mathbf{i} + 6\mathbf{j}) = 3\mathbf{i} + 12\mathbf{j}$

Example 2

If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 8\mathbf{i} + 9\mathbf{j}$ are the position vectors of A and B, find the position vector, **c**, of C which divides AB internally in the ratio 2 : 1.

Solution



$\mathbf{c} = \frac{1}{3}(2\mathbf{b} + 1\mathbf{a}) = \frac{1}{3}(18\mathbf{i} + 21\mathbf{j}) = 6\mathbf{i} + 7\mathbf{j}$

Since $2\mathbf{b} + \mathbf{a} = (16\mathbf{i} + 18\mathbf{i}) + (2\mathbf{i} + 3\mathbf{j})$
$= 18\mathbf{i} + 21\mathbf{j}$

The mid-point theorem

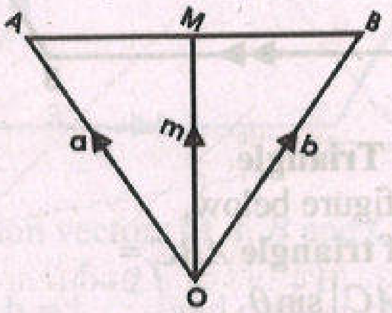
This is a special case of the ratio theorem. If *M* is the mid-point of *AB* with position vector **m** relative to the same origin *O*, then

$\lambda = \mu = 1$ and $\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$

In general if *M* is the midpoint of *AB* then

$\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$

where **a** and **b** are the position vectors of the points *A* and *B* respectively relative to the origin *O*. Thus the position vector of *M* is half the sum of the position vectors of *A* and *B*. This is called the **mid-point theorem**.



Illustrative examples

Example 1

If *P* is the midpoint of *QR* where *Q*(–1, 4) and *R*(3, 6), find the position vector of *P*.

Solution

By the mid-point theorem if *P* is the midpoint of *QR* then;

$\mathbf{p} = \frac{1}{2}(\mathbf{q} + \mathbf{r})$

$$= \tfrac{1}{2}\left[\begin{pmatrix}-1\\4\end{pmatrix} + \begin{pmatrix}3\\6\end{pmatrix}\right] = \tfrac{1}{2}\begin{pmatrix}2\\10\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix}$$

where **p**, **q** and **r** are the position vectors of *P*, *Q* and *R* respectively.

Example 2

A(–2, 4) and *B*(6,2). If *C* is the midpoint of *AB*, find the position vector of *C*.

Solution

By the mid-point theorem if *C* is the midpoint of *AB* then;

$\mathbf{c} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$

$$= \tfrac{1}{2}\left[\begin{pmatrix}-2\\4\end{pmatrix} + \begin{pmatrix}6\\2\end{pmatrix}\right] = \tfrac{1}{2}\begin{pmatrix}4\\6\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$$

where **c**, **a** and **b** are the position vectors of *C*, *A* and *B* respectively.

Example 3

M(5, 6), *N*(–1, 4) and *B*(–1, 5) are three points in a plane. If *A* is the midpoint of *MN*, find $\overrightarrow{AB}$.

Solution

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$

$\therefore$ First find $\overrightarrow{OA} = \mathbf{a} = \frac{1}{2}(\mathbf{m} + \mathbf{n})$

$$= \tfrac{1}{2}\left[\begin{pmatrix}5\\6\end{pmatrix} + \begin{pmatrix}-1\\4\end{pmatrix}\right] = \tfrac{1}{2}\begin{pmatrix}4\\10\end{pmatrix} = \begin{pmatrix}2\\5\end{pmatrix}$$

$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$

$$= \begin{pmatrix}-1\\5\end{pmatrix} - \begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}-3\\0\end{pmatrix}$$

Example 4

If *A*(4, 7) is the vertex of a triangle *ABC*, $\overrightarrow{BA} = \begin{pmatrix}5\\3\end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix}4\\-3\end{pmatrix}$

a) Find the co-ordinates of B and C.

b) If M is the midpoint of the line BC, find $\overrightarrow{AM}$.

Solution

a) The position vector of the point $A(4, 7)$ is

$$\mathbf{a} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$\overrightarrow{BA} = \mathbf{a} - \mathbf{b} \quad \Rightarrow \quad \mathbf{b} = \mathbf{a} - \overrightarrow{BA}$$

$$\mathbf{b} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$\therefore$ The co-ordinates of B is (–1, 4)

$$\text{Also } \overrightarrow{AC} = \mathbf{c} - \mathbf{a} \quad \Rightarrow \quad \mathbf{c} = \mathbf{a} + \overrightarrow{AC}$$

$$\mathbf{c} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

$\therefore$ The co-ordinates C is (8, 4)

b) If M is the midpoint of BC then by the midpoint theorem, $\mathbf{m} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$

$$\mathbf{m} = \tfrac{1}{2}\left[\begin{pmatrix} -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 8 \\ 4 \end{pmatrix}\right] = \tfrac{1}{2}\begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} 3.5 \\ 4 \end{pmatrix}$$

$$\therefore \quad \overrightarrow{AM} = \mathbf{m} - \mathbf{a} = \begin{pmatrix} 3.5 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} -0.5 \\ -3 \end{pmatrix}$$

Quadrilaterals

Some questions require some special properties of quadrilaterals. Below are some quadrilaterals and their special properties.

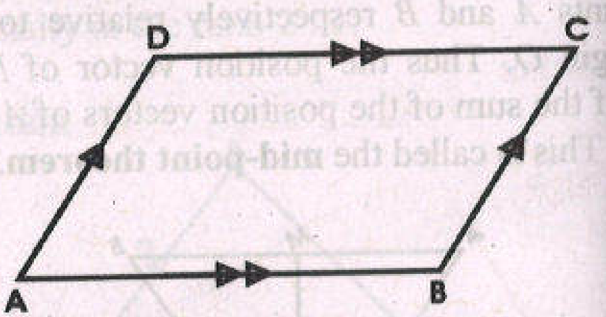
1. A **parallelogram** has opposite angles equal, opposite sides equal and opposite sides parallel. A square, rectangle and rhombus are all parallelograms.
2. A ***Square*** is a parallelogram with all angles equal to 90°, all sides equal and opposite sides parallel
3. A ***Rectangle*** is a parallelogram with all angles equal to 90°, opposite sides equal and opposite sides parallel
4. A ***Rhombus*** is a parallelogram with opposite angles equal, all sides equal and opposite sides parallel. Note that the angles **are not** equal to 90°.
5. A **trapezium** has one pair of opposite sides parallel but unequal.

To show that a given quadrilateral is any of the above figures, use the special property of the figure defined above.

In general, from the figure below, if $ABCD$ is a **parallelogram** then in terms of vectors,

$$\overrightarrow{AB} = \overrightarrow{DC} \text{ and } \overrightarrow{AD} = \overrightarrow{BC}.$$

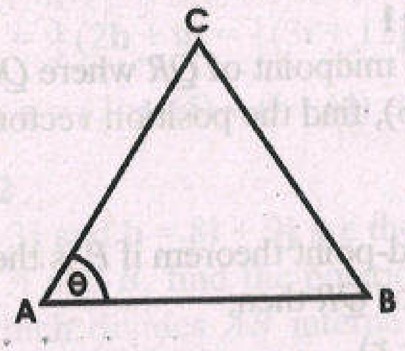
Note this property very well.



Area of a Triangle

From the figure below,
the area of **triangle** $ABC = \frac{1}{2}|AB|\,|AC|\sin\theta$

where θ is the angle between AB and AC.



Area of a Parallelogram

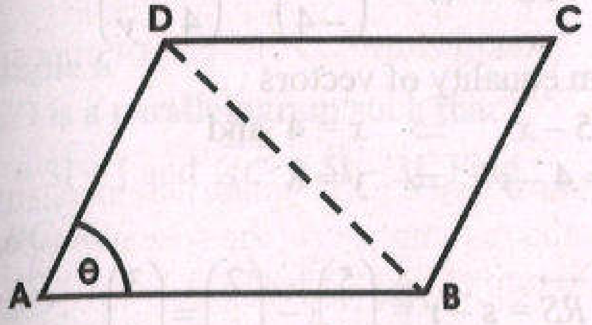
From the figure below, the area of **parallelogram** *ABCD*

$= 2 \times$ area of $\Delta\, ADB$

$= 2(\frac{1}{2}\, |AD|\, |AB| \sin\theta)$

$= |AD|\, |AB| \sin\theta$

where θ is the angle between *AD* and *AB*.

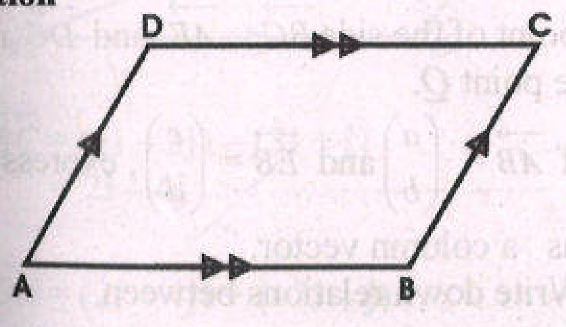


Illustrative examples

Example 1

A parallelogram *ABCD* has vertices *A*(1, 3), *B*(–2 ,5) and *C*(1, 8). Find the co-ordinates of the vertex *D*.

Solution



The position vectors of *A*, *B* and *C* are:

$\mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$ respectively.

Let (*a*, *b*) be the co-ordinates of *D*.

Since *ABCD* is a parallelogram $\overrightarrow{AB} = \overrightarrow{DC}$ [or $\overrightarrow{AD} = \overrightarrow{BC}$].

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$\overrightarrow{DC} = \mathbf{c} - \mathbf{d} = \begin{pmatrix} 1 \\ 8 \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1-a \\ 8-b \end{pmatrix}$$

$$\overrightarrow{AB} = \overrightarrow{DC} \quad \Rightarrow \quad \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1-a \\ 8-b \end{pmatrix}$$

From equality of vectors

$-3 = 1 - a \quad \Rightarrow \quad a = 4$ and

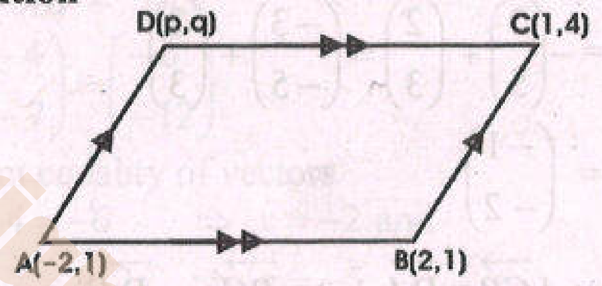
$2 = 8 - b \quad \Rightarrow \quad b = 6$

$\therefore$ The co-ordinates of *D* is (4, 6)

Example 2

The points *A*(–2, 1), *B*(2, 1), *C*(1, 4) and *D*(*p*, *q*) are the vertices of a parallelogram *ABCD*. Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$ and deduce the values of *p* and *q*.

Solution



$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\overrightarrow{DC} = \mathbf{c} - \mathbf{d} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1-p \\ 4-q \end{pmatrix}$$

Since *ABCD* is a parallelogram $\overrightarrow{AB} = \overrightarrow{DC}$

$$\overrightarrow{AB} = \overrightarrow{DC}$$

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-p \\ 4-q \end{pmatrix}$$

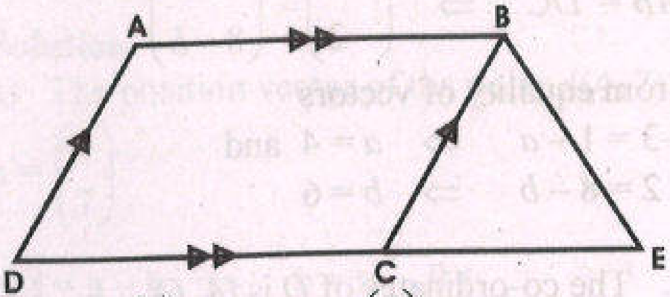
From equality of vectors

$4 = 1 - p \quad \Rightarrow \quad p = -3$ and

$0 = 4 - q \quad \Rightarrow \quad q = 4$

Example 3

In the figure below, *ABCD* is a parallelogram.



If $\overrightarrow{DA} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\overrightarrow{CE} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Find $\overrightarrow{BE}$.

Solution

From the figure,

$$\overrightarrow{BE} = \overrightarrow{BC} + \overrightarrow{CE}$$

$$= -\overrightarrow{DA} + \overrightarrow{CE}$$

$$= -\begin{pmatrix} 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

Note: $[\overrightarrow{CB} = \overrightarrow{DA} \Rightarrow \overrightarrow{BC} = -\overrightarrow{DA}]$

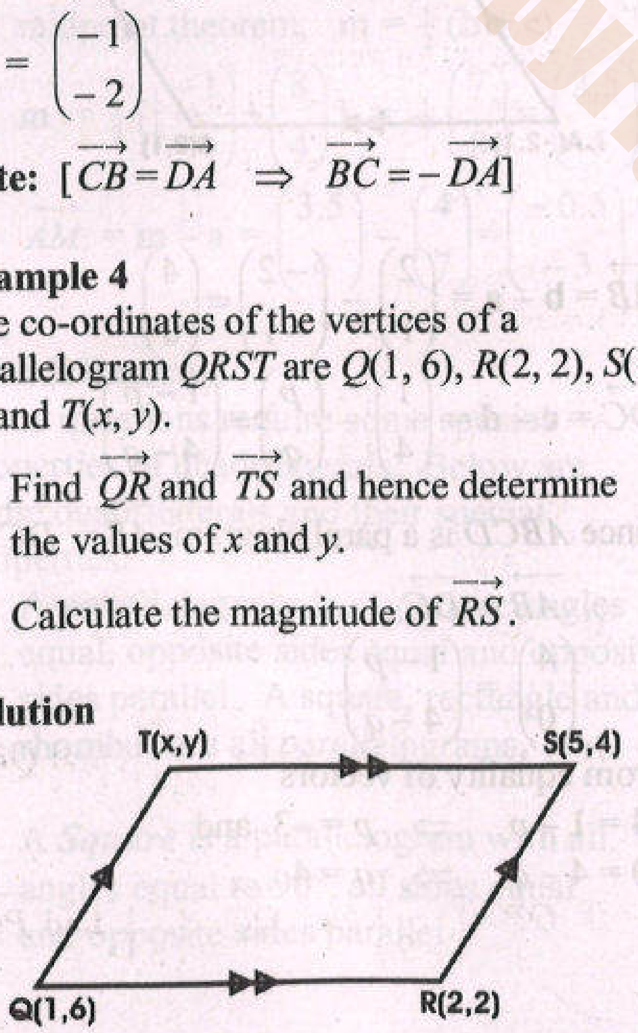
Example 4

The co-ordinates of the vertices of a parallelogram *QRST* are $Q(1, 6)$, $R(2, 2)$, $S(5, 4)$ and $T(x, y)$.

i) Find $\overrightarrow{QR}$ and $\overrightarrow{TS}$ and hence determine the values of x and y.

ii) Calculate the magnitude of $\overrightarrow{RS}$.

Solution



i) $\overrightarrow{QR} = \mathbf{r} - \mathbf{q} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

$\overrightarrow{TS} = \mathbf{s} - \mathbf{t} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5-x \\ 4-y \end{pmatrix}$

Since *QRST* is a parallelogram, $\overrightarrow{QR} = \overrightarrow{TS}$

$$\overrightarrow{QR} = \overrightarrow{TS} \quad \Rightarrow \quad \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 5-x \\ 4-y \end{pmatrix}$$

From equality of vectors

$1 = 5 - x \quad \Rightarrow \quad x = 4$ and

$-4 = 4 - y \quad \Rightarrow \quad y = 8$

ii) $\overrightarrow{RS} = \mathbf{s} - \mathbf{r} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$|\overrightarrow{RS}| = \sqrt{(3)^2 + (2)^2} = \sqrt{(9+4)}$$

$$= \sqrt{13} \text{ units}$$

Example 5

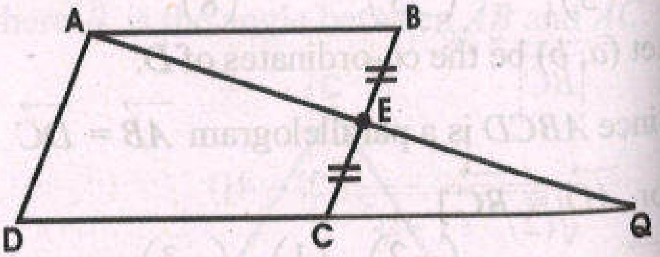
ABCD is a parallelogram and *E* is the midpoint of the side $\overrightarrow{BC}$. $\overrightarrow{AE}$ and $\overrightarrow{DC}$ meet at the point *Q*.

i) If $\overrightarrow{AB} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\overrightarrow{EB} = \begin{pmatrix} c \\ d \end{pmatrix}$, express $\overrightarrow{AE}$ as a column vector.

ii) Write down relations between

a) $\overrightarrow{EQ}$ and $\overrightarrow{AE}$ b) $\overrightarrow{CQ}$ and $\overrightarrow{AB}$.

Solution



i) $\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BE} = \overrightarrow{AB} - \overrightarrow{EB}$

$$= \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a-c \\ b-d \end{pmatrix}$$

ii) $\overrightarrow{CE} = \frac{1}{2}\overrightarrow{CB} = \frac{1}{2}\overrightarrow{DA}$

∴ $\overrightarrow{AE} = \overrightarrow{EQ}$ and $\overrightarrow{DC} = \overrightarrow{CQ}$

But $\overrightarrow{AB} = \overrightarrow{DC}$

a) $\overrightarrow{EQ} = \overrightarrow{AE}$ and

b) $\overrightarrow{CQ} = \overrightarrow{AB}$

Example 6

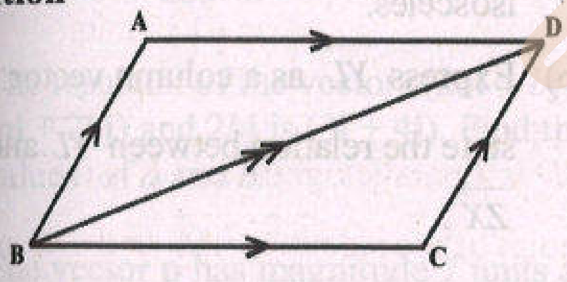
ABCD is a parallelogram such that $\overrightarrow{AB} = 3\mathbf{i} + \mathbf{j}$ and $\overrightarrow{AC} = 5\mathbf{i} - 3\mathbf{j}$. Find

a) $\overrightarrow{BC}$,

b) $\overrightarrow{BD}$,

c) the unit vector in the direction of $\overrightarrow{BC}$.

Solution



a) $\overrightarrow{BC} = (5\mathbf{i} - 3\mathbf{j}) - (3\mathbf{i} + \mathbf{j})$
$= 2\mathbf{i} - 4\mathbf{j}$

b) $\overrightarrow{BD} = (2\mathbf{i} - 4\mathbf{j}) + (-3\mathbf{i} - \mathbf{j})$
$= -\mathbf{i} - 5\mathbf{j}$

c) Unit vector in the direction of $\overrightarrow{BC}$

$$= \frac{1}{|\overrightarrow{BC}|} \times \overrightarrow{BC}$$

$$= \frac{1}{\sqrt{(2)^2 + (-4)^2}}(2\mathbf{i} - 4\mathbf{j})$$

$$= \frac{1}{\sqrt{4 + 16}}(2\mathbf{i} - 4\mathbf{j})$$

$$= \frac{\sqrt{5}}{5}(2\mathbf{i} - 4\mathbf{j})$$

Example 7

The points $A(4, 7)$, $B(x, y)$, $C(-5, -8)$ and $D(1, 4)$ are the vertices of a parallelogram. Find

i) $\overrightarrow{AB}$, $\overrightarrow{DC}$, $\overrightarrow{DA}$

ii) the values of x and y.

Solution

i) $\overrightarrow{AB} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} x-4 \\ y-7 \end{pmatrix}$

$\overrightarrow{DC} = \begin{pmatrix} -5 \\ -8 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -6 \\ -12 \end{pmatrix}$

$\overrightarrow{DA} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

ii) Since *ABCD* is a parallelogram

$\overrightarrow{AB} = \overrightarrow{DC}$

$\begin{pmatrix} x-4 \\ y-7 \end{pmatrix} = \begin{pmatrix} -6 \\ -12 \end{pmatrix}$

From equality of vectors

$x - 4 = -6 \quad \Rightarrow x = -2$ and

$y - 7 = -12 \quad \Rightarrow y = -5$

Review exercise 23

1. Given that $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, evaluate:

   i) $3\mathbf{a} - 4\mathbf{c}$ ii) $2\mathbf{b} + 3\mathbf{c}$

2. If $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\overrightarrow{PR} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, find $\overrightarrow{QR}$.

3. If $\overrightarrow{OP} = \begin{pmatrix} 8 \\ -9 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, find $\overrightarrow{PQ}$.

4. a) If $\mathbf{p}=\begin{pmatrix}3\\4\end{pmatrix}$ and $\mathbf{q}=\begin{pmatrix}-2\\7\end{pmatrix}$ find

i) $4\mathbf{p} - 2\mathbf{q}$ ii) $|4\mathbf{p} - 2\mathbf{q}|$

b) What can you say about your answer in (a) (i) and the vector $\begin{pmatrix}-1\\8\end{pmatrix}$?

5. Given that $\mathbf{p}=\begin{pmatrix}-7\\6\end{pmatrix}$, $\mathbf{q}=\begin{pmatrix}10\\-11\end{pmatrix}$ and

$k(p-q)=\frac{1}{2}\begin{pmatrix}-51\\51\end{pmatrix}$, find k.

6. Given that $\mathbf{a}=\begin{pmatrix}1\\2\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}2\\-1\end{pmatrix}$ and

$\mathbf{c}=\begin{pmatrix}0\\-2\end{pmatrix}$, evaluate:

i) $\mathbf{a}+\mathbf{b}$ ii) $\mathbf{b}-\mathbf{c}$

iii) $\mathbf{a}+\mathbf{b}+\mathbf{c}$ iv) $\mathbf{a}-2\mathbf{b}$

v) $3\mathbf{a}-\mathbf{b}-2\mathbf{c}$ vi) $\mathbf{a}-2\mathbf{b}-3\mathbf{c}$

vii) $2\mathbf{a}+3\mathbf{b}+5\mathbf{c}$

7. *ABCD* is a quadrilateral. *A*, *B* and *D* have coordinates (0, 2), (2, 5) and (8, 0) respectively. If $\overrightarrow{AD}=2\overrightarrow{BC}$, find the coordinates of *C*.

8. If *O*(0, 0), *A*(3, 1) and *B*(2, –1) are the coordinates of a quadrilateral *OABC*. If $\overrightarrow{BC}=\begin{pmatrix}3\\-2\end{pmatrix}$, find:

i) the coordinates of *C*; ii) $\overrightarrow{AC}$.

9. *A*, *B* and *C* are points in the Cartesian plane. The coordinates of *A* are (3, –2), $\overrightarrow{BA}=\begin{pmatrix}4\\-3\end{pmatrix}$ and $\overrightarrow{AC}=\begin{pmatrix}4\\9\end{pmatrix}$. Calculate:

i) the coordinates of *B* and *C*;

ii) the magnitude of $\overrightarrow{BC}$.

10. If $\mathbf{a}=\begin{pmatrix}5\\4\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}-3\\4\end{pmatrix}$, find $|\mathbf{a}-\mathbf{b}|$.

11. *P*(–1, 2) and *Q*(*x*, *y*) are points on the *Oxy* plane such that $\overrightarrow{PQ}=\begin{pmatrix}3\\-4\end{pmatrix}$. Find:

i) the coordinates of *Q*;

ii) the bearing of *P* from *Q*, to the nearest degree.

12. *XYZ* is a triangle with vertices *X*(1, –3), *Y*(7, 5) and *Z*(–3, 5). *L* is the midpoint of the side *XZ*.

a) If *O* is the origin, express $\overrightarrow{XY}$, $\overrightarrow{YZ}$ and $\overrightarrow{ZX}$ as column vectors. Hence show that triangle *XYZ* is isosceles.

b) Express $\overrightarrow{YL}$ as a column vector and state the relation between $\overrightarrow{YL}$ and $\overrightarrow{ZX}$.

13. The points *A*(–2, 1), *B*(2, 1) and *C*(1,4) and *D*(*p*, *q*) are the vertices of a parallelogram *ABCD*.

Write down $\overrightarrow{AB}$, $\overrightarrow{DC}$ and $\overrightarrow{BD}$ as column vectors and deduce the values of *p* and *q*.

14. A triangle *ABC* has vertices *A*(–2, 4), *B*(10, 1) and *C*(3, 8). Show that the triangle is isosceles and find the size of the base angles.

15. *A*(3, 2) *B*(2, 3) and *C*(3, –2) are the vertices of a triangle *ABC*. *D* is the midpoint of *BC* and *H* is (8, 3).

Show that $|\overrightarrow{AH}|=2|\overrightarrow{OD}|$, where *O* is the origin.

16. If $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = 10\mathbf{i} - 2\mathbf{j}$, find $\mathbf{b} - 2\mathbf{a}$.

17. $\overrightarrow{PQ} = \begin{pmatrix} 24 \\ x \end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix} 16 \\ 20 \end{pmatrix}$.

Find x if $\overrightarrow{PQ}$ is paralell to $\overrightarrow{CD}$.

18. If $\begin{pmatrix} 3 \\ -4 \end{pmatrix} = k\begin{pmatrix} 1 \\ 2 \end{pmatrix} + m\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, where k and m are scalars, find the values of k and m.

19. Given that $\begin{pmatrix} 3 \\ 5 \end{pmatrix} = k\begin{pmatrix} 1 \\ 2 \end{pmatrix} + m\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, where k and m are scalars, find the values of k and m.

20. The resultant of the vectors $(2a\mathbf{i} + b\mathbf{j})$, $(a\mathbf{i} + 2b\mathbf{j})$ and $2b\mathbf{j}$ is $(3\mathbf{i} + 4\mathbf{j})$. Find the values of a and b.

21. The vector $\mathbf{p}$ has magnitude 7 units and bearing 052°, and the vector $\mathbf{q}$ has magnitude 12 units and bearing 163°. Calculate, correct to one decimal place, the magnitude of $\mathbf{p} + \mathbf{q}$.

22. Given that $\mathbf{a} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 22 \\ -11 \end{pmatrix}$, find:

i) a unit vector perpendicular to $\mathbf{a}$;

ii) the value of the constants m and n for which $m\mathbf{a} + n\mathbf{b} = \mathbf{c}$.

23. An aeroplane flew from town A, travels 200 km on a bearing of 135° to town B. It then travels 250 km from town B on a bearing of 045° to town C. Find using the vector approach:

a) The distance of A from C;

b) The bearing of A from C.

24. $P(-1, 2)$ and $Q(x, y)$ are points in the Oxy plane such that $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$. Find:

i) the coordinates of Q;

ii) $|\overrightarrow{PQ}|$;

iii) the bearing of P from Q, to the nearest degree.

25. A village P is 10 km from a lorry station, Q, on a bearing 065°. Another village R, is 8 km from Q, on a bearing 155°. Calculate:

i) the distance of R from P, to the nearest kilometre;

ii) the bearing of R from P, to the nearest degree.

26. An aircraft flies from a point A to a point B, x km away on a bearing of 135°. From B the aircraft flies to a point C, $2x$ km away on a bearing of 225°. If the distance from A to C is 350 km, find, correct to the nearest whole number

i) the distance from A to C;

ii) the bearing of C from A..

27. If $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = -10\mathbf{i} - 2\mathbf{j}$, find

i) $\mathbf{b} - 2\mathbf{a}$ ii) $\mathbf{a} - \mathbf{b}$

iii) $\mathbf{b} + 3\mathbf{a}$ iv) $2\mathbf{b} - 5\mathbf{a}$

28. Express the following in magnitude bearing form:

i) $-5\mathbf{i} - 12\mathbf{j}$ ii) $5\mathbf{i} + 12\mathbf{j}$

iii) $4\mathbf{i} - 3\mathbf{j}$ iv) $-\mathbf{i} + \mathbf{j}$

29. Express the following in the form $a\mathbf{i} + b\mathbf{j}$:

i) (7, 052°) ii) (8, 060°)

iii) (6, 150°) iv) (10, 330°)

30. a) OAB is a triangle where O is the origin. Let **a** and **b** be the position vectors of A and B respectively. If X is a point on AB such that $AB = 2XB$ and Y is the mid-point of OX. Show that $\overrightarrow{BY} = \frac{1}{6}\mathbf{a} + \frac{2}{3}\mathbf{b}$

31. A parallelogram $PQRS$ is such that $\overrightarrow{PQ} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and $\overrightarrow{PS} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$. Find the coordinates of R if P is the point (0, 1).

32. Find the unit vector that is perpendicular to the vector $2\mathbf{i} - 3\mathbf{j}$.

33. If $\overrightarrow{OA} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$ and N is the mid-point of $\overrightarrow{AB}$, find $\overrightarrow{ON}$.

34. In triangle ABC, $\overrightarrow{AB} = 3\mathbf{i} - 4\mathbf{j}$ and $\overrightarrow{AC} = 2\mathbf{i} + 7\mathbf{j}$. Find $\overrightarrow{BC}$.

35. Let $\mathbf{a} = 2\mathbf{i} - 5\mathbf{j}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{j}$. Find the resultant of $2\mathbf{a}$ and $-\mathbf{b}$.

36. $PQRS$ is parallelogram and X is a point on the side QR. If $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\overrightarrow{XQ} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, find $\overrightarrow{PX}$.

37. Express (10 m, 240°) as a column vector.

38. If $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find $|\mathbf{a} + \mathbf{b}|$.

39. $ORST$ is a parallelogram. U is the midpoint of RS and V is the midpoint of ST. **r**, **s**, **t**, **u** and **v** are the position vectors of R, S, T, U and V respectively.

a) Express **s** in terms of **r** and **t**

b) Express **v** in terms of **s** and **t**

c) Hence or otherwise, show that $4(\mathbf{u} + \mathbf{v}) = 3(\mathbf{r} + \mathbf{s} + \mathbf{t})$

40. The vectors $\overrightarrow{OK}$ and $\overrightarrow{OL}$ have their initial point at (0, 0) and terminal points are (–2, 1) and (–2, –5) respectively. If M is the mid-point of $\overline{KL}$, find the vector $\overrightarrow{OM}$.

41. Given that $\mathbf{p} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$, find the angle between **p** and **q**.

42. If $m\begin{pmatrix} 2 \\ -1 \end{pmatrix} + n\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$, where m and n are scalars, find the value of $(m + n)$.

43. Find the resultant of the following vectors (10 m, 070°) and (15 m, 260°) by,

a) scale drawing

b) resolution

44. Two ships A and B leave a port simultaneously. A moves at 10 kmh^{-1} on a bearing of 160° while B moves on a bearing of 215°. After 1 hour, the bearing of B from A is 260°. Find the speed of B correct to the nearest whole number.

CHAPTER 24

SCALAR OR DOT PRODUCT

Scalar or dot product

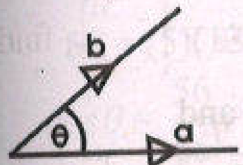
The scalar or dot product of two non-zero vectors **a** and **b** is defined as:

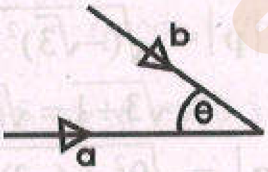
$\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$,

where θ is the angle between **a** and **b**.

Note 1: The two vectors must be directed away from a common point or directed towards a common point (see figure below).

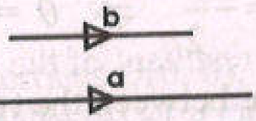
Note 2: θ can be an acute or obtuse angle. If θ is obtuse then **a.b** is negative.





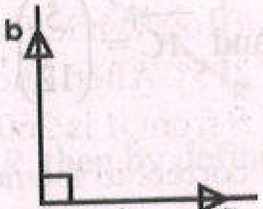
Special cases

1. If **a** and **b** are parallel, i.e. $\theta = 0°$, $\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|$, because $\cos 0° = 1$.



For example **i.i** = **j.j** = 1.

2. If **a** and **b** are perpendicular, i.e. $\theta = 90°$, $\mathbf{a}.\mathbf{b} = 0$, because $\cos 90° = 0$.



For example, **i.j** = **j.i** = 0.

Dot product in terms of components

If $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then in terms of components the dot product of **a** and **b** is

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = x_1x_2 + y_1y_2$$

In the Cartesian form,

$(x_1\mathbf{i} + y_1\mathbf{j}).(x_2\mathbf{i} + y_2\mathbf{j}) = x_1x_2 + y_1y_2$

Note that the dot product of two vectors gives a **number**.

Illustrative examples

Example 1

If $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, find **a.b**.

Solution

$$\mathbf{a}.\mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$
$$= (2\times 3) + (1\times -2) = 6 + (-2) = 4$$

Example 2

Given that $\mathbf{m} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$, find **m.n**.

Solution

$$\mathbf{m}.\mathbf{n} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.\begin{pmatrix} -2 \\ 6 \end{pmatrix}$$
$$= (-2\times -2) + (3\times 6)$$
$$= 4 + 18 = 22$$

Example 3

If $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = -5\mathbf{i} + 2\mathbf{j}$, find **a.b**.

Solution

$$\mathbf{a.b} = (4\mathbf{i} + 3\mathbf{j}).(-5\mathbf{i} + 2\mathbf{j})$$
$$= [4\times(-5)] + [3\times 2] = -20 + 6 = -14.$$

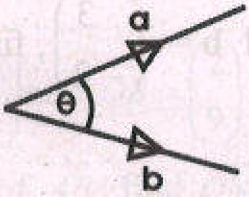
Note: $\mathbf{i.i} = \mathbf{j.j} = 1$ and $\mathbf{i.j} = \mathbf{j.i} = 0$

Properties of the dot product

1. $\mathbf{a.a} = |\mathbf{a}|^2$
2. $\mathbf{a.b} = \mathbf{b.a}$
   (i.e. Dot product is commutative)
3. $\mathbf{a.(b + c)} = \mathbf{a.b} + \mathbf{a.c}$
4. $k(\mathbf{a.b}) = (k\mathbf{a}).\mathbf{b} = \mathbf{a}.(k\mathbf{b})$

Applications of scalar or dot product

1. To find the angle between two vectors



From the definition of dot product,

$$\mathbf{a.b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$$

$$\cos\theta = \frac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|},$$

where θ is the angle between the two vectors **a** and **b**.

Also, $\cos\theta = \dfrac{\mathbf{a}}{|\mathbf{a}|}.\dfrac{\mathbf{b}}{|\mathbf{b}|} = \hat{\mathbf{a}}\,.\,\hat{\mathbf{b}}$

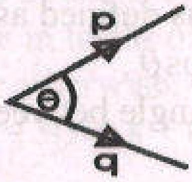
Therefore the angle between two vectors **a** and **b** is the angle between the their direction vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$

Illustrative examples

Example 1

Find the angle between the two vectors $\mathbf{p} = \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$.

Solution



Let θ be the angle between the two vectors, then by the definition of dot product;

$$\mathbf{p.q} = |\mathbf{p}|\,|\mathbf{q}|\cos\theta \quad \text{..................(1)}$$

$$\mathbf{p.q} = \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}.\begin{pmatrix} 0 \\ -3 \end{pmatrix} = 0 + (-3) = -3$$

Also $|\mathbf{p}| = \sqrt{(-\sqrt{3})^2 + 1^2}$
$$= \sqrt{3+1} = \sqrt{4} = 2 \text{ and}$$
$$|\mathbf{q}| = \sqrt{0^2 + (-3)^2} = \sqrt{0+9} = \sqrt{9} = 3$$

$\therefore$ From equation (1); $-3 = (2)(3)\cos\theta$
$$-3 = 6\cos\theta$$

$$\cos\theta = \frac{-3}{6} = -\frac{1}{2} \quad \Rightarrow \quad \theta = 120°$$

$\therefore$ The angle between the two vectors is 120°

Note that θ is an obtuse angle because $\cos\theta$ is negative.

Example 2

If $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$, find

i) $|\overrightarrow{AB}|$

ii) $|\overrightarrow{AC}|$

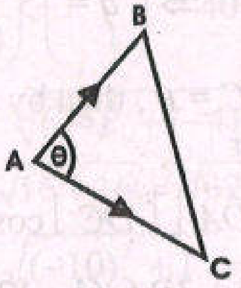
iii) the angle *CAB*.

Solution

i) $|\overrightarrow{AB}| = \sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = 5$

ii) $|\overrightarrow{AC}| = \sqrt{5^2+12^2}$
$= \sqrt{25+144} = \sqrt{169} = 13$

iii)



Let θ = angle CAB, then by definition of dot product

$$\overrightarrow{AB}.\overrightarrow{AC} = |\overrightarrow{AB}||\overrightarrow{AC}|\cos\theta$$

$$\overrightarrow{AB}.\overrightarrow{AC} = \begin{pmatrix}4\\3\end{pmatrix}\cdot\begin{pmatrix}5\\12\end{pmatrix} = 20+36 = 56$$

$$56 = (5)(13)\cos\theta$$

$$\cos\theta = \frac{56}{65} = 0.8615 \Rightarrow \theta = 30.51°$$

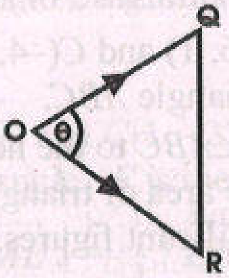
$\therefore$ Angle CAB = 30. 51°

Note that the two vectors must be either directed away from a common point or directed towards a common point.

Example 3

A triangle has vertices $O(0, 0)$, $Q(4, 2)$ and $R(6, -2)$. Find the value of $\angle QOR$.

Solution



Let $\theta = \angle QOR$, then by definition of dot product

$$\overrightarrow{OQ}.\overrightarrow{OR} = |\overrightarrow{OQ}||\overrightarrow{OR}|\cos\theta \ldots\ldots(1)$$

$\overrightarrow{OQ} = \mathbf{q}-\mathbf{o} = \begin{pmatrix}4\\2\end{pmatrix}-\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}$ and

$$\overrightarrow{OR} = \mathbf{r}-\mathbf{o} = \begin{pmatrix}6\\-2\end{pmatrix}-\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}6\\-2\end{pmatrix}$$

$$\overrightarrow{OQ}.\overrightarrow{OR} = \begin{pmatrix}4\\2\end{pmatrix}\cdot\begin{pmatrix}6\\-2\end{pmatrix} = 24+(-4) = 20$$

Also $|\overrightarrow{OQ}| = \sqrt{4^2+2^2} = \sqrt{16+4} = \sqrt{20}$

and $|\overrightarrow{OR}| = \sqrt{6^2+(-2)^2}$
$= \sqrt{36+4} = \sqrt{40}$

$\therefore$ From equation (1);

$$20 = (\sqrt{20})(\sqrt{40})\cos\theta$$

$$20 = \sqrt{800}\cos\theta$$

$$\cos\theta = \frac{20}{\sqrt{800}} = \frac{20}{20\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\theta = 45°$ $\therefore$ $\angle QOR = 45°$

Note that the two vectors must be either directed away from a common point or directed towards a common point.

Example 4

Given that $\mathbf{a} = 3\mathbf{i}-4\mathbf{j}$ and $\mathbf{b} = 6\mathbf{i}+4\mathbf{j}$, find the angle between the two vectors, correct to the nearest degree.

Solution

$\mathbf{a} = 3\mathbf{i}-4\mathbf{j}$ and $\mathbf{b} = 6\mathbf{i}+4\mathbf{j}$

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \Rightarrow \cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\mathbf{a}.\mathbf{b} = (3\mathbf{i}-4\mathbf{j})(6\mathbf{i}+4\mathbf{j}) = 18-16 = 2$$

Note $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = 1$

$$|\mathbf{a}| = \sqrt{(3)^2+(-4)^2} = \sqrt{25} = 5$$

$$|\mathbf{b}| = \sqrt{(6)^2+(4)^2} = \sqrt{52}$$

$$\therefore \cos\theta = \frac{2}{5\times\sqrt{52}} = 0.05547$$

$\therefore \theta = \cos^{-1} 0.05547 = 87°$ to the nearest degree

Example 5

The points $A(4, 7)$, $B(p, q)$, $C(-5, -8)$ and $D(1, 4)$ are the vertices of a parallelogram.

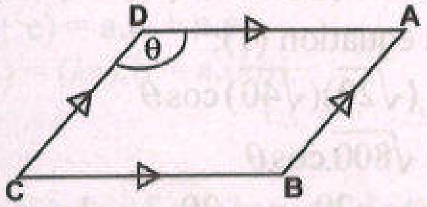
i) Express in Cartesian coordinates, the vectors $\overrightarrow{AB}$, $\overrightarrow{DC}$, $\overrightarrow{DA}$ and write down the unit vectors parallel to $\overrightarrow{DC}$ and $\overrightarrow{DA}$.

ii) Find p and q.

iii) $\angle ADC$ to the nearest degree.

iv) The area of the parallelogram $ABCD$.

Solution



Let **a**, **b**, **c**, and **d**, be the position vectors of the vertices A, B, C and D respectively, then $\mathbf{a} = 4\mathbf{i} + 7\mathbf{j}$, $\mathbf{b} = p\mathbf{i} + q\mathbf{j}$, $\mathbf{c} = -5\mathbf{i} - 8\mathbf{j}$ and $\mathbf{d} = \mathbf{i} + 4\mathbf{j}$

i)
$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (p\mathbf{i} + q\mathbf{j}) - (4\mathbf{i} + 7\mathbf{j}) = (p-4)\mathbf{i} + (q-7)\mathbf{j}$$

$$\overrightarrow{DC} = \mathbf{c} - \mathbf{d} = (-5\mathbf{i} - 8\mathbf{j}) - (\mathbf{i} + 4\mathbf{j}) = -6\mathbf{i} - 12\mathbf{j}$$

$$\overrightarrow{DA} = \mathbf{a} - \mathbf{d} = (4\mathbf{i} + 7\mathbf{j}) - (\mathbf{i} + 4\mathbf{j}) = 3\mathbf{i} + 3\mathbf{j}$$

$$|\overrightarrow{DC}| = \sqrt{(-6)^2 + (-12)^2} = \sqrt{36 + 144} = \sqrt{180} = 6\sqrt{5}$$

$$|\overrightarrow{DA}| = \sqrt{3^2 + 3^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

Unit vector parallel to $\overrightarrow{DC} = \frac{1}{6\sqrt{5}}(-6\mathbf{i} - 12\mathbf{j}) = -\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$

Unit vector parallel to $\overrightarrow{DA} = \frac{1}{3\sqrt{2}}(3\mathbf{i} + 3\mathbf{j}) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

ii) Since $ABCD$ is a parallelogram, we have

$$\overrightarrow{AB} = \overrightarrow{DC}$$

$$(p-4)\mathbf{i} + (q-7)\mathbf{j} = -6\mathbf{i} - 12\mathbf{j}$$

Comparing corresponding components,

$p - 4 = -6 \quad \Rightarrow \quad p = -2$ and

$q - 7 = -12 \quad \Rightarrow \quad q = -5$

iii) Let $\angle ADC = \theta$, then by definition of dot product

$$\overrightarrow{DA}.\overrightarrow{DC} = |\overrightarrow{DA}||\overrightarrow{DC}|\cos\theta \quad \text{........ (1)}$$

$$\overrightarrow{DA}.\overrightarrow{DC} = (3\mathbf{i} + 3\mathbf{j}).(-6\mathbf{i} - 12\mathbf{j}) = -18 + (-36) = -54$$

Also from (i) above, $|\overrightarrow{DA}| = 3\sqrt{2}$ and $|\overrightarrow{DC}| = 6\sqrt{5}$

∴ From equation (1),

$$-54 = (3\sqrt{2})(6\sqrt{5})\cos\theta$$

$$\cos\theta = \frac{-54}{(3\sqrt{2})(6\sqrt{5})} = \frac{-3}{\sqrt{10}} = -0.9486$$

$\theta = 162^\circ$ and $< ADC = 162^\circ$ to the nearest degree.

Note: $\cos\theta$ has a negative value because the angle is obtuse.

iv) The area of parallelogram $ABCD$

$$= |\overrightarrow{DA}||\overrightarrow{DC}|\sin\theta$$

$$= (3\sqrt{2})(6\sqrt{5})\sin 162^\circ = 17.6 \text{ sq units.}$$

Example 6

If $A(2, -3)$, $B(6, 1)$ and $C(-4, 2)$ are the vertices of a triangle ABC.

i) Calculate $\angle ABC$ to the nearest degree.

ii) Deduce the area of triangle ABC correct to two significant figures.

Solution

Let $\theta = \angle ABC$ (see figure above), then by the dot product

$$\overrightarrow{BA}.\overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BC}|\cos\theta$$

Now $\overrightarrow{BA} = \mathbf{a} - \mathbf{b} = \begin{pmatrix}2\\-3\end{pmatrix} - \begin{pmatrix}6\\1\end{pmatrix} = \begin{pmatrix}-4\\-4\end{pmatrix}$

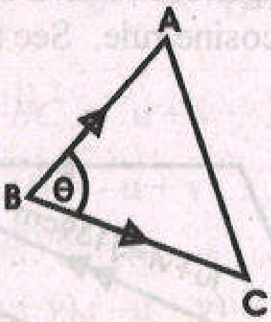
$\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = \begin{pmatrix}-4\\2\end{pmatrix} - \begin{pmatrix}6\\1\end{pmatrix} = \begin{pmatrix}-10\\1\end{pmatrix}$

$\overrightarrow{BA}.\overrightarrow{BC} = \begin{pmatrix}-4\\-4\end{pmatrix}.\begin{pmatrix}-10\\1\end{pmatrix} = 40 + (-4) = 36$

Also $|\overrightarrow{BA}| = \sqrt{(-4)^2 + (-4)^2}$
$= \sqrt{16+16} = \sqrt{32}$

$|\overrightarrow{BC}| = \sqrt{(-10)^2 + 1^2} = \sqrt{101}$

$\therefore$ From equation (1),
$36 = (\sqrt{32})(\sqrt{101})\cos\theta$
$\cos\theta = \dfrac{36}{\sqrt{32}\sqrt{101}} = 0.6332$
$\theta = 50.71^\circ$
$= 51^\circ$ correct to the nearest degree.



iii) Area of triangle ABC
$= \frac{1}{2}|\overrightarrow{BA}||\overrightarrow{BC}|\sin\theta$
$= \frac{1}{2}(\sqrt{32})(\sqrt{101})\sin 50.71^\circ$
$= 44.17$
$= 44$ sq units to 2 significant figures

Example 7

$A(3, 1)$, $B(-1, 9)$ and $C(1, -2)$ are the vertices of a triangle. E is the point on AB such that $|\overrightarrow{AE}| = \frac{1}{3}|\overrightarrow{EB}|$.

i) Express $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{CE}$ as column vectors.

ii) Use the dot product to calculate $\angle ACE$, correct to the nearest degree.

Solution

i) Let **a**, **b**, **c** and **e** be the position vectors of A, B, C, and E respectively.

$\therefore\ \mathbf{a} = \begin{pmatrix}3\\1\end{pmatrix},\ \mathbf{b} = \begin{pmatrix}-1\\9\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}1\\-2\end{pmatrix}$

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix}-1\\9\end{pmatrix} - \begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}-4\\8\end{pmatrix}$

and

$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix}1\\-2\end{pmatrix} - \begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}-2\\-3\end{pmatrix}$

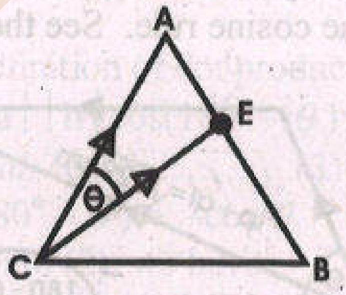
Also $|\overrightarrow{AE}| = \frac{1}{3}|\overrightarrow{EB}|$

$3|\overrightarrow{AE}| = |\overrightarrow{EB}|$
$3(\mathbf{e} - \mathbf{a}) = \mathbf{b} - \mathbf{e}$
$3\mathbf{e} - 3\mathbf{a} = \mathbf{b} - \mathbf{e}$
$4\mathbf{e} = \mathbf{b} + 3\mathbf{a}$
$\mathbf{e} = \frac{1}{4}(\mathbf{b} + 3\mathbf{a})$

$\mathbf{e} = \frac{1}{4}\left[\begin{pmatrix}-1\\9\end{pmatrix} + 3\begin{pmatrix}3\\1\end{pmatrix}\right] = \frac{1}{4}\begin{pmatrix}8\\12\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$

$\therefore\ \overrightarrow{CE} = \mathbf{e} - \mathbf{c} = \begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix}$

ii)



Let $\theta = \angle ACE$, then by the definition of dot product

$\overrightarrow{CA}.\overrightarrow{CE} = |\overrightarrow{CA}||\overrightarrow{CE}|\cos\theta$ .... (1)

$\overrightarrow{CA} = -\overrightarrow{AC} = \begin{pmatrix}2\\3\end{pmatrix}$

$\therefore\ \overrightarrow{CA}.\overrightarrow{CE} = \begin{pmatrix}2\\3\end{pmatrix}.\begin{pmatrix}1\\5\end{pmatrix}$
$= 2 + 15 = 17$

Also $|\overrightarrow{CA}| = \sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13}$

and $|\overrightarrow{CE}| = \sqrt{1^2+5^2} = \sqrt{26}$

$\therefore$ From equation (1),

$$17 = (\sqrt{13})(\sqrt{26})\cos\theta$$

$$\cos\theta = \frac{17}{\sqrt{13}\sqrt{26}} = 0.925$$

$\theta = 22.38^\circ = 22^\circ$ to the nearest degree.

Example 8

Find the angle between **i** + 5**j** and 5**i** – **j**.

Solution

$(\mathbf{i}+5\mathbf{j}).(5\mathbf{i}-\mathbf{j}) = 5 - 5 = 0$

$\therefore$ The angle between the two vectors is 90° since their dot product is zero.

Example 9

Vectors **p** and **q** are such that $|\mathbf{p}| = 5$ cm, $|\mathbf{q}| = 12$ cm and $|\mathbf{p}+\mathbf{q}| = \sqrt{229}$ cm. Find

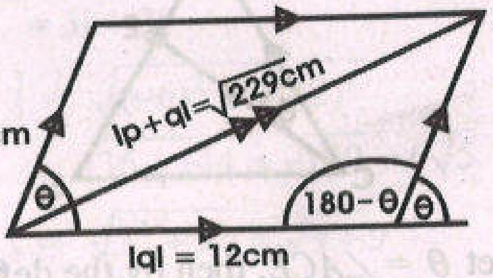
i) the angle between **p** and **q**

ii) the scalar (dot) product of **p** and **q**.

Solution

i) Let θ = angle between **p** and **q**.

Applying the cosine rule. See the figure below.



$$|\mathbf{p}+\mathbf{q}|^2 = |\mathbf{p}|^2 + |\mathbf{q}|^2 - |\mathbf{p}|\,|\mathbf{q}|\cos(180^\circ-\theta)$$

$$|\mathbf{p}+\mathbf{q}|^2 = |\mathbf{p}|^2 + |\mathbf{q}|^2 + 2|\mathbf{p}|\,|\mathbf{q}|\cos\theta,$$

since $\cos(180-\theta) = -\cos\theta$.

$$(\sqrt{229})^2 = 5^2 + 12^2 + (2)(5)(12)\cos\theta$$

$$229 = 25 + 144 + 120\cos\theta$$

$$229 = 169 + 120\cos\theta$$

$$120\cos\theta = 60$$

$$\cos\theta = \tfrac{60}{120} = \tfrac{1}{2}$$

$$\theta = 60^\circ$$

Therefore the angle between **p** and **q** is 60°.

ii) By definition, the scalar (dot) product of **p** and **q** is $\mathbf{p}.\mathbf{q} = |\mathbf{p}|\,|\mathbf{q}|\cos\theta$, where θ is the angle between **p** and **q**.

$\mathbf{p}.\mathbf{q} = 5 \times 12 \times \tfrac{1}{2} = 30$

Example 10

Vectors **u** and **v** are such that $|\mathbf{u}| = 3$ cm, $|\mathbf{v}| = 10$ cm and $|\mathbf{u}+\mathbf{v}| = \sqrt{139}$ cm. Find

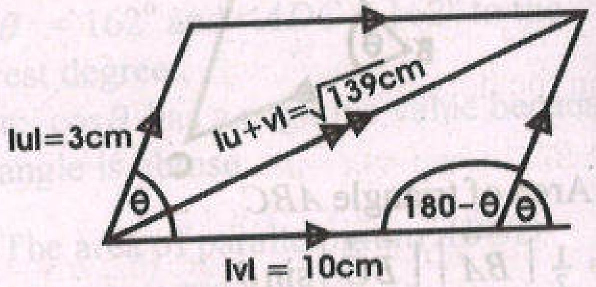
i) the angle between **u** and **v**

ii) the scalar (dot) product of **u** and **v**

Solution

i) Let θ = angle between **u** and **v**.

Applying the cosine rule. See the figure below.



$$|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}|\cos(180^\circ-\theta)$$

$$|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2|\mathbf{u}|\,|\mathbf{v}|\cos\theta$$

since $\cos(180-\theta) = -\cos\theta$

$$(\sqrt{139})^2 = 3^2 + 10^2 + (2)(3)(10)\cos\theta$$

$$139 = 9 + 100 + 60\cos\theta$$

$$139 = 109 + 60\cos\theta$$

$$60\cos\theta = 139 - 109 = 30$$

$$\cos\theta = \tfrac{30}{60} = \tfrac{1}{2}$$

$$\theta = 60^\circ$$

Therefore the angle between **u** and **v** is 60°.

ii) By definition, the scalar (dot) product of **u** and **v** is $\mathbf{u.v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$, where θ is the angle between **u** and **v**.

$\mathbf{u.v} = 3\times10\times\frac{1}{2} = 15$

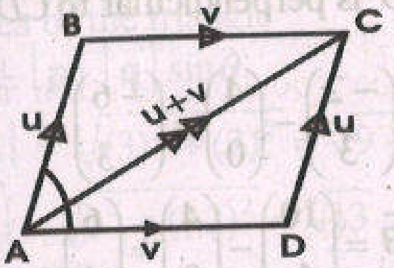
Example 11

Let **u** and **v** denote the vectors along the adjacent sides of a rhombus.

Show that the diagonals of the rhombus are perpendicular.

Solution

The figure below shows the rhombus.



$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \mathbf{u} + \mathbf{v}$ ....(1)

$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = -\mathbf{u} + \mathbf{v}$ ....(2)

Using the dot product;

$$\overrightarrow{AC}\,.\,\overrightarrow{BD} = (\mathbf{u}+\mathbf{v}).(-\mathbf{u}+\mathbf{v})$$
$$= -\mathbf{u.u} + \mathbf{u.v} - \mathbf{v.u} + \mathbf{v.v}$$
$$= -\mathbf{u.u} + \mathbf{v.v} \quad \text{Note that } \mathbf{u.v} = \mathbf{v.u}$$
$$= |\mathbf{v}|^2 - |\mathbf{u}|^2$$

Since all the sides of a rhombus are equal, $|\mathbf{v}| = |\mathbf{u}|$.

$$\overrightarrow{AC}\,.\,\overrightarrow{BD} = |\mathbf{v}|^2 - |\mathbf{v}|^2 = 0$$

Therefore the diagonals of a rhombus are perpendicular.

Example 12

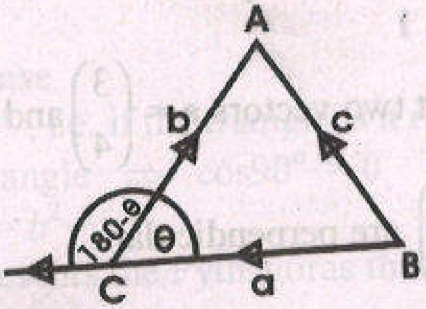
ABC is a triangle in which $\overrightarrow{BC} = \mathbf{a}$, $\overrightarrow{CA} = \mathbf{b}$, $\overrightarrow{BA} = \mathbf{c}$ and $<ACB = \theta$.

i) If one of the vectors can be expressed in terms of the other two, write down the expression.

ii) Using the dot product, show that $c^2 = b^2 + a^2 - 2ba\cos\theta$, where $c = |\mathbf{c}|$, $b = |\mathbf{b}|$ and $a = |\mathbf{a}|$.

iii) For what value of θ will the result in (ii) represent the Pythagoras theorem?

Solution



i) From the figure above,

$\overrightarrow{BA} = \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{a} + \mathbf{b}$

$\mathbf{c} = \mathbf{a} + \mathbf{b}$ ....(1)

ii) $\mathbf{c.c} = (\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})$

$|\mathbf{c}|^2 = |\mathbf{a}|^2 + 2\,|\mathbf{a.b}| + |\mathbf{b}|^2$

$c^2 = a^2 + b^2 + 2\,|\mathbf{a.b}|$ ....(2)

From the definition of dot product,

$\mathbf{a.b} = |\mathbf{a}|\,|\mathbf{b}|\cos(180^\circ - \theta)$

$\mathbf{a.b} = -ab\cos\theta$ ....(3)

since $\cos(180^\circ - \theta) = -\cos\theta$

Putting (3) into (2), we have

$c^2 = a^2 + b^2 - 2ab\cos\theta$

iii) If $\theta = 90^\circ$, i.e. if the triangle is a right-angled triangle

$\cos\theta = \cos 90^\circ = 0$.

$c^2 = b^2 + a^2$

This represents the Pythagoras theorem.

2. To show that two vectors are perpendicular

From the definition of dot product, $\mathbf{a.b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$, if **a** and **b** are perpendicular or orthogonal, i.e. $\theta = 90°$, $\mathbf{a.b} = 0$ because $\cos 90° = 0$. Therefore if the dot or scalar product of two vectors is equal to **zero**, it implies the two vectors are **perpendicular** or **orthogonal.**

Illustrative examples

Example 1

Show that two vectors $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -8 \\ 6 \end{pmatrix}$ are perpendicular.

Solution

$$\mathbf{a.b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} . \begin{pmatrix} -8 \\ 6 \end{pmatrix} = -24 + 24 = 0$$

Since the dot or scalar product of the two vectors is equal to zero, it implies the two vectors **a** and **b** are perpendicular.

Example 2

a) If $\overrightarrow{PQ} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\overrightarrow{RQ} = \begin{pmatrix} c \\ d \end{pmatrix}$, express $\overrightarrow{PR}$ as a column vector.

b) The position vectors of points A, B, C and D are $\begin{pmatrix} -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 10 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 12 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ respectively.

- i) Show that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{CD}$.
- ii) Calculate, correct to the **nearest** degree, angle ADB.

Solution

a) $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$

$$= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} -c \\ -d \end{pmatrix} = \begin{pmatrix} a-c \\ b-d \end{pmatrix}$$

b) i) $\overrightarrow{AB} = \begin{pmatrix} 10 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 1 \end{pmatrix}$

$$\overrightarrow{CD} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \end{pmatrix}$$

$$\overrightarrow{AB} \bullet \overrightarrow{CD} = \begin{pmatrix} 12 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -12 \end{pmatrix} = 12 - 12 = 0$$

$\overrightarrow{AB}$ is perpendicular to $\overrightarrow{CD}$.

ii) $\overrightarrow{DA} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix}$

$$\overrightarrow{DB} = \begin{pmatrix} 10 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$\overrightarrow{DA} . \bullet \overrightarrow{DB} = \begin{pmatrix} -6 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ 4 \end{pmatrix} = -24$$

$$\left|\overrightarrow{DA}\right| = \sqrt{36+9} = 3\sqrt{5}$$

$$\left|\overrightarrow{DB}\right| = \sqrt{16+36} = 2\sqrt{13}$$

From the dot product

$\overrightarrow{DA} . \bullet \overrightarrow{DB} = \left|\overrightarrow{DA}\right| \left|\overrightarrow{DB}\right| \cos\theta$,

where $\theta = \angle ADB$

$$\cos\theta = \frac{-24}{(3\sqrt{5})(2\sqrt{13})} = -0.496$$

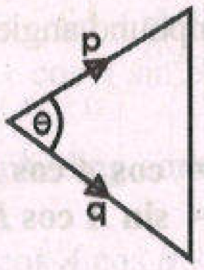
$$\theta = \cos^{-1}(-0.496) = 119.74 = 120°$$

Example 3

Two sides of a triangle are represented by the vectors $\mathbf{p} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

- i) Show that they are perpendicular to each other.
- ii) Hence find the area of the triangle;

Solution



i) **p** and **q** are perpendicular if **p.q** = 0

$$\mathbf{p.q} = \begin{pmatrix}2\\3\end{pmatrix}\cdot\begin{pmatrix}3\\-2\end{pmatrix} = 6+(-6) = 0$$

Therefore **p** and **q** are perpendicular.

ii) a) Area of the triangle

$= \frac{1}{2}\,|\mathbf{p}|\,|\mathbf{q}|\sin\theta$

where $|\mathbf{p}| = \sqrt{(3)^2+(2)^2} = \sqrt{9+4} = \sqrt{13}$,

$|\mathbf{q}| = \sqrt{3^2+(-2)^2} = \sqrt{13}$ and

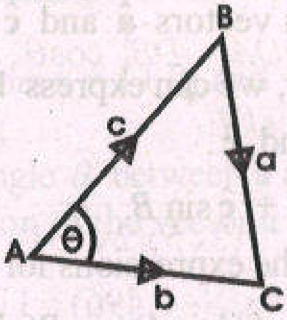
$\theta = 90^\circ$

∴ Area of the triangle

$= \frac{1}{2}(\sqrt{13})(\sqrt{13})\sin 90^\circ$

$= \frac{1}{2}(13) = 6.5$ square units

3. Proving the *cosine rule* using the scalar product



Let $\overrightarrow{AB} = \mathbf{c}$, $\overrightarrow{AC} = \mathbf{b}$ and $\overrightarrow{BC} = \mathbf{a}$ and let θ be the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$, as shown in the figure above.

$\overrightarrow{BC} = \overrightarrow{CA} + \overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{AB}$

$\mathbf{a} = \mathbf{b} - \mathbf{c}$

$\mathbf{a.a} = (\mathbf{b} - \mathbf{c}).(\mathbf{b} - \mathbf{c})$

$\mathbf{a.a} = \mathbf{b.b} - \mathbf{b.c} - \mathbf{c.b} + \mathbf{c.c}$

But $\mathbf{a.a} = |\mathbf{a}|^2$, $\mathbf{b.b} = |\mathbf{b}|^2$, $\mathbf{c.c} = |\mathbf{c}|^2$ and $\mathbf{b.c} = \mathbf{c.b}$

$$|\mathbf{a}|^2 = |\mathbf{b}|^2 + |\mathbf{c}|^2 - 2(\mathbf{b.c})$$

Since $\mathbf{b.c} = |\mathbf{b}|\,|\mathbf{c}|\cos\theta$

$|\mathbf{a}|^2 = |\mathbf{b}|^2 + |\mathbf{c}|^2 - 2\,|\mathbf{b}|\,|\mathbf{c}|\cos\theta$

$$a^2 = b^2 + c^2 - 2bc\cos\theta$$

This is the **cosine rule.**

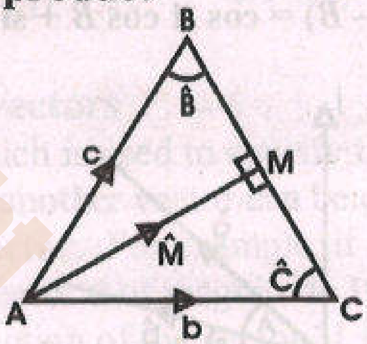
Special case

If $\theta = 90^\circ$, i.e. if the triangle is a right angled triangle ⇒ $\cos 90^\circ = 0$

$a^2 = b^2 + c^2$.

This represents the Pythagoras theorem.

4. Proving the *sine rule* using the scalar product



Let $\overrightarrow{AB} = \mathbf{c}$, $\overrightarrow{AC} = \mathbf{b}$, $\overrightarrow{AM}$ be the perpendicular from A to BC and $\hat{\mathbf{m}}$ be a unit vector from A in the direction of the $\overrightarrow{AM}$, as shown in the figure above,

$\mathbf{c}.\hat{\mathbf{m}} = |\mathbf{c}|\,|\hat{\mathbf{m}}|\cos\angle BAM$

$= |\mathbf{c}|\cos\angle BAM = |AM|$

i.e. the projection of $\overrightarrow{AB}$ on $\overrightarrow{BC}$

$\mathbf{b}.\hat{\mathbf{m}} = |\mathbf{b}|\,|\hat{\mathbf{m}}|\cos\angle CAM$

$= |\mathbf{b}|\cos\angle CAM = |AM|$

i.e. the projection of $\overrightarrow{AC}$ on $\overrightarrow{BC}$

Thus $\mathbf{c}.\,\hat{\mathbf{m}} = |AM| = |\mathbf{c}|\sin\hat{B} = c\sin\hat{B}$
and $\mathbf{b}.\,\hat{\mathbf{m}} = |AM| = |\mathbf{b}|\sin\hat{C} = b\sin\hat{C}$
We can therefore write $c\sin\hat{B} = b\sin\hat{C}$ or

$\dfrac{c}{\sin\hat{C}} = \dfrac{b}{\sin\hat{B}}$. This is the **sine rule.**

By similar method, we can obtain the following:

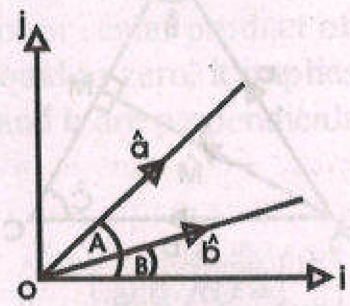
$\dfrac{c}{\sin\hat{C}} = \dfrac{a}{\sin\hat{C}}$ and $\dfrac{a}{\sin\hat{A}} = \dfrac{b}{\sin\hat{B}}$.

Combining these results give:

$$\frac{a}{\sin\hat{A}} = \frac{b}{\sin\hat{B}} = \frac{c}{\sin\hat{C}}$$

5. Proving the *compound angle formulae* using the scalar product

A. Show that
$\cos(A-B) = \cos A\cos B + \sin A\sin B$



From the diagram above, let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be two unit vectors making angles A and B respectively to the **i** direction.

$\hat{\mathbf{a}} = \mathbf{i}\cos A + \mathbf{j}\sin A$ and $\hat{\mathbf{b}} = \mathbf{i}\cos B + \mathbf{j}\sin B$

$$\hat{\mathbf{a}}\,.\,\hat{\mathbf{b}} = (\mathbf{i}\cos A + \mathbf{j}\sin A).(\mathbf{i}\cos B + \mathbf{j}\sin B)$$
$$= \cos A\cos B + \sin A\sin B$$

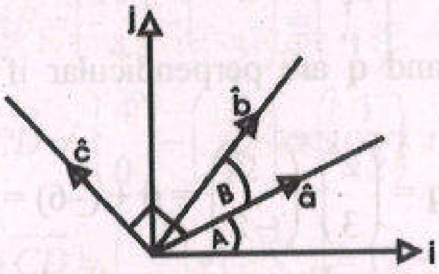
But by definition,

$\hat{\mathbf{a}}\,.\,\hat{\mathbf{b}} = |\hat{\mathbf{a}}|\;|\hat{\mathbf{b}}|\cos(A-B) = \cos(A-B)$,

since $|\hat{\mathbf{a}}| = 1$ and $|\hat{\mathbf{b}}| = 1$.

$\therefore\ \cos(A-B) = \cos A\cos B + \cos A\sin B$
This is a compound angle identity.

B. Show that
$\cos(A+B) = \cos A\cos B - \sin A\sin B$
$\sin(A+B) = \sin A\cos B + \cos A\sin B$



Let **i** and **j** be unit vectors along the x and y axes respectively. Let $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ be unit vectors From the figure above,

$\hat{\mathbf{a}} = \mathbf{i}\cos A + \mathbf{j}\sin A$,
$\hat{\mathbf{b}} = \mathbf{i}\cos(A+B) + \mathbf{j}(\sin A+B)$ and

$\hat{\mathbf{c}} = \mathbf{i}\cos(\frac{\pi}{2}+A) + \mathbf{j}\sin(\frac{\pi}{2}+A)$
$= -\mathbf{i}\sin A + \mathbf{j}\cos A$
since $\cos(\frac{\pi}{2}+A) = -\sin A$ and
$\sin(\frac{\pi}{2}+A) = \cos A$

Since the unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{c}}$ are perpendicular, we can express $\hat{\mathbf{b}}$ in terms of $\hat{\mathbf{a}}$ and $\hat{\mathbf{c}}$.

$\hat{\mathbf{b}} = \hat{\mathbf{a}}\cos B + \hat{\mathbf{c}}\sin B$.

Substituting the expressions for $\hat{\mathbf{a}}$ and $\hat{\mathbf{c}}$,

$\hat{\mathbf{b}} = \cos B(\mathbf{i}\cos A + \mathbf{j}\sin A) + \sin B(-\mathbf{i}\sin A + \mathbf{j}\cos A)$

But $\hat{\mathbf{b}} = \mathbf{i}\cos(A+B) + \mathbf{j}(\sin A+B)$

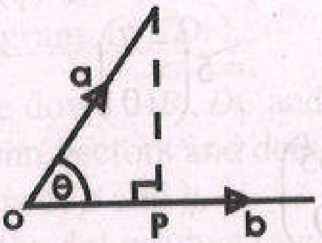
$\therefore\ \mathbf{i}\cos(A+B) + \mathbf{j}(\sin A+B)$
$= \mathbf{i}\cos A\cos B + \mathbf{j}\sin A\cos B -$
$\mathbf{i}\sin A\sin B + \mathbf{j}\cos A\sin B$

$$\mathbf{i}\cos(A+B) + \mathbf{j}(\sin A+B)$$
$$= \mathbf{i}(\cos A \cos B - \sin A \sin B) +$$
$$\mathbf{j}(\sin A \sin B + \cos A \sin B)$$

Comparing corresponding components, we have

1. $\cos(A + B) = \cos A \cos B - \sin A \sin B$
2. $\sin(A + B) = \sin A \cos B + \cos A \sin B$

6. Scalar projection



The **scalar projection** of a vector **a** on **b** i.e. the resolved part or the component of **a** in the direction of **b**, is given by:

$$OP = |\mathbf{a}|\cos\theta$$
$$= \frac{\mathbf{a.b}}{|\mathbf{b}|} = \mathbf{a}.\ \hat{\mathbf{b}} = \mathbf{a}.(\text{unit vector of } \mathbf{b})$$

The vector component of a vector **a** along **u** is given by $(\mathbf{a}.\hat{\mathbf{u}})\hat{\mathbf{u}}$

Illustrative examples

Example 1

Given the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} + \mathbf{j}$, find;

i) the acute angle θ between **a** and **b**

ii) the projection of the vector **a** on the vector **b**.

Solution

i) From the definition of dot product, $\mathbf{a.b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$, where θ is the angle between **a** and **b**

$\mathbf{a.b} = (\mathbf{i} + 2\mathbf{j}).(3\mathbf{i} + \mathbf{j}) = 3 + 2 = 5$

$|\mathbf{a}| = \sqrt{1^2 + 2^2} = \sqrt{5}$ and

$|\mathbf{b}| = \sqrt{3^2 + 1^2} = \sqrt{10}$

$$\cos\theta = \frac{5}{\sqrt{5}.\sqrt{10}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45°$$

ii) The projection of the vector **a** on the vector **b**

$$\text{vector } \mathbf{b} = \frac{\mathbf{a.b}}{|\mathbf{b}|} = \frac{5}{\sqrt{10}} = \frac{\sqrt{10}}{2}$$

Example 2

Let $\mathbf{r} = 7\mathbf{i} + 3\mathbf{j}$ and the vector $\mathbf{u} = 4\mathbf{i} - \mathbf{j}$. Calculate the vector component of **r** along **u**.

Solution

The vector component of **r** along $\mathbf{u} = (\mathbf{r}.\hat{\mathbf{u}})\hat{\mathbf{u}}$

Unit vector of $\mathbf{u} = \hat{\mathbf{u}} = \frac{1}{\sqrt{17}}(4\mathbf{i} - \mathbf{j})$

$(\mathbf{r}.\hat{\mathbf{u}}) = (7\mathbf{i} + 3\mathbf{j}).(\frac{1}{\sqrt{17}}(4\mathbf{i} - \mathbf{j}) = \frac{25}{\sqrt{17}}$

The vector component of **r** along **u**

$= (\mathbf{r}.\hat{\mathbf{u}})\hat{\mathbf{u}} = \frac{25}{\sqrt{17}}[\frac{1}{\sqrt{17}}(4\mathbf{i} - \mathbf{j})] = \frac{25}{17}(4\mathbf{i} - \mathbf{j})$.

Direction vectors

A vector which is used to specify the direction of another vector can be called a **direction vector**. For example, if we are told that a vector **u** of magnitude 10 units is in the direction of the vector $3\mathbf{i} + 4\mathbf{j}$ then $3\mathbf{i} + 4\mathbf{j}$ is the direction vector of **u**.

The unit direction vector, $\hat{\mathbf{u}} = \frac{1}{5}(3\mathbf{i} + 4\mathbf{j})$.

Hence $\mathbf{u} = |\mathbf{u}| \times (\text{unit direction vector})$

$= 10 \times \frac{1}{5}(3\mathbf{i} + 4\mathbf{j}) = 6\mathbf{i} + 8\mathbf{j}$

Finding the vector when given its magnitude and direction vector

From the relation $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$, we can express any vector as a product of its magnitude and its unit vector. Thus from $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$

$\mathbf{a} = |\mathbf{a}| \times \hat{\mathbf{a}}$. Therefore when we know the magnitude of a vector **a** and the vector **u** is the direction vector, then the vector **a** is given by:

$$\mathbf{a} = |\mathbf{a}| \times \hat{\mathbf{u}}$$

where $\hat{\mathbf{u}}$ is the unit vector of **u**.

Illustrative examples

Example 1

If a vector has magnitude 10 units and is in the direction of the vector $4\mathbf{i} - 3\mathbf{j}$, find the vector.

Solution

The unit vector of the direction vector

$= \frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$

$\therefore$ The vector = magnitude × unit vector of direction vector $= 10 \times \frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$

$$= 2(4\mathbf{i} - 3\mathbf{j}) = 8\mathbf{i} - 6\mathbf{j}$$

Example 2

Given that $\mathbf{p} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -12 \\ 5 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, find a vector **s** such that;

i) $|\mathbf{s}| = 35$ and **s** is in the direction of $\mathbf{q} + 5\mathbf{r}$

ii) **s** is parallel to $\mathbf{p} - \mathbf{q}$ and is half the size of $\mathbf{p} - \mathbf{q}$.

Solution

i) Direction vector

$= \mathbf{q} + 5\mathbf{r}$

$$= \begin{pmatrix} -12 \\ 5 \end{pmatrix} + 5\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -12 \\ 5 \end{pmatrix} + \begin{pmatrix} 5 \\ -5 \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \end{pmatrix}$$

Magnitude of direction vector

$= \sqrt{(-7)^2 + 0^2}$

$= \sqrt{49} = 7$

Unit vector of direction vector $= \frac{1}{7}\begin{pmatrix} -7 \\ 0 \end{pmatrix}$ and

$|\mathbf{s}| = 35$

Therefore the vector $\mathbf{s} = |\mathbf{s}| \times$ unit vector in the direction of $\mathbf{s} = 35 \times \dfrac{1}{7}\begin{pmatrix} -7 \\ 0 \end{pmatrix}$

$$= 5\begin{pmatrix} -7 \\ 0 \end{pmatrix}$$

$$\mathbf{s} = \begin{pmatrix} -35 \\ 0 \end{pmatrix}$$

ii) Direction vector $= \mathbf{p} - \mathbf{q}$

$$= \begin{pmatrix} 4 \\ -3 \end{pmatrix} - \begin{pmatrix} -12 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 16 \\ -8 \end{pmatrix}$$

Magnitude of direction vector,

$|\mathbf{p} - \mathbf{q}| = \sqrt{16^2 + (-8)^2}$

$$= \sqrt{256 + 64} = \sqrt{320} = 8\sqrt{5}$$

Unit vector of direction vector $= \frac{1}{8\sqrt{5}}\begin{pmatrix} 16 \\ -8 \end{pmatrix}$

and $|\mathbf{s}| = \frac{1}{2}|\mathbf{p} - \mathbf{q}| = \frac{1}{2}(8\sqrt{5}) = 4\sqrt{5}$.

Therefore the vector $\mathbf{s} = |\mathbf{s}| \times$ unit vector parallel to $\mathbf{s} = 4\sqrt{5} \times \frac{1}{8\sqrt{5}}\begin{pmatrix} 16 \\ -8 \end{pmatrix}$

$$= \tfrac{1}{2}\begin{pmatrix} 16 \\ -8 \end{pmatrix}$$

$$\mathbf{s} = \begin{pmatrix} 8 \\ -4 \end{pmatrix}$$

Review exercise 24

1. $A(-1, 3)$, $B(2, 1)$ and $C(5, 3)$ are the vertices of ABC. Express in column notation, the unit vectors parallel to $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Use the dot product to calculate $<BAC$, correct to the nearest degree.

2. The points $A(-2, 1)$, $B(2, 1)$ and $C(1,4)$ and $D(p, q)$ are the vertices of a parallelogram $ABCD$.
   i) Write down $\overrightarrow{AB}$, $\overrightarrow{DC}$ and $\overrightarrow{BD}$ as column vectors and deduce the values of p and q.
   ii) Use the dot product to calculate $<CBD$, correct to the nearest degree.
   iii) Calculate the area of parallelogram $ABCD$ correct to one decimal place.

3. $P(-2, 3)$, $Q(10, 4)$, $R(3, 12)$ and $S(4, 0)$ are four points in the xy – plane. Use the scalar (dot) product of vectors to:
   i) Show that $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{RS}$.
   ii) Find, correct to the nearest degree, the measure of $<PSQ$.

4. Using the scalar (dot) product, find the angle between the vectors $\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$.

5. The resultant of the vectors $(2a\mathbf{i} + b\mathbf{j})$, $(a\mathbf{i} + 2b\mathbf{j})$ and $2b\mathbf{j}$ is $(3\mathbf{i} + 4\mathbf{j})$. Find the values of a and b.

6. If the vectors $x\mathbf{i} + y\mathbf{j}$ and $3\mathbf{i} - \mathbf{j}$ are perpendicular, find y when $x = 2$.

7. Find the magnitude and direction of the vector $8\mathbf{i} - 15\mathbf{j}$.

8. Given that $\mathbf{a} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.
   Evaluate the scalar product of **a** and **b** and hence find the cosine of the angle between the direction of **a** and the direction of **b**.

9. The points $A(3, 4)$, $B(1, 1)$ and $C(6, 2)$ and $D(x, y)$ are the vertices of a parallelogram $ABCD$.
   i) Write down $\overrightarrow{AB}$, $\overrightarrow{DC}$ and $\overrightarrow{BD}$ as column vectors and deduce the values of x and y.
   ii) Use the dot product to calculate $<CBD$, correct to the nearest degree.
   iii) Calculate the area of parallelogram $ABCD$ correct to one decimal place.

10. Using the scalar (dot) product, find, correct to the nearest degree the angle between the vectors
    $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 8\mathbf{i} + 9\mathbf{j}$.

11. Let $\mathbf{r} = 7\mathbf{i} + 3\mathbf{j}$ and $\mathbf{s} = 4\mathbf{i} - \mathbf{j}$. Calculate the vector component of **r** along **s**.

12. Find the angle between the vectors
    $\text{P} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\text{q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

13. Given that $\mathbf{x} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$,
    find the dot product **x.y**.

14. Given that $\mathbf{k} = (4, 030°)$, $\mathbf{i} = (1, 180°)$, and $\mathbf{m} = (3, 300°)$, find the scalar (dot) product $\mathbf{k}.(\mathbf{i} + \mathbf{m})$.

15. $P(3, 2)$, $Q(-1, 5)$ and $R(2, 1)$ are the vertices of a triangle.
  i) Find the lengths of the sides of the triangle.
  ii) Use the scalar product to find the angles of the triangle.

16. If $\mathbf{p} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{q} = k\mathbf{i} - 6\mathbf{j}$ are perpendicular, determine the value of the constant k.

17. PQR is a triangle. $|PQ| = 2\,\text{cm}$, $|QR| = 4\,\text{cm}$ and $|PR| = 3$ cm. Calculate, correct to **three** decimal places, the cosine of the **smallest** angle.

18 Given that $\mathbf{p} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$, find the angle between **p** and **q**.

19. Let $\mathbf{a} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$. Given that $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$ and that **c** is perpendicular to **a**, find the ratio of λ to μ.

20. Use the dot product to obtain
  i) the cosine rule
  ii) sine rule
  iii) the compound angle formulae for sine and cosine

CHAPTER 25

STATICS

Scalar and vector quantities

Some physical quantities have *magnitude only* and are called **scalar quantities** or **scalars**. Examples of scalars include time, speed, distance, mass, area, temperature, energy and volume.

Scalar quantities can be manipulated by the laws of arithmetic applicable to natural numbers. Many physical quantities can be added together in the same way as natural numbers. For example, if we first put 100 cm^3 of water into a cup and then put in an additional 150 cm^3, the cup will contain 250 cm^3 of water. Similarly, if you were to run around a square field having a side length 100 m, you would have run a total distance of 400 m.

Other physical quantities have *both magnitude* and *direction* and these are called **vector quantities** or **vectors**. Examples of vectors include displacement, acceleration, momentum, velocity, force, electric field and moment. For example, a speed of 20 ms^{-1} in the direction of 030° is the velocity vector (20 ms^{-1}, 030°). Similarly, a distance of 15 m in the direction of 060° is the displacement vector (15 m, 060°). Also, (10 N, 090°) is a force vector.

Vector quantities cannot be added in the simple way described for scalars.

Special arithmetic rules must be obeyed when adding vectors together.

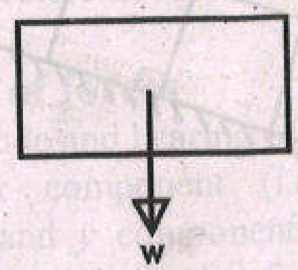
The units of these quantities shall be discussed at the appropriate sections.

Types of forces

Forces occur in mechanics in various ways. Some of the most common are described below.

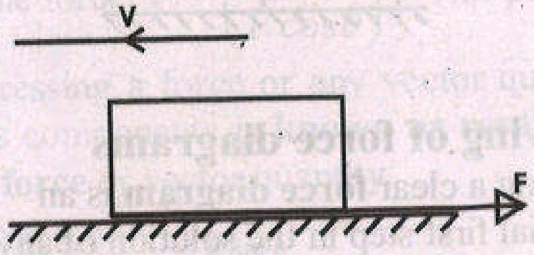
Weight *W*

The weight of a body is the force with which the earth attracts the body. It acts at the body's *centre of gravity* and it *always acts vertically downwards.*



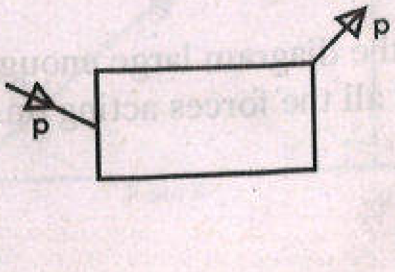
Friction *F*

Friction is a force which acts on a body in contact with a *rough surface*. It acts tangentially between the two surfaces and in a direction to oppose or resist the motion of the body. **Note:** *Smooth surfaces* are considered to be *frictionless.*

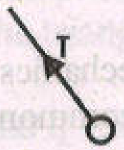


Push and pull *P*

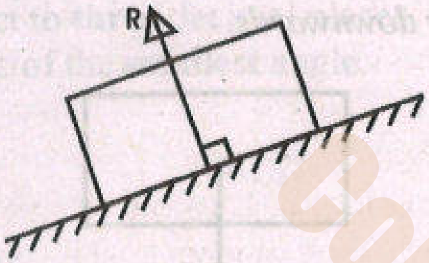
Pushes and pulls are forces which act on a body at the points where they are applied.



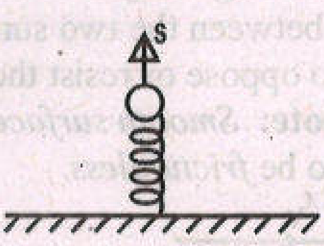
Tension T
The tension in a string is a force which acts on a body to which the string is attached. Tensions can also come from springs, rods etc. They act in a direction *away from the body*.



Normal reaction R
A normal reaction is a force which acts on a body in contact with a surface. It acts in a direction at *right angles* to the surfaces in contact.



Thrust S
The thrust from a spring or rod is similar to tension but acts in opposite direction.



Drawing of force diagrams

Drawing a clear **force diagram** is an essential first step in the solution of any problem in mechanics which is concerned with the action of forces on a body. The following are important points to remember when drawing such a force diagram.

1. Make the diagram large enough to show clearly all the forces acting on the body and to enable any necessary geometry and trigonometry to be done.

2. Show only forces which are *acting on the body* being considered. A common fault is to include forces which the body is applying to its surroundings (i.e. other bodies).

3. Weight always acts on a body unless the body is described as **light**.

4. Contact with another body or surface gives rise to a normal reaction and sometimes friction.

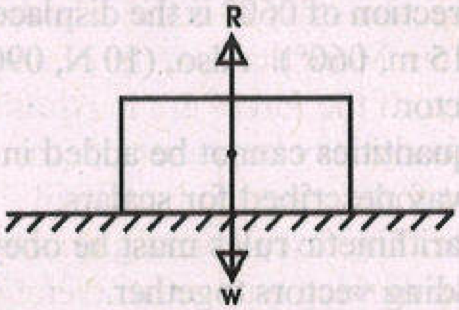
5. Attachment to another object (by string, spring, hinge, etc.) gives rise to a force on the body at the point of attachment.

6. Forces acting on a **particle** act at the same point. Forces acting on other bodies may act at different points

7. Check that no forces have been omitted or included more than once.

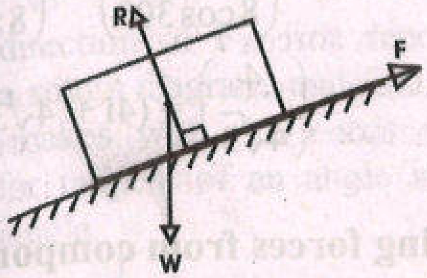
The following simple force diagrams illustrate the above.

1. *Forces acting on a block at rest on a smooth horizontal plane*



W – weight (vertically downwards)
R – normal reaction (at *right angles* to the surfaces in contact)

2. *Forces acting on a block at rest on a rough inclined plane*

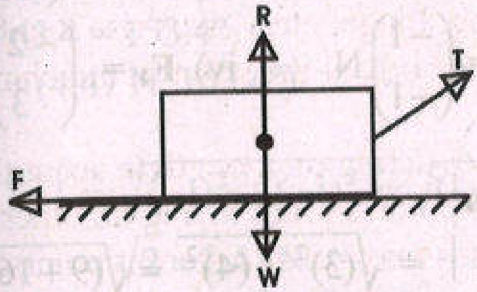


W – weight (vertically downwards)

R – normal reaction (at *right angles* to the surfaces in contact)

F – friction (acting to resist motion)

3. *Forces acting on a block being pulled (by a string) on a rough horizontal plane*



W – weight (vertically downwards)

R – normal reaction (at *right angles* to the surfaces in contact)

F – friction (acting to resist motion)

T – tension in string (acting away from the body)

Forces as vectors

A **force** is necessary to cause a body to **accelerate**. More than one force may act on a body. If the forces are in **equilibrium**, then the body may be at rest. If there is a **resultant** force, then the body will accelerate. The force F acting on a body of mass m moving with an acceleration a is given by the relation:

Force = mass × acceleration i.e. $\mathbf{F} = m\mathbf{a}$

Note that mass is in kilograms (kg) and acceleration is in ms^{-2}. Force is a **vector** quantity and its SI unit is newton (N). When there is a system of forces acting on the particle, **F** is the **resultant force** on the body.

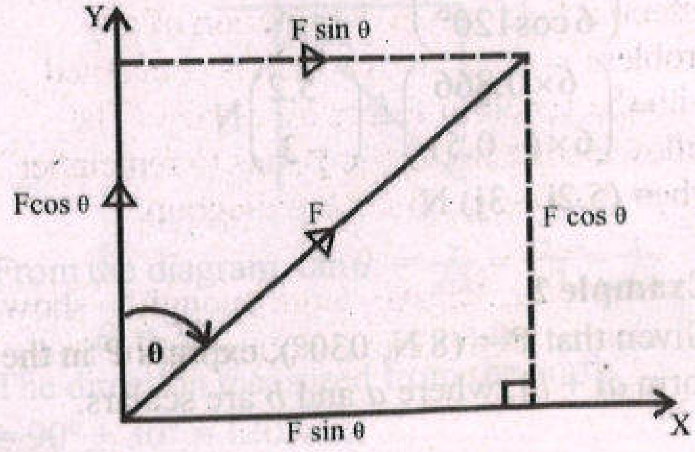
Representation of forces

Forces acting at a point may be represented by their magnitude and direction from the north (bearing) i.e. $\mathbf{F} = (F\,\text{N},\ \theta°)$, where F is the magnitude of the force and θ is the direction. When forces are given in this form it can be expressed in the **column components** form, i.e. $\begin{pmatrix} a \\ b \end{pmatrix}$ or the **Cartesian coordinate** form i.e. $x\mathbf{i} + y\mathbf{j}$, where **i** and **j** are the unit vectors in the x and y directions.

Resolution of forces

If the magnitude and bearing of any force are given, its x component (i.e. horizontal component) and y component (i.e. vertical component) can easily be found. In the figure below, if F is the magnitude of the force **F** and θ is the direction or bearing of the force [i.e. $\mathbf{F} = (F, \theta)$], then the *horizontal* component of the force is $F\sin\theta$ and the *vertical* component is $F\cos\theta$.

Thus the force, $\mathbf{F} = \begin{pmatrix} F\sin\theta \\ F\cos\theta \end{pmatrix}$. This process of expressing a force or any vector quantity into its components is **known as resolution of the force** or vector quantity.



Illustrative examples

Example 1

Express the following as column vectors

i) $\mathbf{F}_1 = (10\text{ N}, 030°)$

ii) $\mathbf{F}_2 = (20\text{ N}, 240°)$

iii) $\mathbf{F}_3 = (12\text{ N}, 330°)$

iv) $\mathbf{F}_4 = (6\text{ N}, 120°)$

Solution

i) $\mathbf{F}_1 = (10\text{ N}, 030°)$

$$\mathbf{F}_1 = \begin{pmatrix}10\sin 30°\\ 10\cos 30°\end{pmatrix} = \begin{pmatrix}10\times 0.5\\ 10\times 0.866\end{pmatrix} = \begin{pmatrix}5\\ 8.67\end{pmatrix}\text{N}$$

$= (5\mathbf{i} + 8.67\mathbf{j})$ N

ii) $\mathbf{F}_2 = (20\text{ N}, 240°)$

$$\mathbf{F}_2 = \begin{pmatrix}20\sin 240°\\ 20\cos 240°\end{pmatrix}$$

$$= \begin{pmatrix}20\times(-0.866)\\ 20\times(-0.5)\end{pmatrix} = \begin{pmatrix}-17.32\\ -10\end{pmatrix}\text{N}$$

$= (-17.32\mathbf{i} - 10\mathbf{j})$ N

iii) $\mathbf{F}_3 = (12\text{ N}, 330°)$

$$\mathbf{F}_3 = \begin{pmatrix}12\sin 330°\\ 12\cos 330°\end{pmatrix}$$

$$= \begin{pmatrix}12\times(-0.5)\\ 12\times 0.866\end{pmatrix} = \begin{pmatrix}-6\\ 10.40\end{pmatrix}\text{N}$$

$= (-6\mathbf{i} + 10.40\mathbf{j})$ N

iv) $\mathbf{F}_4 = (6\text{ N}, 120°)$

$$\mathbf{F}_4 = \begin{pmatrix}6\sin 120°\\ 6\cos 120°\end{pmatrix}$$

$$= \begin{pmatrix}6\times 0.866\\ 6\times(-0.5)\end{pmatrix} = \begin{pmatrix}5.2\\ -3\end{pmatrix}\text{N}$$

$= (5.2\mathbf{i} - 3\mathbf{j})$ N

Example 2

Given that $\mathbf{P} = (8\text{ N}, 030°)$, express $\mathbf{P}$ in the form $a\mathbf{i} + b\mathbf{j}$, where a and b are scalars.

Solution

$$\mathbf{P} = (8\text{ N}, 030°) = \begin{pmatrix}8\sin 30°\\ 8\cos 30°\end{pmatrix} = \begin{pmatrix}8\times\frac{1}{2}\\ 8\times\frac{\sqrt{3}}{2}\end{pmatrix}$$

$$= \begin{pmatrix}4\\ 4\sqrt{3}\end{pmatrix} = (4\mathbf{i} + 4\sqrt{3}\,\mathbf{j})\text{ N}$$

Expressing forces from component form to magnitude–bearing form

Illustrative examples

Example 1

Find the magnitudes and directions (bearings) of the following forces.

i) $\mathbf{F}_1 = \begin{pmatrix}3\\ 4\end{pmatrix}\text{N}$ ii) $\mathbf{F}_2 = \begin{pmatrix}2\\ -4\end{pmatrix}\text{N}$

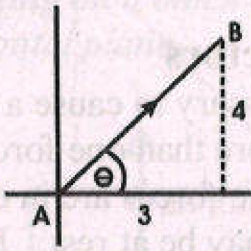
iii) $\mathbf{F}_3 = \begin{pmatrix}-1\\ -1\end{pmatrix}\text{N}$ iv) $\mathbf{F}_4 = \begin{pmatrix}-2\\ 3\end{pmatrix}\text{N}$

Solution

i) $|\,\mathbf{F}_1\,| = \sqrt{(3)^2 + (4)^2} = \sqrt{(9+16)}$

$= \sqrt{25} = 5$ N

For the direction of $\mathbf{F}_1$, first represent the force on a rough diagram, and find the acute angle it makes with the x-axis using the relation for tangent of an angle in a right-angled triangle.



From the diagram, $\tan\theta = \frac{4}{3} = 1.33$

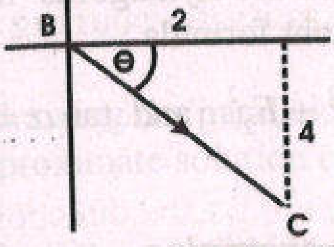
$\theta = 53°$

The direction of $\mathbf{F}_1$ measured from the north $= 90° - 53° = 37°$

$\therefore\ \mathbf{F}_1 = (5\text{ N}, 37°)$

ii) $|\mathbf{F}_2| = \sqrt{(2)^2 + (-4)^2}$
$= \sqrt{(4+16)} = \sqrt{20} = 4.47$ N

For the direction of $\mathbf{F}_2$, first represent the force on a rough diagram, and find the acute angle it makes with the *x*-axis using the relation for tangent of an angle in a right-angled triangle.



From the diagram, $\tan\theta = \frac{4}{2} = 2$

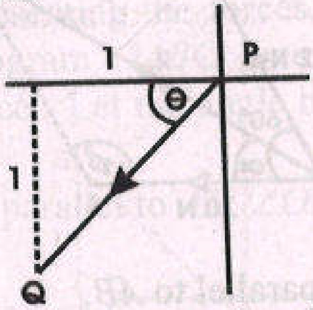
$\theta = 63.4°$

The direction of $\mathbf{F}_2$ measured from the north
$= 90° + 63.4 = 153.4°$

$\therefore \quad \mathbf{F}_2 = (4.47 \text{ N}, 153.4°)$

iii) $|\mathbf{F}_3| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{(1+1)}$
$= \sqrt{2} = 1.41$ N

For the direction of $\mathbf{F}_3$, first represent the force on a rough diagram, and find the acute angle it makes with the *x*-axis using the relation for tangent of an angle in a right-angled triangle.



From the diagram, $\tan\theta = \frac{1}{1} = 1$

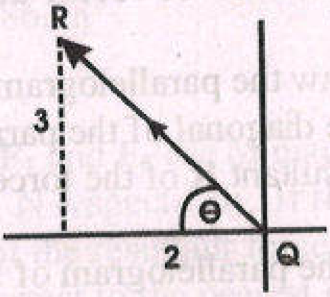
$\theta = 45°$

The direction of $\mathbf{F}_3$ measured from the north
$= 270° - 45° = 225°$

$\therefore \quad \mathbf{F}_3 = (1.41 \text{ N}, 225°)$

iv) $|\mathbf{F}_4| = \sqrt{(-2)^2 + (3)^2} = \sqrt{(4+9)}$
$= \sqrt{13} = 3.6$ N

For the direction of $\mathbf{F}_4$, first represent the force on a rough diagram, and find the acute angle it makes with the *x*-axis using the relation for tangent of an angle in a right-angled triangle.



From the diagram, $\tan\theta = \frac{3}{2} = 1.5$

$\theta = 56°$ to the nearest degree

The direction of $\mathbf{F}_4$ measured from the north
$= 270° + 56° = 326°$

$\therefore \quad \mathbf{F}_4 = (3.6 \text{ N}, 326°)$

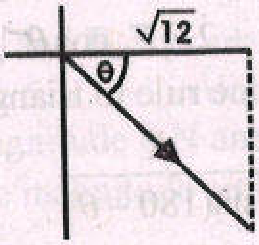
Example 2

Find the magnitude and direction of the force $[(\sqrt{12})\,\mathbf{i} - 2\mathbf{j}]$ N.

Solution

Magnitude $= \sqrt{(\sqrt{12})^2 + (-2)^2}$ N
$= \sqrt{12+4} = \sqrt{16} = 4$

For the direction, first represent the force on a rough diagram, and find the acute angle it makes with the *x*-axis.



From the diagram, $\tan\theta = \frac{2}{\sqrt{12}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\theta = 30°$

The direction measured from the north
$= 90° + 30° = 120°$.

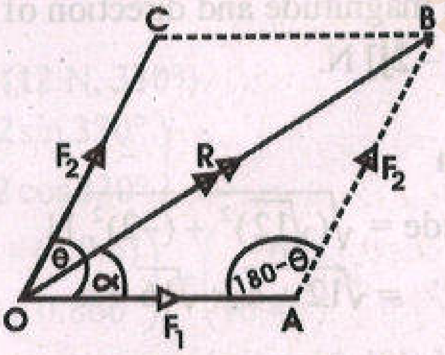
Resultant of forces

1. Resultant of two concurrent forces

If two concurrent forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act at a point O and the angle between them is θ, we can find their **resultant** using the **parallelogram law of vector addition**.

Step 1: Draw the parallelogram of forces $OABC$. The diagonal of the parallelogram gives the resultant $\mathbf{R}$ of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$.

Step 2: If the parallelogram of forces is drawn to scale, then the resultant can be obtained by direct measurement of the diagonal, OB.
However if the parallelogram of forces is not drawn to scale then the resultant can be obtained by calculation using the sine and cosine rules. Since OC is parallel to AB, $\angle OAB = 180 - \theta$



Applying the cosine rule to triangle OAB gives: $R^2 = F_1^{\ 2} + F_2^{\ 2} - 2F_1F_2\cos(180-\theta)$

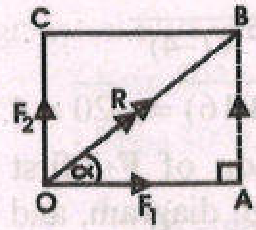
$= F_1^{\ 2} + F_2^{\ 2} + 2F_1F_2\cos\theta$

Applying the sine rule to triangle OAB

gives: $\dfrac{F_2}{\sin\alpha} = \dfrac{R}{\sin(180-\theta)}$

$$\sin\alpha = \frac{F_2\sin(180-\theta)}{R} = \frac{F_2\sin\theta}{R}$$

where α is the angle between $\mathbf{R}$ and the $\mathbf{F}_1$.



In the special case where the angle between the two forces is 90° as shown above, we can use Pythagoras' theorem and the tangent formula.

$$R^2 = F_1^{\ 2} + F_2^{\ 2} \quad \text{and} \quad \tan\alpha = \frac{F_2}{F_1}$$

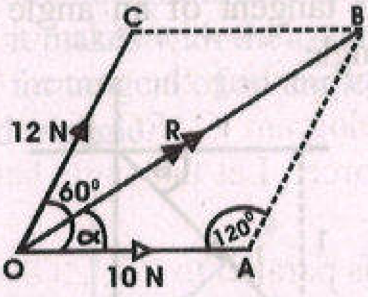
Illustrative examples

Example 1

Two forces of magnitudes 10 N and 12 N act on a particle. If the angle between them is 60°, find the magnitude and direction of their resultant.

Solution

In the figure below, OA and OC are the vectors representing the forces. Completing the parallelogram $OABC$ gives OB as the resultant force.



Since OC is parallel to AB,
$\angle OAB = 180° - 60° = 120°$
Applying the cosine rule to triangle OAB gives:

$$R^2 = 10^2 + 12^2 - 2(10)(12)\cos 120°$$
$$= 100 + 144 + 2(10)(12)\cos 60°$$
$$= 244 + 240 \times \tfrac{1}{2} = 364$$
$$\therefore \quad R = \sqrt{364} = 19.08 \text{ N}$$

Applying the sine rule to triangle *OAB* gives:

$$\frac{12}{\sin\alpha}=\frac{19.08}{\sin 120^\circ}$$

$$\sin\alpha=\frac{12\times\sin 120^\circ}{19.08}=0.5447 \text{ and } \alpha=33^\circ$$

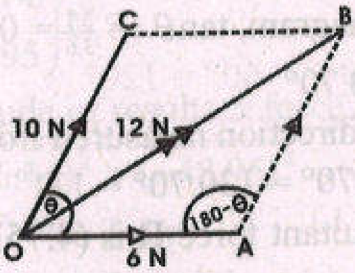
The resultant force is 19.08 N acting at an angle of 33° with the force of magnitude 10 N.

Note: By drawing the parallelogram to scale, an approximate solution can be obtained.
Using a scale of 1 cm to 2 N, draw $OA = 5$ cm and $OC = 6$ cm, making $\angle COA = 60^\circ$. Then complete the parallelogram *OABC* and measure the diagonal *OB* and $\angle BOA$. Work it out and compare your answer with the one above.

Example 2
Two forces of magnitudes 6 N and 10 N act on a particle. Their resultant is of magnitude 12 N. Find the angle between them.

Solution
In the figure below, *OA* and *OC* are the vectors representing the forces. Completing the parallelogram *OABC* gives *OB* as the resultant force. Let the angle between them be θ

Since *OC* is parallel to *AB*, $\angle OAB = 180^\circ - \theta$



Applying the cosine rule to triangle *OAB* gives:

$$12^2=6^2+10^2-2(6)(10)\cos(180^\circ-\theta)$$

$$144=36+100+120\cos\theta$$

$$120\cos\theta=144-136=8$$

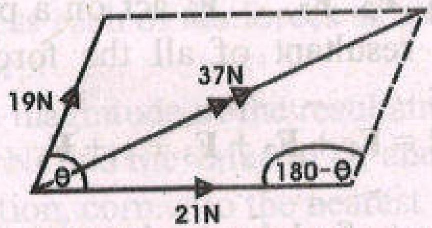
$$\cos\theta=\frac{8}{120}=0.0667$$

$$\theta=86.18^\circ.$$

Therefore the angle between the forces 6 N and 10 N is 86.18°.

Example 3
Two forces $\mathbf{F_1}$ and $\mathbf{F_2}$ are of magnitudes 19 N and 21 N respectively. If the magnitude of the resultant force is 37 N, calculate, correct to the nearest degree, the angle between $\mathbf{F_1}$ and $\mathbf{F_2}$.

Solution



Let the angle between $\mathbf{F_1}$ and $\mathbf{F_2}$ be θ
Using the cosine rule,

$$37^2=19^2+21^2-2(19)(21)\cos(180-\theta)$$

$$37^2=19^2+21^2+2(19)(21)\cos\theta$$

$$1369=361+441+798\cos\theta$$

$$798\cos\theta=567$$

$$\theta=\cos^{-1}\left(\tfrac{567}{798}\right)=44.7^\circ$$

$\therefore$ The angle between $\mathbf{F_1}$ and $\mathbf{F_2}$ to the nearest degree is 45°.

Example 4
Two forces $\mathbf{F_1}$ and $\mathbf{F_2}$ act on a particle. $\mathbf{F_1}$ has magnitude 5 N and in direction 030°, and $\mathbf{F_2}$ has magnitude 8 N and in direction 090°. Find the magnitude and direction of their resultant.

Solution
Find the resultant of the two forces by calculation and drawing. Resultant, $R = 11.36$ N and direction of R is 067.6°.

Example 5

Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on a particle. $\mathbf{F}_1$ has magnitude 6 N and in direction 045°, and $\mathbf{F}_2$ has magnitude 7 N and in direction 154°. Find the magnitude and direction of their resultant.

Solution

Resultant, $R = 7.59$ N and direction of R is 105.6°

2. Resultant of a system of coplanar forces acting at a point

The resultant of a system of coplanar forces acting at a point (concurrent forces) is equal to the **vector sum** of all the forces. If the forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots, \mathbf{F}_n$ act on a particle P, then the resultant of all the forces $\mathbf{R}$, is given by:

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots + \mathbf{F}_n$$

Therefore, to find the resultant of a system of forces, first resolve the forces into **column vectors** or the **Cartesian coordinate** form if they are given in the form (F N, θ°), where F is the magnitude of the force and θ is the direction.

Example 1

A body of mass 500 g lying on a horizontal floor is acted upon by the forces

$\mathbf{F}_1$ (10N, 090°), $\mathbf{F}_2$ (16N, 180°),
$\mathbf{F}_3$ (8N, 030°) and $\mathbf{F}_4$ (12N, 300°)

a) Express all the forces acting on the body as column vectors.

b) Find the magnitude and direction of the resultant force.

c) Find the magnitude and direction of the resulting acceleration.

Solution

a) Resolving the forces as column vectors, we have

$$\mathbf{F}_1 = \begin{pmatrix} 10\sin 90^\circ \\ 10\cos 90^\circ \end{pmatrix} = \begin{pmatrix} 10\times 1 \\ 10\times 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} \text{N}$$

$$\mathbf{F}_2 = \begin{pmatrix} 16\sin 180^\circ \\ 16\cos 180^\circ \end{pmatrix} = \begin{pmatrix} 16\times 0 \\ -16\times 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -16 \end{pmatrix} \text{N}$$

$$\mathbf{F}_3 = \begin{pmatrix} 8\sin 30^\circ \\ 8\cos 30^\circ \end{pmatrix} = \begin{pmatrix} 4 \\ 6.9 \end{pmatrix} \text{N}$$

$$\mathbf{F}_4 = \begin{pmatrix} 12\sin 300^\circ \\ 12\cos 300^\circ \end{pmatrix} = \begin{pmatrix} -10.4 \\ 6 \end{pmatrix} \text{N}$$

b) The resultant of the forces $\mathbf{R}$ is given by $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4$

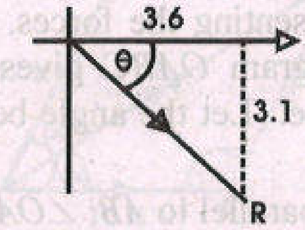
$$\mathbf{R} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -16 \end{pmatrix} + \begin{pmatrix} 4 \\ 6.9 \end{pmatrix} + \begin{pmatrix} -10.4 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 3.6 \\ -3.1 \end{pmatrix} \text{N}$$

The magnitude of the resultant force,

$$|\mathbf{R}| = \sqrt{(3.6)^2 + (-3.1)^2} = \sqrt{22.57} = 4.75\text{N}$$

To find the direction of $\mathbf{R}$, first represent the resultant force $\mathbf{R}$ on a rough diagram and calculate the acute angle θ, it makes with the x-axis.



From the diagram, $\tan\theta = \frac{3.1}{3.6} = 0.8611$

$\theta = 40.70°$

The actual direction measured from the north $= 90° + 40.70° = 130.70° = 131°$

$\therefore$ The resultant force $\mathbf{R}$ is (4.75N, 131°).

c) Resultant force = mass × acceleration, i.e. $\mathbf{R} = m\mathbf{a}$

$$|\mathbf{a}| = \frac{|R|}{m} = \frac{4.75}{0.5} = 9.5 \text{ m/s}^2$$

(Note that 500 g = 0.5 kg).

$\therefore$ The acceleration is $(9.5 \text{ m/s}^2, 131^\circ)$.

Note that the acceleration acts in the same direction as the force.

Example 2

A body is acted on by forces $\mathbf{F}_1 = (16 \text{ N}, 180^\circ)$, $\mathbf{F}_2 = (50 \text{ N}, 040^\circ)$ and $\mathbf{F}_3 = (28 \text{ N}, 150^\circ)$. Calculate the magnitude (correct to one decimal place) and the direction (correct to the nearest degree) of the resultant of the three forces.

Solution

Expressing the forces into column vectors, we have

$$\mathbf{F}_1 = \begin{pmatrix} 16\sin 180^\circ \\ 16\cos 180^\circ \end{pmatrix} = \begin{pmatrix} 0 \\ -16 \end{pmatrix} \text{N},$$

$$\mathbf{F}_2 = \begin{pmatrix} 50\sin 40^\circ \\ 50\cos 40^\circ \end{pmatrix} = \begin{pmatrix} 32.14 \\ 38.30 \end{pmatrix} \text{N}$$

$$\mathbf{F}_3 = \begin{pmatrix} 28\sin 150^\circ \\ 28\cos 150^\circ \end{pmatrix} = \begin{pmatrix} 14 \\ -24.25 \end{pmatrix} \text{N}$$

The resultant force, $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$

$$\mathbf{R} = \begin{pmatrix} 0 \\ -16 \end{pmatrix} + \begin{pmatrix} 32.14 \\ 38.30 \end{pmatrix} + \begin{pmatrix} 14 \\ -24.25 \end{pmatrix}$$

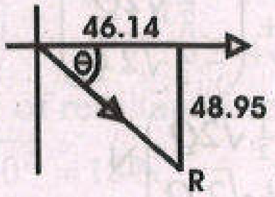
$$= \begin{pmatrix} 46.14 \\ -48.95 \end{pmatrix} \text{N}$$

The magnitude of resultant force,

$$|\mathbf{R}| = \sqrt{(46.14)^2 + (-48.95)^2} = \sqrt{4525}$$

$$= 67.3 \text{ N (to 1 d.p)}$$

To find the direction of **R**, first represent the resultant force **R** on a rough diagram and calculate the acute angle θ, it makes with the x-axis.



From the diagram, $\tan\theta = \frac{48.95}{46.14} = 1.0609$

$$\theta = 47^\circ$$

The actual direction measured from the north $= 90^\circ + 47^\circ = 137^\circ$

$\therefore$ The resultant force **R** is $(67.3\text{N}, 137^\circ)$.

Example 3

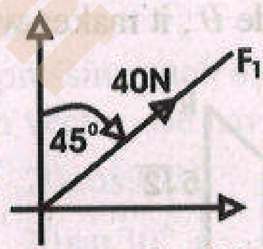
A body at rest is acted upon by two forces $\mathbf{F}_1 = (40 \text{ N}, 045^\circ)$ and $\mathbf{F}_2 = (Q \text{ N}, 135^\circ)$.

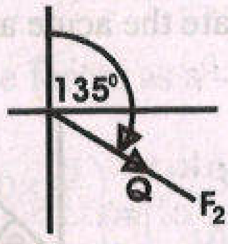
i) Express each of the forces as a column vector.

ii) If the magnitude of the resultant force is 50 N, find the value of Q, and the direction, correct to the nearest degree of the resultant force.

Solution

i)





Expressing each of the forces as a column vector, we have;

$$\mathbf{F}_1 = \begin{pmatrix} 40\sin 45^\circ \\ 40\cos 45^\circ \end{pmatrix}$$

$$= \begin{pmatrix} 40 \times \frac{1}{2}\sqrt{2} \\ 40 \times \frac{1}{2}\sqrt{2} \end{pmatrix} = \begin{pmatrix} 20\sqrt{2} \\ 20\sqrt{2} \end{pmatrix} \text{N}$$

$$\mathbf{F}_2 = \begin{pmatrix} Q\sin 135^\circ \\ Q\cos 135^\circ \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2}Q \\ -\frac{1}{2}\sqrt{2}\,Q \end{pmatrix} \text{N}$$

ii) The resultant force, $\mathbf{R}=\mathbf{F}_1+\mathbf{F}_2$

$$\mathbf{R}=\begin{pmatrix}20\sqrt{2}\\20\sqrt{2}\end{pmatrix}+\begin{pmatrix}\frac{1}{2}\sqrt{2}Q\\-\frac{1}{2}\sqrt{2}\,Q\end{pmatrix}$$

$$=\begin{pmatrix}20\sqrt{2}+\frac{1}{2}\sqrt{2}Q\\20\sqrt{2}-\frac{1}{2}\sqrt{2}\,Q\end{pmatrix}\text{N}$$

The magnitude of **R**,

$$|\mathbf{R}|=\sqrt{(20\sqrt{2}+\tfrac{1}{2}\sqrt{2}Q)^2+(20\sqrt{2}-\tfrac{1}{2}\sqrt{2}Q)^2}$$

$$=50$$

$$(20\sqrt{2}+\tfrac{1}{2}\sqrt{2}Q)^2+(20\sqrt{2}-\tfrac{1}{2}\sqrt{2}Q)^2$$

$$=2500$$

{i.e. squaring both sides)

$$800+40Q+\tfrac{1}{2}Q^2+800-40Q+\tfrac{1}{2}Q^2$$

$$=2500$$

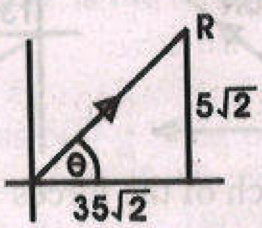
$$1600+Q^2=2500 \quad\Rightarrow\quad Q^2=900$$

$$Q=30$$

Substituting the value of Q into **R**, we have

$$\mathbf{R}=\begin{pmatrix}20\sqrt{2}+15\sqrt{2}\\20\sqrt{2}-15\sqrt{2}\end{pmatrix}=\begin{pmatrix}35\sqrt{2}\\5\sqrt{2}\end{pmatrix}\text{N}$$

To find the direction of **R**, first represent the resultant force **R** on a rough diagram and calculate the acute angle θ, it makes with the x-axis.



From the diagram, $\tan\theta=\frac{5\sqrt{2}}{35\sqrt{2}}=0.143$

$$\theta=8.14^\circ=8^\circ$$

The actual direction measured from the north $=90^\circ-8^\circ=82^\circ$, correct to the nearest degree.

$\therefore$ The resultant force **R** is (50 N, 082°).

Example 4

A vehicle which has broken down is pushed by three forces (220 N, 030°), (315 N, 090°) and (240 N, 120°). Calculate the magnitude correct to the nearest N and the direction correct to the nearest degree, of the resultant of the three forces.

Solution

Expressing each of the forces as a column vector, we have;

$\mathbf{F}_1=(220\text{ N}, 030^\circ)$

$$=\begin{pmatrix}220\sin 30^\circ\\220\cos 30^\circ\end{pmatrix}$$

$$=\begin{pmatrix}220\times 0.500\\220\times 0.866\end{pmatrix}=\begin{pmatrix}110\\190.52\end{pmatrix}\text{N}$$

$\mathbf{F}_2=(315\text{ N}, 090^\circ)$

$$=\begin{pmatrix}315\sin 90^\circ\\315\cos 90^\circ\end{pmatrix}$$

$$=\begin{pmatrix}315\times 1\\315\times 0\end{pmatrix}=\begin{pmatrix}315\\0\end{pmatrix}\text{N}$$

$\mathbf{F}_3=(240\text{ N}, 120^\circ)$

$$=\begin{pmatrix}240\sin 120^\circ\\240\cos 120^\circ\end{pmatrix}$$

$$=\begin{pmatrix}240\times 0.866\\240\times(-0.500)\end{pmatrix}=\begin{pmatrix}207.84\\-120\end{pmatrix}\text{N}$$

ii) The resultant force, $\mathbf{R}=\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3$

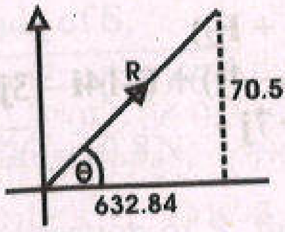
$$\mathbf{R}=\begin{pmatrix}110\\190.52\end{pmatrix}+\begin{pmatrix}315\\0\end{pmatrix}+\begin{pmatrix}207.84\\-120\end{pmatrix}$$

$$=\begin{pmatrix}632.84\\70.52\end{pmatrix}$$

The magnitude of resultant force,

$$|\mathbf{R}|=\sqrt{(632.84^2+(70.52)^2}$$

$$=637\text{ N to the nearest N.}$$



From the diagram above,

$$\tan\theta = \frac{70.52}{632.84} = 0.1114 \Rightarrow \theta = 6.36^\circ$$

The direction of **R** is

$(90 - 6.36)^\circ = 83.64^\circ$

$= 84^\circ$ to the nearest degree

∴ The resultant force **R** is (637 N, 084°).

Example 5

Three forces $\mathbf{F}_1 = (8\text{ N}, 030^\circ)$, $\mathbf{F}_2 = (10\text{ N}, 090^\circ)$ and $\mathbf{F}_3 = (8\text{ N}, 150^\circ)$

a) Express all the forces acting on the body as column vectors.

b) Find the magnitude and direction of the resultant force.

Solution

i) Expressing each of the forces as a column vector, we have;

$$\mathbf{F}_1 = (8\text{ N}, 030^\circ) = \begin{pmatrix} 8\sin 30^\circ \\ 8\cos 30^\circ \end{pmatrix} = \begin{pmatrix} 8\times\frac{1}{2} \\ 8\times\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 4 \\ 4\sqrt{3} \end{pmatrix}\text{N}$$

$$\mathbf{F}_2 = (10\text{ N}, 090^\circ) = \begin{pmatrix} 10\sin 90^\circ \\ 10\cos 90^\circ \end{pmatrix} = \begin{pmatrix} 10\times 1 \\ 10\times 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}\text{N}$$

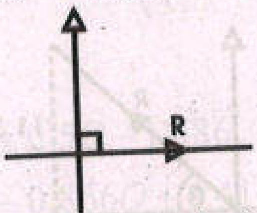
$$\mathbf{F}_3 = (8\text{ N}, 150^\circ) = \begin{pmatrix} 8\sin 150^\circ \\ 8\cos 150^\circ \end{pmatrix} = \begin{pmatrix} 8\times\frac{1}{2} \\ 8\times(-\frac{\sqrt{3}}{2}) \end{pmatrix} = \begin{pmatrix} 4 \\ -4\sqrt{3} \end{pmatrix}\text{N}$$

ii) The resultant force, $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$

$$\mathbf{R} = \begin{pmatrix} 4 \\ 4\sqrt{3} \end{pmatrix} + \begin{pmatrix} 10 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ -4\sqrt{3} \end{pmatrix} = \begin{pmatrix} 18 \\ 0 \end{pmatrix}$$

The magnitude of resultant force,

$$|\mathbf{R}| = \sqrt{18^2 + 0^2} = 18\text{ N}$$

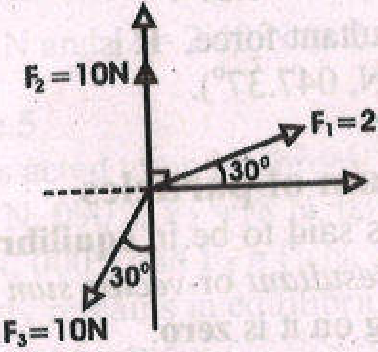


The direction of **R** is clearly 090°

∴ The resultant force **R** is (18 N, 090°).

Example 6

Find the resultant of the system of coplanar forces shown in the diagram below.



Solution

i) Expressing each of the forces as a column vector, we have;

$$\mathbf{F}_1 = \begin{pmatrix} 20\cos 30^\circ \\ 20\sin 30^\circ \end{pmatrix} = \begin{pmatrix} 20\times\frac{\sqrt{3}}{2} \\ 20\times\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 17.32 \\ 10 \end{pmatrix}\text{N}$$

$$\mathbf{F}_2 = \begin{pmatrix} 0 \\ 10 \end{pmatrix}\text{N}$$

$$\mathbf{F}_3 = \begin{pmatrix} -10\sin 30^\circ \\ -10\cos 30^\circ \end{pmatrix} = \begin{pmatrix} -10\times(\frac{1}{2}) \\ -10\times\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -5 \\ -8.66 \end{pmatrix}\text{N}$$

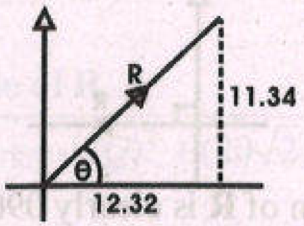
The resultant force, $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$

$$\mathbf{R} = \begin{pmatrix} 17.32 \\ 10 \end{pmatrix} + \begin{pmatrix} 0 \\ 10 \end{pmatrix} + \begin{pmatrix} -5 \\ -8.66 \end{pmatrix} = \begin{pmatrix} 12.32 \\ 11.34 \end{pmatrix}$$

The magnitude of resultant force,

$$|\mathbf{R}| = \sqrt{12.32^2 + 11.34^2} = 16.74\,\text{N}$$

To find the direction of **R**, first represent the resultant force **R** on a rough diagram and calculate the acute angle θ, it makes with the x-axis.



From the diagram, $\tan\theta = \frac{11.34}{12.32} = 0.9205$

$\theta = 42.63^\circ$

The actual direction measured from the north $= 90^\circ - 42.63^\circ = 47.37^\circ$.

$\therefore$ The resultant force, **R** is (16.74 N, 047.37°).

Equilibrium of particles

A particle is said to be in **equilibrium** or at **rest** if the *resultant* or *vector sum* of all the forces acting on it is **zero**.

i.e. $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots + \mathbf{F}_n = \mathbf{O}$

where $\mathbf{F}_1$, $\mathbf{F}_2$,... $\mathbf{F}_n$ are the forces acting on the body and **R** is the resultant force.

Note that the forces are first resolved into *components* or *column vectors* or the *Cartesian coordinate* form if they are given in form (F N, θ°), where F is the magnitude of the force and θ is the direction.

Illustrative examples

Example 1

Two forces $\mathbf{F}_1 = (3\mathbf{i} - 4\mathbf{j})$, $\mathbf{F}_2 = (-14\mathbf{i} - 3\mathbf{j})$ act on a particle. Find the third force $\mathbf{F}_3$ that will keep the particle in equilibrium.

Solution

If the forces are in equilibrium, then

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{O}$$

$$\mathbf{F}_3 = -(\mathbf{F}_1 + \mathbf{F}_2)$$

$$\mathbf{F}_3 = -[(3\mathbf{i} - 4\mathbf{j}) + (-14\mathbf{i} - 3\mathbf{j})]$$

$$= 11\mathbf{i} + 7\mathbf{j}$$

Example 2

Three forces $\mathbf{F}_1 = (3\mathbf{i} + 5\mathbf{j})$, $\mathbf{F}_2 = (6\mathbf{i} - 10\mathbf{j})$ and $\mathbf{F}_3 = (\mathbf{i} + 15\mathbf{j})$ are in equilibrium with a fourth force **F**. Find the force **F**.

Solution

If the forces are in equilibrium, then

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F} = \mathbf{O}$$

$$\mathbf{F} = -(\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3)$$

$$\mathbf{F} = -[(3\mathbf{i} + 5\mathbf{j}) + (6\mathbf{i} - 10\mathbf{j}) + (\mathbf{i} + 15\mathbf{j})]$$

$$= -10\mathbf{i} - 10\mathbf{j}$$

Example 3

A particle is at rest under the action of three forces, two of which are **P**(4N, 030°) and **Q**(10N, 300°). Find the third force.

Solution

Resolving the forces **P** and **Q** into column vectors, we have

$$\mathbf{P} = \begin{pmatrix} 4\sin 30^\circ \\ 4\cos 30^\circ \end{pmatrix} = \begin{pmatrix} 2 \\ 3.46 \end{pmatrix}\text{N}$$

$$\mathbf{Q} = \begin{pmatrix} 10\sin 300^\circ \\ 10\cos 300^\circ \end{pmatrix} = \begin{pmatrix} -8.67 \\ 5 \end{pmatrix}\text{N}$$

Let the third force be **S**. Therefore since the particle is at rest, $\mathbf{P} + \mathbf{Q} + \mathbf{S} = \mathbf{O}$

$$\begin{pmatrix} 2 \\ 3.46 \end{pmatrix} + \begin{pmatrix} -8.67 \\ 5 \end{pmatrix} + \mathbf{S} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -6.67 \\ 8.46 \end{pmatrix} + \mathbf{S} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

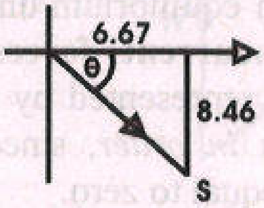
$$\mathbf{S} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -6.67 \\ 8.46 \end{pmatrix} = \begin{pmatrix} 6.67 \\ -8.46 \end{pmatrix}$$

Note: **O** is the zero vector.

The magnitude of **S**,

$$|\mathbf{S}| = \sqrt{(6.67)^2 + (-8.46)^2}$$
$$= \sqrt{116.06} = 10.8\,\text{N}$$

To find the direction of **S**, first represent the force **S** on a rough diagram and calculate the acute angle θ, it makes with the x-axis.



From the diagram, $\tan\theta = \frac{8.46}{6.67} = 1.27$

$\theta = 52^\circ$

The actual direction measured from the north $= 90^\circ + 52^\circ = 142^\circ$.

$\therefore$ The third force is (10.8 N, 142°).

Example 4

Five forces (P, 030°), (2N, 045°), (5N, 090°), (Q, 150°) and (8N, 270°) are in equilibrium. Calculate, correct to two decimal places, the magnitude of P and Q.

Solution

Resolving the forces into column vectors, we have

$$\mathbf{F}_1 = \begin{pmatrix} P\sin 30^\circ \\ P\cos 30^\circ \end{pmatrix} = \begin{pmatrix} 0.5P \\ 0.866P \end{pmatrix}\text{N},$$

$$\mathbf{F}_2 = \begin{pmatrix} 2\sin 45^\circ \\ 2\cos 45^\circ \end{pmatrix} = \begin{pmatrix} 1.414 \\ 1.414 \end{pmatrix}\text{N}$$

$$\mathbf{F}_3 = \begin{pmatrix} 5\sin 90^\circ \\ 5\cos 90^\circ \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}\text{N},$$

$$\mathbf{F}_4 = \begin{pmatrix} Q\sin 150^\circ \\ Q\cos 150^\circ \end{pmatrix} = \begin{pmatrix} 0.5Q \\ -0.866Q \end{pmatrix}\text{N}$$

$$\mathbf{F}_5 = \begin{pmatrix} 8\sin 270^\circ \\ 8\cos 270^\circ \end{pmatrix} = \begin{pmatrix} -8 \\ 0 \end{pmatrix}$$

Since the forces are in equilibrium,

$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 + \mathbf{F}_5 = \mathbf{O}$,

where **O** is the zero vector.

$$\begin{pmatrix} 0.5P \\ 0.866P \end{pmatrix} + \begin{pmatrix} 1.414 \\ 1.414 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.5Q \\ -0.866Q \end{pmatrix} + \begin{pmatrix} -8 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0.5P + 0.5Q - 1.586 \\ 0.866P - 0.866Q + 1.414 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From equality of vectors i.e.. equating corresponding components, we have;

$0.5P + 0.5Q - 1.586 = 0$ ---------- (1)

$0.866P - 0.866Q + 1.414 = 0$ ---- (2)

Solving equations 1 and 2 simultaneously, we have

$P = 0.77$ N and $Q = 2.40$ N to 2 d.p.

Example 5

A body is acted upon by forces

$\mathbf{F_1}$ = (10 N, 090°), $\mathbf{F_2}$ = (8 N, 030°), $\mathbf{F_3}$ = (x N, 000°) and $\mathbf{F_4}$ = (y N, 240°).

If the body remains in equilibrium, find values of x and y.

Solution

Since the body is in equilibrium,

$\mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} + \mathbf{F_4} = \mathbf{O}$

$$\begin{pmatrix} 10\sin 90^\circ \\ 10\cos 90^\circ \end{pmatrix} + \begin{pmatrix} 8\sin 30^\circ \\ 8\cos 30^\circ \end{pmatrix} + \begin{pmatrix} x\sin 0^\circ \\ x\cos 0^\circ \end{pmatrix} + \begin{pmatrix} y\sin 240^\circ \\ y\cos 240^\circ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 10 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 4\sqrt{3} \end{pmatrix} + \begin{pmatrix} 0 \\ x \end{pmatrix} - \begin{pmatrix} y\frac{\sqrt{3}}{2} \\ \frac{y}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \quad 14 - \frac{y}{2}\sqrt{3} = 0 \Rightarrow y = \frac{28\sqrt{3}}{3} = 16.166\ \text{N}$$

$$\text{Also } 4\sqrt{3} + x - \frac{y}{2} = 0$$

$$x = \frac{14\sqrt{3}}{3} - 4\sqrt{3} = \frac{2\sqrt{3}}{3} = 1.1547\,\text{N}$$

Example 6

Three forces $\mathbf{F_1} = (48\text{ N}, 060^\circ)$, $\mathbf{F_2} = (12\text{ N}, 120^\circ)$ and $\mathbf{F_3} = (18\text{ N}, 240^\circ)$ act on a body. Find, correct to **one** decimal place, the force that will keep the system in equilibrium.

Solution

Let **P** be the required force, then

$$\mathbf{P} + \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} = \mathbf{O}$$

$$\mathbf{P} + \begin{pmatrix} 48\sin 60^\circ \\ 48\cos 60^\circ \end{pmatrix} + \begin{pmatrix} 12\sin 120^\circ \\ 12\cos 120^\circ \end{pmatrix} + \begin{pmatrix} 18\sin 240^\circ \\ 18\cos 240^\circ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{P} + \begin{pmatrix} 21\sqrt{3} \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} -21\sqrt{3} \\ -9 \end{pmatrix} = \begin{pmatrix} -36.37 \\ -9 \end{pmatrix}$$

$$|P| = \sqrt{(-21\sqrt{3})^2 + (-9)^2} = 37.5\text{ N}$$

$$\theta = \tan^{-1}\left(\frac{9}{36.37}\right) = 13.9^\circ$$

Required angle = 270° − 13.9° = 256.1°

$\therefore$ The force to keep the particle in equilibrium is (37.5 N, 256.1°)

Three forces in equilibrium

When a body is in equilibrium under a system of **three** coplanar forces ***only***, the following can be used to make the 'three force problem' easier.

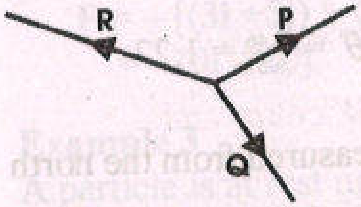
1. *The triangle of forces*
2. *Lami's theorem* and
3. *Vector approach*

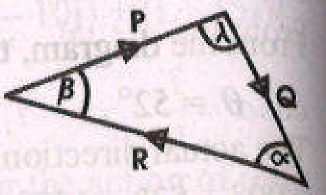
All the methods have been used in some of the examples below. In some questions you will be asked specifically to use the vector approach.

Note: Lami's theorem and triangle of forces are used only when we have **three forces** keeping the body in equilibrium. However the vector approach can be used for any number of forces keeping a body in equilibrium.

The triangle of forces

If a body is in equilibrium under the action of **three concurrent forces**, then these forces can be represented by the *sides of a triangle taken in order*, since their vector sum must be equal to zero.





Note: The sides of the triangle must be *parallel* to the forces they represent, and equal in length to the magnitudes of the forces.

Notice how the arrows 'chase' each other round the triangle of forces, showing zero resultant i.e. the particle in equilibrium.

The **cosine rule** can then be applied to the triangle of forces to calculate any unknown.

1. $P^2 = Q^2 + R^2 - 2QR\cos\alpha$
2. $Q^2 = P^2 + R^2 - 2PR\cos\beta$
3. $R^2 = P^2 + Q^2 - 2PQ\cos\lambda$

You have to work out the angles α, β and λ from a sketch of the problem. α, β and λ are the **acute angles** between the forces in the triangle.

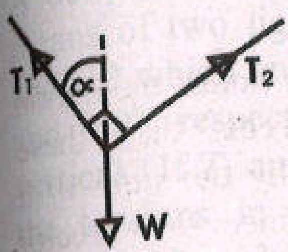
Note: *Angle between forces in the triangle* = 180° − (*angle between the forces in the force diagram*)

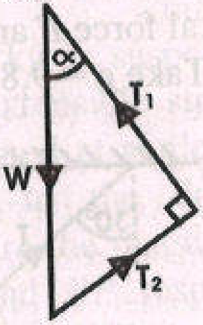
If the triangle is **right angled**, *Pythagoras' theorem* and the *trigonometric ratios* can be used to find any unknown quantity instead of the cosine rule.

Illustrative examples

Example 1

A particle of weight W is supported by two light strings as shown in the diagram below. Find the tensions T_1 and T_2.





Solution

From the triangle of forces:

$$\cos\alpha = \frac{T_1}{W} \Rightarrow T_1 = W\cos\alpha \text{ and}$$

$$\sin\alpha = \frac{T_2}{W} \Rightarrow T_2 = W\sin\alpha$$

Note: The triangle is right-angled

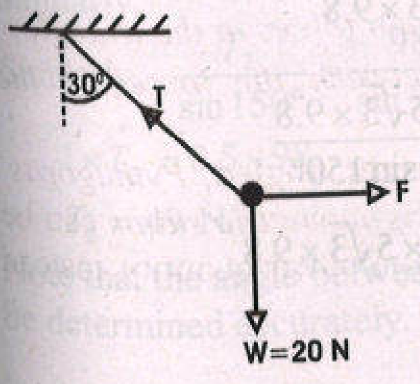
Example 2

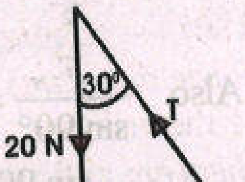
An object of mass 2 kg is suspended vertically by means of an inextensible string. The object is then drawn aside by means of a horizontal force of magnitude F newtons until the string makes an angle of 30° with the downward vertical. If T is the tension in the string, find the values of F and T. (Take $g = 10\ \text{ms}^{-2}$)

Solution

The diagrams below show a sketch of the forces acting on the particle and the triangle of forces.

Weight of object, $W = mg = 2\times 10 = 20$ N.





From the right-angled triangle,

$$\tan 30^\circ = \frac{F}{20}$$

$$F = 20\times\tan 30^\circ$$

$$= 20\times 0.5774 = 11.54\text{ N}$$

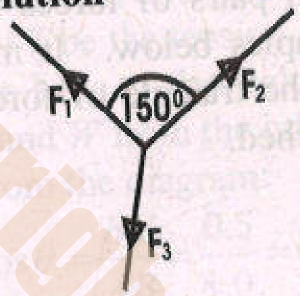
Also $\cos 30^\circ = \dfrac{20}{T}$

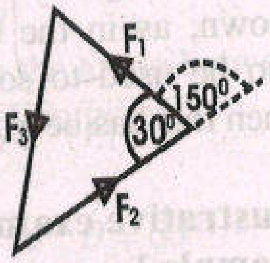
$$T = \frac{20}{\cos 30^\circ} = \frac{20}{0.866} = 23.09\text{ N}$$

Example 3

Three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ are in equilibrium. The magnitude of $\mathbf{F}_1$ is 10 N and that of $\mathbf{F}_2$ is 20 N. If the angle between $\mathbf{F}_1$ and $\mathbf{F}_2$ is 150°, calculate the magnitude of $\mathbf{F}_3$.

Solution





Note: Angle between $\mathbf{F}_1$ and $\mathbf{F}_2$ in the triangle = 180° – 150° = 30°.

In this case, the triangle is not right angled. Therefore applying the cosine rule:

$$F_3^{\,2} = F_1^{\,2} + F_2^{\,2} - 2F_1F_2\cos\alpha,$$

where $\alpha = 180° - 150° = 30°$ (see diagram)

$$F_3^{\,2} = 10^2 + 20^2 - 2(10)(20)\cos 30^\circ$$

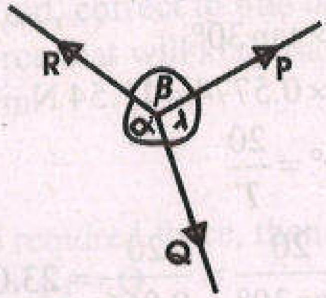
$$F_3^{\,2} = 100 + 400 - 400(0.866)$$

$$F_3^{\,2} = 500 - 346.40 = 153.6$$

$$F_3 = \sqrt{153.6} = 12.4\text{ N}$$

Lami's theorem

Lamis theorem is a special version of the sine rule.



If three concurrent coplanar forces P, Q and R are in equilibrium and the angles between Q and R, R and P, P and Q are α, β and λ respectively as shown in the figure above, then Lami's theorem states that:

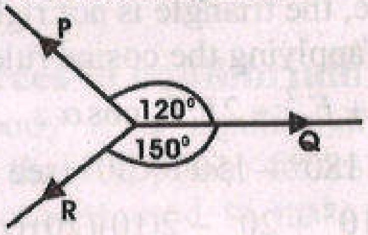
$$\frac{P}{\sin\alpha}=\frac{Q}{\sin\beta}=\frac{R}{\sin\lambda}$$

This theorem is useful when one of the forces and the angles between pairs of forces are known, as in the examples below. It may also be used to solve the triangle of forces when this has been sketched.

Illustrative examples

Example 1

The diagram below shows three forces acting on a particle at rest. If the magnitude of $R = 3$ N, find the magnitude of P.



Solution

Applying Lami's theorem, we have

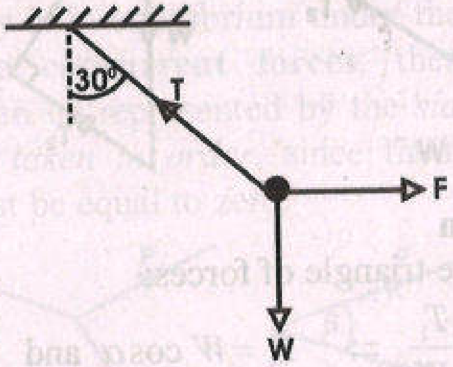
$$\frac{P}{\sin 150°}=\frac{R}{\sin 120°}$$

$$P=\frac{\sin 150°}{\sin 120°}\times 3=\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\times 3$$

$$P=\frac{3}{\sqrt{3}}=\sqrt{3}. \qquad \text{Note that } R=3\text{ N}$$

Example 2

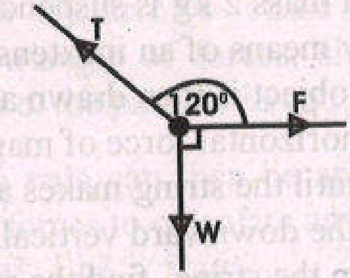
The diagram below shows an object of mass $5\sqrt{3}$ kg held at rest by a horizontal force F N and an inextensible string inclined at 30° to the downward vertical. Calculate the horizontal force F and tension, T in the string. (Take $g = 9.8\ \text{ms}^{-2}$)



Solution

The angle between T and F is 120°, T and W is 150° and the angle between W and F is 90°

Weight of object, $W = mg = 5\sqrt{3}\times 9.8$ N.



Applying Lami's theorem, we have:

$$\frac{T}{\sin 90°}=\frac{5\sqrt{3}\times 9.8}{\sin 120°}$$

$$T=\frac{\sin 90°}{\sin 120°}\times 5\sqrt{3}\times 9.8$$

$$=\frac{1}{\frac{\sqrt{3}}{2}}\times 5\sqrt{3}\times 9.8$$

Also $$\frac{F}{\sin 90°}=\frac{5\sqrt{3}\times 9.8}{\sin 150°}$$

$$T=\frac{\sin 90°}{\sin 150°}\times 5\sqrt{3}\times 9.8$$

$$=\frac{1}{\frac{1}{2}}\times 5\sqrt{3}\times 9.8$$

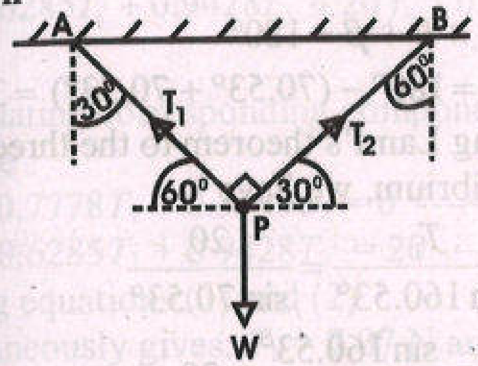
$$=2\times 5\sqrt{3}\times 9.8=169.74\text{ N}$$

Example 3

A body P of mass 10 kg is suspended by means of two light inextensible strings AP and BP which are inclined at angles of 30° and 60° respectively to the downward vertical. If T_1 and T_2 are the magnitudes of the tensions in AP and BP respectively, calculate the values of T_1 and T_2.

(Take $g = 9.8\text{ ms}^{-2}$)

Solution



Weight of body P,

$W = mg = 10 \times 9.8 = 98$ N

The angle between T_1 and T_2 is 90°, the angle between T_1 and W is 150° and the angle between T_2 and W is 120°.

Therefore applying Lami's theorem:

$$\frac{T_1}{\sin 120°}=\frac{98}{\sin 90°}$$

$$1\times T_1 = 0.866\times 98$$

$$T_1 = 84.87\text{ N}$$

Similarly,

Similarly, $\dfrac{T_2}{\sin 150°}=\dfrac{98}{\sin 90°}$

$$1\times T_2 = 0.5\times 98$$

$$T_2 = 49\text{ N}$$

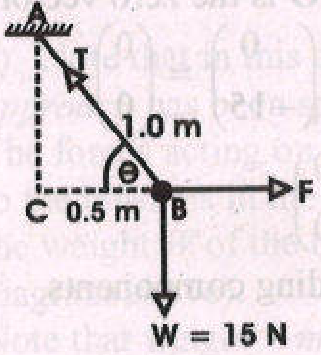
Note that the angle between the forces must be determined accurately.

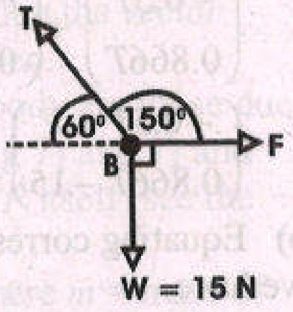
Example 4

One end of a string 1.0 m long is fixed to a point A and the other end is fastened to a body B of mass 1.5 kg. The body is pulled aside by a horizontal force F until it is 0.5 m from the vertical through A. Calculate the horizontal force, F and the tension in AB.

(Take $g = 10\text{ ms}^{-2}$)

Solution





Weight of body B, $W = mg = 1.5\times 10 = 15$ N

Let T be the tension in AB

We have to first calculate the angle between T and W from the information given.

From the diagram:

$$\cos\theta=\frac{BC}{AB}=\frac{0.5}{1.0}=0.5 \quad\Rightarrow\quad \theta=60°$$

Applying Lami's theorem, we have

$$\frac{T}{\sin 90°}=\frac{15}{\sin 120°}$$

$$T=\frac{\sin 90°}{\sin 120°}\times 15$$

$$T=\frac{15}{0.866}=17.32\text{ N}$$

Similarly,

$$\frac{F}{\sin 150°}=\frac{15}{\sin 120°}$$

$$T=\frac{\sin 150°}{\sin 120°}\times 15$$

$$T=\frac{0.5\times 15}{0.866}=8.66\text{ N}$$

Vector Approach

Resolving the forces in column vectors, we have;

$$\mathbf{T} = \begin{pmatrix} -T\cos 60^\circ \\ T\sin 60^\circ \end{pmatrix} = \begin{pmatrix} -0.5T \\ 0.866T \end{pmatrix}\text{N},$$

$$\mathbf{F} = \begin{pmatrix} F\cos 0^\circ \\ F\sin 0^\circ \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}\text{N and } \mathbf{W} = \begin{pmatrix} 0 \\ -15 \end{pmatrix}\text{N}$$

Therefore since mass C is at rest

$\mathbf{T} + \mathbf{F} + \mathbf{W} = \mathbf{O}$, where $\mathbf{O}$ is the zero vector.

$$\begin{pmatrix} -0.5T \\ 0.866T \end{pmatrix} + \begin{pmatrix} F \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -15 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.5T + F \\ 0.866T - 15 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

b) Equating corresponding components, we have

$-0.5T + F = 0$ -------(1)

$0.866T - 15 = 0$----- (2)

Solving equations (1) and (2) simultaneously gives:

$T = 17.32$ N and $F = 8.66$ N.

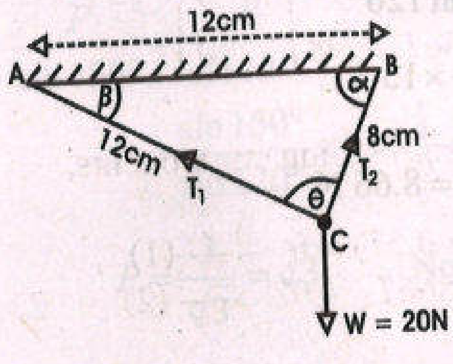
Note: Use the triangle of forces to arrive at the same answer.

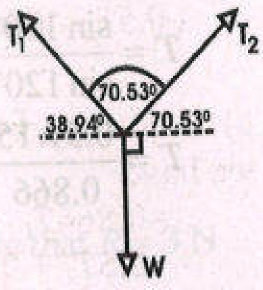
Example 5

A mass C of weight 20 N is suspended by two light strings AC and BC from points A and B on the same horizontal level above C. $AB = AC = 12$ cm and $CB = 8$ cm. Find the tensions in AC and BC.

Solution

Let the tension in string AC be T_1 and the tension in string CB be T_2





In order to solve this question using the vector approach or Lami's theorem or triangle of forces, the angle between the forces must first be determined. Thus applying the cosine rule for the isosceles triangle ABC

$$AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos\alpha$$

$$12^2 = 12^2 + 8^2 - 2(12)(8)\cos\alpha$$

$$144 = 144 + 64 - 192\cos\alpha$$

$$192\cos\alpha = 64$$

$$\cos\alpha = \tfrac{64}{192} = 0.3333$$

$$\alpha = 70.53^\circ$$

Triangle ABC is isosceles ($AB = AC$)

$\therefore\ \alpha = \theta = 70.53^\circ$

Also $\alpha + \theta + \beta = 180^\circ$

$\beta = 180^\circ - (70.53^\circ + 70.53^\circ) = 38.94^\circ$

Applying Lami's theorem to the three forces in equilibrium, we have

$$\frac{T_1}{\sin 160.53^\circ} = \frac{20}{\sin 70.53^\circ}$$

$$T_1 = \frac{\sin 160.53^\circ}{\sin 70.53^\circ} \times 20$$

$$T_1 = \frac{0.3333}{0.9428} \times 20 = 7.07\text{ N}$$

$$\frac{T_2}{\sin 128.94^\circ} = \frac{20}{\sin 70.53^\circ}$$

$$T_2 = \frac{\sin 128.94^\circ}{\sin 70.53^\circ} \times 20$$

$$T_2 = \frac{0.7778 \times 20}{0.9428} = 16.5\text{ N}$$

Therefore the tensions in string AC is 7.07 N and the tension in string BC is 16.5 N.

Vector Approach

Resolving the forces in column vectors, we have;

$$\mathbf{T}_1 = \begin{pmatrix} -T_1\cos 38.94^\circ \\ T_1\sin 38.94^\circ \end{pmatrix} = \begin{pmatrix} -0.7778T_1 \\ 0.6285T_1 \end{pmatrix}\text{N},$$

$$\mathbf{T}_2 = \begin{pmatrix} T_2 \cos 70.53^\circ \\ T_2 \sin 70.53^\circ \end{pmatrix} = \begin{pmatrix} 0.3333T_2 \\ 0.9428T_2 \end{pmatrix} \text{N}$$

and $\mathbf{W} = \begin{pmatrix} 0 \\ -20 \end{pmatrix} \text{N}$

Therefore since mass *C* is at rest $\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{W} = \mathbf{O}$, where **O** is the zero vector.

$$\begin{pmatrix} -0.7778T_1 \\ 0.6285T_1 \end{pmatrix} + \begin{pmatrix} 0.3333T_2 \\ 0.9428T_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -20 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.7778T_1 + 0.3333T_2 \\ 0.6285T_1 + 0.9428T_2 - 20 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

b) Equating corresponding components, we have

$-0.7778T_1 + 0.3333T_2 = 0$ -------(1)

$0.6285T_1 + 0.9428T_2 = 20$ ----- (2)

Solving equations (1) and (2) simultaneously gives: $T_1 = 7.07$ N and $T_2 = 16.5$ N.

Example 6

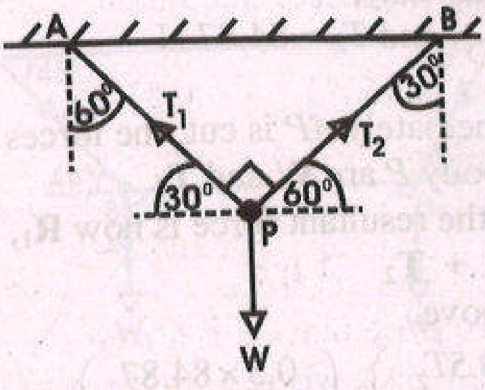
A body *P* of mass 10 kg is suspended by means of two light inextensible strings *AP* and *BP* which are inclined at angles of 60° and 30° respectively to the downward vertical. If T_1 and T_2 are the magnitudes of the tensions in *AP* and *BP* respectively

a) write down a vector equation connecting all the forces on the body is at rest.

b) use the vector equation in a) to calculate the values of T_1 and T_2.

c) find the direction in which *P* begins to move when the string *AP* is cut. (Take $g = 9.8$ m/s^2).

Solution



a) Note that in this question the *vector approach* has been specified.

The forces acting on the body are those due to the tensions in the string T_1 and T_2 and the weight *W* of the body *P* itself (see the diagram above).

Note that weight = *mg* where *m* = mass of body and *g* = acceleration due to gravity.

Resolving the forces in column vectors, we have;

$$\mathbf{T}_1 = \begin{pmatrix} -T_1 \cos 30^\circ \\ T_1 \sin 30^\circ \end{pmatrix} = \begin{pmatrix} -0.866T_1 \\ 0.5T_1 \end{pmatrix}$$

$$\mathbf{T}_2 = \begin{pmatrix} T_2 \cos 60^\circ \\ T_2 \sin 60^\circ \end{pmatrix} = \begin{pmatrix} 0.5T_2 \\ 0.866T_2 \end{pmatrix} \text{ and}$$

$$\mathbf{W} = \begin{pmatrix} 0 \\ -98 \end{pmatrix} \text{N}$$

Therefore since *P* is at rest $\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{W} = \mathbf{O}$, where **O** is the zero vector.

$$\begin{pmatrix} -0.866T_1 \\ 0.5T_1 \end{pmatrix} + \begin{pmatrix} 0.5T_2 \\ 0.866T_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -98 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.866T_1 + 0.5T_2 \\ 0.5T_1 + 0.866T_2 - 98 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

b) Equating corresponding components, we have

$-0.866T_1 + 0.5T_2 = 0$ -------(1)

$0.5T_1 + 0.866T_2 = 98$ ------ (2)

Solving equations (1) and (2) simultaneously,
$T_1 = 49$ N and $T_2 = 84.87$ N

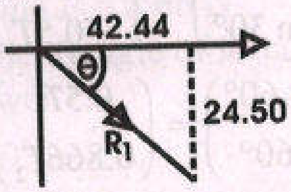
c) Immediately AP is cut, the forces acting on the body P are **W** and $\mathbf{T}_2$.
Thus if the resultant force is now $\mathbf{R}_1$, then
$\mathbf{R}_1 = \mathbf{W} + \mathbf{T}_2$
From above,

$$\mathbf{T}_2 = \begin{pmatrix} 0.5T_2 \\ 0.866T_2 \end{pmatrix} = \begin{pmatrix} 0.5 \times 84.87 \\ 0.866 \times 84.87 \end{pmatrix}$$

$$= \begin{pmatrix} 42.44 \\ 73.50 \end{pmatrix} \text{N}$$

$$\mathbf{R}_1 = \begin{pmatrix} 0 \\ -98 \end{pmatrix} + \begin{pmatrix} 42.44 \\ 73.50 \end{pmatrix} = \begin{pmatrix} 42.44 \\ -24.50 \end{pmatrix} \text{N}$$

To find the direction of $\mathbf{R}_1$, first represent the resultant force $\mathbf{R}_1$ on a rough diagram and calculate the acute angle θ it makes with the x-axis.



From the diagram, $\tan\theta = \frac{24.50}{42.44} = 0.578$

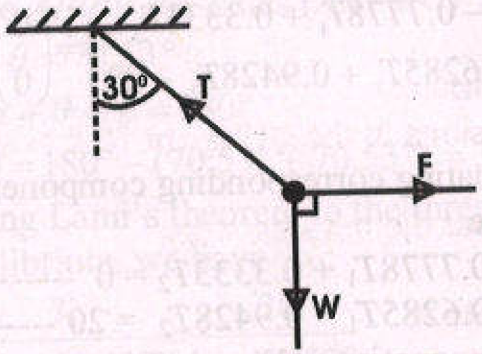
$\theta = 30°$

The actual direction measured from the north $= 90° + 30° = 120°$
Therefore the direction of P when the string AP is cut is the same as the direction of $\mathbf{R}_1$ which is 120°.

Example 7
An object of mass 2 kg is suspended vertically by means of an in extensible string. The object is then drawn aside by means of a horizontal force of magnitude F newtons until the string makes an angle of 30° with the downward vertical.
If T is the tension in the string;

i) Express the vectors in the component form for all the forces acting on the body.
ii) Write down the vector equation connecting all the forces in (i) if the object is at rest.
iii) Use your vector equation in ii) to obtain the values of F and T, each correct to one decimal place. (Take $g = 9.8\ \text{ms}^{-2}$).

Solution
Note that in this question the vector approach has been specified.



Let W be the weight of the object
i.e. $W = mg = 2 \times 9.8 = 19.6$ N
Let **T**, **F** and **W** represent the tension, horizontal force and weight respectively
Expressing the vectors into components, we have

i) $$\mathbf{T} = \begin{pmatrix} -T\cos 60° \\ T\sin 60° \end{pmatrix} = \begin{pmatrix} -0.5T \\ 0.866T \end{pmatrix} \text{N},$$

$$\mathbf{F} = \begin{pmatrix} F \\ 0 \end{pmatrix} \text{N and } \mathbf{W} = \begin{pmatrix} 0 \\ -19.6 \end{pmatrix} \text{N}$$

ii) Since the body is at rest $\mathbf{T} + \mathbf{F} + \mathbf{W} = \mathbf{O}$, where **O** is the zero vector.

$$\begin{pmatrix} -0.5T \\ 0.866T \end{pmatrix} + \begin{pmatrix} F \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -19.6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.5T + F \\ 0.866T - 19.6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

iii) Equating corresponding components, we have;

$-0.5T + F = 0$ ....(1)

$0.866T - 19.6 = 0$ ....(2)

Solving equations (1) and (2) simultaneously gives: $T = 22.6$ N and $F = 11.3$ N.

Example 8

A body of mass 5 kg is held in equilibrium by two strings inclined at 40° and 15° respectively to the vertical. Find, correct to **one** decimal place, the tensions in the strings.

Solution

the tensions in the string are 15.8 N and 39.2 N.

Work out the missing steps.

Example 9

A light inextensible string is suspended from two points A and D at the same level. A particle of mass 5 kg is attached to the string at B and a particle of mass m kg is attached to the string at C. If AB and CD make angles of 45° and 60° respectively with the horizontal, and if BC is horizontal, find

i) the tension in each of the three parts of the string

ii) the value of m (Take $g = 9.8$ ms^{-2})

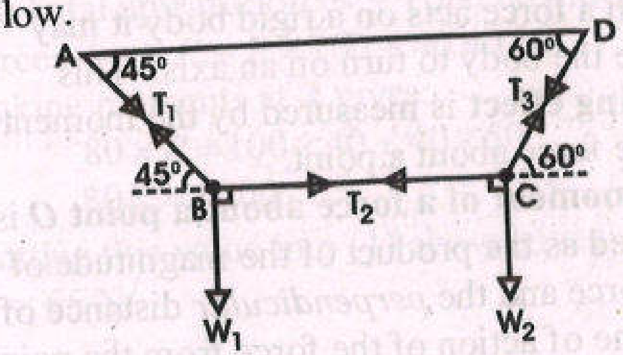
Solution

Let the weight of the particle at $B = W_1$ and the weight of the particle at $C = W_2$

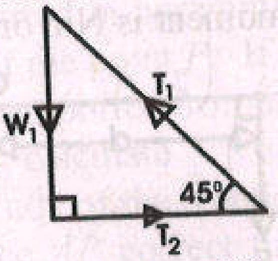
Also let the tensions in the three parts of the string be T_1, T_2 and T_3.

Consider the equilibrium of each particle separately.

The diagram of the problem is shown below.



Using the triangle of forces for the forces acting on particle at B

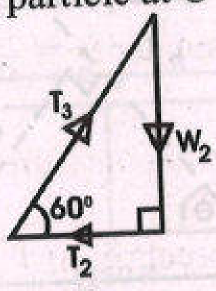


$$\tan 45° = \frac{W_1}{T_2} \quad \Rightarrow T_2 = \frac{W_1}{\tan 45°} = \frac{49}{1} = 49$$

$$\text{N } \cos 45° = \frac{W_1}{T_1} \Rightarrow T_1 = \frac{W_1}{\sin 45°}$$

$$T_1 = \frac{49}{\frac{1}{\sqrt{2}}} = 49\sqrt{2}\text{ N}$$

Similarly, using the triangle of forces for the forces acting on particle at C



$$\tan 60° = \frac{W_2}{T_2} \;\Rightarrow\; W_2 = T_2 \tan 60° = 49\sqrt{3}\text{ N}$$

$$\text{and } \cos 60° = \frac{T_2}{T_3} \;\Rightarrow T_3 = \frac{T_2}{\cos 60°} = \frac{49}{\frac{1}{2}}$$

$$T_3 = 49 \times 2 = 98\text{ N}$$

b) Weight of particle at C = mass (m) × g

$$49\sqrt{3} = 9.8m \quad \Rightarrow \quad m = 5\sqrt{3}$$

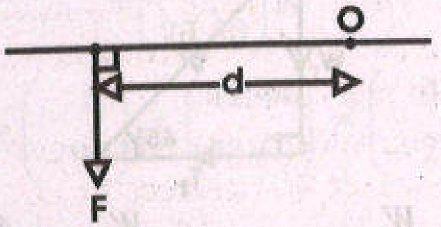
Note that you can use the vector approach or Lami's theorem to get the same answers.

Moments of forces

When a force acts on a rigid body it may cause the body to turn on an axis. This **turning effect** is measured by the moment of the force about a point.

The **moment of a force about a point *O*** is defined as the product of the magnitude of the force and the *perpendicular* distance of the line of action of the force from the point *O*. In the figure below, the moment of force F about $O = F \times d$

The units of moment is Nm or N cm.

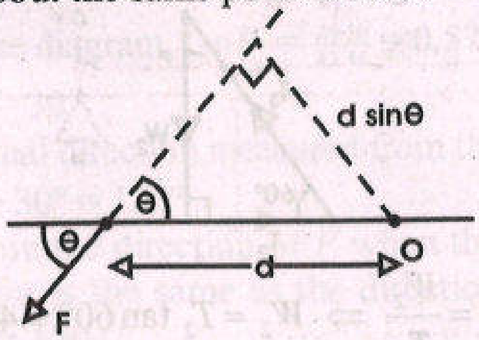


If the force F is inclined at angle θ as shown in the figure below, the component of the force perpendicular to the distance is used in finding the moment.

The moment of force F about $O = F\sin\theta \times d$

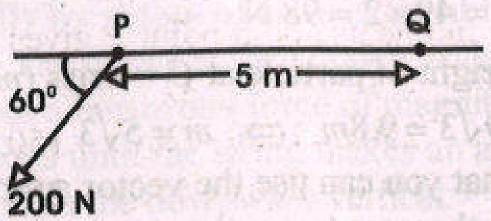
$$= Fd\sin\theta$$

Note: The moment of force acting at a point about the same point is *zero*.



Illustrative example

Find the moment of 200 N about P and Q.



Solution

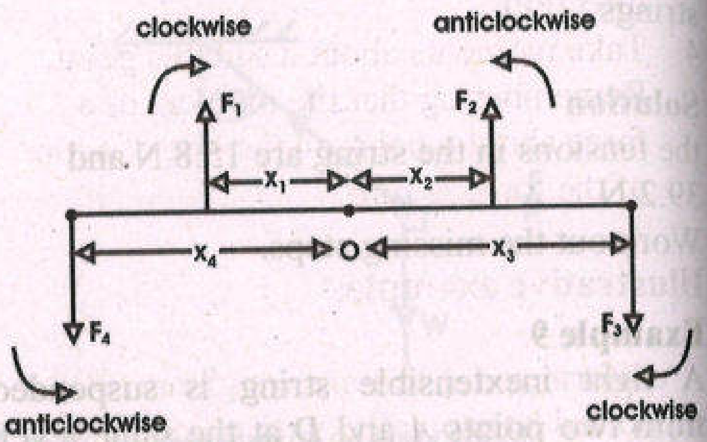
Moment about $P = 0$, since the line of action of 200 N passes through P

Moment about $Q = 200 \times 5\sin 60°$

$$= 500\sqrt{3}\ \text{Nm}$$

Principle of moments

When a body is in equilibrium under a set of coplanar forces, the sum of the ***clockwise moments*** is equal to the sum of the ***anticlockwise moments*** about the same point.



From the figure above:

Clockwise moments = $F_1x_1 + F_3x_3$

Anticlockwise moments = $F_2x_2 + F_4x_4$

If the body is in equilibrium then:

Clockwise moments = Anticlockwise moments

$$F_1x_1 + F_3x_3 = F_2x_2 + F_4x_4$$

Equilibrium of rigid bodies

When a rigid body is in equilibrium under a system of coplanar forces, the following conditions are satisfied:

1. The sum of all the forces in one direction is equal to the sum of the forces in the opposite direction. In terms of vectors, the **vector sum** of all the forces is equal to zero.

2. The sum of the clockwise moments is equal to the sum of the anticlockwise moments about the same point.

When solving problems about rigid bodies in equilibrium:

1. Draw a clear force diagram
2. Choose two directions at right angles for resolving the forces.
3. Resolve the forces in the chosen direction. Remembering that sum of all the forces in one direction is equal to the sum of the forces in the opposite direction.
4. Take moments about a suitable point. Remembering that the moment of a force about a point on the line of action of the force is zero.

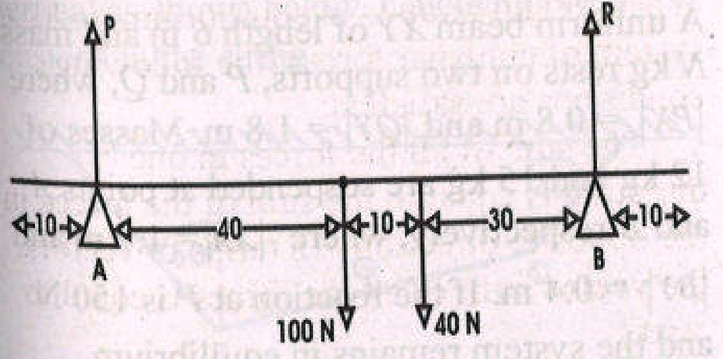
Illustrative examples

Example 1

A uniform rod 1 m long weighing 100 N is supported horizontally on two knife edges A and B placed at 10 cm from its ends. What will be the reactions at the supports when a 40 N weight is suspended 10 cm from the midpoint of the rod?

Solution

Let P and R be the reactions. The weight of the rod acts at the 50 cm (since the rod is uniform and the weight of a rigid body acts at the centre of gravity).



The rod is in equilibrium:

$\therefore$ Total upward forces = Total downward forces i.e. $P + R = 100 + 40 = 140$ ……(1)

Taking moments at A gives:

$$80 \times R = 100 \times 40 + 40 \times 50$$
$$80R = 6000 \Rightarrow R = 75\text{ N}$$

Putting this value into (1) above gives $P = 65$ N

Example 2

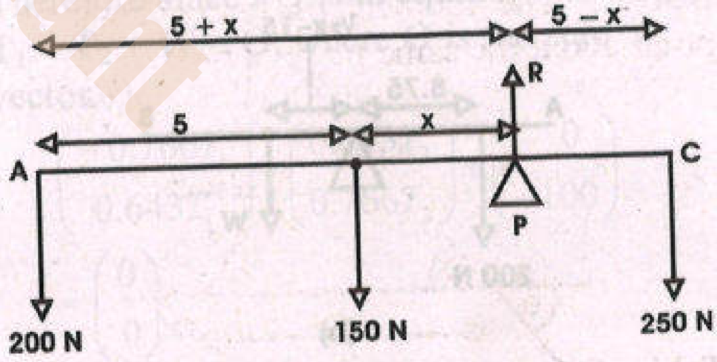
A uniform rod APC, 10 m long of mass 15 kg is supported in horizontal position on a knife pivot at the point P. If masses 20 kg and 25 kg are supported from its ends, A and C respectively, calculate

i) the magnitude of the reaction at P

ii) the distance AP correct to two decimal places. [Take $g = 10\text{ ms}^{-2}$]

Solution

i) Let R be the reaction at P and let the pivot be at a distance, x from the centre of the rod. Convert the masses to weight i.e. weight = mg



Total upward forces
= Total downward forces

i.e. $R = 200 + 150 + 250 = 600$ N

ii) Let the distance AP be x m.

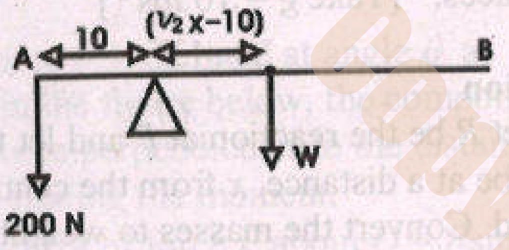
Taking moments about point P gives:

$$250(5 - x) = 200(5 + x) + 150x$$
$$1250 - 250x = 1000 + 200x + 150x$$
$$1250 - 1000 = 200x + 150x + 250x$$
$$250 = 600x \quad \Rightarrow \quad x = \tfrac{250}{600} = 0.4167\text{ m}$$

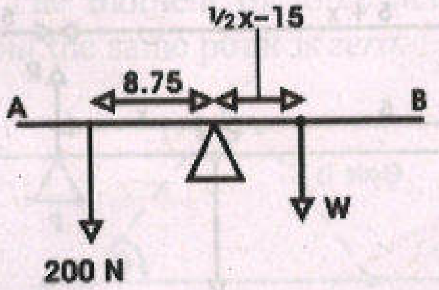
$\therefore\ AP = 5 + x = 5 + 0.4167$
$= 5.4167 \text{ m} = 5.42 \text{ m}$

Example 3
A uniform stick AB can be balanced on a knife edge 10 cm from one end when a weight of 200 N is hung from that end. When the knife edge is removed 5 cm further from that end, the 200 N weight has to be moved to a point 8.75 cm from the knife edge to obtain a balance. Find the length of the stick and its weight.

Solution
Let x be the length of the stick and W the weight. Refer to the figures (a and b) below for the two situations.



(a)

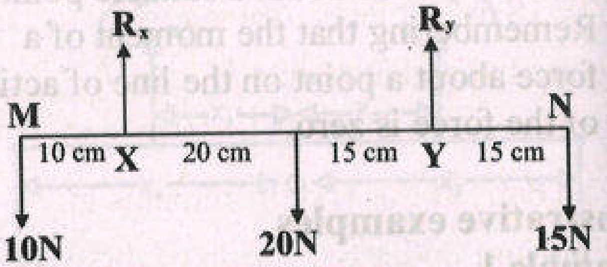


(b)

Taking moments about the knife edge gives:
$W(0.5x - 10) = 200 \times 10 = 2000 \ldots\ldots(1)$
$W(0.5x - 15) = 200 \times 8.75 = 2000 \ldots\ldots(2)$
Eqn(1) – eqn(2)
$5W = 250$
$W = 50 \text{ N}$
Putting $W = 50$ N into eqn (1) gives $x = 100$ cm.

Example 4
A uniform beam MN of weight 20 N and length 60 cm is supported at two points X and Y such that $|MX| = 10 \text{ cm}$ and $|YN| = 15 \text{ cm}$. Two forces 10 N and 15 N are placed at M and N respectively. If the system remains in equilibrium under the action of these forces, calculate the reactions at X and Y. [Take $g = 10 \text{ ms}^{-2}$]

Solution
The problem is represented by the diagram below.



Taking moments at Y
$35R_x + 15 \times 15 = 20 \times 15 + 10 \times 45$
$35R_x = 750 - 225$
$R_x = 15 \text{ N}$ = Reaction at X

Total upward forces = Total downward forces
$R_x + R_y = 45 \text{ N} \Rightarrow 15 + R_y = 45$
$R_y = 30 \text{ N}$ = Reaction at Y

Example 5
A uniform beam XY of length 6 m and mass N kg rests on two supports, P and Q, where $|PX| = 0.8$ m and $|QY| = 1.8$ m. Masses of 12 kg and 15 kg are suspended at points A and B respectively, where $|XA| = 1.7$ m and $|BY| = 0.4$ m. If the reaction at P is 150 N and the system remains in equilibrium under the action at P is 150 N and the

system remains in equilibrium under the action of these forces, find the

a) value of n;
b) reaction at Q.
[Take $g = 10 \text{ ms}^{-2}$]

Solution

$n = 3.5$ and $R = 470$ N

Work out the missing steps

Example 6

A uniform plank XY of mass 40 kg and length 30 m rests horizontally on two supports P and Q, where $|PX| = 8$ m and $|QX| = 18$ m. Objects of masses 10 kg and 5 kg are hanged at points M and N respectively, where $|MX| = 10$ m and $|NY| = 6$ m. If the system remains in equilibrium under the action of three forces, calculate the pressure exerted on

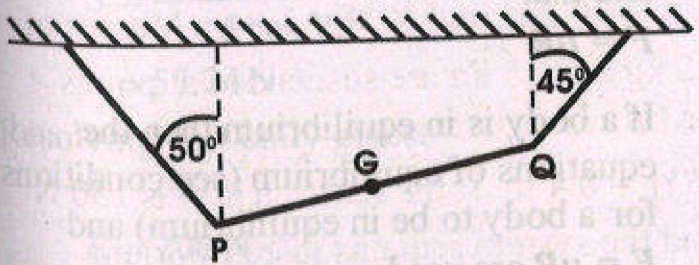
i) P ii) Q

Solution

Pressure exerted on P = 170 N
Pressure exerted on Q = 380 N
Work out the missing steps

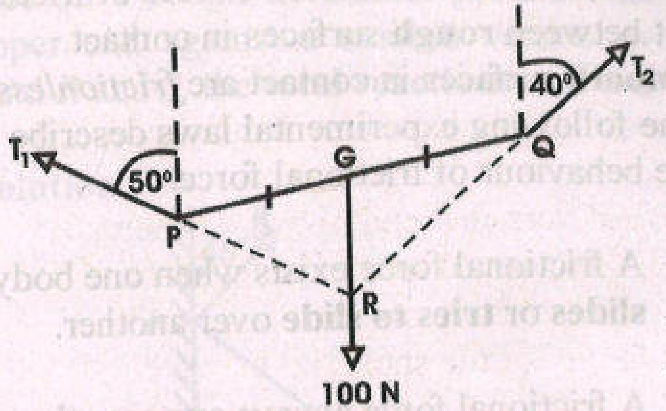
Example 7

The diagram below shows a uniform rod PQ of mass 10 kg which is held in equilibrium by two strings attached to the ends P and Q of the rod. The strings are inclined to the vertical as shown below. Calculate the tensions in the string.



Solution

The force diagram showing the forces acting on the rod PQ is shown below.



The rod is in equilibrium under the action of the forces. Using the vector approach:

$$\mathbf{T_1} = \begin{pmatrix} -T_1 \sin 50° \\ T_1 \cos 50° \end{pmatrix} = \begin{pmatrix} -0.766T_1 \\ 0.643T_1 \end{pmatrix},$$

$$\mathbf{T_2} = \begin{pmatrix} T_2 \sin 40° \\ T_2 \cos 40° \end{pmatrix} = \begin{pmatrix} 0.643T_2 \\ 0.766T_2 \end{pmatrix} \text{ and}$$

$$\mathbf{W} = \begin{pmatrix} 0 \\ -100 \end{pmatrix}$$

Therefore since PQ is in equilibrium:
$\mathbf{T_1} + \mathbf{T_2} + \mathbf{W} = \mathbf{O}$, where $\mathbf{O}$ is the zero vector.

$$\begin{pmatrix} -0.766T_1 \\ 0.643T_1 \end{pmatrix} + \begin{pmatrix} 0.643T_2 \\ 0.766T_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -100 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.766T_1 + 0.643T_2 \\ 0.643T_1 + 0.766T_2 - 100 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Equating corresponding components, we have

$$-0.766T_1 + 0.643T_2 = 0 \text{ ---------(1)}$$
$$0.643T_1 + 0.766T_2 = 100 \text{ ------ (2)}$$

Solving equations (1) and (2) simultaneously gives:
$T_1 = 64.3$ N and $T_2 = 76.6$ N

Forces of friction

When a body slides or attempts to slide over another, **forces of friction** usually exist between the two surfaces. Forces of friction act between **rough** surfaces in contact. **Smooth** surfaces in contact are *frictionless*. The following experimental laws describe the behaviour of frictional forces.

1. A frictional force exists when one body **slides** or **tries to slide** over another.

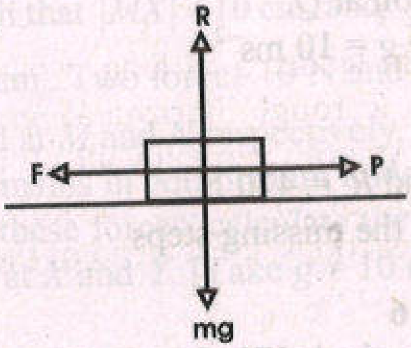
2. A frictional force always **opposes** the tendency of one body to slide over another.

3. The magnitude of the frictional force may vary, always just being sufficient to prevent motion, until it reaches a **maximum value** called the **limiting value**.

4. The **limiting value** of the frictional force is given by: $F = \mu R$
   Where μ is called the **coefficient of friction** and R is the normal reaction for the surfaces in contact. μ is a measure of the degree of roughness of the two surfaces in contact and is different for two pairs of surfaces

5. When one body slides over another, the frictional force between them equals the limiting value μR. A consequence of laws 3 and 4 is that the frictional force obeys the relation $F \leq \mu R$

From the diagram below, a body of mass m, is resting on a rough horizontal plane and is being acted upon by a horizontal force P.



As the magnitude of P is gradually increased from zero, the magnitude of F will also increase from zero to attempt to prevent motion. When the motion begins F has reached its maximum, called the limiting value μR, and cannot increase any more to prevent motion. So the frictional force remains constant i.e. $F = \mu R$ whatever the increase in P.

Problem solving

The following points are important when solving problems involving frictional force.

1. Draw a clear force diagram. Show the frictional force as F, do not use μR. Remember that F tends to oppose motion.

2. In general $F \leq \mu R$. If F has reached limiting value then $F = \mu R$ may be used in the solution

3. Limiting equilibrium means that the body is at rest but at the point of moving and that
   $F = \mu R$

4. If a body is in equilibrium then the equations of equilibrium (see conditions for a body to be in equilibrium) and $F = \mu R$ are used.

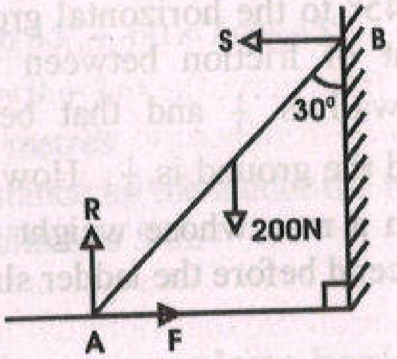
Ladder Problems

Illustrative examples

Example 1

A uniform ladder 5 m long, weight 200 N, rests on a rough horizontal ground and against a smooth vertical wall. It is inclined at an angle of 30° to the vertical. Find the **normal** reactions at each end of the ladder.

Solution



AB = ladder
R = normal reaction at A
S = normal reaction at B
F = frictional force

The wall is smooth, so the reaction at B is perpendicular to the wall and there is no frictional force. The ground is rough, so there is frictional force and normal reaction at A. The forces are more than three so we cannot use Lami's theorem or triangle of forces.

Taking moments about A gives:

$$S \times 5\cos 30° = 200 \times 2.5\cos 60°$$

$$S = \frac{200 \times 2.5\cos 60°}{5\cos 30°}$$

$$= 57.74\,\text{N}$$

Resolving vertically gives:

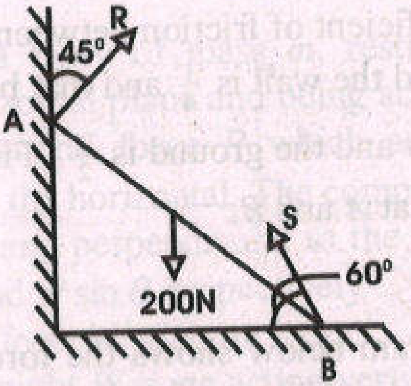
$$R - 200 = 0$$

$$R = 200\,\text{N}$$

Example 2

The diagram shows a uniform ladder AB of weight 200 N, resting in equilibrium with its foot on a rough horizontal ground and its upper end against a smooth vertical wall. Find R and S, the **total reactions** at A and B.

Solution



Note that there is no frictional force.
The ladder is in equilibrium under three concurrent forces R, S and 200 N.
Lami's theorem can therefore be applied.

$$\frac{R}{\sin 150°} = \frac{200}{\sin 75°}$$

$$R = \frac{\sin 150°}{\sin 75°} \times 200 = 103.53\ \text{N}$$

Similarly, $$\frac{S}{\sin 135°} = \frac{200}{\sin 75°}$$

$$R = \frac{\sin 135°}{\sin 75°} \times 200 = 146.41\ \text{N}$$

Difference between normal reaction and total reaction at a point

Note that in this example, R and S are total reactions not normal reactions. The total reactions are the *resultant* of the frictional forces and the normal reactions at a point. The total reactions are at an angle to the horizontal when the point of contact is *rough*. However they are equal to the normal reactions when the point of contact is *smooth*.

Illustrative examples

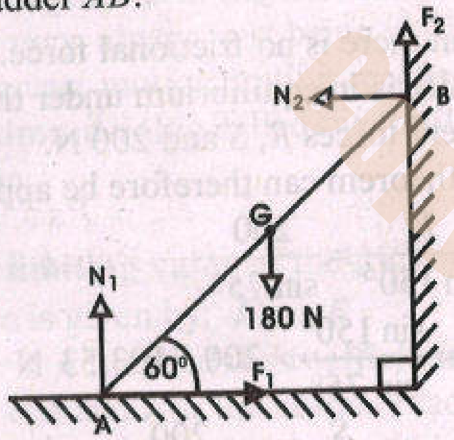
Example 1

A uniform ladder *AB* of weight 180 N and length 8 m, rests in equilibrium with its foot on a rough horizontal ground and its upper end against a rough vertical wall. The ladder makes an angle of 60° with the horizontal ground.

The coefficient of friction between the ladder and the wall is $\frac{1}{4}$ and that between the ladder and the ground is $\frac{1}{2}$. Find normal reactions at *A* and *B*.

Solution

The diagram below shows the forces acting on the ladder *AB*.



$AG = GB = 4$ metres

F_1 and F_2 are the frictional forces

N_1 and N_2 are the normal reactions

Frictional force = μR when the ladder is about to slip

$\therefore \quad F_1 = \frac{1}{2}N_1$ .....(1) and

$F_2 = \frac{1}{4}N_2$ .....(2)

Resolving vertically :

$N_1 + F_2 = 180$ ......(3)

Resolving horizontally :

$F_1 = N_2$............(4)

From the four equations above, substituting (1) and (2) into (3) and (4) respectively, we have:

$N_1 + \frac{1}{4}N_2 = 180$ .....(5) and

$\frac{1}{2}N_1 = N_2$ ....(6)

Solving (5) and (6) simultaneously, gives

$N_1 = 160$ N and $N_2 = 80$ N

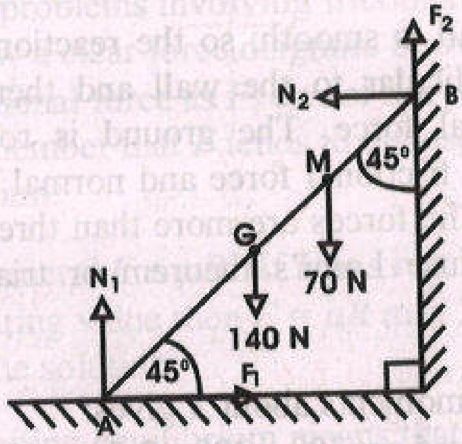
$\therefore$ The normal reactions at *A* and *B* are 180 N and 80 N respectively.

Example 2

A uniform ladder of weight 140 N and length 7 m leans against a vertical wall at an angle of 45° to the horizontal ground. The coefficient of friction between the ladder and the wall is $\frac{1}{3}$ and that between the ladder and the ground is $\frac{1}{2}$. How far up the ladder can a man whose weight is half the ladder, ascend before the ladder slips?

Solution

Assuming that the man has climbed to such a position that the ladder is about to slip. The diagram below shows the forces acting on the ladder *AB*, *M* is the man.



Let the distance $AM = x$ metres

$AG = GB = 3.5$ metres

F_1 and F_2 are the frictional forces

N_1 and N_2 are the normal reactions

Frictional force = μR when ladder is about to slip

$\therefore \quad F_1 = \frac{1}{2}N_1$ .....(1) and

$F_2 = \frac{1}{3}N_2$ .....(2)

Resolving vertically :

$N_1 + F_2 = 140 + 70 = 210 \ldots.(3)$

Resolving horizontally :

$F_1 = N_2 \ldots\ldots\ldots\ldots(4)$

From the four equations above

$N_1 = 360\text{ N}; \quad F_1 = 180\text{ N};$

$N_2 = 180\text{ N}; \quad F_2 = 60\text{ N}$

Taking moments about A gives:

$140 \times 3.5\cos 45° + 70 \times x\cos 45°$

$= 180 \times 7\sin 45° + 60 \times 7\cos 45°$

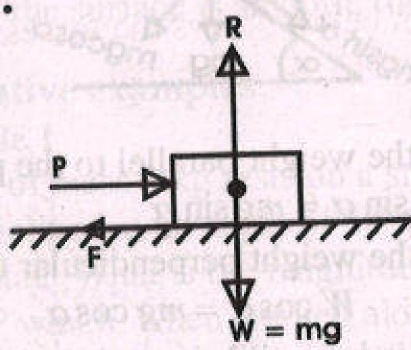
$7 + x = 9 + 3$

$x = 5$ metres

∴ The distance up the ladder the man can safely climb is 5 m.

Equilibrium on a plane horizontal surface

Case 1:



Consider a body of mass m, resting on a rough horizontal plane and being acted upon by a horizontal force P. The other forces acting on it are:

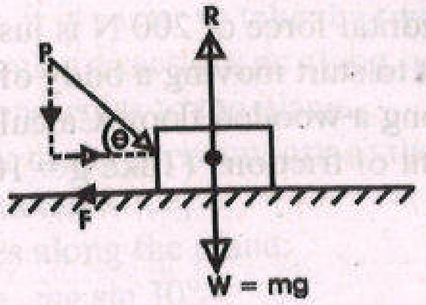
1. The weight $W = mg$ acting vertically downwards
2. The Normal reaction R
3. The Frictional force $F = \mu R$

For vertical forces: $R = mg$

For horizontal forces: $F = P$

If the *surface is smooth*, then $F = 0$

Case 2:



Consider a body of mass m, resting on a rough horizontal plane and being acted upon by a horizontal force P which acts at an angle θ to the horizontal. The components of P parallel and perpendicular to the plane are $P\cos\theta$ and $P\sin\theta$ respectively

The other forces acting on it are:

1. The weight $W = mg$ acting vertically downwards
2. The Normal reaction R
3. The Frictional force $F = \mu R$

For vertical forces: $R = mg + P\sin\theta$

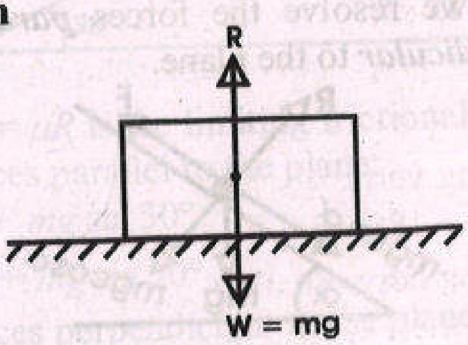
For horizontal forces: $F = P\cos\theta$

Illustrative examples

Example 1

A small body of mass 5 kg is placed on a horizontal table. What is the normal reaction between the table and the body? (Take g = 10 ms^{-2}).

Solution



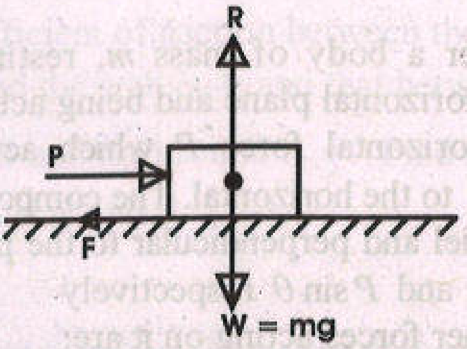
For vertical forces:

$R = mg = 5\text{ kg} \times 10\text{ ms}^{-2} = 50\text{ N}$

Example 2

If a horizontal force of 200 N is just sufficient to start moving a body of mass 50 kg along a wooden floor. Calculate the coefficient of friction. (Take $g = 10\ ms^{-2}$).

Solution



For vertical forces:

$R = mg = 50 \times 10$

For horizontal forces: $F = P$

$\mu R = P$

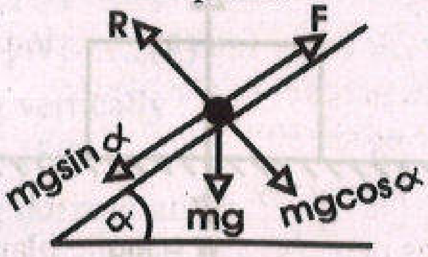
$\mu \times 50 \times 10 = 200$

$$\mu = \frac{200}{500} = 0.4$$

Equilibrium on an inclined plane

Case 1:

A body of mass m is at rest on a rough inclined plane making an angle α to the horizontal. For equilibrium on an inclined plane, we resolve the forces *parallel* and *perpendicular* to the plane.



Resolving the weight parallel to the plane gives: $W \sin\alpha = mg\sin\alpha$

Resolving the weight perpendicular to the plane gives: $W\cos\alpha = mg\cos\alpha$

F = The frictional force (i.e. if the plane is rough)

If the body is in equilibrium, then there is **no acceleration** so the forces balance:

For forces parallel to the plane:

$F = mg\sin\alpha$

For forces perpendicular to the plane:

$R = mg\cos\alpha$

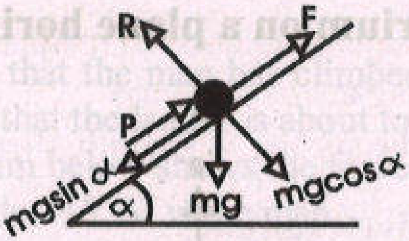
$\boldsymbol{F = \mu R} = \tan\alpha$

= limiting friction for inclined planes

Note: If the surface is smooth, then $F = 0$

Case 2:

If a force P parallel to the plane holds a body of mass m at rest (or prevents the body from sliding down) on an inclined plane making an angle α to the horizontal.



Resolving the weight parallel to the plane gives: $W\sin\alpha = mg\sin\alpha$

Resolving the weight perpendicular to the plane gives: $W\cos\alpha = mg\cos\alpha$

If the body is in equilibrium, then there is **no acceleration** so the forces balance:

For forces parallel to the plane:

$F + P = mg\sin\alpha$

For forces perpendicular to the plane:

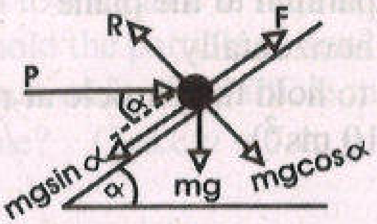
$R = mg\cos\alpha$

$\boldsymbol{F = \mu R}$

Note: If the surface is smooth, then $F = 0$

Note carefully: If the force is required to just move the body up the plane, the frictional force will act down along the plane.

Case 3:
If the force *P in case 2 above* acts horizontally, we find the components of *P* parallel and perpendicular to the plane.



Resolving *P* parallel to the plane gives: $P\cos\alpha$

Resolving *P* perpendicular to the plane gives: $P\sin\alpha$

If the body is in equilibrium, then there is no acceleration so the forces balance:

For forces parallel to the plane:

$$P\cos\alpha + F = mg\sin\alpha$$

For forces perpendicular to the plane:

$$R = mg\cos\alpha + P\sin\alpha$$

Note: If the surface is smooth, then $F = 0$

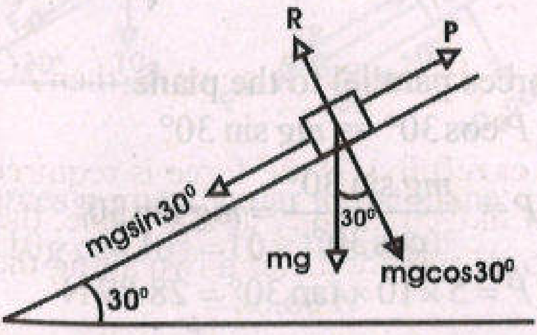
Illustrative examples

Example 1

A body of mass 2 kg rests on a smooth plane inclined at an angle of 30° to the horizontal. What is the magnitude of the force, *P* which, when applied along the plane, keeps it from sliding? What is the normal reaction between the body and plane? (Take $g = 10\ \text{ms}^{-2}$).

Solution

The three forces *R*, *F* and *mg* acting on the body are shown in the diagram below.



As mentioned earlier, in all inclined plane problems, it is easier to take the two perpendicular directions as along (parallel) and perpendicular to the plane.

[Note that the plane is smooth so there is no frictional force]

For forces along the plane:

$$P = mg\sin 30°$$

$$P = 2\times10\times\tfrac{1}{2} = 10\ \text{N}$$

For forces perpendicular to the plane:

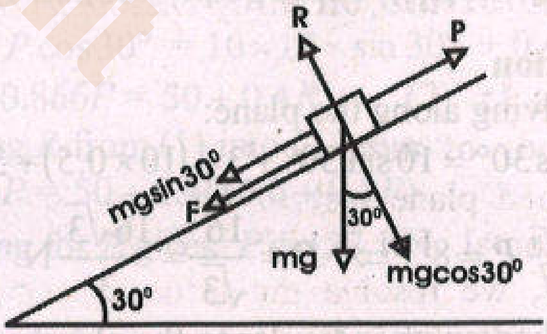
$$R = mg\cos 30° = 2\times10\times0.866 = 17.32\ \text{N}$$

Example 2

A body of mass 8 kg is on the point of slipping down a rough plane of inclination 30° to the horizontal. What is the magnitude of the force, *P* which, when applied along the plane, will just move it up? (Take $g = 10\ \text{ms}^{-2}$).

Solution

Let *P* be the force required to just move it up the plane. The three other forces acting on the body are *R*, *F* and *mg*.



Note $F = \mu R$ is the limiting frictional force.

For forces parallel to the plane:

$$P = F + mg\sin 30°$$

$$= \mu R + mg\sin 30° \ldots\ldots\ldots\ldots(1)$$

For forces perpendicular to the plane:

$$R = mg\cos 30°$$

$$= 8\times10\times\tfrac{\sqrt{3}}{2} = 40\sqrt{3}\ \text{N} \ldots\ldots(2)$$

Also since the body is about to move up the plane of slope 30°;

$\mu = \tan 30° = \frac{1}{\sqrt{3}}$ Substituting the values of R and μ into (1)

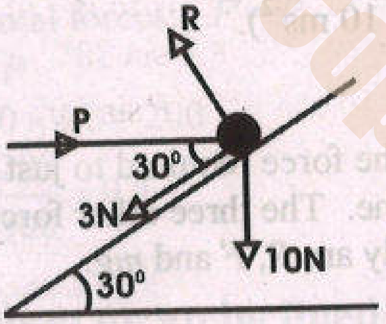
$$\therefore \quad P = \mu R + mg\sin 30°$$

$$= \left(\frac{1}{\sqrt{3}}\right)\left(40\sqrt{3}\right) + \left(8 \times 10 \times \frac{1}{2}\right) = 80\,\text{N}$$

Note: $\mu = \tan\alpha$, where α is the angle of the plane with the horizontal when the body is about to move.

Example 3

A particle on a smooth inclined plane is subjected to the forces shown in the diagram and is in equilibrium. Find the forces R and P.



Solution

Resolving along the plane:

$$P\cos 30° = 10\sin 30° + 3 = (10 \times 0.5) + 3 = 8$$

$$\tfrac{\sqrt{3}}{2}P = 8 \Rightarrow P = \frac{16}{\sqrt{3}} = \frac{16\sqrt{3}}{3}\,\text{N}$$

Resolving perpendicular to the plane:

$$P\sin 30° + 10\cos 30 = R$$

$$R = \left(\frac{16\sqrt{3}}{3}\right) \times \frac{1}{2} + 10 \times \frac{\sqrt{3}}{2}$$

$$R = \frac{23\sqrt{3}}{3}\,\text{N}$$

Example 4

A particle of mass 5 kg is placed on a smooth plane inclined at an angle of 30° to the horizontal. What force

i) acting parallel to the plane

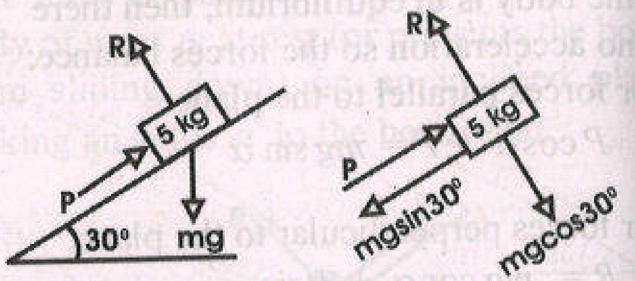
ii) acting horizontally

is required to hold the particle at rest? (Take $g = 10\ \text{ms}^{-2}$).

Solution

i) The plane is *smooth*, therefore, friction $F = 0$

The diagram below shows the forces acting on the particle.

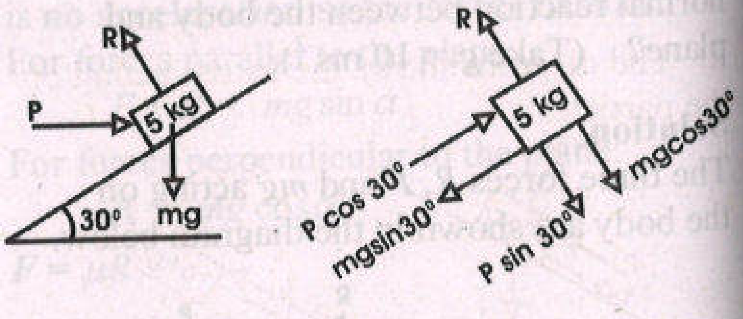


For forces parallel to the plane:

$$P = mg\sin 30°$$

$$P = 5 \times 10 \times \sin 30° = 25\ \text{N}$$

ii) The diagram shows the forces acting on the particle, where P is a horizontal force. The diagram below shows the forces acting on the particle.



For forces parallel to the plane:

$$P\cos 30° = mg\sin 30°$$

$$P = \frac{mg\sin 30°}{\cos 30°} = mg\tan 30°$$

$$P = 5 \times 10 \times \tan 30° = 28.87\,\text{N}$$

Example 5

a) A particle of mass 10 kg is placed on a rough plane inclined at an angle of 30° to the horizontal. What force acting parallel to the plane is required?

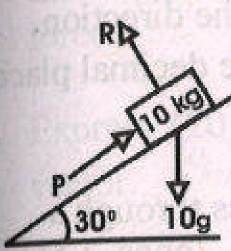
i) to hold the particle at rest?

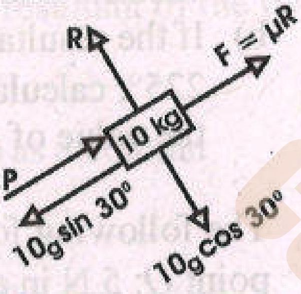
ii) to just make the particle move up the plane? (Take $\mu = 0.4$, $g = 10\ \text{ms}^{-2}$)

b) What horizontal force is required to just make it move up the plane?

Solution

a) i) The diagram below shows the forces acting on the particle.





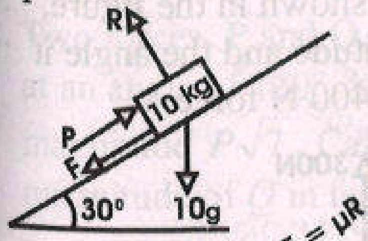
For forces perpendicular to the plane:

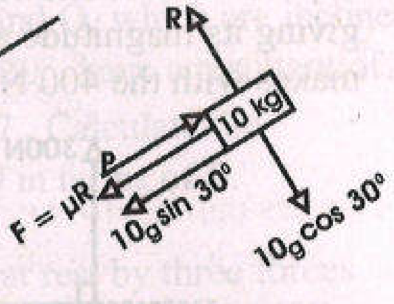
$$R = 10g\cos 30° = 10\times 10\times \cos 30°$$
$$= 86.60\text{ N}$$

For forces parallel to the plane:

$$F + P = 10g\sin 30°$$
$$\mu R + P = 10g\sin 30°$$
$$0.4(86.60) + P = 10\times 10\times \sin 30°$$
$$P = 50 - 36.4 = 15.4\text{ N}$$

ii) The diagram shows the forces acting on the particle.





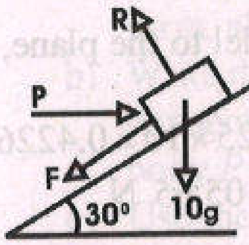
For forces perpendicular to the plane:

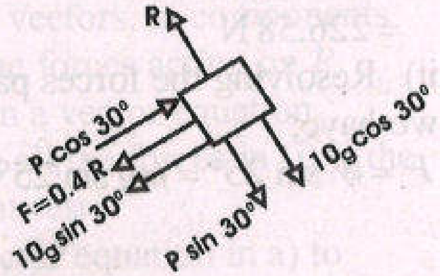
$$R = 10g\cos 30° = 10\times 10\cos 30°$$
$$= 86.60\text{ N}$$

For forces parallel to the plane:

$$P = 10g\sin 30° + F$$
$$P = 10g\sin 30° + \mu R$$
$$P = 10\times 10\times \sin 30° + 0.4(86.60)$$
$$P = 50 + 36.4 = 86.6\text{ N}$$

b) If the particle is just about to move up the plane, friction will act downwards. The diagram below shows the forces acting on the particle, where P is a horizontal force.





For forces perpendicular to the plane:

$$R = 10g\cos 30° + P\sin 30°$$
$$= 10\times 10\cos 30° + P\sin 30°$$
$$= 86.6 + 0.5P\ \ldots.(1)$$

For forces parallel to the plane:

$$P\cos 30° = 10g\sin 30° + F$$
$$P\cos 30° = 10g\sin 30° + \mu R$$
$$P\cos 30° = 10\times 10\times \sin 30° + 0.4R$$
$$0.866P = 50 + 0.4R\ \ldots\ldots(2)$$

Putting R from (1) into (2) gives:

$0.866P = 50 + 0.4(86.6 + 0.5P)$,

Solving for P gives: $P = 127.1$ N

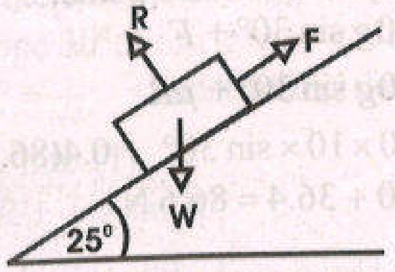
Example 6

A block of mass 25 kg is placed on a plane inclined at an angle of 25° to the horizontal as shown in the diagram below. If the block slides down the plane, calculate the

i) normal reaction perpendicular to the plane

ii) frictional force F acting on the block 3 (Take g = 10 ms^{-2}).

Solution



i) Resolving the forces perpendicular to the plane, we have;

$R = W\cos 25° = mg\cos 25°$

$= 25 \times 10 \times 0.9063$

$= 226.58\text{ N}$

ii) Resolving the forces parallel to the plane, we have;

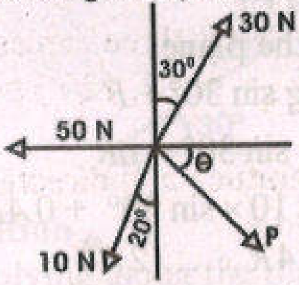
$F = W\sin 25° = mg\sin 25° = 25 \times 10 \times 0.4226$

$= 105.65\text{ N}$

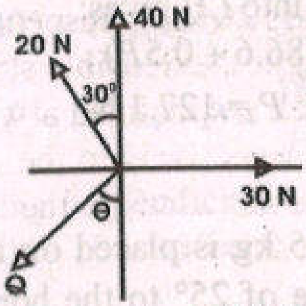
Review exercise 25

1. The system of forces below are in equilibrium, find the unknown forces in magnitude and direction.

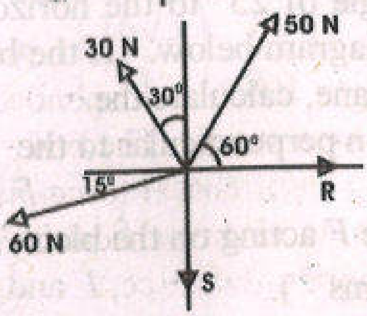
(a)



(b)



(c)

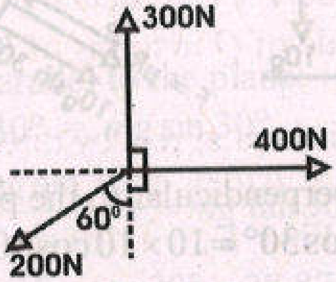


2. Three forces $(3\mathbf{i} + 5\mathbf{j})$ N, $(4\mathbf{i} + 11\mathbf{j})$ N, $(2\mathbf{i} + \mathbf{j})$ N act at a point. Given that **i** and **j** are perpendicular unit vectors, find;
   a) the resultant of the forces in the form $a\mathbf{i} + b\mathbf{j}$,
   b) the magnitude of this resultant,
   c) the cosine of the angle that the resultant makes with the unit vector **i**.

3. A particle is acted on by three forces, $\mathbf{F}_1 = (120N, 180°)$, $\mathbf{F}_2 = (50N, 030°)$ and $\mathbf{F}_3 = (KN, 315°)$.
   a) Express each force as a column vector.
   b) If the resultant is in the direction 225°, calculate to one decimal place the value of K.

4. The following forces pass through a point O: 5 N in a direction 000°, 1 N in a direction 090°, 4 N in a direction 225° and 6 N in a direction 315°. Find the magnitude and direction of their resultant. Two further horizontal forces are introduced to act at O: P N in a direction 135° and Q N in a direction 225°. If the complete set of forces is now in equilibrium, calculate the values of P and Q.

5. Find the resultant of the system of coplanar forces shown in the figure, giving its magnitude and the angle it makes with the 400 N force.



6. The components of a force **F** are $\begin{pmatrix} \sqrt{3}\text{ N} \\ 1\text{ N} \end{pmatrix}$. Find the magnitude and direction of **F**.

7. The resultant of two forces **P** and **Q** of magnitude 3 N and 4 N respectively has magnitude of $\sqrt{37}$ N. Find the angle between **P** and **Q**.

8. A body is acted upon by forces $\mathbf{F}_1 = (4N, 045°)$ and $\mathbf{F}_2 = (8N, 120°)$. Find, correct to three significant figures, the magnitude of the resultant of the two forces.

9. Express $(20\,N, 240°)$ as a column vector.

10. Find the resultant of the forces (in newton) $\mathbf{F}_1 = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$, $\mathbf{F}_2 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and $\mathbf{F}_3 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

11. A particle is kept at rest under the action of the three forces (4N, 090°), (*X*N, 120°) and ($4\sqrt{3}$ N, 270°). Calculate the value of *X*.

12. Two forces, **P** and **Q**, which are inclined at an angle of 120°, have a resultant of magnitude $P\sqrt{7}$. Calculate the magnitude of *Q* in terms of *P*.

13. A body is kept at rest by three forces $\mathbf{F}_1 = (10N, 030°)$, $\mathbf{F}_2 = (10N, 150°)$ and $\mathbf{F}_3$. Find $\mathbf{F}_3$.

14. Express in the form $a\mathbf{i} + b\mathbf{j}$N, where *a* and *b* are real numbers, the force **F** = (3N, 060°).

15. A body *P* of mass 5 kg is suspended by means of two light inextensible strings *AP* and *BP* which are inclined at angles of 30° and 60° respectively to the downward vertical. T_1 and T_2 are the magnitudes of the tensions in *AP* and *BP* respectively.

a) Express as vectors, in components form, all the forces acting on *P*.

b) Write down a vector equation connecting all the forces in (i) if the body *P* is at rest.

c) Use the vector equation in a) to calculate the values of T_1 and T_2.

d) Find the direction in which *P* begins to move when the string *AP* is cut or snapped. (Take $g = 9.8\text{ m/s}^2$)

16. A body is acted upon by three forces $\mathbf{F}_1$ = (10N, 030°), $\mathbf{F}_2$ = (15N, 120°) and $\mathbf{F}_3$ = (15N, 240°). Determine the magnitude and direction of the resultant force.

17. A body *P* of mass 20 kg is suspended by means of two light inextensible strings *XP* and *YP* which are inclined at angles of 30° and 60° respectively to the downward vertical. Calculate the tensions in both strings. (Take $g = 10\text{ ms}^{-2}$).

18. A body of weight 100 N is suspended from a ceiling by a string. The body is pulled aside by a horizontal force *F* so that the inclination to the vertical is 30°. Calculate the horizontal force, *F* and the tension in the string.

19. A body of mass 5 kg is suspended from a roof by an inextensible string. The body is pulled aside by a horizontal force F so that the inclination to the vertical is 60°. Calculate the horizontal force, F and the tension in the string. (Take $g = 10\ \text{ms}^{-2}$)

20. The foot of a uniform ladder of length 10 m and weight 50 N, rests on a rough horizontal ground, and the top of the ladder rest against a smooth vertical wall. The ladder is inclined at an angle of 30 ° to the vertical. Find the magnitude of the force exerted by the ladder on the wall.

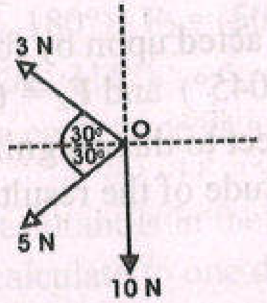
21. A body of mass 10 kg rests on a smooth plane inclined at an angle of 60° to the horizontal. What is the magnitude of the force, P which, when applied along the plane, keeps it from sliding? What is the normal reaction between the body and plane? (Take $g = 10\ \text{ms}^{-2}$).

22. A ladder of length 4 m and weight 200 N rests with one end on a rough horizontal ground and the other end against a smooth vertical wall. The foot of the ladder is 2 m from the wall and the ladder is on the point of slipping. The centre of gravity of the ladder is 1 m from its foot. Find coefficient of friction between the ladder and the ground.

23. A particle of mass 1.5 kg rests on a rough plane inclined at 45° to the horizontal. It is maintained in equilibrium by a horizontal force of P N. Given that the coefficient of friction between the particle and the plane is $\frac{1}{4}$, calculate the value of P when the particle is on the point of moving

i) down the plane ii) up the plane.

24. The diagram below shows three coplanar horizontal forces acting on a particle at O. Calculate the magnitude and direction of the additional force which, acting on the particle, will result in equilibrium.



25. A uniform ladder of length 8 m and mass 20 kg is inclined at 60° to the horizontal against a smooth vertical wall. A 60 kg man is standing on the ladder x m from its lower end. The horizontal floor has coefficient of friction 0.4 with the base of the ladder. The ladder is about to slip.

i) Show that the frictional force on the ladder is $32g$ N.

ii) Find the reaction of the wall on the ladder.

iii) By taking moments about the base of the ladder, or otherwise, show that x is about 6.06 m

26. The ends P and Q of an inextensible string 17 m long are attached to two fixed points 13 m apart on the same horizontal level. A body of mass 20 kg is suspended from a point C on the string, 5 m from P.

a) Calculate, correct to the nearest 0.1° the angle which each part of the string makes with the horizontal.

b) Express as column vectors the tension T_1 and T_2 in the string $\overline{CP}$ and $\overline{CQ}$ respectively and hence find the values of T_1 and T_2. (Take $g = 10\text{m/s}^2$).

27. A uniform plank AB, 3 m long, of mass 10 kg, is supported in a horizontal position by two vertical strings attached at distances 1 m and 2 m from end A. Calculate the mass which should be placed at the end A of the plank so that
i) the tension in the string nearer end B just vanishes,
ii) the tension in the string nearer A is three times that in the string nearer B.

28. A uniform rod AB, of length 150 mm and mass 50 g, has a load of X grams attached at a point C of the rod such that $AC = 30$ mm. The rod rests horizontally on a smooth support placed at a point D where $AD = 60$ mm. Calculate the numerical value of X.

29. A uniform bar AB of length 6 m and mass 15 kg lies horizontally on supports C and D where $AC = 1$ m and $DB = 2$ m. Find the magnitude of the vertical forces at C and D.

30. A particle P, of mass 7 m is placed on a rough horizontal table, the coefficient of friction between P and the table being μ. A force of magnitude 2 mg, acting upwards at an acute angle α to the horizontal, is applied to P and equilibrium is on the point of being broken by the particle sliding on the table. Given that $\tan\alpha = \frac{5}{12}$, find the value of μ.

31. A block of mass 1.5 kg lying on a rough inclined plane is prevented from slipping down the plane by a string end, to a fixed point. The string is parallel to a line of greatest slope of the plane. The angle of inclination of the plane to the horizontal is α, where $\tan\alpha = \frac{4}{3}$, and the coefficient of friction between the block and the plane is 0.5. Calculate the tension of the string in newtons. If the string is cut, find the acceleration with which the block slides down the plane.

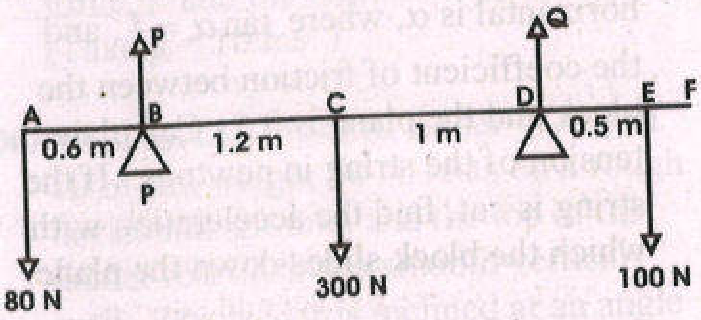
32. The foot of a uniform ladder, of length l and weight W, rests on rough horizontal ground, and the top of the ladder rests against a smooth vertical wall. The ladder is inclined at 30° to the vertical. Find the magnitude of the force exerted by the ladder on the wall.

33. Forces of magnitude 2 N, 3 N and 4 N act at a point in the directions, W, SW and SE respectively. Find the magnitude and direction of their resultant.

34. A body of mass 20 kg is placed on a rough surface inclined at an angle of 30° to the horizontal. It is just about to slip downwards. Find the coefficient of friction. Using this value, find the least force parallel to the plane which is just sufficient to make the body move up the plane.

35. A uniform ladder, weight 150 N and length 6 m, is placed on a rough horizontal ground at an angle of 60° with its top resting against a smooth vertical wall. Find the reactions at the ends of the ladder.

36. In the figure below, the beam *ABCDEF* is shown resting on two pivots *B* and *D*. The weight of the beam is 300 N and weights are attached at *A* and *E* at the distances shown. Find the reactions at the pivots.



37. Find the force which must act with the force 5 N due West to produce a force of 12 N due North

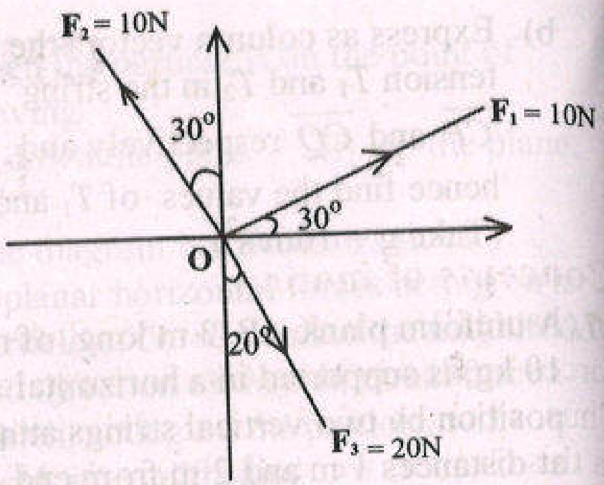
38. Two forces, 10 N and 6 N act in the direction 060° and 330° respectively. Find the component of the resultant along the *x*-axis.

39. A particle is acted on by forces $\mathbf{F}_1(5\text{ N}, 240°)$ and $\mathbf{F}_2(4\text{ N}, 315°)$. Find, correct to **two** decimal places, the magnitude of the resultant force.

40. A uniform plank *AB*, 10 m long and of mass *M* kg, rests on two supports *A* and *B*. A load of 8 kg is placed on the plank at point *C*, such that $|AC| = 3.5$ m. The reaction at *B* is 100N. If the system remains in equilibrium under the action of these forces, calculate

i) the value of *M*;
ii) the reaction at *A*.
[Take $g = 10$ ms^{-1}]

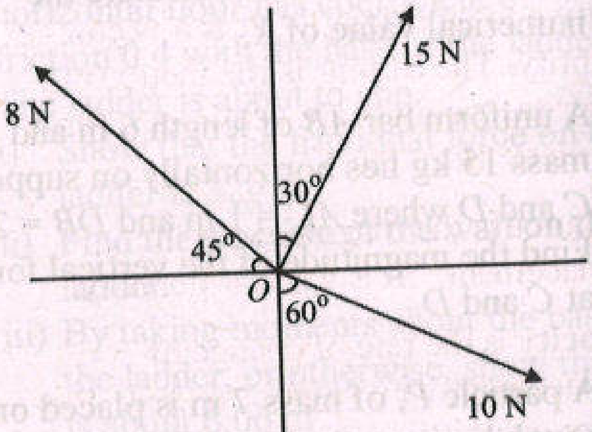
41. Forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ act on a body at point *O* as shown in the diagram. Calculate, correct to two decimal places i) the magnitude, ii) the direction of the resultant force.



42. A body *N* of mass 10 kg is suspended by means of two inextensible strings *AN* and *BN* which are inclined at angles of 60° and 30° respectively to the downward vertical, where *A* and *B* are on the same horizontal line. If T_1 and T_2 are the magnitude of the tensions in *AN* and *BN* respectively,

i) write down a vector equation connecting all the forces acting on the body when *N* is at rest.
ii) evaluate the values of T_1 and T_2.

43.



Forces of magnitudes 15 N, 10 N and 8N act on a body at *O* as shown in the diagram above. Find, correct to **three** significant figures, the magnitude of the force required to keep the system in equilibrium.

CHAPTER 26

DYNAMICS

Concepts of motion

In chapter 25, we discussed the effects of forces on a body in equilibrium or at rest. Thus, a branch of *Mechanics* called **Statics**. In this chapter, we shall discuss the effects of forces on a body in motion. This is also a branch of mechanics called **Dynamics.** Dynamics is the study of a body in motion. In mechanics, a **body** is any object to which force can be applied. The branch of dynamics which studies the motion of a particle is called **Kinematics.** A **particle** is a body whose dimensions, except mass are negligible. In the study of kinematics, the cause of the motion is not taken into consideration.

Definition of terms

Displacement is the position of a point relative to an origin O. It is a vector quantity. The SI unit is the metre (m). Other metric units are centimetre (cm), kilometre (km) etc.

Distance is the magnitude of the displacement. It is a scalar quantity.

Velocity is the rate of change of displacement with respect to time. It is a vector quantity.
The SI unit is metre per second (ms^{-1} or m/s). Other metric units are cms^{-1}, kmh^{-1} or kms^{-1} etc.

Speed is the magnitude of the velocity.
It is a scalar quantity.
We have **uniform velocity** if the rate of change of displacement with respect to time is constant.

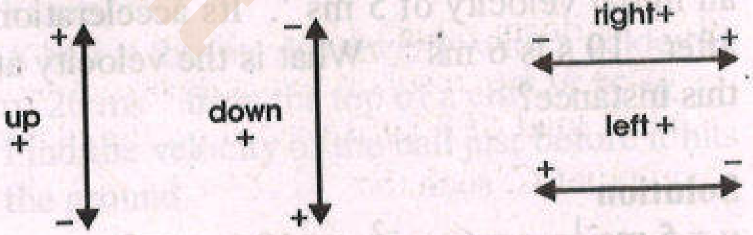
$$\textbf{Average velocity} = \frac{\text{change in displacement}}{\text{time taken}}$$

$$\text{and } \textbf{Average speed} = \frac{\text{change in distance}}{\text{time taken}}$$

Acceleration is the rate of change of velocity with respect to time. It is a vector quantity. The SI unit is metre per second squared (ms^{-2} or m/s^2). Other metric units are cms^{-2}, kms^{-2} etc.
Negative acceleration is sometimes called **retardation** or **deceleration**. We have **uniform acceleration** if the rate of change of velocity with respect to time is constant.

Motion in one dimension

When a particle moves in **one dimension,** i.e. along a straight line, it has only two possible directions in which to move. *Positive* and *negative* signs are used to identify the two directions as shown below.



Equations of motion – uniform acceleration

Consider a particle which moves from a fixed point O along a straight line with a uniform acceleration, a. its initial velocity is u and its velocity and displacement from the fixed point at time, t are v and s respectively. The equations of motion in a straight line are:

1. $v = u + at$
2. $s = \frac{1}{2}(u + v)t$
3. $v^2 = u^2 + 2as$
4. $s = ut + \frac{1}{2}at^2$

The notation used is:

s – displacement ; t – time;

u – initial velocity; v – final velocity;

a – acceleration

Note that the units used must be consistent. For example, if s is in metres and t is in seconds, then u and v must be in ms^{-1} and a in ms^{-2}. To solve problems using these equations of motion:

1. Choose the positive direction.
2. List the five quantities (s, t, u, v, a), fill in the known values and mark which are to be found.
3. Use the appropriate equation(s) to find the required unknown(s). If any three of the quantities are known, then the other can always be found.

Note: If a particle moves from rest, then the initial velocity, $u = 0$. Also if a particle comes to rest then the final velocity, $v = 0$.

Illustrative examples

Example 1

A body moving in a linear motion starts with an initial velocity of 5 ms^{-1}. Its acceleration after 10 s is 6 ms^{-2}. What is the velocity at this instance?

Solution

$u = 5\ \text{ms}^{-1},\ a = 6\ \text{ms}^{-2},\ t = 10\ \text{s},\ v = ?$

Using $v = u + at$

$$v = 5 + 6(10) = 5 + 60 = 65\,\text{ms}^{-1}$$

Example 2

A particle moves in a straight line with a constant acceleration. Its initial velocity is 6 ms^{-1} and its velocity after 8 s is 10 ms^{-1}. Find the acceleration and the displacement of the particle after 16 s.

Solution

First stage

$u = 6\ \text{ms}^{-1},\ v = 10\ \text{ms}^{-1}, t = 8\ \text{s},\ a = ?\ s = ?$

Using $v = u + at$

$$10 = 6 + a(8) \quad \Rightarrow \quad a = 0.5\ \text{ms}^{-2}$$

So the constant acceleration is 0.5 ms^{-2} This is the same as the acceleration after 16 s since the acceleration is constant.

Second stage

$u = 6\ \text{ms}^{-1}, v = ?, t = 16\ \text{s}, a = 0.5\ \text{ms}^{-2}, s = ?$

Using $s = ut + \frac{1}{2}at^2$

$$s = 6(16) + \tfrac{1}{2}(0.5)(16)^2 = 160\,\text{m}$$

So the displacement after 16 s is 160 m.

Example 3

A car changes its velocity from 10 ms^{-1} for 10 s at a rate of 0.5 ms^{-2}. Find the distance covered.

Solution

$u = 10\ \text{ms}^{-1},\ a = 0.5\ \text{ms}^{-2}, t = 10\ \text{s},\ s = ?$

Using $s = ut + \frac{1}{2}at^2$, we have

$$s = 10(10) + \tfrac{1}{2}(0.5)(10)^2$$

$$s = 100 + 25 \quad \Rightarrow \quad s = 125\,\text{m}$$

Example 4

A car is moving along a straight road with uniform acceleration. The car passes a checkpoint M with a velocity of 12 ms^{-1} and another checkpoint N with velocity of 32 ms^{-1} The distance between M and N is 1200 m. If Q is the midpoint of MN, find

i) correct to the nearest whole number, the time taken by the car to move from M to N.

ii) correct to one decimal place, the velocity of the car with which the car passes Q.

Solution

i) $s = 1200$ m i.e. distance between M and N, $t = ?, u = 12\ \text{ms}^{-1}, v = 32\ \text{ms}^{-1},\ a = ?$

First we find a, using $v^2 = u^2 + 2as$

$$32^2 = (12)^2 + 2(a)(1200)$$

$$a = \frac{32^2 - (12)^2}{2 \times 1200} \Rightarrow a = \tfrac{11}{30}\ \text{ms}^{-2}$$

Next using $v = u + at$

$$32 = 12 + \tfrac{11}{30}t$$

$$t = \tfrac{30}{11} \times 20 = 54.54\ \text{s}$$

$\therefore$ The time taken by the car to move from M to N is 55 s to the nearest whole number.

ii) $s = \frac{1200}{2} = 600$ m i.e. distance between M and Q, $t = ?$, $u = 12\ \text{ms}^{-1}$, $v = 32\ \text{ms}^{-1}$, $a = \frac{11}{30}\ \text{ms}^{-2}$

Using $v^2 = 12^2 + 2(\frac{11}{30})(600)$

$$v^2 = 584$$

$$v = \sqrt{584} = 24.166\,\text{ms}^{-1}$$

$\therefore$ The velocity of the car with which the car passes Q is 24.2 ms^{-1} to one decimal place.

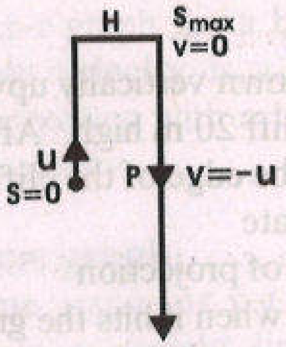
Vertical motion under gravity

The motion of a body thrown **vertically upward** or falling **freely downward** (ignoring air resistance) is a special case of uniform acceleration in a straight line. This uniform acceleration is due to **gravity** and acts vertically downwards towards the centre of the earth. It is denoted by 'g' and the common approximate values are **10 ms^{-2}** and **9.8 ms^{-2}**. If a body is thrown vertically downwards, $\boldsymbol{a = g}$ and if a body is thrown vertically upwards, $\boldsymbol{a = -g}$. If a body is thrown vertically upwards, the following conditions are true:

Taking upward direction as positive, $a = -g$

At the highest point H, s is maximum and $v = 0$

At the point of projection P, $s = 0$ and $v = \pm u$



Illustrative examples

Example 1

A ball is thrown vertically upwards with a velocity of 25 ms^{-1}. Find its height when it is moving with a velocity of 15 ms^{-1}. (Take $g = 10\ \text{ms}^{-2}$).

Solution

$u = 25\ \text{ms}^{-1}$, $a = -g = -10\ \text{ms}^{-2}$, $v = 15\ \text{ms}^{-1}$, $s = h = ?$

Using $v^2 = u^2 + 2as$

$$(15)^2 = (25)^2 + 2(-10)s$$

$$225^2 = 625 - 20s$$

$$20s = 400 \Rightarrow s = 20\ \text{m}$$

So the height when it is moving with a velocity of 15 ms^{-1} is 20 m.

Example 2

A ball is thrown downwards with a velocity of 20 ms^{-1} from the top of a cliff of 35 m. Find the velocity of the ball just before it hits the ground.

Solution

$u = 20\ \text{ms}^{-1}$, $a = \text{g} = 10\ \text{ms}^{-2}$, $v = ?$, $s = h = 35$ m

Using $v^2 = u^2 + 2as$

$$v^2 = (20)^2 + 2(10)(35)$$

$$v^2 = 400 + 700 = 1100$$

$$v = \sqrt{1100} = 33.2\ \text{ms}^{-1}$$

So the velocity of the ball just before it hits the ground is 33.2 ms^{-1}.

Example 3

A stone is thrown vertically upwards from the top of a cliff 20 m high. After a time of 3 s it passes the edge of the cliff on its way down. Calculate

a) the speed of projection

b) the speed when it hits the ground

c) the times when it is 10 m above the top of the cliff

d) the time it is 15 m above the ground

Solution

Take up as positive

a) $s = 0\text{ m},\ t = 3\text{ s},\ u = ?,\ v = ?,$
$a = -10\text{ ms}^{-2}$

Using $s = ut + \frac{1}{2}at^2$, we have

$$0 = u(3) + \tfrac{1}{2}(-10)(3)^2$$

$$0 = 3u - 45 \quad \Rightarrow 3u = 45 \Rightarrow \ u = 15\text{ ms}^{-1}$$

b) $s = -20\text{ m},\ t = ?,\ u = 15\text{ ms}^{-1},\ v = ?,$
$a = -10\text{ ms}^{-2}$

Note: $a = -g$ for downward motion

Using $v^2 = u^2 + 2as$

$$v^2 = (15)^2 + 2(-10)(-20)$$

$$v^2 = 625$$

$v = \sqrt{625} = -25\ \text{ms}^{-1}$ i.e. minus means particle is moving downwards with speed 25 ms^{-1}

c) $s = 10\text{ m},\ t = ?,\ u = 15\text{ ms}^{-1},\ v = ?,$
$a = -10\text{ ms}^{-2}$

Using $s = ut + \frac{1}{2}at^2$, we have

$$10 = 15t + \tfrac{1}{2}(-10)t^2 \quad \Rightarrow \quad 10 = 15t - 5t^2$$

$$t^2 - 3t + 2 = 0$$

$$(t-1)(t-2) = 0 \quad \Rightarrow \quad t = 1\text{ s or } t = 2\text{ s}$$

The times when it is 10 m above the top of the cliff are 1 s and 2 s.

d) $s = -15\text{ m},\ t = ?,\ u = 15\text{ ms}^{-1},\ v = ?,$
$a = -10\text{ ms}^{-2}$

Using $s = ut + \frac{1}{2}at^2$, we have

$$-15 = 15t + \tfrac{1}{2}(-10)t^2$$

$$-15 = 15t - 5t^2$$

$$t^2 - 3t - 3 = 0$$

$$t = \frac{3 \pm \sqrt{9 - 4(1)(-3)}}{2}$$

$t \approx 3.7\text{ s}$ neglecting the negative root.

Example 4

A particle is projected vertically upwards with a velocity of 60 ms^{-1} from a point O. Find the

a) maximum height reached by the particle.

b) time taken to return to the point of projection. [Take $g = 10\text{ ms}^{-1}$]

Solution

a) $v^2 = u^2 + 2gs$

At maximum height $v = 0$

$0 = 60^2 - 20s$ [Note: g is negative)

$$20s = 3600$$

$$s = 180\text{ m}$$

b) $v = u + gt$ At maximum height $v = 0$

$$0 = 60 - 10t \ \Rightarrow \quad t = 6\text{ s}$$

Required time $= 2 \times 6\text{ s} = 12\text{ s}$

Example 5

A stone is thrown vertically upwards with a speed of 20 ms^{-1}. A second stone is thrown vertically upwards from the same point with the same initial speed 20 ms^{-1} but 2 s later than the first. Show that the two stones collide at a distance of 15 m above the point of projection.

Solution

Take up as positive

Let $s = h$ m be the displacement of each stone when they collide.

For the first stone,
$s = h,\ t = T$ s (say), $u = 20\ \text{ms}^{-1},\ v = ?,$
$a = -10\ \text{ms}^{-2}$

For the second stone,
$s = h,\ t = (T-2)$ s, $u = 20\ \text{ms}^{-1},\ v = ?,$
$a = -10\ \text{ms}^{-2}$

Use $s = ut + \frac{1}{2}at^2$ for each stone.

Equate the two expressions for h and find T. (i.e. at the point of contact the stones are at the same height).

Use $s = ut + \frac{1}{2}at^2$ again with the value of T found and calculate h.

[$T = 3$ s and $h = 15$ m] Work out the solution.

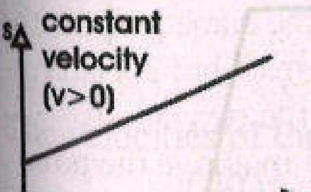
Graphs of motion

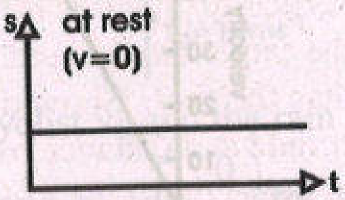
Displacement – time graph

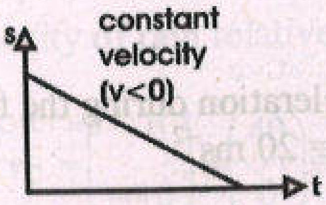
A displacement-time graph (or ***s–t* graph**) for a body moving in a straight line shows the displacement s from a fixed point plotted against time t.

In general, the **velocity** v of the body at time t is given by the **gradient** of the ***s–t* graph** at t since $v = \frac{ds}{dt}$. The ***s–t* graph** for a body moving with a **constant velocity** is a **straight line**.

The velocity v of the body is the **gradient** of the line.







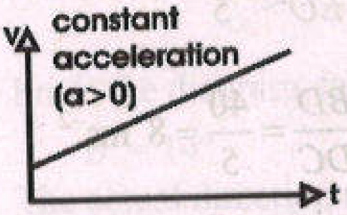
Note that the $s–t$ graph for a body moving with a **variable velocity** is a **curve.** We shall restrict ourselves with a body moving with constant velocity.

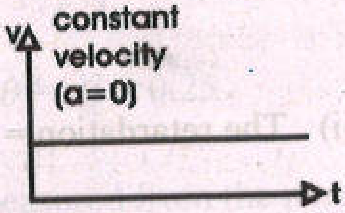
Velocity – time graph

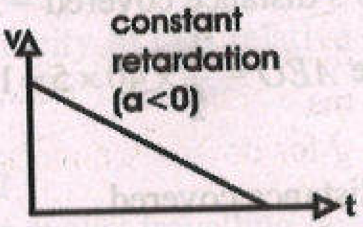
A **velocity time** graph (or $v–t$ graph) for a body moving in a straight line shows its velocity v plotted against time t.

The ***v–t* graph** for a body moving with **uniform acceleration** is a **straight line.**

The acceleration a of the body is given by the **gradient** of the line.







The **total displacement** s of a body can be found from a ***v–t* graph** of the above type by calculating the **area under the graph.**

Illustrative examples

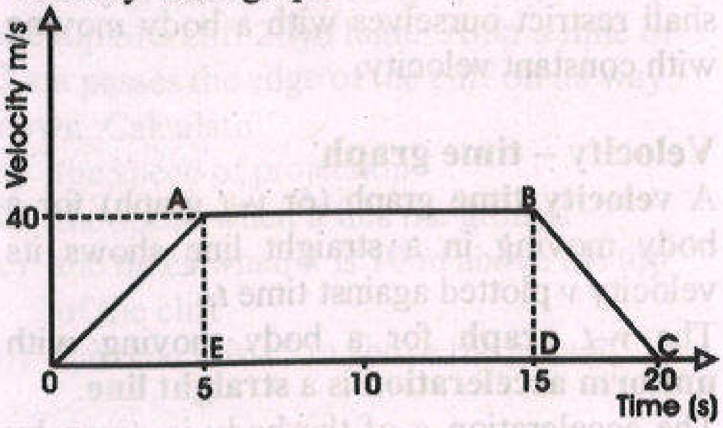
Example 1

A car starts from rest and accelerates uniformly until it reaches a velocity of 40 ms^{-1} after 5 s. It then travels with constant velocity for 10 s, and is brought to rest in 5 s with a uniform retardation. Draw a velocity–time graph and use it to determine

i) the acceleration
ii) the retardation
iii) the distance covered after 5 s
iv) the total distance covered

Solution

Velocity–time graph



i) The acceleration $= \dfrac{AE}{EO} = \dfrac{40}{5} = 8\ \text{ms}^{-2}$

ii) The retardation $= \dfrac{BD}{DC} = \dfrac{40}{5} = 8\ \text{ms}^{-2}$

iii) After 5 s distance covered = Area of triangle $AEO = \dfrac{1}{2} \times 40 \times 5 = 100\,\text{m}$

iv) Total distance covered
= Area of trapezium $OABC$
$= \dfrac{1}{2} \times (20+10) \times 40 = 600\,\text{m}$

Example 2

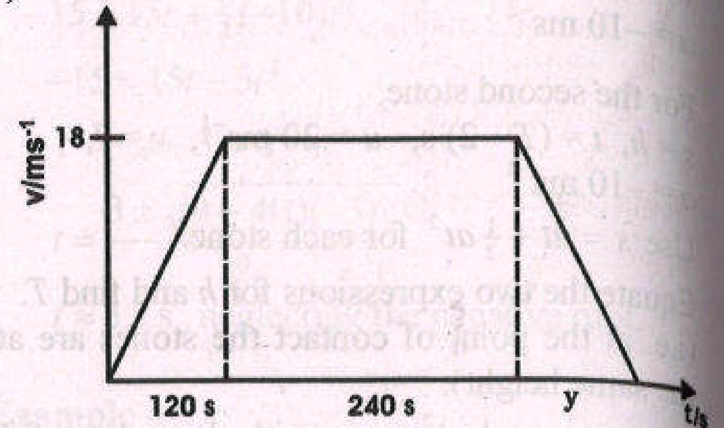
A train starts from rest and moves with a uniform acceleration of $0.15\ \text{ms}^{-2}$ for 2 minutes. It travels at a uniform speed for the next 4 minutes and retards uniformly to rest after covering a distance of 720 m.

i) sketch a velocity-time graph of the motion.

ii) calculate
  (α) the total distance travelled.
  (β) the magnitude of the uniform retardation.

Solution

i)



ii) (α) Total distance covered
$= \frac{1}{2} \times 18 \times 120 + 18 \times 240 + 720$
$= 5400 + 720$
$= 6120\ \text{m}$

(β) $\frac{1}{2} \times 18 \times y = 720$
$y = 80\ \text{s}$

$\therefore$ Retardation $= \dfrac{18}{80} = 0.225\ \text{m s}^{-2}$

Example 3

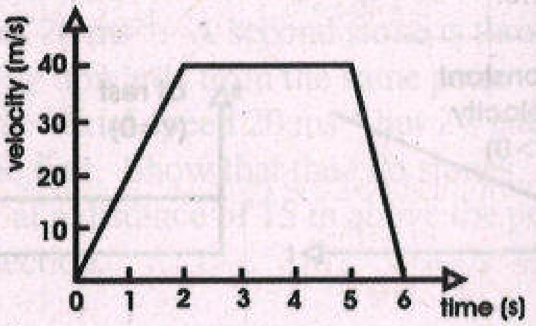
The velocity – time graph for a moving vehicle is shown below.

Use the graph to calculate

i) the acceleration during the first two seconds

ii) the retardation during the last second

iii) the total displacement of the vehicle



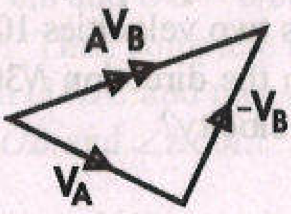
Solution

i) The acceleration during the first two seconds = $20\ \text{ms}^{-2}$

ii) The retardation during the last second $= 40\ ms^{-2}$

iii) The total displacement of the vehicle $= 180$ m
[Work out the solution]

Relative velocity



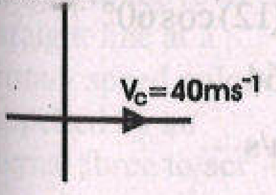
If $\mathbf{v}_A$ and $\mathbf{v}_B$ are the velocities of two moving objects relative to a fixed point, then the velocity of A relative to B denoted by $_A\mathbf{v}_B = \mathbf{v}_{AB} = \mathbf{v}_A - \mathbf{v}_B$

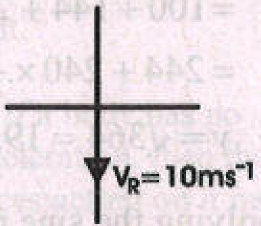
Similarly, $\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A$ is the velocity of B relative to A.
Note that the two velocities are referred to the same point.

Illustrative examples

Example 1
A cyclist is travelling eastwards at a velocity of $40\ ms^{-1}$ and rain is falling vertically at a speed of $10\ ms^{-1}$. Find the velocity of the rain relative to the cyclist.

Solution





The velocities of the cyclist $\mathbf{v}_C$ and the rain $\mathbf{v}_R$ are $\mathbf{v}_C = \begin{pmatrix} 40 \\ 0 \end{pmatrix} ms^{-1}$ and $\mathbf{v}_R = \begin{pmatrix} 0 \\ -10 \end{pmatrix} ms^{-1}$

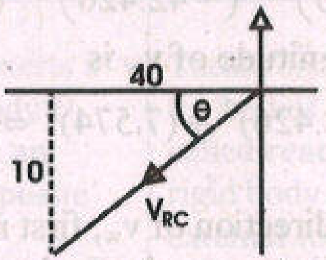
$\therefore$ The velocity of rain relative to cyclist is $\mathbf{v}_{RC}$, where

$$\mathbf{v}_{RC} = \mathbf{v}_R - \mathbf{v}_C = \begin{pmatrix} 0 \\ -10 \end{pmatrix} - \begin{pmatrix} 40 \\ 0 \end{pmatrix} = \begin{pmatrix} -40 \\ -10 \end{pmatrix} ms^{-1}$$

$\therefore$ The magnitude of $\mathbf{v}_{RC}$ is

$$|\mathbf{v}_{RC}| = \sqrt{(-40)^2 + (-10)^2} = \sqrt{1700}$$
$$= 41.2\ ms^{-1}$$

To find the direction of $\mathbf{v}_{RC}$, first represent $\mathbf{v}_{RC}$ on a rough diagram and calculate the acute angle θ, it makes with the x-axis.



From the diagram, $\tan\theta = \frac{10}{40} = 0.25$

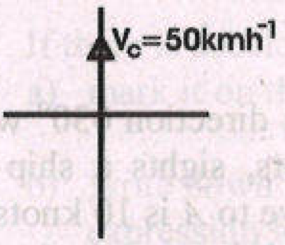
$\theta = 14°$

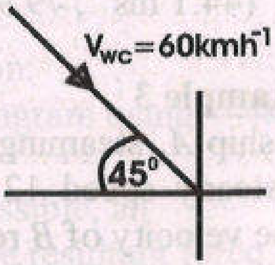
The actual direction measured from the north $= 270° - 14° = 256°$

$\therefore$ The velocity of rain relative to cyclist is $(41.2\ ms^{-1}, 256°)$.

Example 2
To a motor-cyclist travelling due north at $50\ kmh^{-1}$ the wind appears to come from north-west at $60\ kmh^{-1}$. What is the true velocity of the wind?

Solution





The velocity of the cyclist $\mathbf{v}_C = \begin{pmatrix} 0 \\ 50 \end{pmatrix} kmh^{-1}$

and velocity of wind $= \mathbf{v}_w$

The velocity of wind relative to cyclist is

$$\mathbf{v}_{wc} = \begin{pmatrix} 60\cos 45^\circ \\ -60\sin 45^\circ \end{pmatrix} = \begin{pmatrix} 42.426 \\ -42.426 \end{pmatrix} \text{kmh}^{-1}$$

$\therefore$ The velocity of wind relative to cyclist is $\mathbf{v}_{wc} = \mathbf{v}_w - \mathbf{v}_C$

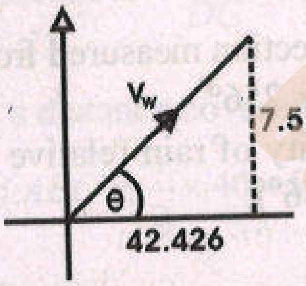
$$\mathbf{v}_w = \mathbf{v}_{wc} + \mathbf{v}_c$$

$$\mathbf{v}_w = \begin{pmatrix} 0 \\ 50 \end{pmatrix} + \begin{pmatrix} 42.426 \\ -42.426 \end{pmatrix} = \begin{pmatrix} 42.426 \\ 7.574 \end{pmatrix}$$

$\therefore$ The magnitude of $\mathbf{v}_w$ is

$$|\mathbf{v}_w| = \sqrt{(42.426)^2 + (7.574)^2} = 44.1 \text{ kmh}^{-1}$$

To find the direction of $\mathbf{v}_w$, first represent $\mathbf{v}_w$ on a rough diagram and calculate the acute angle θ, it makes with the x-axis.



From the diagram, $\tan\theta = \frac{7.574}{42.426} = 0.1785$

$\theta = 10.1^\circ$

The actual direction measured from the north $= 90^\circ - 10.1^\circ = 79.9^\circ$

$\therefore$ The true velocity of the wind is $(44.1 \text{ ms}^{-1}, 79.9^\circ)$.

Example 3

A ship A, steaming in a direction 030° with a steady speed 12 knots, sights a ship B. The velocity of B relative to A is 10 knots in a direction of 270°. Find the magnitude and direction of the velocity of ship B.

Solution

Write $\mathbf{v}_A$ and $\mathbf{v}_{AB}$ in column vectors.

$$\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A$$

$$\mathbf{v}_B = \mathbf{v}_{BA} + \mathbf{v}_A$$

$\mathbf{v}_B = (11.1 \text{ knots}, 339^\circ)$ [work it out]

Resultant velocity

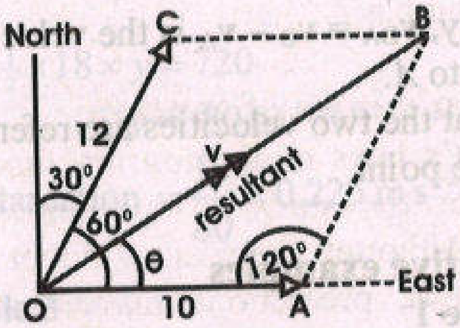
The resultant of **two velocities** can be found using the parallelogram law of vector addition. The vector approach can also be used for **two or more velocities.**

Illustrative example

A particle has two velocities 10 m/s due east and 12 m/s in the direction $N30°E$. What is its resultant velocity?

Solution

In the figure, OA and OC are the vectors representing the velocities. Completing the parallelogram $OABC$ gives OB as the resultant velocity (v).



Since OC is parallel to AB,

$\angle OAB = 180^\circ - 60^\circ = 120^\circ$

Applying the cosine rule to triangle OAB:

$$v^2 = 10^2 + 12^2 - 2(10)(12)\cos 120^\circ$$
$$= 100 + 144 + 2(10)(12)\cos 60^\circ$$
$$= 244 + 240 \times \tfrac{1}{2} = 364$$

$$\therefore \quad v = \sqrt{364} = 19.08 \text{ m/s}$$

Applying the sine rule to triangle OAB:

$$\frac{12}{\sin\theta} = \frac{19.08}{\sin 120^\circ}$$

$$\sin\theta = \frac{12 \times \sin 120^\circ}{19.08}$$
$$= 0.5446$$
$$\theta = 33^\circ$$

The resultant velocity is 19.08 m/s acting at an angle of 33° with the velocity of magnitude 10 m/s or in the direction $N57°E$.

Note: By drawing the parallelogram to scale, an approximate solution can be obtained. Using a scale of 1 cm to 2 m/s, draw $OA = 5$ cm and $OC = 6$ cm, making $\angle COA = 60°$. Then complete the parallelogram $OABC$ and measure the diagonal OB and $\angle BOA$.

Particle dynamics

This is concerned with the action of forces on bodies. In mechanics, a **body** is any object to which a force can be applied. A **particle** is a body whose dimensions, except mass, are negligible.

Newton's laws of motion

Newton's laws of motion are fundamental basis for the study of mechanics at this level. Although there is no direct proof of these laws, predictions made using them agree very closely with observations.

Statement	***Consequence(s)***
First law Every body will remain at rest or continue to move in a straight line at a constant speed unless compelled by an external force to act otherwise.	a) If a body has acceleration, then there must be a force acting on it. b) If a body has no acceleration, then the forces acting on it must be in equilibrium.
Second law The rate of change of momentum of a moving body is directly proportional to the external force(s) acting on it and takes place in the direction of the force or forces. So when an external force acts on a body of constant mass, the force produces an acceleration which is directly proportional to the force.	a) The basic equation of motion for constant mass is: **force=mass× acceleration** Force is in N, mass is in kg and acceleration is in ms^{-2}. b) The force and acceleration of the body are both in the same direction c) A constant force on a mass gives a constant acceleration.
Third law If a body A exerts a force on a body B, then B exerts an equal and opposite force on A. i.e. To every action, there is an equal and opposite reaction.	These forces between the bodies are often called **reactions**. In a rigid body only the external forces need to be considered.

Problem solving

The following are important points to remember when solving problems using Newton's laws of motion

1. Draw a clear force diagram (see force diagrams under statics)
2. If there is no acceleration. i.e. if the body is either at rest or moving with uniform velocity, then the forces balance in each direction
3. If there is acceleration:
   a) mark it on the diagram using $\xrightarrow{a}$
   b) write down if possible, an expression for the resultant force
   c) use Newton's second law i.e. write the equation of motion: *Resultant force = mass × acceleration*

Mass and weight

Mass and **weight** are different. The mass of a body is a measure of the matter contained in the body. The mass of a body is constant regardless of the position of the body. Mass is a **scalar quantity** i.e. it has magnitude only. SI unit of mass is the **kilogram** (kg).

The weight of a body is the force with which the earth attracts it. It is dependent on the body's distance from the earth. Weight is a **vector quantity** since it is a force. SI unit of weight is the **newton** (N). The relation between weight and mass is:

Weight = mass × acceleration due to gravity
i.e. $W = mg$

Common approximate values for g are 10 ms^{-2} and 9.8 ms^{-2}.

Force

A **force** is necessary to cause a body to **accelerate**. More than one force may act on a body. If the forces are in **equilibrium**, then the body may be at *rest* or moving in a straight line with a *constant velocity*.
If there is a **resultant** force, then the body will *accelerate*.
The force F acting on a body of mass m moving with acceleration a is given by the relation:

$$\boxed{\textbf{Force} = \textbf{mass} \times \textbf{acceleration i.e. } F = ma}$$

When there is a system of forces acting on a body, F is the **resultant force** on the body.

Note that mass is in kilograms (kg) and acceleration is in ms^{-2}. Force is a vector quantity and its SI unit is newton (N).
Higher units are the kilonewtons (kN).
1 kN = 1 000 N.
meganewtons (MN) = 1 000 000 N etc.

Note also that, when the mass is given in grams, convert it to kilograms.
i.e. 1 000 g = 1 kg.

Illustrative examples

Example 1
A force of 5 kilonewtons (kN) acts on a mass of 15 tonnes. Calculate the acceleration.
(1 tonne = 1000 kg).

Solution
Force, $F = ma \Rightarrow 5\,000 = 15\,000 \times a$

$$a = \frac{5000}{15000} = 0.33 \text{ ms}^{-2} \quad [1 \text{ kN} = 1\,000 \text{ N}]$$

Example 2
A body of mass 500 g is acted upon by a force of 4 N. Find its acceleration.

Solution
Force, $F = ma \Rightarrow 4 = 0.5 \times a$

$$a = \frac{4}{0.5} = 8 \text{ ms}^{-2}$$

Note: convert the mass from g to kg
i.e. 500 g = 0.5 kg [1 000 g = 1 kg]

Example 3
A force of 30 N acts on a particle of mass 6 kg which is at rest. How far will it move in 3 s?

Solution
$F = ma \Rightarrow 30 = 6 \times a$

$$a = \frac{30}{6} = 5 \text{ ms}^{-2}$$

$u = 0 \text{ ms}^{-1}$ (particle is at rest), $a = 5 \text{ ms}^{-2}$,
$t = 3$ s, $s = ?$
Using $s = ut + \frac{1}{2}at^2$, we have

$$s = 0(3) + \tfrac{1}{2}(5)(3)^2$$

$$s = 0 + 22.5 \quad \Rightarrow \quad s = 22.5 \text{ m}$$

Example 4

A force of magnitude 12 N acting on a body of mass 6 kg changes its speed from 10 ms^{-1} to 30 ms^{-1}. Find the time during which the force acts on the body.

Solution

$F = ma$

$12 = 6a \quad \Rightarrow \quad a = 2$

From $v = u + at$

$30 = 10 + 2t$

$$t = \frac{30-10}{2} = 10\,\text{s}$$

Example 5

A body of mass 10 kg is moving under a force (6 N, 090°). Find its acceleration

Solution

$\mathbf{F} = m\mathbf{a}$

$$\mathbf{a} = \frac{\mathrm{F}}{m} = \left(\tfrac{6}{10}\ \text{ms}^{-2}, 090°\right)$$

$$= (0.6\,\text{ms}^{-2}, 090°)$$

Note that acceleration is in the same direction as the force.

Example 6

Calculate the magnitude of the force required to move a particle of mass 3 kg from rest, through a distance of 4 m in 1 second.

Solution

$u = 0\ \text{ms}^{-1}$ (particle moved from rest), $a = ?$, $t = 1$ s, $s = 4$ m, $F = ma = ?$, $m = 3$ kg

First we have to find a using $s = ut + \frac{1}{2}at^2$

i.e. $4 = 0(1) + \frac{1}{2}a(1)^2 \quad \Rightarrow \quad 4 = \frac{1}{2}a$

$a = 8\,\text{ms}^{-2}$

Therefore, $F = ma$

$= 3(8) = 24$ N

Example 7

A body of mass 96 kg moving with a velocity (3 ms^{-1}, 090°) is brought to rest in 12 s. Find the retarding force.

Solution

$u = 3\ \text{ms}^{-1}$, $v = 0\ \text{ms}^{-1}$ (particle is brought to rest), $t = 12$ s, $a = ?$, $F = ma = ?$, $m = 96$ kg

First we have to find a using $v = u + at$

i.e. $0 = 3 + a(12) \quad \Rightarrow \quad a = -0.25\ \text{ms}^{-2}$

Therefore, $F = ma = 96(-0.25) = -24$ N

i.e. in the direction opposite to 090° i.e. 270°.

The retarding force is (24 N, 270°).

Example 8

What force will produce a velocity of 1.2 ms^{-1} in 6 seconds on a particle of mass 0.5 kg which is at rest?

Solution

$u = 0\ \text{ms}^{-1}$ (particle is at rest), $v = 1.2\ \text{ms}^{-1}$, $t = 6$ s, $a = ?$, $F = ma = ?$, $m = 0.5$ kg

First we have to find a using $v = u + at$

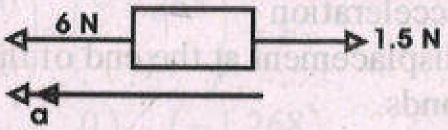
i.e. $1.2 = 0 + a(6) \quad \Rightarrow \quad a = 0.2\ \text{ms}^{-2}$

Therefore, $F = ma = 0.5(0.2) = 0.1$ N

Example 9

A horizontal force of 6 N acts on a body of mass 3 kg. There is a resistance of 1.5 N opposing the first force. What acceleration will be produced?

Solution



The resultant force, $F = 6 - 1.5 = 4.5$ N

$F = ma$

$$a = \frac{F}{m} = \frac{4.5}{5} = 1.5\,\text{ms}^{-2}$$

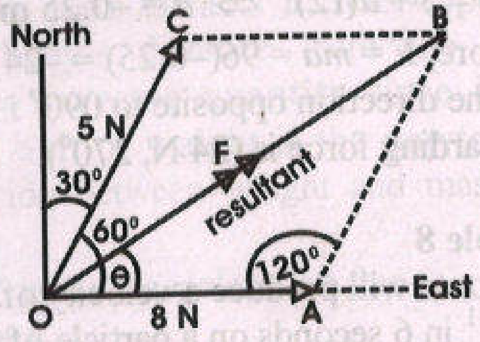
$\therefore$ Acceleration is $1.5\,\text{ms}^{-2}$

Example 10

Two forces $\mathbf{F}_1 = (5\text{ N}, N30°E)$ and $\mathbf{F}_2 = (8\text{ N}, N90°E)$ act on a body of mass 2.5 kg. Find the acceleration on the body.

Solution

Since we have only two forces, we can use the parallelogram of forces or vector approach to find the resultant force **F**.



Using the parallelogram of forces, the resultant force, $F = 11.36$ N and $\theta = 22.4°$ (or in the $N\,67.6°\,E$ in terms of bearings)

$\mathbf{F} = m\mathbf{a}$

$$\mathbf{a} = \frac{\text{F}}{m} = \left(\frac{11.36}{2.5}\text{ ms}^{-2},\ N67.6°E\right)$$

$$= (4.54\text{ ms}^{-2},\ N67.6°E)$$

Use the vector approach to arrive at the same answer.

Example 11

A body of mass 6 kg, initially at rest, is acted upon by two forces $\mathbf{F}_1 = (12\text{ N}, 090°)$ and $\mathbf{F}_2 = (3\text{ N}, 330°)$.

Find the magnitudes of

i) its acceleration

ii) its displacement at the end of three seconds.

Solution

$$\mathbf{F}_1 = (12\text{ N}, 090°) = \begin{pmatrix} 12\sin 90° \\ 12\cos 90° \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

$$\mathbf{F}_2 = (3\text{ N}, 330°) = \begin{pmatrix} 12\sin 330° \\ 12\cos 330° \end{pmatrix} = \begin{pmatrix} -6 \\ 10.392 \end{pmatrix}$$

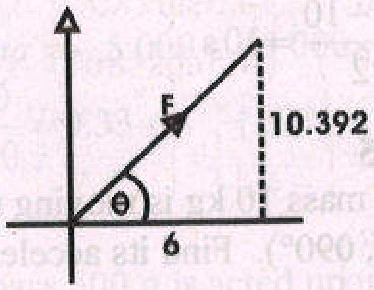
Resultant force,

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \begin{pmatrix} 12 \\ 0 \end{pmatrix} + \begin{pmatrix} -6 \\ 10.392 \end{pmatrix} = \begin{pmatrix} 6 \\ 10.392 \end{pmatrix}$$

$\therefore$ The magnitude of **F**

$$= \sqrt{(6)^2 + (10.392)^2} = 12\text{ N}$$

To find the direction of **F**, first represent **F** on a rough diagram and calculate the acute angle θ, it makes with the x-axis.



From the diagram, $\tan\theta = \frac{10.392}{6} = 1.732$

$\theta = 60°$

The actual direction measured from the north $= 90° - 60° = 30°$

$\therefore$ The resultant force is (12 N, 030°).

i) $\mathbf{F} = m\mathbf{a}$

$$\mathbf{a} = \frac{\text{F}}{m} = \left(\frac{12}{6}\text{ ms}^{-2}, 030°\right)$$

$$= (2\text{ ms}^{-2},\ 030°)$$

Note that acceleration is in the same direction as the force.

ii) $u = 0\text{ ms}^{-1}$ (body is initially at rest), $t = 3$ s, $a = 2\text{ ms}^{-2}$, $s = ?$

Using $s = ut + \frac{1}{2}at^2$, $s = 0 + \frac{1}{2}(2)(3)^2 = 9$ m

Displacement at the end of three seconds is (9 m, 030°).

Note that displacement has the same direction as the acceleration.

Example 12

Three forces $\mathbf{F_1}$ (2 N, 060°), $\mathbf{F_2}$ (4.5 N, 180°) and $\mathbf{F_3}$ (5 N, 300°) act on a body of mass 2 kg, initially at rest. Find

i) the magnitude and direction of the resultant force.

ii) the acceleration with which the body begins to move.

Solution

Resultant force

$$= \begin{pmatrix} 2\sin 60^\circ \\ 2\cos 60^\circ \end{pmatrix} + \begin{pmatrix} 4.5\sin 180^\circ \\ 4.5\cos 180^\circ \end{pmatrix} + \begin{pmatrix} 5\sin 300^\circ \\ 5\cos 300^\circ \end{pmatrix}$$

$$= \begin{pmatrix} 1.732 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -4.5 \end{pmatrix} + \begin{pmatrix} -4.33 \\ 2.5 \end{pmatrix}$$

$$= \begin{pmatrix} -2.598 \\ -1 \end{pmatrix}.$$

i) Magnitude of the resultant force

$$= \sqrt{(-2.598)^2 + (-1)^2} = 2.7832\text{ N}$$

$$\tan\theta = \frac{1}{2.598} = 0.3849$$

$$\theta = 21.0522 = 21.05^\circ$$

$$\therefore\ \text{Direction} = 270^\circ - 21.05^\circ = 248.95^\circ$$

ii) $F = ma$

$$a = \frac{F}{m} = \frac{2.7838}{2} = 1.392\text{ ms}^{-2}$$

Acceleration is $(1.392\text{ ms}^{-2}, 248.95^\circ)$

Example 13

Three forces $8\mathbf{i} - 3\mathbf{j}$, $3\mathbf{i} + 3\mathbf{j}$ and $\mathbf{i} - 4\mathbf{j}$ act on a body of mass 4 kg. Find:

i) the resultant force;

ii) the acceleration of the body.

Solution

i) Resultant force,

$$\mathbf{F} = (8\mathbf{i} - 3\mathbf{j}) + (3\mathbf{i} + 3\mathbf{j}) + (\mathbf{i} - 4\mathbf{j})$$
$$= 12\mathbf{i} - 4\mathbf{j}$$

ii) $\mathbf{F} = m\mathbf{a} \Rightarrow \mathbf{a} = \dfrac{\mathbf{F}}{m}$

$$= \tfrac{1}{4}(12\mathbf{i} - 4\mathbf{j})$$
$$= 3\mathbf{i} - \mathbf{j}$$

Example 14

Three forces $\mathbf{F}_1 = (2\text{ N}, 060^\circ)$, $\mathbf{F}_2 = (3\text{ N}, 270^\circ)$ and $\mathbf{F}_3$ act on a body of mass 1 kg. If the body accelerates at 3 ms^{-2} on a bearing of 180°, find the force $\mathbf{F}_3$.

Solution

Magnitude of resultant force,

$\mathbf{F} = ma = 1\text{ kg} \times 3\text{ ms}^{-2} = 3\text{ N}$

$\therefore\ \mathbf{F} = (3\text{ N}, 180^\circ)$

Note: The direction of **F** is 180° since the force is in the same direction as the acceleration.

Resolving the forces into components, we have

$$\mathbf{F} = (3\text{ N}, 180^\circ) = \begin{pmatrix} 3\sin 180^\circ \\ 3\cos 180^\circ \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix},$$

$$\mathbf{F}_1 = (2\text{N}, 060^\circ) = \begin{pmatrix} 2\sin 60^\circ \\ 2\cos 60^\circ \end{pmatrix} = \begin{pmatrix} 1.732 \\ 1 \end{pmatrix}$$

$$\mathbf{F}_2 = (3\text{N}, 270^\circ) = \begin{pmatrix} 3\sin 270^\circ \\ 3\cos 270^\circ \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix},$$

Resultant force, $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$

$$\begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 1.732 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix} + \mathbf{F}_3$$

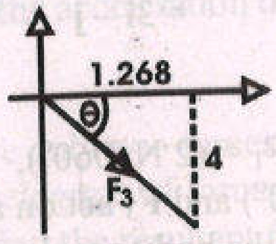
$$\begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} -1.268 \\ 1 \end{pmatrix} + \mathbf{F}_3$$

$$\mathbf{F}_3 = \begin{pmatrix} 0 \\ -3 \end{pmatrix} - \begin{pmatrix} -1.268 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.268 \\ -4 \end{pmatrix}$$

$\therefore$ The magnitude of $\mathbf{F}_3$

$$= \sqrt{(1.268)^2 + (-4)^2} = 4.2\text{ N}$$

To find the direction of $\mathbf{F}_3$, first represent $\mathbf{F}_3$ on a rough diagram and calculate the acute angle θ, it makes with the x-axis.



From the diagram, $\tan\theta = \frac{4}{1.268} = 3.1546$

$\theta = 72.4°$

The actual direction measured from the north $= 90° + 72.4° = 162.4°$

$\therefore$ $\mathbf{F}_3 = (4.2\text{ N}, 162.4°)$.

Example 15

Two forces $\mathbf{F}_1 = (3\text{ N}, 090°)$ and $\mathbf{F}_2 = (4\text{ N}, 150°)$ act on a mass of 2 kg, find the magnitude and direction of the acceleration of the body.

Solution

$$\mathbf{F}_1 = (3\text{ N}, 090°) = \begin{pmatrix} 3\sin 90° \\ 3\cos 90° \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\mathbf{F}_2 = (4\text{ N}, 150°) = \begin{pmatrix} 4\sin 150° \\ 4\cos 150° \end{pmatrix} = \begin{pmatrix} 2 \\ -2\sqrt{3} \end{pmatrix}$$

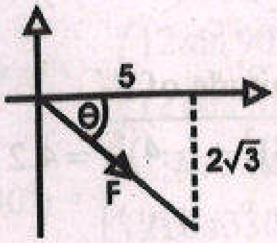
Resultant force,

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -2\sqrt{3} \end{pmatrix} = \begin{pmatrix} 5 \\ -2\sqrt{3} \end{pmatrix}$$

$\therefore$ The magnitude of $\mathbf{F} = \sqrt{(5)^2 + (-2\sqrt{3})^2}$

$= 6.08\text{ N}$

To find the direction of $\mathbf{F}$, first represent $\mathbf{F}$ on a rough diagram and calculate the acute angle θ, it makes with the x-axis.



From the diagram, $\tan\theta = \frac{2\sqrt{3}}{5} = 0.6928$

$\theta = 34.71°$

The actual direction measured from the north $= 90° + 34.71° = 124.71°$

$\therefore$ The resultant force is $(6.08\text{ N}, 124.71°)$.

$\mathbf{F} = m\mathbf{a}$

$$\mathbf{a} = \frac{\mathbf{F}}{m} = \left(\frac{6.08}{2}\text{ ms}^{-2}, 124.71°\right)$$

$$= (3.04\text{ ms}^{-2}, 124.71°)$$

Note that the acceleration has the same direction as the force.

Example 16

A bucket of water of mass 20 kg is being pulled out of a well. Find the magnitude of the tension in the rope when:

i) it is moving with a constant speed;

ii) it has an acceleration of 0.3 ms^{-2} (Take $g = 9.8\text{ ms}^{-2}$).

Solution

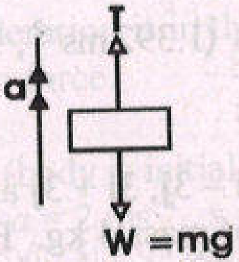
Let T = tension in the rope and W = weight of the bucket = mg

i) If the bucket is moving with a constant speed, the acceleration, $a = 0$

$T - W = ma = 0$

$T = W = mg = 20(9.8) = 196\text{ N}$

ii)



If the bucket has an acceleration of 0.3 ms^{-2} then, $a = 0.3\text{ ms}^{-2}$

Equation of motion: $T - mg = ma$

$T = ma + mg = 20(0.3) + 20(9.8)$

$= 6 + 196 = 202\text{ N}$

Motion in a lift

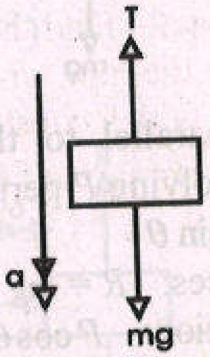
Illustrative examples

Example 1

A lift of mass 500 kg is descending with an acceleration of 1.5 ms^{-2}. A man of mass 80 kg is inside the lift. Calculate the tension in the cable connected to the lift. (Take $g = 10\ \text{ms}^{-2}$).

Solution

Consider the man and lift to be one mass subject to a force of T upward (the tension in the cable) and the weight force downwards.



Write down the equation of motion downwards.

Equation of motion: $mg - T = ma$

where $m = 500\text{ kg} + 80\text{ kg} = 580\text{ kg}$

$$\begin{aligned} T &= mg - ma \\ &= m(g - a) \\ &= 580(10 - 1.5) \\ &= 580 \times 8.5 \\ &= 4930\text{ N} \end{aligned}$$

Example 2

A man of mass 70 kg is standing in a lift. What force does the floor of the lift exert on the man if the lift is

i) moving with a uniform velocity ?

ii) accelerating at 3 ms^{-2} upwards?

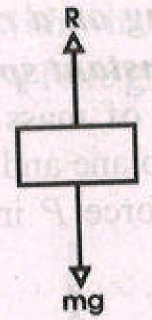
iii) accelerating at 3 ms^{-2} downwards?

(Take $g = 10\ \text{ms}^{-2}$)

Solution

R is the normal reaction from the floor on the man

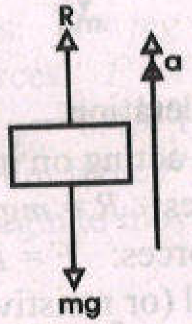
i)



Since the lift is moving with a uniform velocity, the resultant force is zero:

$R = mg = 70 \times 10 = 700\text{ N}$

ii)

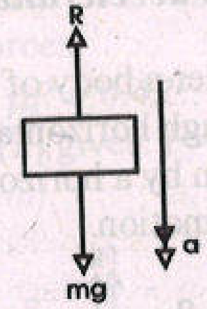


Since the lift is accelerating upwards

Equation of motion: $R - mg = ma$

$$\begin{aligned} R &= mg + ma = m(g + a) \\ &= 70(10 + 3) \\ &= 70 \times 13 = 910\text{ N} \end{aligned}$$

iii)



Since the lift is accelerating downwards

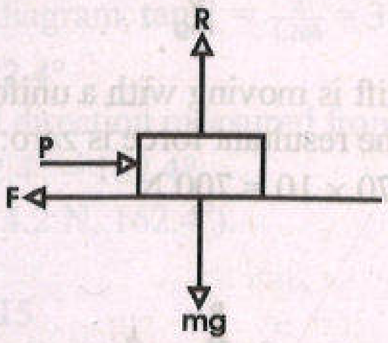
Equation of motion: $mg - R = ma$

$$\begin{aligned} R &= mg - ma = m(g - a) \\ &= 70(10 - 3) \\ &= 70 \times 7 = 490\text{ N} \end{aligned}$$

Motion on a horizontal plane surface

1. *Body moving on a rough horizontal plane at constant speed*

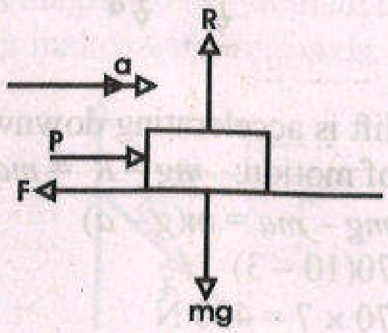
Consider a body of mass m, moving on a rough horizontal plane and being acted upon by a horizontal force P in the direction of motion.



There is no acceleration.
The other forces acting on it are:
For vertical forces: $R = mg$
For horizontal forces: $F = P$
F is the frictional (or resistive) force between the surfaces and R is the normal reaction.
[Note that : $F = \mu R$]
If the surface is *smooth*, then the frictional force, $F = 0$

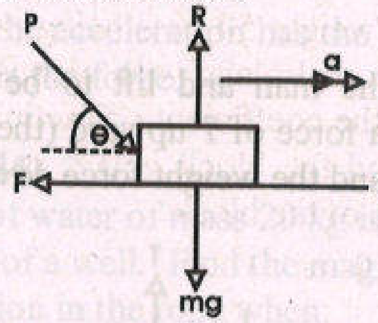
2. *Body moving on a rough horizontal plane with acceleration*

Case1: Consider a body of mass m, moving on a rough horizontal plane and is being acted upon by a horizontal force P in the direction of motion.



For vertical forces: $R = mg$
Equation of motion: $P - F = ma$
If the surface is *smooth*, then the frictional force, $F = 0$
[Note that : $F = \mu R$]

Case 2: Consider a body of mass m, moving on a rough horizontal plane and is being acted upon by a force P acting at an angle θ to the horizontal.



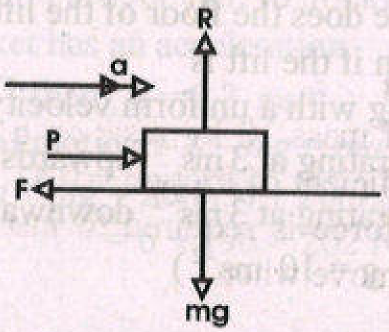
Resolving P parallel to the plane gives: $P\cos\theta$ and resolving P perpendicular to the plane gives: $P\sin\theta$
For vertical forces: $R = mg + P\sin\theta$
Equation of motion: $P\cos\theta - F = ma$
If the surface is smooth, then the frictional force, $F = 0$ [Note that : $F = \mu R$]
Note the direction of forces when the body is being pulled (refer to example 27.33)

Illustrative examples

Example 1

A constant force of 600 N acts on a body of mass 40 kg on a horizontal floor and gives it an acceleration of $10\,\text{ms}^{-2}$. Calculate the coefficient of friction between the floor and the body. (Take $g = 10\ \text{ms}^{-2}$)

Solution



Let P be the constant force.
For vertical forces: $R = mg$
Equation of motion: $P - F = ma$
[**Note:** $F = \mu R = F = \mu mg$]

$$P - \mu R = ma \quad \Rightarrow \quad P - \mu mg = ma$$

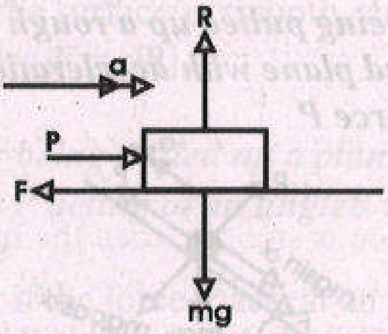
Substituting values, we have

$$600 - \mu(40)(10) = 40(10)$$
$$400\mu = 200 \quad \Rightarrow \quad \mu = 0.5$$

Example 2

A block of weight 50 N is being pulled on a rough horizontal plane with a constant force of 20 N. If the coefficient of friction between the floor and the block is 0.2, calculate the acceleration of the block.
(Take $g = 10\ \text{ms}^{-2}$)

Solution



Let P be the constant force.
For vertical forces: $R = mg = 50$ N
[Note: weight, $W = mg$

$$m = \frac{W}{g} = \frac{50}{10} = 5\text{ kg}$$

Equation of motion: $P - F = ma$

$$P - \mu R = ma \quad \Rightarrow \quad P - \mu mg = ma$$

Substituting values, we have

$$20 - (0.2)(50) = 5a$$
$$5a = 10 \quad \Rightarrow \quad a = 2\ \text{ms}^{-2}.$$

The acceleration of the block is $2\ \text{ms}^{-2}$.

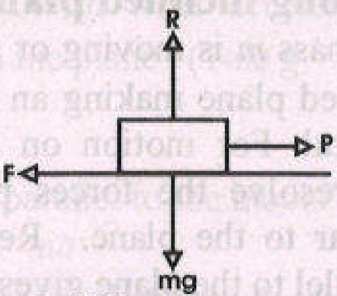
Example 3

A block of mass 10 kg rests on a horizontal floor (coefficient of friction = 0.4)

a) What force is required to just make the block move when
   i) pulling horizontally?
   ii) pulling horizontally at an angle of 60° to the horizontal?
b) If the block is pulled with a horizontal force of 50 N, with what acceleration does it move? (Take $g = 10\ \text{ms}^{-2}$)

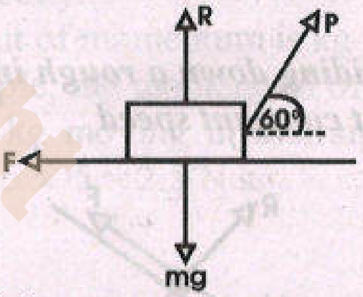
Solution

i)



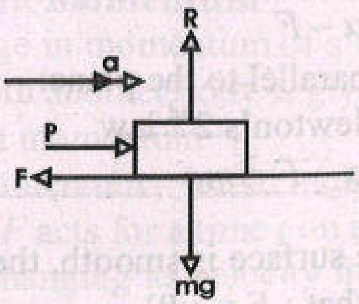
On the point of sliding,
For vertical forces: $R = mg$
For horizontal forces: $F = P$
$P = F = \mu R = \mu mg = 0.4 \times 10 \times 9.8 = 39$ N
This is the minimum force P required to make the block begin to move.

ii)



For vertical forces:
$mg = R + P\sin 60°$ ………...(1)
For horizontal forces:
$P\cos 60° = F = \mu R = 0.4R$…...(2)
Solving (1) and (2) gives: $P = 46$ N

b)



For vertical forces: $R = mg$
Equation of motion: $P - F = ma$
$F = \mu R = \mu mg = 0.4 \times 10 \times 9.8 = 39$ N and
$P = 50$ N

$$50 - 39 = 10a$$
$$10a = 11$$
$$a = 1.1 \text{ ms}^{-2}$$

Motion along inclined planes

A body of mass m is moving or sliding on a rough inclined plane making an angle α to the horizontal. For motion on an inclined plane, we resolve the forces parallel and perpendicular to the plane. Resolving the weight parallel to the plane gives:

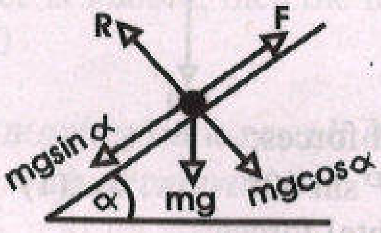
$$W \sin \alpha = mg \sin \alpha$$

Resolving the weight perpendicular to the plane gives: $W \cos \alpha = mg \cos \alpha$

F is the frictional (or resistive) force between the surfaces and R is the reaction force.

Consider the following cases:

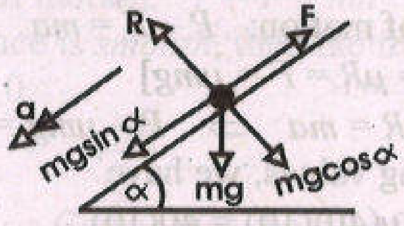
1. ***Body sliding down a rough inclined plane at constant speed***



No acceleration perpendicular to the plane:
$R = mg \cos \alpha$
Resultant force down the plane is:
$mg \sin \alpha - F$
For forces parallel to the plane:
Applying Newton's 2nd law
$mg \sin \alpha - F = ma$

Note: If the surface is smooth, then $F = 0$
[Note also that : $F = \mu R$]

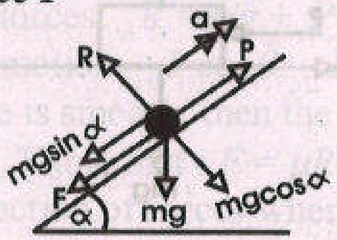
2. ***Body sliding down rough inclined plane with acceleration***



No acceleration perpendicular to the plane:
$R = mg \cos \alpha + P \sin \theta$
Resultant force up the plane is:
$P \cos \theta - mg \sin \alpha - F$
For forces parallel to the plane:
Applying Newton's 2nd law
$P \cos \theta - mg \sin \alpha - F = ma$

Note: If the surface is smooth, then $F = 0$
[Note that : $F = \mu R$]

3. ***Body being pulled up a rough inclined plane with acceleration by a force P***



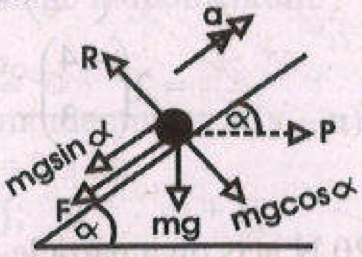
If P acts along the plane as shown above.
No acceleration perpendicular to the plane:
$R = mg \cos \alpha$
Resultant force up the plane is:
$P \cos \alpha - mg \sin \alpha - F$
For forces parallel to the plane:
Applying Newton's 2nd law
$P - mg \sin \alpha - F = ma$

Note: If the surface is smooth, then $F = 0$
[Note that : $F = \mu R$]

4. ***Body being pulled up a rough plane by a horizontal force P with acceleration***

If the force P acts horizontally, we find the components of P along and perpendicular to the plane. Resolving P parallel to the plane gives: $P\cos\alpha$

Resolving P perpendicular to the plane gives: $P\sin\alpha$



No acceleration perpendicular to the plane:

$$R = mg\cos\alpha + P\sin\alpha$$

Resultant force up the plane is:

$$P\cos\alpha - mg\sin\alpha - F$$

For forces parallel to the plane:

Applying Newton's 2nd law

$$P\cos\alpha - mg\sin\alpha - F = ma$$

Note: If the surface is smooth, then $F = 0$

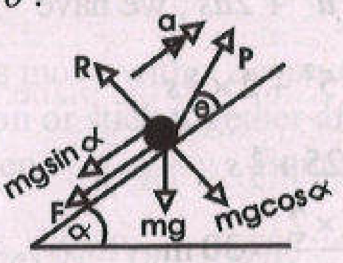
[Note that : $F = \mu R$]

5. ***Body being pulled up a plane by a force P acting at an angleθ with the plane***

Note that if the force P acts at an angle θ with the plane, we find the components of P along and perpendicular to the plane.

Resolving P parallel to the plane gives: $P\cos\theta$

Resolving P perpendicular to the plane gives: $P\sin\theta$.



No acceleration perpendicular to the plane:

$$R = mg\cos\alpha + P\sin\theta$$

Resultant force up the plane is:

$$P\cos\theta - mg\sin\alpha - F$$

For forces parallel to the plane:

Applying Newton's 2nd law

$$P\cos\theta - mg\sin\alpha - F = ma$$

Note: If the surface is smooth, then $F = 0$

[Note that : $F = \mu R$]

Illustrative example

A block of mass 10 kg is placed on a smooth plane inclined at an angle of 30° to the horizontal. With what acceleration does it slide down the plane? (Tale $g = 10\,\text{ms}^{-2}$)

Solution

Resolving parallel to the block:

$$mg\sin\alpha = ma$$

$$a = g\sin 30° = 10 \times 0.5 = 5\text{ N}$$

Momentum and Impulse

The **momentum** of a moving body is defined as the product of mass (m) and velocity (v) i.e. momentum = mv

It is a vector quantity whose direction is that of the velocity.

The SI unit of momentum is kg ms^{-1} or N s.

For example, the momentum of a particle of mass 1.5 kg moving in a straight line at 5 ms^{-1} is $1.5 \times 5 = 2.5$ N s.

Impulse

Impulse is the time effect of a force acting on a body. It is a vector quantity. For a constant force F acting for time t, the impulse is given by Impulse = Ft = change in momentum. SI unit of impulse is N s.

Change in momentum

The change in momentum of a body after an impact with another body is given by:

Change in momentum

= final momentum – initial momentum

If a force F acts for a time t on a body of mass m, changing its velocity from u to v, then

Change in momentum = $mv - mu = m(v - u)$, and Impulse = change in momentum

$$\boxed{\therefore \quad Ft \;=\; m(v-u)}$$

Illustrative examples

Example 1

A constant force acts on a particle of mass 0.5 kg changing its speed from 3 ms^{-1} to 7 ms^{-1}. What is the change in momentum and its impulse?

Solution

Change in momentum

$= 0.5(7 - 3)$

$= 0.5(4) = 2$ N s = Impulse

Example 2

A particle of mass 2 kg has an initial velocity $\mathbf{u} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ m/s and final velocity $\mathbf{v} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ m/s. Find the change in momentum.

Solution

Change in momentum = $m(\mathbf{v} - \mathbf{u})$

$$= 2\left[\begin{pmatrix} 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix}\right] = 2\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}\text{Ns}$$

Example 3

A particle of mass 2 kg has an initial velocity $\mathbf{u} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ m/s. It is acted upon by a force which causes it to move with a velocity $\mathbf{v} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ m/s. Find the change in momentum.

Solution

Change in momentum = $m(\mathbf{v} - \mathbf{u})$

$$= 2\left[\begin{pmatrix} -2 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix}\right]$$

$$= 2\begin{pmatrix} -4 \\ -3 \end{pmatrix} = \begin{pmatrix} -8 \\ -6 \end{pmatrix}\text{Ns}$$

Example 4

A force of 20 N acts on a particle of mass 2 kg travelling at 4 m/s for 0.3 s in the direction of its motion. What is the final velocity of the particle?

Solution

Impulse of the force = $Ft = 20 \times 0.3 = 6$ Ns

Impulse = Change in momentum = $mv - mu$

$$6 = 2v - 2 \times 4 \;\Rightarrow\; 6 = 2v - 8$$

$$2v = 14 \;\Rightarrow\; v = 7 \text{ m/s}$$

Example 5

An object of mass 8 kg, moving with a velocity of 5 ms^{-1} attains within 4 seconds a velocity of 10 ms^{-1}. Calculate

i) the distance covered.

ii) the change in momentum.

Solution

i) $a = \dfrac{v-u}{t} = \dfrac{10-5}{4} = \dfrac{5}{4}\,\text{m}\,\text{s}^{-2} = 1.25\,\text{m}\,\text{s}^{-2}$

Using $v^2 = u^2 + 2as$, we have

$$10^2 = 5^2 + 2 \times \tfrac{5}{4}s$$

$$100 = 25 + \tfrac{5}{2}s$$

$$s = \frac{75 \times 2}{5} = 30 \text{ m}$$

ii) Change in momentum

$= mv - mu = m(v - u)$

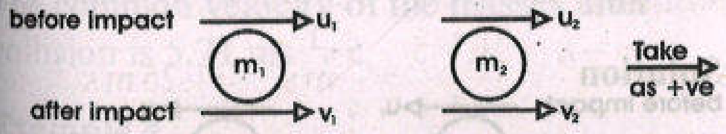
$= 8(10 - 5) = 40$ N s

Example 6
The velocity of a body of mass 300 g changes from $-4\mathbf{i} + 6\mathbf{j}$ to $12\mathbf{i} + 2\mathbf{j}$. Find the change in momentum.

Solution
Change in momentum $= m(\mathbf{v} - \mathbf{u})$
$m = 300 \text{ g} = 0.3 \text{ kg}$
$\mathbf{v} = 12\mathbf{i} + 2\mathbf{j}; \quad \mathbf{u} = -4\mathbf{i} + 6\mathbf{j}$
Change in momentum
$= 0.3\,[(12\mathbf{i} + 2\mathbf{j}) - (-4\mathbf{i} + 6\mathbf{j})]$
$= 0.3\,(16\mathbf{i} - 4\mathbf{j}) = 4.8\mathbf{i} - 1.2\mathbf{j}$

Principle of conservation of linear momentum

The principle of conservation of linear momentum states that provided no external force acts on a system of colliding bodies, the total momentum before collision is equal to the total momentum after collision.
Suppose two bodies of masses m_1 and m_2 moving with initial velocities u_1 and u_2 respectively ($u_1 > u_2$) collide and after collision move with velocities v_1 and v_2 (see figure below), then by the principle of conservation of linear momentum
$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$ ………(1)



Case 1
If the bodies move with a *common velocity* after collision or stick together after collision, then $v_1 = v_2 = v$. Equation (1) becomes
$$m_1 u_1 + m_2 u_2 = (m_1 + m_2)v$$

Case 2
If the bodies move *in opposite directions* before collision i.e. if m_2 moves opposite to m_1, then u_2 = negative, and equation (1) becomes
$$m_1 u_1 - m_2 u_2 = m_1 v_1 + m_2 v_2$$

Case 3
If the mass, m_2 was at rest before collision, then $u_2 = 0$. Equation (1) becomes
$$m_1 u_1 = m_1 v_1 + m_2 v_2$$

Note that there can be a combination of the above cases.

Direct Impact

Direct impact takes place when two similar spheres moving along the same straight line collide. To solve problems involving the direct impact of two smooth spheres or bodies of masses m_1 and m_2 moving with initial velocities u_1 and u_2 respectively ($u_1 > u_2$) and final velocities v_1 and v_2.

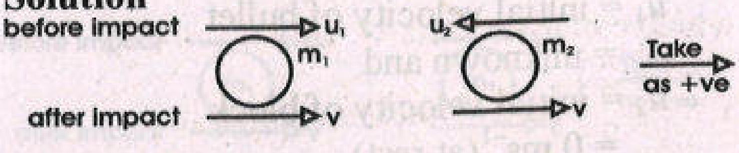
1. draw a clear diagram
2. use the conservation of linear momentum in the chosen direction:
$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

Note: The sign convention is that bodies moving purely **eastwards** are considered as having positive momentum and bodies moving purely **westwards** are considered as having negative momentum.

Illustrative examples

Example 1
A car P of mass 4×10^3 kg travelling at 15 ms^{-1} collides with another car Q of mass 10^3 kg travelling at 20 ms^{-1}. Assuming they move together after collision, find the common velocity if the cars were travelling in opposite directions.

Solution



Let m_1 = mass of car $P = 4 \times 10^3$ kg and
m_2 = mass of car $Q = 10^3$ kg

u_1 = initial velocity of car $P = 15 \text{ ms}^{-1}$
u_2 = initial velocity of car $Q = 20 \text{ ms}^{-1}$
(in opposite direction P)
v = common velocity of cars P and Q
after collision, which is unknown.

Total momentum before collision
$= m_1 u_1 - m_2 u_2$
$= (4\times10^3)15 - (10^3)20 = 60\times10^3 - 20\times10^3$
$= 40\times10^3$

Total momentum after collision
$= (m_1 + m_2)v = (4\times10^3 + 10^3)v = (5\times10^3)v$

By the principle of conservation of linear momentum
Total momentum before collision
= Total momentum after collision

$$40\times10^3 = (5\times10^3)v$$

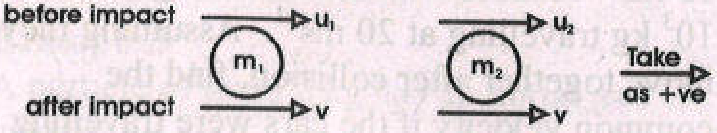
$$v = \frac{40\times10^3}{5\times10^3} = 8 \text{ ms}^{-1}$$

Therefore the common velocity of the cars is 8 ms^{-1}.

Example 2

A bullet of mass 0.096 kg is fired horizontally into a 12 kg block of wood on a smooth horizontal floor. If the velocity of the block and bullet after impact is 0.48ms^{-1} calculate to the nearest whole number the initial velocity of the bullet.

Solution



Let m_1 = mass of bullet = 0.096 kg and
m_2 = mass of block = 12 kg
u_1 = initial velocity of bullet
= unknown and
u_2 = initial velocity of block
$= 0 \text{ ms}^{-1}$(at rest)
v = common velocity of bullet and
block after impact $= 0.48 \text{ ms}^{-1}$

Total momentum before collision
$= m_1 u_1 + m_2 u_2$
$= 0.096\, u_1 + 12(0) = 0.096\, u_1$

Total momentum after collision
$= (m_1 + m_2)v = (0.096 + 12)0.48$
$= 12.096\times0.48$
$= 5.806$

By the principle of conservation of linear momentum
Total momentum before collision
= Total momentum after collision

$$0.096\, u_1 = 5.806$$

$$u_1 = \frac{5.806}{0.096} = 60.48 \text{ ms}^{-1} = 60 \text{ ms}^{-1}$$

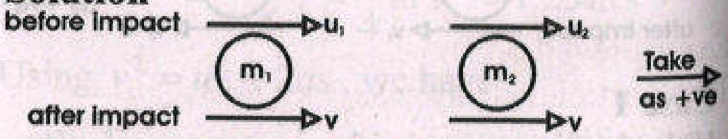
The initial velocity of the bullet is 60 ms^{-1} to the nearest whole number.

Example 3

Two masses 20 kg and 15 kg moving with velocities 25 ms^{-1} and 20 ms^{-1} respectively collide. They move together after collision, find their common velocity, if they were moving in

i) the same direction.
ii) opposite directions.

Solution



Let $m_1 = 20$ kg and $u_1 = 25 \text{ ms}^{-1}$ be the mass and velocity of the heavier mass 1 respectively and $m_2 = 15$ kg and $u_2 = 20 \text{ ms}^{-1}$ be the mass and velocity of the lighter mass 2 respectively.

Let v be the common velocity of the masses after collision. This is unknown.

i) If they were moving in the same direction before collision then their velocities before collision have the same signs. Note that the mass with the higher velocity will move to collide with the mass with smaller velocity.

By the principle of conservation of linear momentum, $m_1 u_1 + m_2 u_2 = (m_1 + m_2)v$

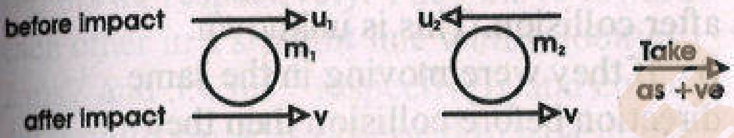
$$20(25) + 15(20) = (20 + 15)v$$

$$500 + 300 = 35v \quad \Rightarrow \quad 800 = 35v$$

$$v = \frac{800}{35} = 22.86 \text{ ms}^{-1}$$

The common velocity of the masses after collision is 22.86 ms^{-1}.

ii) If they were moving in opposite directions before collision then their velocities before collision have different signs.



By the law of conservation of linear momentum, $m_1 u_1 - m_2 u_2 = (m_1 + m_2)v$

$$20(25) - 15(20) = (20 + 15)v$$

$$500 - 300 = 35v$$

$$200 = 35v \quad \Rightarrow \quad v = \frac{200}{35} = 5.71 \text{ ms}^{-1}$$

The common velocity of the masses after collision is 5.71 ms^{-1}.

Example 4

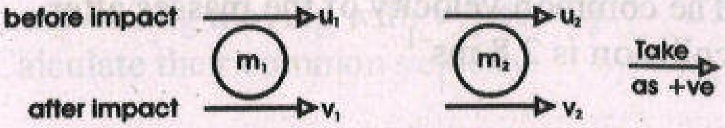
Two particles of masses 3 kg and 5 kg move with velocities 4 ms^{-1} and 2 ms^{-1} respectively in the same direction. After collision, the bigger mass moves with a velocity of 0.8 ms^{-1}. Find the velocity of the smaller mass.

Solution

Let $m_1 = 3$ kg and $u_1 = 4 \text{ ms}^{-1}$ be the mass and velocity of the smaller mass 1 respectively and $m_2 = 5$ kg and $u_2 = 2 \text{ ms}^{-1}$ be the mass and velocity of the bigger mass 2 respectively.

Let v_1 = unknown final velocity of smaller mass 1 and $v_2 = 0.8 \text{ ms}^{-1}$ be the final velocity of bigger mass 2.

If they were moving in the same direction before collision then their velocities before collision have the same signs.



By the principle of conservation of linear momentum, $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$

$$3(4) + 5(2) = 3v_1 + 5(0.8)$$

$$12 + 10 = 3v_1 + 4$$

$$22 - 4 = 18 = 3 v_1$$

$$v_1 = \frac{18}{3} = 6 \text{ ms}^{-1}$$

The velocity of the smaller mass after collision is 6 ms^{-1} in the same direction as the initial direction of motion.

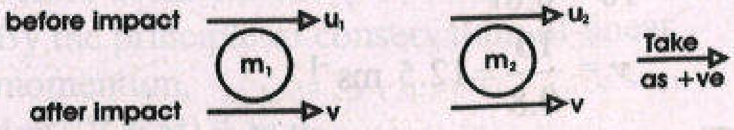
Example 5

Two bodies having masses 6 kg and 4 kg move in the same direction with velocities 2 ms^{-1} and 4 ms^{-1} respectively. They collide and stick together. Find their common velocity after collision.

Solution

Let $m_1 = 4$ kg and $u_1 = 4 \text{ ms}^{-1}$ be the mass and velocity of the smaller mass 1 respectively and $m_2 = 6$ kg and $u_2 = 2 \text{ ms}^{-1}$ be the mass and velocity of the bigger mass 2 respectively.

Let v be the common velocity of the masses after collision. This is unknown.



By the principle of conservation of linear momentum, $m_1 u_1 + m_2 u_2 = (m_1 + m_2)v$

$$4(4) + 6(2) = (4 + 6)v$$

$$16 + 12 = 10v \Rightarrow 28 = 10v$$

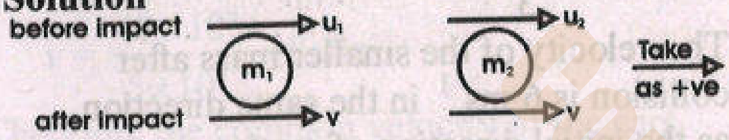
$$v = \frac{28}{10} = 2.8 \text{ ms}^{-1}$$

The common velocity of the masses after collision is 2.8 ms^{-1}.

Example 6

A bullet of mass 0.04 kg is fired with velocity of 250 ms^{-1} into a 0.76 kg block of wood. Find the common velocity of the block and bullet after impact if the bullet got stuck in the wood.

Solution



Let m_1 = mass of bullet = 0.04 kg and
m_2 = mass of block = 0.76 kg
u_1 = initial velocity of bullet = 250 ms^{-1}

and u_2 = initial velocity of block
= 0 ms^{-1} (at rest)
v = common velocity of bullet and block
= unknown

Total momentum before collision
$= m_1 u_1 + m_2 u_2 = 0.04(250) + 0.76(0) = 10$

Total momentum after collision
$= (m_1 + m_2)v = (0.04 + 0.76)v = 0.8v$

By the principle of conservation of linear momentum

Total momentum before collision
= Total momentum after collision

$$10 = 0.8v$$

$$v = \frac{10}{0.8} = 12.5 \text{ ms}^{-1}$$

The common velocity of the bullet and block is 12.5 ms^{-1}.

Example 7

A particle of mass 3 kg moving along a track at 8 ms^{-1} towards another particle of mass 5 kg travelling 5 ms^{-1} on the same track. If the particles move together after impact, find their common velocity, if they were travelling in:

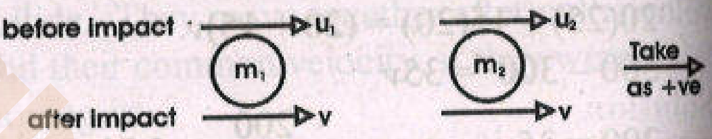
i) the same direction;
ii) opposite directions.

Solution

Let m_1 = 3 kg and u_1 = 8 ms^{-1} be the mass and velocity of the smaller mass 1 respectively and m_2 = 5 kg and u_2 = 5 ms^{-1} be the mass and velocity of the bigger mass 2 respectively.

Let v be the common velocity of the masses after collision. This is unknown.

i) If they were moving in the same direction before collision then their velocities before collision have the same signs.



By the principle of conservation of linear momentum, $m_1 u_1 + m_2 u_2 = (m_1 + m_2)v$

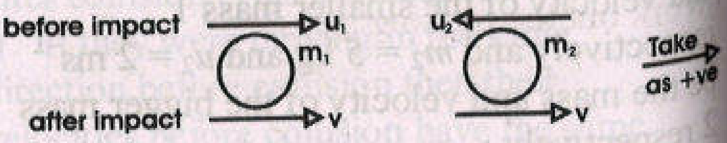
$$3(8) + 5(5) = (3 + 5)v$$

$$24 + 25 = 8v \Rightarrow 49 = 8v$$

$$v = \frac{49}{8} = 6.125 \text{ ms}^{-1}$$

The common velocity of the masses after collision is 6.125 ms^{-1}.

ii) If they were moving in opposite directions before collision then their velocities before collision have different signs.



By the law of conservation of linear momentum,

$$m_1 u_1 - m_2 u_2 = (m_1 + m_2)v$$
$$3(8) - 5(5) = (3 + 5)v$$
$$24 - 25 = 8v$$
$$-1 = 8v$$

$v = \dfrac{-1}{8} = -0.125\ \text{ms}^{-1}$ (i.e. in the direction of bigger mass).

The common velocity of the masses after collision is 0.125 ms^{-1} in the direction of bigger mass.

Example 8

Two objects, *P* and *Q* are of masses 0.8 kg and 0.5 kg respectively. They move towards each other in a straight line with velocities 2 ms^{-1} and 3.4 ms^{-1} respectively and collide. After collision, *P* is observed to have velocity of 1.5 ms^{-1} in the opposite direction. Calculate the magnitude of the velocity of *Q*.

Solution

2.2 ms^{-1}

(Work out the missing steps)

Example 9

A body of mass 8 kg moving with a speed of 8 ms^{-1} collides with another body of mass 12 kg with a speed of 4 ms^{-1} moving in the same direction. After collision the body of mass 8 kg moves with a speed of 5 ms^{-1} while the other body moves with a speed, V ms^{-1}. Calculate the value of V.

Solution

$V = 6\ \text{ms}^{-1}$

(Work out the missing steps)

Impacts with velocities given in vector form

Illustrative examples

Example 1

A body of mass 1 kg moving with a velocity $\mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$ collides and combines with a mass of 2 kg moving with $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$. Calculate their common velocity.

Solution

Total momentum before collision
$= 1(4\mathbf{i} + 3\mathbf{j}) + 2(4\mathbf{i} - 3\mathbf{j})$
$= (4\mathbf{i} + 3\mathbf{j}) + (8\mathbf{i} - 6\mathbf{j}) = 12\mathbf{i} - 3\mathbf{j}$

Total momentum after collision
$= (1 + 2)\mathbf{v} = 3\mathbf{v}$, where $\mathbf{v}$ = their common velocity after collision. By the principle of conservation of linear momentum,
$12\mathbf{i} - 3\mathbf{j} = 3\mathbf{v} \Rightarrow \mathbf{v} = 4\mathbf{i} - \mathbf{j}$

Example 2

Two smooth spheres *A* and *B* collide. The mass of *A* is four times that of *B*. The final velocities of *A* and *B* after collision are $\mathbf{i} + 2\mathbf{j}$ and $-4\mathbf{i} + 7\mathbf{j}$ respectively. If the initial velocity of *A* is $-2\mathbf{i} + 5\mathbf{j}$ and the linear momentum is conserved, find the initial velocity of *B*.

Solution

Let the mass of *B*, $m_B = m$, then the mass of *A*, $m_A = 4m$

Total momentum before collision
$= m_A \mathbf{u}_A + m_B \mathbf{u}_B = 4m(-2\mathbf{i} + 5\mathbf{j}) + m\,\mathbf{u}_B$

Total momentum after collision
$= m_A \mathbf{v}_A + m_B \mathbf{v}_B = 4m(\mathbf{i} + 2\mathbf{j}) + m(-4\mathbf{i} + 7\mathbf{j})$

Note that $\mathbf{u}_B$ = the initial velocity of *B* before collision.

By the principle of conservation of linear momentum,
$4m(-2\mathbf{i} + 5\mathbf{j}) + m\,\mathbf{u}_B$
$= 4m(\mathbf{i} + 2\mathbf{j}) + m(-4\mathbf{i} + 7\mathbf{j})$

$$\mathbf{u}_B = 4(\mathbf{i} + 2\mathbf{j}) + (-4\mathbf{i} + 7\mathbf{j}) - 4(-2\mathbf{i} + 5\mathbf{j})$$
$$= 8\mathbf{i} - 5\mathbf{j}$$

Therefore the initial velocity of *B* before collision is $8\mathbf{i} - 5\mathbf{j}$.

Example 3

A particle of mass 4 kg moving with a velocity $\mathbf{u} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ ms^{-1} collides with another particle of mass 6 kg moving with a velocity $\mathbf{v} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$ ms^{-1}. If they move together after collision, find:

i) the magnitude of their common velocity to the nearest whole number;

ii) the direction, correct to the nearest degree, of their common velocity.

Solution

Let mass of lighter particle be $m_A = 4$ kg and mass of heavy particle be $m_B = 6$ kg

Total momentum before collision

$$= m_A\,\mathbf{u}_A + m_B\,\mathbf{u}_B = 4\begin{pmatrix} 5 \\ 3 \end{pmatrix} + 6\begin{pmatrix} 5 \\ 12 \end{pmatrix} = \begin{pmatrix} 50 \\ 84 \end{pmatrix}$$

Total momentum after collision

$= (m_A + m_B)\mathbf{v} = (4 + 6)\mathbf{v} = 10\mathbf{v}$

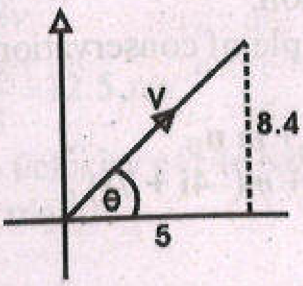
Note that $\mathbf{v}$ = their common velocity

By the principle of conservation of momentum, $\begin{pmatrix} 50 \\ 84 \end{pmatrix} = 10\mathbf{v}$

$$\mathbf{v} = \tfrac{1}{10}\begin{pmatrix} 50 \\ 84 \end{pmatrix} = \begin{pmatrix} 5 \\ 8.4 \end{pmatrix}$$

The magnitude of $\mathbf{v} = |\mathbf{v}| = \sqrt{(5)^2 + (8.4)^2}$

$$= 8 \text{ ms}^{-1}$$



From the diagram, $\tan\theta = \frac{8.4}{5} = 1.68$

$\theta = 59°$

The actual direction measured from the north $= 90° - 59° = 31°$

$\therefore$ The common velocity is (8 ms^{-1}, 031°).

Example 4

A vehicle *P* of mass 3×10^5 kg moving with a velocity (15 ms^{-1}, 090°) collides with a vehicle *Q* of mass 4×10^5 kg moving with a velocity (12 ms^{-1}, 120°).

a) Write down the velocities as column vectors.

b) Calculate the magnitude and direction of the common velocity with which the vehicle begin to move immediately after the collision.

Solution

Writing down the velocities as column vectors, we have;

a) The velocity of vehicle *P* before collision

$$\mathbf{u}_p = \begin{pmatrix} 15\sin 90° \\ 15\cos 90° \end{pmatrix} = \begin{pmatrix} 15 \\ 0 \end{pmatrix} \text{ms}^{-1}$$

The velocity of vehicle *Q* before collision

$$\mathbf{u}_Q = \begin{pmatrix} 12\sin 120° \\ 12\cos 120° \end{pmatrix} = \begin{pmatrix} 10.4 \\ -6 \end{pmatrix} \text{ms}^{-1}$$

b) Let $\mathbf{v}$ be the common velocity of the vehicles after collision

Total momentum before collision

$= m_p\,\mathbf{u}_p + m_Q\,\mathbf{u}_Q$

$$= 3 \times 10^5\begin{pmatrix} 15 \\ 0 \end{pmatrix} + 4 \times 10^5\begin{pmatrix} 10.4 \\ -6 \end{pmatrix}$$

Note that $m_p = 3 \times 10^5$ kg and m_Q
$= 4 \times 10^5$ kg

Total momentum after collision

$= m_p\,\mathbf{v} + m_Q\mathbf{v} = (m_p + m_Q)\mathbf{v}$

$= (3 \times 10^5 + 4 \times 10^5)\mathbf{v} = 7 \times 10^5\mathbf{v}$

Therefore by the principle of conservation of linear momentum,

$$m_P\,\mathbf{u}_P + m_Q\,\mathbf{u}_Q = (m_P + m_Q)\mathbf{v}$$

$$3 \times 10^5\begin{pmatrix}15\\0\end{pmatrix} + 4 \times 10^5\begin{pmatrix}10.4\\-6\end{pmatrix} = 7 \times 10^5\mathbf{v}$$

$$3\begin{pmatrix}15\\0\end{pmatrix} + 4\begin{pmatrix}10.4\\-6\end{pmatrix} = 7\mathbf{v}$$

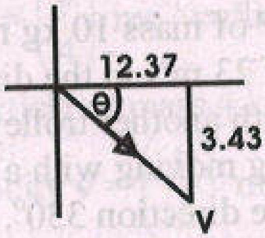
$$\begin{pmatrix}86.6\\-24\end{pmatrix} = 7\mathbf{v}$$

$$\mathbf{v} = \begin{pmatrix}12.37\\-3.43\end{pmatrix}\text{ms}^{-1}$$

$\therefore$ The magnitude of **v** is

$$|\mathbf{v}| = \sqrt{(12.37)^2 + (-3.43)^2}$$
$$= 12.84\text{ ms}^{-1}$$

To find the direction of **v**, first represent the velocity, **v** on a rough diagram and calculate the acute angle θ, it makes with the x-axis.



From the diagram, $\tan\theta = \frac{3.43}{12.37} = 0.2773$

$\theta = 15.5°$

The actual direction measured from the north $= 90° + 15.5° = 105.5°$

$\therefore$ The common velocity is $(12.84\text{ ms}^{-1}, 105.5°)$.

Example 5

A truck A of mass 5×10^4 kg moving with a velocity $(4\text{ ms}^{-1}, 060°)$ collides with another truck B of mass 3×10^4 kg moving with a velocity $(6\text{ ms}^{-1}, 000°)$. Immediately after collision, A begins to move with a velocity $(3\text{ ms}^{-1}, 090°)$ while B begins to move with a velocity $(v\text{ ms}^{-1}, 004°)$.

a) Express each of the above four velocities as a column vector.

b) Using the fact that the total momentum in the direction 090° is conserved, calculate, correct to two decimal places, the value of v.

Solution

a) Let the mass of truck A be $m_A = 5 \times 10^4$ kg and that of truck B be $m_B = 3 \times 10^4$ kg

Let the velocities of trucks A and B before collision be $\mathbf{u}_A$ and $\mathbf{u}_B$ respectively.

$$\mathbf{u}_A = \begin{pmatrix}4\sin 60°\\4\cos 60°\end{pmatrix} = \begin{pmatrix}3.46\\2\end{pmatrix}\text{ms}^{-1}$$

$$\mathbf{u}_B = \begin{pmatrix}6\sin 0°\\6\cos 0°\end{pmatrix} = \begin{pmatrix}0\\6\end{pmatrix}\text{ms}^{-1}$$

Let the velocities of trucks A and B after collision be $\mathbf{v}_A$ and $\mathbf{v}_B$ respectively

$$\mathbf{v}_A = \begin{pmatrix}3\sin 90°\\3\cos 90°\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix}\text{ms}^{-1}$$

$$\mathbf{v}_B = \begin{pmatrix}v\sin 4°\\v\cos 4°\end{pmatrix} = \begin{pmatrix}0.07v\\1.00v\end{pmatrix}\text{ms}^{-1}$$

b) Total momentum before collision

$$= m_A\,\mathbf{u}_A + m_B\,\mathbf{u}_B$$
$$= 5 \times 10^4\begin{pmatrix}3.46\\2\end{pmatrix} + 3 \times 10^4\begin{pmatrix}0\\6\end{pmatrix}$$

Total momentum after collision

$$= m_A\,\mathbf{v}_A + m_B\,\mathbf{v}_B$$
$$= 5 \times 10^4\begin{pmatrix}3\\0\end{pmatrix} + 3 \times 10^4\begin{pmatrix}0.07v\\1.00v\end{pmatrix}$$

Since momentum is conserved in the direction 090° i.e. on the x-axis, we can only equate the x components of the total momentum before collision and the total momentum after collision.

$$\therefore\ 5 \times 10^4(3.46) + 3 \times 10^4(0)$$
$$= 5 \times 10^4(3) + 3 \times 10^4(0.07v)$$

$$5(3.46) + 3(0) = 5(3) + 3(0.07v)$$
$$17.3 = 15 + 0.21v$$
$$2.3 = 0.21v$$
$$v = 2.3/0.21 = 10.95$$
$$\therefore \quad v = 10.95 \text{ ms}^{-1}.$$

Example 6

A car of mass 4×10^3 kg travelling with a velocity of (12 ms^{-1}, 120°) collides with another car of mass 2×10^3 kg travelling with a velocity of (6 ms^{-1}, 300°). If the velocity of the first car changes to (4 ms^{-1}, 120°) immediately after the collision, find the velocity of the second car.

Solution

Let $m_1 = 4 \times 10^3$; $m_2 = 2 \times 10^3$

By the principle of conservation of linear momentum

$$m_1 \mathbf{u}_1 + m_2 \mathbf{u}_2 = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2$$

$$\mathbf{u}_1 = \begin{pmatrix} 12\sin 120^\circ \\ 12\cos 120^\circ \end{pmatrix} = \begin{pmatrix} 10.39 \\ -6.00 \end{pmatrix}$$

$$\mathbf{u}_2 = \begin{pmatrix} 6\sin 300^\circ \\ 6\cos 300^\circ \end{pmatrix} = \begin{pmatrix} -5.20 \\ 3.00 \end{pmatrix}$$

$$\mathbf{v}_1 = \begin{pmatrix} 4\sin 120^\circ \\ 4\cos 120^\circ \end{pmatrix} = \begin{pmatrix} 3.66 \\ -2.00 \end{pmatrix}$$

$$4 \times 10^3 \begin{pmatrix} 10.39 \\ -6.00 \end{pmatrix} + 2 \times 10^3 \begin{pmatrix} -5.20 \\ 3.00 \end{pmatrix}$$

$$= 4 \times 10^3 \begin{pmatrix} 3.66 \\ -2.00 \end{pmatrix} + 2 \times 10^3 \mathbf{v}_2$$

$$\begin{pmatrix} 31.16 \\ -18 \end{pmatrix} = \begin{pmatrix} 13.84 \\ -8 \end{pmatrix} + 2\mathbf{v}_2$$

$$2\mathbf{v}_2 = \begin{pmatrix} 17.32 \\ -10 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} 8.66 \\ -5.00 \end{pmatrix}$$

$$|\mathbf{v}_2| = \sqrt{(8.66)^2 + (-5)^2}$$
$$= \sqrt{99.9956} = 10.0 \text{ ms}^{-1}$$

$$\text{Direction} = 90^\circ + \tan^{-1}\frac{5}{8.66}$$
$$= 90^\circ + 30^\circ = 120^\circ$$

$\therefore$ $\mathbf{v}_2 = (10 \text{ ms}^{-1}, 120^\circ)$ = velocity of second car

Review exercise 26

1. A particle is projected vertically upwards with a speed of 25 ms^{-1} from a point on the ground. [Take $g = 10 \text{ ms}^{-2}$]. Find
   - i) the position of the particle after 4 s;
   - ii) the maximum height reached;
   - iii) the time taken to reach the maximum height;
   - iv) the time(s) when the particle is 30 m above the ground.

2. A trolley *P* of mass 10 kg moving with a velocity of 23 m/s in the direction 040° collides with another trolley *Q* of mass 1.5×10^3 kg moving with a velocity of 9 m/s in the direction 330°. Immediately after the collision, the trolley *P* begins to move with a velocity of 8 m/s in the direction 300°. Assuming conservation of linear momentum, calculate the magnitude and direction (correct to the nearest degree) of the velocity of *Q* immediately after collision.

3. A body *P* of mass 100 kg moving with velocity 16 m/s due south collides with a body *Q* of mass 120 kg moving with a velocity 5 m/s due north. After collision *Q* moves with a velocity of 5 m/s due south. Find:
   - a) the velocity of *P* immediately after collision;

b) the change in momentum of *P* and of *Q*.

4. A particle of mass 2 kg has an initial velocity $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ms^{-1} and final velocity $\mathbf{v} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ ms^{-1}. Find the change in momentum.

5. A particle of mass 2 kg has an initial velocity $\mathbf{u} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$ ms^{-1} and final velocity $\mathbf{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ ms^{-1}. Find the change in momentum

6. A particle of mass 6 kg moving with a velocity $\mathbf{u} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$ ms^{-1} collides with another particle of mass 4 kg moving with a velocity $\mathbf{v} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$ ms^{-1}. They move together after collision. If they were moving in opposite directions, find correct to two decimal places:
   i) the magnitude of their common velocity;
   ii) the direction, correct to the nearest degree, of their common velocity.

7. A particle of mass 4 kg moving with a velocity $\mathbf{u} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ ms^{-1} collides with another particle of mass 6 kg moving with a velocity $\mathbf{v} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$ ms^{-1}. They move together after collision. If they were moving in opposite directions, find:
   i) the magnitude of their common velocity to the nearest whole number;
   ii) the direction, correct to the nearest degree, of their common velocity.

8. Two particles of masses 3 kg and 2 kg move with velocities 5ms^{-1} and 2ms^{-1} respectively in the opposite direction. After collision, the bigger mass moves with a velocity of $1.4\ \text{ms}^{-1}$. Find the velocity of the smaller mass.

9. A pile driver of mass 2 kg, moving with velocity $7\ \text{ms}^{-1}$ before impact, hits a stationary pile of mass 0.5 kg. Find their common velocity after impact.

10. Two masses 20 kg and 15 kg moving with velocities $9\ \text{ms}^{-1}$ and $4\ \text{ms}^{-1}$ respectively collide. They move together after collision, find their common velocity, if they were moving
    i) in the same direction.
    ii) in opposite directions.

11. A bullet of mass 50 g, moving horizontally, strikes a stationary target at 486 m/s and becomes embedded in it. The target is of mass 4 kg and is free to move, calculate
    i) the speed at which the target and the embedded bullet move initially.
    ii) the impulse imparted by the target by the bullet.

12. At time $t = 0$ a particle is projected vertically upwards from a point *O* with speed $19.6\ \text{ms}^{-1}$ and two seconds later, a second particle is projected vertically upwards from *O* with the same speed.

Assuming that the only force acting is that due to gravity, express the heights above O of both particles in terms of t and hence, or otherwise, find the value of t when they collide. Find the speeds of the particles at the instant of collision.

13. A particle moves along a straight line ABC. It starts from rest at A and moves with constant acceleration 2 ms^{-2} until its reaches B. It then moves from B to C with constant acceleration 1 ms^{-2}. The times taken to travel from A to B and from B to C are each equal to T seconds. Show that $5AB = 2AC$.
Given that the distance AC is 56 m, calculate T.

14. A car starts from rest at time $t = 0$ seconds and moves with a uniform acceleration of magnitude 2.3ms^{-2} along a straight horizontal road. After T seconds, when its speed is V ms^{-1} it immediately stops acceleration and maintains this steady speed until it hits a brick wall when it comes instantly to rest. The car has the travelled a distance of 776.25 m in 30 s.

a) Sketch a speed-time graph to illustrate this information.

b) Write down an expression for V in terms of T.

c) Show that $T^2 - 60T + 675 = 0$.

15. A man of 80 kg stands on the floor of a lift which is ascending vertically. Given that the reaction of the floor on the man is 830 N, calculate the acceleration of the lift. Given that the lift starts from rest at ground level and ascends with this acceleration for 8 s, show that the lift attains a speed of 3 ms^{-1}.

16. A stone is thrown vertically upwards with a speed of 20 ms^{-1}. One second later, a second stone is thrown vertically upwards with speed of 30 ms^{-1}. At what height above the ground do they collide?

17. A small ball, of mass 0.3 kg, moving with velocity $(5\mathbf{i} - 3\mathbf{j})$ ms^{-1} receives an impulse $(-1.2\mathbf{i} + 5.7\mathbf{j})$ N s. Find the speed of the ball immediately afterwards.

18. A lift of mass 500 kg is descending with an acceleration of 1.5 m s^{-2}. A man of mass 80 kg is inside the lift. Calculate

a) the tension in the cable connected to the lift.

b) the force between the man and the floor of the lift.

19. A truck of mass 2×10^4 kg moving with velocity of (5m/s, 000°) collides with another of mass 1.5×10^4 kg moving with a velocity of (8m/s, 090°).

a) Write down the two velocities as column vectors.

b) If the trucks move on with a common velocity, find the magnitude and direction (to the nearest degree) of the common velocity.

20. a) If a constant force acts on a toy car of mass 5 kg and increases its velocity from 5 m/s to 9 m/s in 2 s, calculate

i) the magnitude of the force.

ii) the velocity of the car 3s after attaining the velocity of 9 m/s.

21. A train moves from rest and reaches a speed of 72 km/h after 4 minutes and 90 km/h after a further 6 minutes. The train maintains a steady speed for the next 3 minutes and decelerates to rest in the last 3 minutes.

a) Indicate the motion of the train on a speed-time graph.

b) Calculate
   - i) the accelerations in m/s^2
   - ii) the total distance covered in the whole journey
   - iii) the average speed of the train in km/h

22. A constant force of 35 N, always acting in the same horizontal direction, causes a particle of mass 2 kg to move over a rough horizontal plane. The particle passes two points *X* and *Y*, 4 m apart, with speeds of 5 m/s and 10 m/s respectively. The friction resistance to motion is constant. Calculate:

i) the acceleration of the particle,

ii) the magnitude of the frictional resistance,

iii) the distance of the particle from *X*, 4 s after it has passed *X*.

23. A force $\mathbf{F_1} = (4\mathbf{i} + 2\mathbf{j})$N. State the magnitude of $\mathbf{F_1}$. A second force $\mathbf{F_2}$ has magnitude $8\sqrt{5}$ N and acts in a direction given by the vector $\mathbf{i} + 2\mathbf{j}$. State the force vector $\mathbf{F_2}$. Hence calculate the resultant force $\mathbf{F_R}$ of these two forces and the unit vector in the direction of this resultant. What is the acceleration that this resultant would produce on a mass of 3 kg.

24. Two smooth balls of masses 20 g and 15g moving with speeds 9 ms^{-1} and 4 ms^{-1} respectively collide. If they were moving in the same direction and the speed of the first ball reduces to 6 ms^{-1}, find the speed of the second ball after collision.

25. Two particles of masses 3 g and 5 g, move with velocities 4 ms^{-1} and 2 ms^{-1} respectively in the same direction. After collision the bigger mass moves with a velocity of 0.8 ms^{-1}. Find the velocity of the smaller mass.

26. The velocity of a ball of mass 5 kg increases from (3 ms^{-1}, 180°) to (18 ms^{-1}, 180°) in 3 s. Find the force acting on it.

27. A particle of mass 0.5 kg is connected to a particle of mass 0.2 kg by a light string passing over a smooth pulley. The system is released from rest with the string taut.

a) Find the acceleration of either particle and the tension in the string.

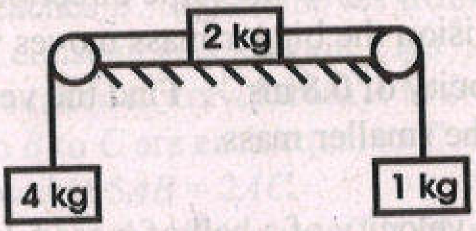
b) After 2 seconds the ascending particle picks up a stationary particle of mass 0.42 kg. Find the velocity of the particles immediately afterwards.

28. A car of mass 800 kg is towing a trailer of mass 200 kg on a level ground. The resistances to motion are 4 N per kg of mass for each vehicle. Find the tension in the towing bar, and the tractive force of the car when the car and the trailer are:

a) travelling at a steady speed

b) accelerating at 0.8 ms^{-2}

29. Three particles are connected by a light inextensible string. The 2 kg particle is placed on a smooth table and the string passes over two small smooth light pulleys which are fixed at the ends of the table as shown below. The system is released from rest. Calculate the acceleration of each particle and the tensions in the strings. (Take $g = 10\,ms^{-2}$).



30. A railway truck *A* of mass 4000 kg travelling at 2 ms^{-1} collides with another truck *B* of mass 6000 kg travelling at 1 ms^{-1} in the same direction. The speed of truck *A* after collision is 1.25 ms^{-1} in the same direction. Calculate the speed of truck *B* after collision.

31. An orange falls from a height of 18 m above the ground. Find the speed with which the orange hits the ground. [Take $g = 10\ ms^{-2}$]

32. A car accelerates uniformly from 8.3 ms^{-1} to 13.8 ms^{-1} within a distance of 60 m. Calculate, correct to the **nearest** second, the time it takes to cover the distance.

33. Find the constant force that will give a body of mass 5 kg at rest a velocity of 6.4 ms^{-1} in 4 s.

34. A body of mass 4 kg moving with a speed of 12 ms^{-1}, collides with another body of mass 2 kg initially at rest. After collision, the two bodies coalesce.

i) Calculate the common velocity after collision.

ii) If the two bodies come to rest after covering a distance of 16 m, find the acceleration.

35. A stone is projected vertically upwards with a speed of 20 ms^{-1}. Three seconds later, a second stone is projected at the same point and with the same initial speed. At what height will the two stones meet? [Take $g = 10\ ms^{-2}$]

36. A stone is released vertically down from the top of a roof 40 m high. Calculate, correct to **three** significant figures, the speed with which it hits the ground. [Take $g = 10\ ms^{-2}$]

37. Two bodies of masses 80 kg and 50 kg moving in the same direction with speeds 4 ms^{-1} and 2 ms^{-1} respectively, collide. If they move together find the common velocity, correct to one decimal place.

CHAPTER 27

MATRICES

Idea of a matrix

A matrix is a rectangular array of numbers, usually encased in parentheses.

Suppose a car dealer sells two brands of cars Hondas (*H*) and Toyotas (*T*), and three models for each of brands Sedan(*S*), Pickup(*P*) and Coupe (*C*).

The inventory of the cars is shown below.

	S	*P*	*C*
H	5	3	2
T	8	4	1

This inventory can be put in a 2 × 3 matrix where the rows are indexed as brands and the columns are indexed as models.

i.e. $\mathbf{V} = \begin{pmatrix} 5 & 3 & 2 \\ 8 & 4 & 1 \end{pmatrix}$

If the dealer were to get a shipment of cars as is shown in the table below.

	S	*P*	*C*
H	5	2	3
T	1	3	3

Then, this new shipment can also be represented by a 2 × 3 matrix as

$$\mathbf{S} = \begin{pmatrix} 5 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix}$$

Definition of a matrix

A **matrix** is a rectangular array of numbers. The numbers in a matrix are called **elements**.

The **order** of a matrix is the "number of rows × the number of columns". Thus a $(m \times n)$ matrix is a matrix with m rows and n columns. A (2×3) matrix has two rows and three columns.

A matrix is denoted by a bold letter.

For example, $\mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2 × 2 matrix

i.e. a matrix with 2 rows and 2 columns.

The numbers or entries a, b, c, and d are the **elements** of the matrix .

Also the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ is a 3 × 3 matrix i.e. a matrix with 3 rows and 3 columns.

Note: Commas are not used in matrices. Gaps are left between the columns.

Types of matrices

Column or row matrix

A **column matrix** has only one column of elements. For example, $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is a 2 × 1 matrix or a column matrix.

A **row matrix** has only one row of elements. For example, $\mathbf{A} = (3 \;\; 2)$ is a 1 × 2 matrix or a row matrix. We shall restrict ourselves to 2 × 2 matrices and in some cases 3 × 3 matrices .

The null or zero matrix

The **null or zero matrix** is a square matrix in which every element is zero.
For example, the 2 × 2 zero matrix is

$$\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and the } 3 \times 3 \text{ zero matrix is}$$

$$\mathbf{O} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ The zero matrix is the identity matrix for addition.

Note that a zero matrix may or may not be a square matrix.

Rectangular matrix

A matrix in which the number of rows is not equal to the number of columns is called a rectangular matrix. For example,

1. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \end{bmatrix}$ is a rectangular matrix of order 2 × 3.
2. $\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$ is a rectangular matrix of order 3 × 2.

Square matrix

A **square matrix** is a matrix which has the same number of rows as columns.
For example,

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 0 & -3 \\ 4 & 5 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

are square matrices.

The unit matrix or identity matrix

A square matrix having every non-diagonal element equal to zero and every diagonal element equal to one is called a unit or an identity matrix.

For example, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are unit matrices of order 2 and 3 respectively.

Note that a unit matrix is always a diagonal matrix.
Any matrix multiplied by the unit matrix gives the same matrix i.e. $\mathbf{AI = A}$.

Diagonal matrix

A square matrix in which every non-diagonal element is equal to zero is called a diagonal matrix.
Note that the definition of a diagonal matrix does not specify anything about the diagonal elements. All it says is that the non-diagonal elements are zeros. Hence its diagonal elements may or may not be zeros

Examples: $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

Equal matrices

Two matrices are said to be **equal** if their corresponding entries are the same.

For example, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, then
$a = e;\ b = f;\ c = g;\ d = h.$

Also if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 0 & 7 \end{pmatrix}$, then
$a = 3;\ b = 5;\ c = 0;\ d = 7.$

Addition of matrices

Two matrices can be added if they are of the same order. Thus we can add a 2 × 2 matrix to another 2 × 2 matrix. To **add** two or more matrices, simply add corresponding elements. This gives a matrix of the *same* order.

For example, for these 2 × 2 matrices:

$$\underset{\text{order}(2\times2)}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} + \underset{\text{order}(2\times2)}{\begin{pmatrix} e & f \\ g & h \end{pmatrix}} = \underset{\text{order}(2\times2)}{\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}}$$

Note the following:

1. The zero matrix is the identity matrix under addition i.e. $\mathbf{O}+\mathbf{A}=\mathbf{A}+\mathbf{O}=\mathbf{A}$

2. Matrix addition is commutative and associative i.e.
   $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
   $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$

Illustrative examples

Example 1

A shopkeeper has two separate shops which she opens on Mondays, Wednesdays and Fridays. In a particular week, she made the following sales in the two shops. Represent the total sales she made in the week in a matrix.

Shop *A*

	Fanta	Malta	Cola
Monday	10	8	6
Wednesday	8	7	9
Friday	11	9	12

Shop *B*

	Fanta	Malta	Cola
Monday	9	10	8
Wednesday	7	9	10
Friday	13	11	15

Solution

This sales in each shop can be put in a 3 × 3 matrix where the rows are indexed as days of the week and the columns are indexed as types of drinks.

i.e. $\mathbf{A}=\begin{pmatrix} 10 & 8 & 6 \\ 8 & 7 & 9 \\ 11 & 9 & 12 \end{pmatrix}$, $\mathbf{B}=\begin{pmatrix} 9 & 10 & 8 \\ 7 & 9 & 10 \\ 13 & 11 & 15 \end{pmatrix}$

The total sales of the two shops is given by the sum of the two matrices.

i.e. $\mathbf{T}=\mathbf{A}+\mathbf{B}$

$$\mathbf{T}=\begin{pmatrix} 10 & 8 & 6 \\ 8 & 7 & 9 \\ 11 & 9 & 12 \end{pmatrix}+\begin{pmatrix} 9 & 10 & 8 \\ 7 & 9 & 10 \\ 13 & 11 & 15 \end{pmatrix}$$

$$\mathbf{T}=\begin{pmatrix} 19 & 18 & 14 \\ 15 & 16 & 19 \\ 24 & 20 & 27 \end{pmatrix}$$

Example 2

The matrices **A** and **B** are given by

$\mathbf{A}=\begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} 3 & -5 \\ -7 & 9 \end{pmatrix}$.

Find **A** + **B**.

Solution

$$\mathbf{A}+\mathbf{B}=\begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix}+\begin{pmatrix} 3 & -5 \\ -7 & 9 \end{pmatrix}$$

$$=\begin{pmatrix} 2+3 & 6+(-5) \\ 4+(-7) & 8+9 \end{pmatrix}=\begin{pmatrix} 5 & 1 \\ -3 & 17 \end{pmatrix}$$

Take care with the *signs* when adding the numbers.

Example 3

The matrices **A** and **B** are given by

$\mathbf{A}=\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$.

Find **A** + **B**.

Solution

$$\mathbf{A}+\mathbf{B}=\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}+\begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$$

$$=\begin{pmatrix} 2+(-3) & -1+0 \\ -7+7 & 4+(-4) \end{pmatrix}=\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$$

Take care with the *signs* when adding the numbers.

Subtraction of matrices

Two matrices can be subtracted if they are of the same order. Thus we can subtract a 2 × 2 matrix from another 2 × 2 matrix. To **subtract** matrices, simply subtract corresponding elements. This gives a matrix of the *same* order. For example, for these 2 × 2 matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a-e & b-f \\ c-g & d-h \end{pmatrix}$$

order(2×2) order(2×2) order(2×2)

Note: $\mathbf{A}-\mathbf{B} \neq \mathbf{B}-\mathbf{A}$
Matrix subtraction is not commutative.

Illustrative examples

Example 1

A car dealer sells two brands of cars Hondas (*H*) and Toyotas (*T*), and three models for each of brands Sedan(*S*), Pickup(*P*) and Coupe (*C*). The inventory of the cars is shown below.

	S	*P*	*C*
H	5	3	2
T	8	4	1

If the dealer makes sales the following month as given by the table below, find the new inventory.

	S	*P*	*C*
H	1	2	0
T	3	0	1

Solution

The inventory can be put in a 2 × 3 matrix where the rows are indexed as brands and the columns are indexed as models.

i.e. $\mathbf{V} = \begin{pmatrix} 5 & 3 & 2 \\ 8 & 4 & 1 \end{pmatrix}$

The sales can also be put in a 2 × 3 matrix as $\mathbf{S} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{pmatrix}$.

The new inventory will be

$$\mathbf{N} = \mathbf{V} - \mathbf{S} = \begin{pmatrix} 5 & 3 & 2 \\ 8 & 4 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 & 2 \\ 5 & 4 & 0 \end{pmatrix}$$

Example 2

The matrices **A**, **B** and **C** are given by

$\mathbf{A} = \begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -5 \\ -7 & 9 \end{pmatrix}$ and

$\mathbf{C} = \begin{pmatrix} -5 & -1 \\ 3 & 2 \end{pmatrix}$. Find

i) **A** – **B**
ii) **B** – **C**.

Solution

$$\text{i) } \mathbf{A}-\mathbf{B} = \begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix} - \begin{pmatrix} 3 & -5 \\ -7 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 2-3 & 6-(-5) \\ 4-(-7) & 8-9 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 11 \\ 11 & -1 \end{pmatrix}$$

$$\text{ii) } \mathbf{B}-\mathbf{C} = \begin{pmatrix} 3 & -5 \\ -7 & 9 \end{pmatrix} - \begin{pmatrix} -5 & -1 \\ 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & -4 \\ -10 & 7 \end{pmatrix}$$

Take care with the *signs* when subtracting the numbers.

Multiplication of a matrix by a scalar

To **multiply** a matrix by a **number** (**a scalar**), multiply each element of the matrix by the number. For example, given the number or scalar k and the matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ then}$$

$$k\mathbf{A} = k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}.$$

If $k = -1$, then $-\mathbf{A} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ is called the **negative of A.**

Example 1

Find i) $2\begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}$ ii) $-3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution

i) $2\begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -4 & 6 \end{pmatrix}$

ii) $-3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$

Example 2

If $\mathbf{A} = \begin{pmatrix} 5 & -2 \\ 3 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}$, find $\mathbf{C} = 3\mathbf{A} - \mathbf{B}$.

Solution

$$3\mathbf{A} = 3\begin{pmatrix} 5 & -2 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 15 & -6 \\ 9 & 0 \end{pmatrix}$$

$$\mathbf{C} = 3\mathbf{A} - \mathbf{B}$$

$$= \begin{pmatrix} 15 & -6 \\ 9 & 0 \end{pmatrix} - \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 15-5 & -6-(-1) \\ 9-1 & 0-3 \end{pmatrix} = \begin{pmatrix} 10 & -5 \\ 8 & -3 \end{pmatrix}$$

②

Example 3

If $\mathbf{A} = \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix}$, find $2\mathbf{A} - 3\mathbf{B} + 4\mathbf{C}$.

Solution

$$2\mathbf{A} = 2\begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ 14 & -8 \end{pmatrix},$$

$$3\mathbf{B} = 3\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -21 & 12 \end{pmatrix} \text{ and}$$

$$4\mathbf{C} = 4\begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ -8 & -16 \end{pmatrix}$$

$$\therefore 2\mathbf{A} - 3\mathbf{B} + 4\mathbf{C}$$

$$= \begin{pmatrix} -6 & 0 \\ 14 & -8 \end{pmatrix} - \begin{pmatrix} 6 & -3 \\ -21 & 12 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ -8 & -16 \end{pmatrix}$$

$$= \begin{pmatrix} -8 & 3 \\ 27 & -36 \end{pmatrix}$$

Multiplication of two (2x2) matrices

Two matrices **A** and **B** can **only** be multiplied together if the *number of columns in the first matrix* **A** **equals** the *number of the rows in the second matrix* **B**. When this happens the matrix is said to be **compatible.**

To find the product of two matrices, multiply every row of the first matrix by every column of the second. Thus the first row of the first matrix multiplies the first column of the second matrix. In general, when multiplying a matrix of order $(m \times n)$ by a matrix of order $(n \times p)$, the resulting matrix has order $(m \times p)$.

i.e: $(m \times \underbrace{n) \times (n}_{} \times p) = (m \times p)$

Thus a (3×2) matrix multiplied by a (2×1) matrix gives a matrix of order (3×1).

Given the matrices $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and

$\mathbf{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, the product of **A** and **B** is

$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$= \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

Illustrative examples

Example 1

The matrices **A**, **B** and **C** are given by

$\mathbf{A} = \begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -5 \\ -7 & 9 \end{pmatrix}$ and

$\mathbf{C} = \begin{pmatrix} -5 & -1 \\ 3 & 2 \end{pmatrix}$, find

i) **AB** ii) **BC** iii) **CB**

Solution

i) $$\mathbf{AB} = \begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -7 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} (2\times 3)+(6\times -7) & (2\times -5)+(6\times 9) \\ (4\times 3)+(8\times -7) & (4\times -5)+(8\times 9) \end{pmatrix}$$

$$= \begin{pmatrix} -36 & 44 \\ -44 & 52 \end{pmatrix}$$

ii) $$\mathbf{BC} = \begin{pmatrix} 3 & -5 \\ -7 & 9 \end{pmatrix}\begin{pmatrix} -5 & -1 \\ 3 & 2 \end{pmatrix}$$

$$\mathbf{BC} = \begin{pmatrix} (3\times -5)+(-5\times 3) & (3\times -1)+(-5\times 2) \\ (-7\times -5)+(9\times 3) & (-7\times -1)+(9\times 2) \end{pmatrix}$$

$$\mathbf{BC} = \begin{pmatrix} -30 & -13 \\ 62 & 25 \end{pmatrix}$$

iii) $$\mathbf{CB} = \begin{pmatrix} -5 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -7 & 9 \end{pmatrix}$$

$$\mathbf{CB} = \begin{pmatrix} (-5\times 3)+(-1\times -7) & (-5\times -5)+(-1\times 9) \\ (3\times 3)+(2\times -7) & (3\times -5)+(2\times 9) \end{pmatrix}$$

$$\mathbf{CB} = \begin{pmatrix} -8 & 16 \\ -5 & 3 \end{pmatrix}$$

From the results of (ii) and (iii) above, it can be seen that **BC ≠ CB.** This shows that the order in which you do matrix multiplication is very important. Multiplying two matrices in a different order does not give the same answer. Therefore, in general, matrix multiplication **is not commutative**. However matrix multiplication **is associative**.

Example 2

Given the matrices $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$ and

$\mathbf{B} = \begin{pmatrix} -1 & 2 \\ 5 & -2 \end{pmatrix}$, find **AB** and **BA**.

Solution

$$\mathbf{AB} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 5 & -2 \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} (4\times -1)+(2\times 5) & (4\times 2)+(2\times -2) \\ (1\times -1)+(3\times 5) & (1\times 2)+(3\times -2) \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 4 \\ 14 & -4 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} -1 & 2 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} (-1\times 4)+(2\times 1) & (-1\times 4)+(2\times 3) \\ (5\times 4)+(-2\times 1) & (5\times 2)+(-2\times 3) \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 4 \\ 18 & 4 \end{pmatrix}$$

Example 3

Given that $\mathbf{A}=\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}$, find $\mathbf{A}^2$

Solution

$$\mathbf{A}^2=\mathbf{AA}=\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1+2 & 2+2\\ 1+1 & 2+1\end{pmatrix}$$

$$=\begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix}$$

Note that if **B** is any 2 x 2 matrix, then **BI = IB = B** and $\mathbf{B}^2=\mathbf{B}\times\mathbf{B}$.

Example 4

If $\mathbf{A}=\begin{pmatrix}-2 & 4\\ 5 & 3\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}3 & -2\\ 1 & -3\end{pmatrix}$, find the values of p and q such that

$$\mathbf{AB}=\begin{pmatrix}p & 1\\ -3 & -4\end{pmatrix}+3\begin{pmatrix}-1 & -3\\ 7 & q\end{pmatrix}$$

Solution

$$\mathbf{AB}=\begin{pmatrix}-2 & 4\\ 5 & 3\end{pmatrix}\begin{pmatrix}3 & -2\\ 1 & -3\end{pmatrix}=\begin{pmatrix}-2 & -8\\ 18 & -19\end{pmatrix}$$

and

$$\begin{pmatrix}p & 1\\ -3 & -4\end{pmatrix}+3\begin{pmatrix}-1 & -3\\ 7 & q\end{pmatrix}$$

$$=\begin{pmatrix}p & 1\\ -3 & -4\end{pmatrix}+\begin{pmatrix}-3 & -9\\ 21 & 3q\end{pmatrix}$$

$$=\begin{pmatrix}p-3 & -8\\ 18 & -4+3q\end{pmatrix}$$

$$\therefore\begin{pmatrix}-2 & -8\\ 18 & -19\end{pmatrix}=\begin{pmatrix}p-3 & -8\\ 18 & -4+3q\end{pmatrix}$$

Since the matrices are equal

$-2=p-3 \quad\Rightarrow\quad p=1$ and

$-19=-4+3q \quad\Rightarrow\quad q=-5$

Multiplication of a matrix by a vector

The method used is the same as multiplying two matrices. Note that a vector is a (2 × 1) matrix. Given the matrix $\mathbf{A}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and the vector $\mathbf{X}=\begin{pmatrix}x\\ y\end{pmatrix}$, then

$$\mathbf{AX}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}ax+by\\ cx+dy\end{pmatrix}$$

Note that when a matrix is multiplied by a vector, the result is a vector i.e. a (2 × 1) matrix.

Illustrative examples

35.

Example 1

If $\mathbf{A}=\begin{pmatrix}3 & 1\\ 2 & 3\end{pmatrix}$ and $\mathbf{X}=\begin{pmatrix}3\\ 4\end{pmatrix}$, find **AX**.

Solution

$$\mathbf{AX}=\begin{pmatrix}3 & 1\\ 2 & 3\end{pmatrix}\begin{pmatrix}3\\ 4\end{pmatrix}$$

$$=\begin{pmatrix}(3\times3)+(1\times4)\\ (2\times3)+(3\times4)\end{pmatrix}$$

$$=\begin{pmatrix}13\\ 18\end{pmatrix}$$

Example 2

⑥; Given that $\begin{pmatrix}2 & 1\\ 4 & 2y\end{pmatrix}\begin{pmatrix}3\\ 1\end{pmatrix}=\begin{pmatrix}7\\ 8\end{pmatrix}$, find y.

Solution

$$\begin{pmatrix}2 & 1\\ 4 & 2y\end{pmatrix}\begin{pmatrix}3\\ 1\end{pmatrix}=\begin{pmatrix}7\\ 8\end{pmatrix}$$

$$\begin{pmatrix}6+1\\ 12+2y\end{pmatrix}=\begin{pmatrix}7\\ 8\end{pmatrix}$$

Comparing corresponding elements

$12+2y=8 \ \Rightarrow\ 2y=-4 \ \Rightarrow\ y=-2$

The determinant of a (2 × 2) matrix

The **determinant** of a (2 × 2), $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

is defined as det **A** = $|\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

For example, the determinant of the matrix

$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$ is given by:

$|\mathbf{A}| = (3 \times 5) - (2 \times 1) = 13$

Also the determinant of $\mathbf{B} = \begin{pmatrix} 3 & -4 \\ 1 & 6 \end{pmatrix}$ is

given by:

det $\mathbf{B} = (3 \times 6) - (-4 \times 1) = 18 - (-4) = 22$

Hence the determinant of a matrix can be expressed as a single numerical value.

Illustrative example

Find the determinants of the following matrices

4.N i) $\mathbf{A} = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$ ii) $\mathbf{B} = \begin{pmatrix} 3 & 2 \\ 5 & 8 \end{pmatrix}$

iii) $\mathbf{C} = \begin{pmatrix} 4 & -2 \\ 1 & 7 \end{pmatrix}$ iv) $\mathbf{D} = \begin{pmatrix} 6 & \frac{1}{4} \\ 8 & \frac{1}{2} \end{pmatrix}$

Solution

i) 1 ii) 14 iii) 30 iv) 1

Work out the missing steps

The determinant of a (3 × 3) matrix

The **determinant** of a 3 × 3 matrix is the **sum** of the products of the element and their cofactors of **any row** or **column** of the 3 × 3 matrix.

What is the **cofactor of an element**? The cofactor of an element is the signed minor of the element. The **minor** of an element of a 3 × 3 matrix is the value of the 2 × 2 matrix determinant obtained by covering up the row and column containing that element.

For the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, the minor of the element 4 is obtained by covering the row and column containing 4 as shown above to obtain the 2 × 2 matrix determinant $\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}$, i.e. the minor of the element 4 is $(2 \times 9) - (3 \times 8) = -6$. The sign of a minor depends on its position within the matrix, the sign pattern being

$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$. Thus the sign of element 4 in the

matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ is – (minus).

Therefore the cofactor of the element 4 is $-(-6) = 6$. Knowing how to determine the cofactor of an element we can now find the determinant of a 3 × 3 matrix.

Example 1.

Find the determinant of the matrix

$\mathbf{A} = \begin{pmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{pmatrix}$

Solution

The determinant is the sum of the products of the elements and their cofactors of any row or column of the 3 × 3 matrix.

Choosing the first row, the minor of 3 is

$\begin{vmatrix} 0 & 7 \\ -3 & -2 \end{vmatrix} = 0 - (-21) = 21.$

∴ the cofactor of 3 is 21

The minor of 4 is $\begin{vmatrix} 2 & 7 \\ 1 & -2 \end{vmatrix} = -4 - 7 = -11.$

∴ the cofactor of 4 is –(11) = 11

The minor of −1 is $\begin{vmatrix}2 & 0\\ 1 & -3\end{vmatrix} = -6 - 0 = -6$.

∴ the cofactor of 4 is − 6

Hence the determinant of **A**

$= (3 \times 21) + (4 \times 11) + (-1 \times -6)$

$= 63 + 44 + 6 = 113$

The same result will be obtained whichever row or column is selected. It is easier to choose the row or column containing a zero.

Example 2.

Find the determinant of the matrix

i) $\mathbf{B} = \begin{pmatrix} 4 & -7 & 6\\ -2 & 4 & 0\\ 5 & 7 & -4\end{pmatrix}$

ii) $\mathbf{C} = \begin{pmatrix} 1 & 1 & 1\\ 2 & -3 & -2\\ 3 & -2 & -2\end{pmatrix}$

Solution

i) Using the second row, since it contains zero. Note the signs by using the sign pattern.

The determinant of **B**

$$= -(-2)\begin{vmatrix}-7 & 6\\ 7 & -4\end{vmatrix} + (4)\begin{vmatrix}4 & 6\\ 5 & -4\end{vmatrix} + (0)\begin{vmatrix}4 & -7\\ 5 & 7\end{vmatrix}$$

$= 2(28 - 42) + 4(-16 - 30) + 0$

$= -28 - 184$

$= -212$

ii) Using the first row, (Note the signs using the sign pattern)

The determinant of **C**

$$= +(1)\begin{vmatrix}-3 & 4\\ -2 & -2\end{vmatrix} - (1)\begin{vmatrix}2 & 4\\ 3 & -2\end{vmatrix} + (1)\begin{vmatrix}2 & -3\\ 3 & -2\end{vmatrix}$$

$= 1[6 - (-8)] - 1(-4 - 12) + 1[-4 - (-9)]$

$= 14 + 16 + 5 = 35$

Use different rows to arrive at the answer. Be careful about the signs.

The inverse of a (2 × 2) matrix

The **inverse** of a matrix **A** is the matrix $\mathbf{A}^{-1}$. such that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, where **I** is the unit matrix. Thus when any matrix is multiplied by its inverse matrix the result is the unit matrix.

It is only possible to calculate the inverses of square matrices. If the determinant of a square matrix is **zero** then the matrix is called a **singular** matrix and has no inverse.

If the determinant of a square matrix is **not zero** then the matrix is called a **non-singular** matrix and has an inverse. At this level only the inverses of (2 × 2) matrices shall be considered.

The inverse of a (2 × 2) matrix $\mathbf{A} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$

is given by: $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}$

where $\det \mathbf{A} = ad - bc$. Thus to find the inverse of any matrix $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$,

1. interchange the positions of *a* and *d*
   i.e. the elements in the leading diagonal

2. change the signs of *b* and *c*
   i.e. the elements of the other diagonal

3. multiply this new matrix by $\frac{1}{ad-bc}$ i.e. the reciprocal of the determinant of the matrix.

This is a quicker method of finding the inverse of a (2×2) matrix.

Another method which uses $\mathbf{AA}^{-1} = \mathbf{I}$, as illustrated in the example 1 below can also be used.

Note that the matrix **A** *has no inverse if the* ***determinant*** *of* **A** *is zero. Also if* **A** *is a singular matrix then the* ***determinant*** *of* **A** *is zero*

Illustrative examples

Example 1

Given the matrix $\mathbf{A} = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$, find the inverse matrix $\mathbf{A}^{-1}$ of **A**.

Solution

Quicker method

det $\mathbf{A} = 4 - 6 = -2$

$$\mathbf{A}^{-1} = \tfrac{1}{-2}\begin{pmatrix}4 & -2\\ -3 & 1\end{pmatrix} = \begin{pmatrix}\frac{4}{-2} & \frac{-2}{-2}\\ \frac{-3}{-2} & \frac{1}{-2}\end{pmatrix} = \begin{pmatrix}-2 & 1\\ \frac{3}{2} & -\frac{1}{2}\end{pmatrix}$$

Alternative method

Let the inverse matrix, $\mathbf{A}^{-1} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$,

then since $\mathbf{AA}^{-1} = \mathbf{I}$

$$\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}a + 2c & b + 2d\\ 3a + 4c & 3b + 4d\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Equating corresponding elements

$b + 2d = 0$ i.e. $b = -2d$

and $3a + 4c = 0$ i.e. $a = -\frac{4}{3}c$

Substituting for a and b gives:

$$\begin{pmatrix}-\frac{4}{3}c + 2c & -2d + 2d\\ 3(-\frac{4}{3}c) + 4c & 3(-2d) + 4d\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}\frac{2}{3}c & 0\\ 0 & -2d\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Equating corresponding elements,

$\frac{2}{3}c = 1 \Rightarrow c = \frac{3}{2}$ and

$-2d = 1 \Rightarrow d = -\frac{1}{2}$

Since $b = -2d$, $b = 1$ and

since $a = -\frac{4}{3}c$, $a = -2$

Therefore the inverse matrix,

$$\mathbf{A}^{-1} = \begin{pmatrix}-2 & 1\\ \frac{3}{2} & -\frac{1}{2}\end{pmatrix}$$

Note that $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}-2 & 1\\ \frac{3}{2} & -\frac{1}{2}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

Example 2

Find the determinants and inverses of the following matrices:

i) $\mathbf{A} = \begin{pmatrix}3 & 1\\ 2 & 5\end{pmatrix}$ ii) $\mathbf{B} = \begin{pmatrix}3 & -2\\ 7 & 4\end{pmatrix}$

iii) $\mathbf{C} = \begin{pmatrix}3 & 2\\ -5 & 4\end{pmatrix}$ iv) $\mathbf{P} = \begin{pmatrix}3 & 5\\ -2 & -4\end{pmatrix}$

v) $\mathbf{Q} = \begin{pmatrix}4 & -3\\ -6 & 5\end{pmatrix}$

Solution

i) det $\mathbf{A} = 15 - 2 = 13$ and

$$\mathbf{A}^{-1} = \tfrac{1}{13}\begin{pmatrix}5 & -1\\ -2 & 3\end{pmatrix} = \begin{pmatrix}\frac{5}{13} & -\frac{1}{13}\\ -\frac{2}{13} & \frac{3}{13}\end{pmatrix}$$

ii) det **B** = 12 – (–14) = 26 and

$$\mathbf{B}^{-1} = \tfrac{1}{26}\begin{pmatrix} 4 & 2 \\ -7 & 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{13} & \frac{1}{13} \\ -\frac{7}{26} & \frac{3}{26} \end{pmatrix}$$

iii) det **C** = 12 – (–10) = 22 and

$$\mathbf{C}^{-1} = \tfrac{1}{22}\begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{11} & -\frac{1}{11} \\ \frac{5}{22} & \frac{3}{22} \end{pmatrix}$$

iv) det **P** = –12 – (–10) = –2 and

$$\mathbf{P}^{-1} = \tfrac{1}{-2}\begin{pmatrix} -4 & -5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & \frac{5}{2} \\ -1 & \frac{3}{2} \end{pmatrix}$$

v) det **Q** = 20 – 18 = 2 and

$$\mathbf{Q}^{-1} = \tfrac{1}{2}\begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} & \frac{3}{2} \\ 3 & 2 \end{pmatrix}$$

Always ***check*** that your inverse is correct. Multiply the matrix by your inverse and check that it gives $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Example 3

Find the determinants and inverses of the following matrices:

i) $\mathbf{A} = \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}$ ii) $\mathbf{B} = \begin{pmatrix} 8 & 2 \\ 11 & 3 \end{pmatrix}$

iii) $\mathbf{C} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$ iv) $\mathbf{D} = \begin{pmatrix} 3 & -2 \\ 0 & 6 \end{pmatrix}$

Solution

Work out the solution and compare answers.

Example 4

If the matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & x \end{pmatrix}$ has no inverse, find x.

Solution

Since the matrix has no inverse, the determinant of **A** is zero i.e. det **A** = 0

$$2x - (-4) = 0 \quad \Rightarrow 2x + 4 = 0$$
$$2x = -4 \quad \Rightarrow x = -2$$

Example 5

Find the values of k for which the following matrices have no inverse.

i) $\begin{pmatrix} k & 7 \\ 8 & 2 \end{pmatrix}$ ii) $\begin{pmatrix} k-2 & 1 \\ 2 & k-3 \end{pmatrix}$

Solution

i) $k = 28$ ii) $k = 1,\ 4$

Work out the solution

Example 6

Given $\mathbf{A} = \begin{pmatrix} -2 & x & -2 \\ 2 & -3 & x \\ -7 & 3 & -4 \end{pmatrix}$, find the values of x, for which **A** has no inverse.

Solution

$$\begin{vmatrix} -2 & x & -2 \\ 2 & -3 & x \\ -7 & 3 & -4 \end{vmatrix} = 0$$

$$-2\begin{vmatrix} -3 & x \\ 3 & -4 \end{vmatrix} - x\begin{vmatrix} 2 & x \\ -7 & -4 \end{vmatrix} - 2\begin{vmatrix} 2 & -3 \\ -7 & 3 \end{vmatrix} = 0$$

$$-24 + 6x + 8x - 7x^2 + 30 = 0$$
$$7x^2 - 14x - 6 = 0$$
$$x = \frac{14 \pm \sqrt{196 - 4(7)(-6)}}{14}$$
$$x = 2.36 \text{ or } -0.36$$

Using the inverse to solve matrix equation

Given any matrix equation **AX** = **b**, we can use the identities $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and **AI** = **I** to solve the equation.

i.e. **AX** = **b** Premultiply both sides by $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{b}$$
$$\mathbf{IX} = \mathbf{A}^{-1}\mathbf{b} \quad \text{since } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$
$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{b} \quad \text{since } \mathbf{IX} = \mathbf{XI}$$

where $\mathbf{A}^{-1}$ is the inverse of **A** and **I** is the unit matrix.

Similarly, if $\mathbf{AB} = \mathbf{C}$

$\mathbf{A} = \mathbf{CB}^{-1}$ i.e. if we post multiply both sides by $\mathbf{B}^{-1}$.

Illustrative examples

Example 1

Solve the simultaneous equations:

$$5x - 3y = 28$$
$$-2x + 4y = 14$$

Solution

Writing the equation in the matrix form, we have

$$\begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 28 \\ 14 \end{pmatrix}$$

i.e. $\mathbf{AX} = \mathbf{b} \Rightarrow \mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{b}$,

$\mathbf{IX} = \mathbf{A}^{-1}\mathbf{b}$ since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

$\mathbf{X} = \mathbf{A}^{-1}\mathbf{b}$ since $\mathbf{IX} = \mathbf{XI}$,

where $\mathbf{A} = \begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 28 \\ 14 \end{pmatrix}$

But $\mathbf{A}^{-1} = \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}$

$$\mathbf{X} = \tfrac{1}{14}\begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 28 \\ 14 \end{pmatrix} = \begin{pmatrix} 11 \\ 9 \end{pmatrix}$$

$x = 11$ and $y = 9$

Example 2

Given the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$, find a 2×2 matrix $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{XA} = \mathbf{I}$ where $\mathbf{I}$ is the 2×2 unit matrix.

Solution

First method

If $\mathbf{AX} = \mathbf{XA} = \mathbf{I}$, then $\mathbf{X}$ is the inverse of $\mathbf{A}$

To find the inverse of $\mathbf{A}$, first find the determinant of $\mathbf{A}$ i.e. $\det \mathbf{A} = 4 - (-6) = 10$

$\therefore \mathbf{X} =$ inverse of $\mathbf{A}$

$$= \tfrac{1}{10}\begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{3}{10} & \frac{1}{10} \end{pmatrix}$$

Second method

Let $\mathbf{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{AX} = \mathbf{I}$

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a + 2c & b + 2d \\ -3a + 4c & -3b + 4d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Comparing corresponding elements

$a + 2c = 1$ ---------(1)

$-3a + 4c = 0$ ---------(2)

Solving eqns (1) and (2) simultaneously gives:

$a = \frac{2}{5}$ and $c = \frac{3}{10}$

$b + 2d = 0$ --------(1)

$-3b + 4d = 1$ --------(2)

Solving eqns (1) and (2) simultaneously gives:

$b = -\frac{1}{5}$ and $d = \frac{1}{10}$. Hence $\mathbf{X} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{3}{10} & \frac{1}{10} \end{pmatrix}$

Using determinants to solve simultaneous equations – Crammer's Rule

When solving linear simultaneous equations in **two or three unknowns** using determinants, follow the steps below:

Step 1: Write the equations in the form

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned} \quad \Rightarrow \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

The matrix $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ formed by the coefficients of the variables is called the **coefficient matrix**. The matrix $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ formed by the constants on the right side of the system of equations is called the **constant matrix**.

Step 2: The solution is given by:

$$x = \frac{|A_x|}{|A|} \text{ and } y = \frac{|A_y|}{|A|}, \text{ provided } |A| \neq 0.$$

Where $|A|$ is the determinant of the matrix formed by the coefficients of x and y.

$|A_x|$ is the determinant of the matrix formed by replacing the first column with $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

i.e. $|A_x| = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$. Similarly, $|A_y|$ is the determinant of the matrix formed by replacing the second column with $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

i.e. $|A_y| = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

Illustrative examples

Example 1

Solve the simultaneous equations

$$5x - 3y = 28$$
$$-2x + 4y = 14$$

Solution

i) $5x - 3y = 28$

$-2x + 4y = 14$

$$\begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 28 \\ 14 \end{pmatrix}$$

$$A = \begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix}$$

$|A| = 20 - 6 = 14$;

$|A_x| = \begin{vmatrix} 28 & -3 \\ 14 & 4 \end{vmatrix} = 154$ and

$|A_y| = \begin{vmatrix} 5 & 28 \\ -2 & 14 \end{vmatrix} = 126$

∴ The solution is given by:

$$x = \frac{|A_x|}{|A|} = \frac{154}{14} = 11 \text{ and}$$

$$y = \frac{|A_y|}{|A|} = \frac{126}{14} = 9$$

Example 2

b) i) Evaluate:

$$\begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}$$

ii) Using the result in (i), find, correct to two decimal places the value of x in the system of equations:

$2x - y + 2z + 5 = 0$
$x + 3y + 4z - 1 = 0$
$x + 2y + z + 2 = 0$

Solution

b) i) $\begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}$

$$= 2\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= 2(3-8) + 1(1-4) + 2(2-3)$$

$$= -10 - 3 - 2$$

$$= -15$$

ii) $2x - y + 2z = -5$
$x + 3y + 4z = 1$
$x + 2y + z = -2$

$$\Delta x = \begin{vmatrix} -5 & -1 & 2 \\ 1 & 3 & 4 \\ -2 & 2 & 1 \end{vmatrix}$$

$$\Delta x = -5\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ -2 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ -2 & 2 \end{vmatrix}$$

$$\begin{aligned}\Delta x &= -5(3-8)+(1+8)+2(2+6)\\ &= 25+9+16\\ &= 50\\ x &= \frac{\Delta x}{\Delta}\\ &= \frac{50}{-15}\\ x &= -3.33\end{aligned}$$

Example 3
Solve the following simultaneous equations using determinants.

i) $\begin{aligned}2x+3y&=-5\\ 5x-2y&=16\end{aligned}$ ii) $\begin{aligned}2x-3y&=18\\ x+2y&=-5\end{aligned}$

iii) $\begin{aligned}3x+4y&=0\\ x+2y&=2\end{aligned}$ iv) $\begin{aligned}4x+8y&=0\\ x-2y&=1\end{aligned}$

Solution
Work out the solution and verify whether they satisfy the equations.

Review exercise 27

1. i) Matrices **A**, **B** and **C** are given by

$\mathbf{A}=\begin{pmatrix}2 & 3\\ 4 & 5\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ and

$\mathbf{C}=\begin{pmatrix}4 & -2\\ -3 & 1\end{pmatrix}$

Find a) **CB** b) **A** – **B** c) **A** + **B**

2. Which of the following matrices has no multiplicative inverse?

i) $\begin{pmatrix}1 & 2\\ 3 & 5\end{pmatrix}$ ii) $\begin{pmatrix}3 & 4\\ -1 & 7\end{pmatrix}$

iii) $\begin{pmatrix}4 & -2\\ -2 & 1\end{pmatrix}$ iv) $\begin{pmatrix}5 & 2\\ 1 & 2\end{pmatrix}$

v) $\begin{pmatrix}2 & 3\\ 4 & 6\end{pmatrix}$

3. Given that $\mathbf{A}=\begin{pmatrix}4 & 2\\ 2 & 3\end{pmatrix}$ and

$\mathbf{B}=\begin{pmatrix}1 & -2\\ -1 & 4\end{pmatrix}$, find **2A + 3B**

4. If $\mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$, find $\mathbf{A}^2-2\mathbf{I}$, where **I** is the 2 × 2 identity matrix.

5. If $\mathbf{M}=\begin{pmatrix}2 & -1\\ 1 & 3\end{pmatrix}$, find the values of $\mathbf{M}^2$, $\mathbf{M}^3$ and $\mathbf{M}^{-1}$.

Find x and y, given that $\mathbf{M}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}3\\ 5\end{pmatrix}$

6. Find the values of k for which the matrix $\mathbf{A}=\begin{pmatrix}k+1 & 6\\ 1 & k\end{pmatrix}$ has no inverse.

7. Given that $\mathbf{X}=\begin{pmatrix}4 & 0\\ 2 & -1\end{pmatrix}$ and

$\mathbf{Y}=\begin{pmatrix}1 & -2\\ 5 & 3\end{pmatrix}$, find i) **XY** ii) **YX**

What do your answers in i) and ii) show you about matrix multiplication?

8. The matrices **A**, **B** and **C** are given by

$\mathbf{A}=\begin{pmatrix}2 & 6\\ 4 & 8\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}3 & -5\\ -7 & 9\end{pmatrix}$,

$\mathbf{C}=\begin{pmatrix}-5 & -1\\ 3 & 2\end{pmatrix}$

a) Evaluate
   i) **(A – B)**
   ii) **(A + B)**
   iii) **A(B – C)**
   iv) **(B – C)A**

b) Find i) **(A – B)(A + B)**
   ii) $(\mathbf{A}^2-\mathbf{B}^2)$

9. Given that $\begin{pmatrix} 3x+1 & 7 \\ 2x+y & 3 \end{pmatrix} = \begin{pmatrix} 10 & 3x-y \\ 8 & 3 \end{pmatrix}$, find the values of x and y.

10. $A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Find AB.

11. Given that $\begin{pmatrix} 1 & 0 \\ 2 & x \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 6 & 3 \end{pmatrix}$, find x.

12. Given that $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}$. Find $2\mathbf{A} + 3\mathbf{B}$.

13. Find the inverse of the matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

14. A matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$.

What is $\mathbf{A}^{-1}$?

15. Let $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Find the image of the point (2, 3) under the matrix $\mathbf{B}^2 + \mathbf{B} + \mathbf{I}$, where I is the 2 × 2 unit matrix.

16. Let $\mathbf{P} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 5 & 2 \\ -4 & 0 \end{pmatrix}$.

Determine $\mathbf{P}^2 - \mathbf{Q}^2$.

17. Determine the values of x and y such that $\begin{pmatrix} 2 & -3 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

18. a) Find 2 × 2 matrices **A** and **B** such that $\mathbf{A} + 2\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ and

$$2\mathbf{A} + 3\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

19. Use determinants to solve the simultaneous equations

$$3x + 5y = 7$$
$$4x - 3y = 19$$

20. If $\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 3x & 5 \end{pmatrix}$, find the value of x if the determinant of $\mathbf{A} = 32$

21. a) Calculate the determinant of $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$.

b) Use your result to solve the system of simultaneous equations

$$2x + 3y = 4$$
$$4x + 5y = 3$$

22. If $\begin{vmatrix} 2x-1 & x+7 & x+4 \\ x & 6 & 2 \\ x-1 & x+1 & 3 \end{vmatrix} = 0$,

find the value of x.

23. find the determinants of the following (3 x 3) matrices.

i) $A = \begin{pmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{pmatrix}$ ii) $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 3 & -1 \end{pmatrix}$

iii) $A = \begin{pmatrix} 5 & -2 & 1 \\ 0 & 3 & -1 \\ 2 & 0 & 7 \end{pmatrix}$

CHAPTER 28

LINEAR TRANSFORMATIONS

Definition

A **linear transformation** is one which is of the form $T : (x, y) \rightarrow (x', y')$, where $x' = ax + by$ and $y' = cx + dy$ and such that:

1. the image of (0, 0) is always (0, 0)
2. straight lines are transformed into straight lines

Linear transformations and matrices

Linear transformations in a plane, other than translations, can be produced and described using (2 × 2) matrices. Any point $P(x, y)$ can be mapped to a new point $Q(x', y')$ using a (2 × 2) matrix **M**, where $\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix}$.

This new point, $Q(x', y')$ is said to be the image of the point $P(x, y)$ under the linear transformation by the matrix **M**.

Thus if the position vector $\begin{pmatrix} x \\ y \end{pmatrix}$ of the point $P(x, y)$ is *pre-multiplied* by a (2 × 2) matrix **M**, it is transformed to another point $Q(x', y')$.

In general, the matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

transforms $\begin{pmatrix} x \\ y \end{pmatrix}$ like this :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

The matrix **M** that transforms the point $P(x, y)$ to the point $Q(x', y')$ is known as the **transformation matrix** and such transformations are called **linear transformations.**

Stating the matrix of a linear transformation

Given the linear transformation $\begin{matrix} x' = ax + by \\ y' = cx + dy \end{matrix}$,

we can find the linear transformation as follows: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax & by \\ cx & dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$,

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the **matrix of the linear transformation.**

Illustrative examples

Example 1

Write the matrix of the linear transformation

$x' = 4x - 3y$

$y' = 5x + 2y$

Solution

The transformation can be expressed in the form

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4x & -3y \\ 5x & 2y \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Hence the matrix of the transformation is $\begin{pmatrix} 4 & -3 \\ 5 & 2 \end{pmatrix}$.

Example 2

Write the matrix **Q** of the linear transformation **Q**:$(x, y) \rightarrow (x - 2y, -3x + 4y)$.

Solution

The transformation can be expressed in the form:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x & -2y \\ -3x & 4y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Hence the matrix **Q** is $\begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}$

Example 3

Write the matrices of the linear transformations

i) $(x, y) \rightarrow (-x, 2y)$

ii) $(x, y) \rightarrow (2x + 3y, -x + 5y)$

iii) $(x, y) \rightarrow (5y, -x - y)$

Solution

i) $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

ii) $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}$

iii) $\begin{pmatrix} 0 & 5 \\ -1 & -1 \end{pmatrix}$

[Work out the solution]

To find the image of a given point

The matrix of the linear transformation will be given and you may be asked to find the image of a given point.

Illustrative examples

Example 1

If $\mathbf{Q} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, find the image of the vertices of a triangle $A(3, 1)$, $B(-2, -1)$ and $C(-4, 2)$ under the linear transformation **Q**.

Solution

The image of the point $A(3, 1)$ is

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} \text{ i.e. } (6, 1)$$

The image of the point $B(-2, -1)$ is

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} \text{ i.e. } (-4, -1)$$

The image of the point $C(-4, 2)$ is

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -8 \\ 2 \end{pmatrix} \text{ i.e. } (-8, 2)$$

Example 2

Find the image of the point $P(5, 3)$ under the linear transformation $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Solution

The image of the point $P(5, 3)$ under the linear transformation **M** is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \text{ i.e. } (5, -3)$$

Example 3

The vertices of a square $O(0, 0)$, $A(1, 0)$, $B(1, 1)$ and $C(0, 1)$ are transformed by the matrix **M** into $O'\,A'\,B'\,C'$, where

$$\mathbf{M} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}.$$

Find the coordinates of O', A', B' and C'.

Solution

The image of the point $O(0, 0)$ is

$$\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ i.e. } O'(0, 0)$$

The image of the point $A(1, 0)$ is

$$\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \text{ i.e. } A'(3, -1)$$

The image of the point $B(1, 1)$ is

$$\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \text{ i.e. } B'(4, 0)$$

The image of the point $C(0, 1)$ is

$$\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ i.e. } C'(1, 1)$$

To find the object when its image is given

The matrix of the linear transformation and the image will be given and you may be asked to find the object (i.e. the transformed point).

Illustrative examples

Example 1

A transformation **T** is given by $\mathbf{T} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. If $Q(5, 6)$ is the image of the point P under the transformation **T**, find the coordinates of P.

Solution

Let the coordinates of the point P be (a, b) then

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} a+2b \\ 3a+4b \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$a + 2b = 5 \text{ ---------(1)}$$

and $$3a + 4b = 6 \text{ ---------(2)}$$

We then solve eqn(1) and (2) simultaneously to obtain the values of a and b.

From eqn (2) – 2 x eqn(1) $\Rightarrow a = -4$

Putting $a = -4$ in eqn(1)

$$-4 + 2b = 5 \quad \Rightarrow \quad b = \tfrac{9}{2}$$

Hence the coordinates of the point P is $(-4, \frac{9}{2})$.

Alternatively, multiply the inverse of the transformation matrix **T** by the image to get the object.

Thus using $\mathbf{TX} = \mathbf{b} \Rightarrow \mathbf{X} = \mathbf{T}^{-1}\mathbf{b}$,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix},$$

where $\mathbf{T}^{-1} = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

$$\begin{pmatrix} a \\ b \end{pmatrix} = -\tfrac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = -\tfrac{1}{2}\begin{pmatrix} 8 \\ -9 \end{pmatrix} = \begin{pmatrix} 4 \\ -\frac{9}{2} \end{pmatrix}$$

Hence the coordinates of the point P is $(-4, \frac{9}{2})$.

Example 2

A transformation **M** is represented by the matrix $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$. If $Q(11, 13)$ is the image of the point $P(a, b)$ under the transformation **M**, find the value of a and b.

Solution

If the coordinates of the point P is (a, b) then

$$\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix}$$

$$\begin{pmatrix} 4a+b \\ 2a+3b \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix}$$

$$4a + b = 11 \text{ ---------(1)}$$

and $$2a + 3b = 13 \text{ --------(2)}$$

We then solve eqn(1) and (2) simultaneously to obtain the values of a and b.

From 2 × eqn (2) – eqn(1)

$$5b = 15 \quad \Rightarrow \quad b = 3$$

Putting $b = 3$ in eqn(1)

$$4a + 3 = 11 \quad \Rightarrow \quad a = 2$$

Hence the coordinates of the point P is(2, 3).

Note : Multiply the inverse of the transformation matrix **M** by the image to get the object.

To find the matrix of a linear transformation when two point and their images are given

Two points and their images will be given and you may be asked to find the matrix of the linear transformation.

Illustrative examples

Example 1

Under a linear transformation **P** of a plane, the image of (2, 1) is (4, 2) and that of (–3, 1) is (–1, 7). Find the matrix **P** of the transformation.

Solution

Let $\mathbf{P}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix of the transformation.

Now $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}=\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 2a+b \\ 2c+d \end{pmatrix}=\begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$2a+b=4$ ---------(1)

and $2c+d=2$ ---------(2)

Also $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix}=\begin{pmatrix} -1 \\ 7 \end{pmatrix}$

$$\begin{pmatrix} -3a+b \\ -3c+d \end{pmatrix}=\begin{pmatrix} -1 \\ 7 \end{pmatrix}$$

$-3a+b=-1$ ---------(3)

and $-3c+d=7$ ----------(4)

Now solving eqns(1) and (3) simultaneously to obtain the values of a and b, we have

$2a+b=4$ -----------(1)

$-3a+b=-1$ ----------(3)

Eqn(1) – eqn(3) $\Rightarrow$ $5a=5$ $\Rightarrow$ $a=1$

Putting $a=1$ in eqn(1)

$2(1)+b=4 \Rightarrow b=2$

Next solving eqns(2) and (4) simultaneously to obtain the values of c and d.

$2c+d=2$ -----------(2)

$-3c+d=7$ -----------(4)

eqn(2) – eqn(4) $\Rightarrow$ $5c=-5$ $\Rightarrow$ $c=-1$

Putting $c=-1$ in eqn(2)

$2(-1)+d=2 \quad \Rightarrow d=4$

Hence the matrix of the linear transformation is $\mathbf{P}=\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$

Alternative method

From the question, the matrix $\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$ is the object and the matrix $\begin{pmatrix} 4 & -1 \\ 2 & 7 \end{pmatrix}$ is the image. Note the order carefully.

Therefore, $\mathbf{P}\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 4 & -1 \\ 2 & 7 \end{pmatrix}$.

Using $\mathbf{PA}=\mathbf{B} \quad \Rightarrow \quad \mathbf{P}=\mathbf{BA}^{-1}$

$$\mathbf{P}=\begin{pmatrix} 4 & -1 \\ 2 & 7 \end{pmatrix}\tfrac{1}{5}\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$$

$$\mathbf{P}=\tfrac{1}{5}\begin{pmatrix} 5 & 10 \\ -5 & 20 \end{pmatrix}=\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$$

Example 2

Under a linear transformation **T** of a plane, the image of (1, 3) is (2, 4) and that of (–2, –3) is (–3, –11). Find the matrix **T** of the transformation.

Solution

Let $\mathbf{T}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix of the transformation

Now $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}=\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$$\begin{pmatrix} a+3b \\ c+3d \end{pmatrix}=\begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$a + 3b = 2 \text{ ---------(1)}$$
$$c + 3d = 4 \text{ ---------(2)}$$

Also, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ -11 \end{pmatrix}$

$$\begin{pmatrix} -2a-3b \\ -2c-3d \end{pmatrix} = \begin{pmatrix} -3 \\ -11 \end{pmatrix}$$

$$-2a - 3b = -3 \text{ ---------(3)}$$
$$-2c - 3d = -11 \text{ -------(4)}$$

Now solving eqns(1) and (3) simultaneously to obtain the values of a and b, we have

$$a + 3b = 2 \text{ -------------(1)}$$
$$-2a - 3b = -3 \text{ ----------(3)}$$

Eqn(1) + eqn(3) $\Rightarrow \ -a = -1 \ \Rightarrow \ a = 1$

Putting $a = 1$ in eqn(1) $\Rightarrow \ 1 + 3b = 2$

$$b = \tfrac{1}{3}$$

Next solving eqns(2) and (4) simultaneously to obtain the values of c and d.

$$c + 3d = 4 \text{ -------------(2)}$$
$$-2c - 3d = -11 \text{ ---------(4)}$$

eqn(2) + eqn(4) $\Rightarrow \ -c = -7 \ \Rightarrow \ c = 7$

Putting $c = 7$ in eqn(2)

$$7 + 3d = 4 \quad \Rightarrow \ d = -1$$

Hence the matrix of the linear transformation is $\mathbf{T} = \begin{pmatrix} 1 & \frac{1}{3} \\ 7 & -1 \end{pmatrix}$.

Use the alternative method to arrive at the same answer.

To write the linear transformation when the matrix is given

You may be given the matrix and be asked to write down the linear transformation.

Illustrative example

Write down the linear transformation **Q** whose matrix is $\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$.

Solution

The linear transformation is given by

$$\mathbf{Q}: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\mathbf{Q}: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x-2y \\ -2x+3y \end{pmatrix}$$

Writing it in the row form we have

$\mathbf{Q}: (x, y) \to (x - 2y, -2x + 3y)$

Inverse transformations

The **inverse of a transformation** moves a point *back* to its original position. It is represented by the inverse of its transformation matrix. Thus if **M** is the matrix of a linear transformation, then the inverse matrix $\mathbf{M}^{-1}$ is the matrix of the **inverse transformation.**

For example if the matrix of a linear transformation is $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the inverse transformation is given by the matrix

$\mathbf{M}^{-1} = \begin{pmatrix} \frac{d}{k} & -\frac{b}{k} \\ -\frac{c}{k} & \frac{a}{k} \end{pmatrix}$, where $k = ad - bc \neq 0$ is the determinant of **M**.

To find the inverse of linear transformations

1. Find the matrix of the transformation
2. Find the inverse of the matrix
3. Use the inverse matrix to write the inverse of the linear transformation.

   i.e. $\mathbf{M}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$.

Illustrative example

Find the inverse of the linear transformation $\mathbf{M}: (x, y) \to (x, 2x + y)$ and use it to find the point whose image is (2, 1) under **M**.

Solution

The transformation can be expressed in the form:

$$\begin{pmatrix}x\\y\end{pmatrix}\rightarrow\begin{pmatrix}x & 0y\\2x & y\end{pmatrix}=\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

Hence the matrix of the transformation is

$$\mathbf{M}=\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}$$

The determinant of **M** = 1 − 0 = 1.

$$\text{Hence } \mathbf{M}^{-1}=\begin{pmatrix}1 & 0\\-2 & 1\end{pmatrix}$$

(refer to inverse of a matrix)

Therefore the inverse of the transformation is:

$$\begin{pmatrix}x\\y\end{pmatrix}\rightarrow\begin{pmatrix}1 & 0\\-2 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\-2x+y\end{pmatrix} \text{ or}$$

$\mathbf{M}^{-1}: (x, y) \rightarrow (x, -2x + y)$.

Multiplying the inverse matrix by (1, 2) gives the required point.

$$\text{i.e. } \begin{pmatrix}1 & 0\\-2 & 1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}.$$

Therefore the point whose image is (2, 1) under **M** is (1, 0)

Note: The inverse transformation can be used to find the point whose image is given instead of using the method described earlier.

Composition of linear transformations

If **A** and **B** are the matrices of two linear transformations then the **combined transformation A o B** (i.e. **B** followed by **A**) is given by the product **AB**

i.e. **A o B = AB.**

Also the combined transformation **B o A** (i.e. **A** followed by **B**) is given by the product **BA** i.e. **B o A = BA.**

Since matrix multiplication is not commutative **A o B ≠ B o A**

To find the composition of a linear transformation

1. Write down the matrices of the linear transformations **A** and **B**.
2. Find the matrix product **AB**
3. Write the linear transformation for **AB**

$$\text{i.e. } \mathbf{AB}\begin{pmatrix}x\\y\end{pmatrix}$$

Illustrative example

$$\text{If } \mathbf{A}=\begin{pmatrix}6 & 1\\-8 & -7\end{pmatrix} \text{ and } \mathbf{B}=\begin{pmatrix}5 & 3\\4 & 2\end{pmatrix}.$$

Find

i) **A o B**

ii) **B o A**

iii) the image of (−1, 1) under the transformation **A o B** (i.e. **B** followed by **A**)

Solution

$$\text{i) } \mathbf{A \circ B = AB} = \begin{pmatrix}6 & 1\\-8 & -7\end{pmatrix}\begin{pmatrix}5 & 3\\4 & 2\end{pmatrix}$$

$$=\begin{pmatrix}30+4 & 18+2\\-40-28 & -24-14\end{pmatrix}$$

$$=\begin{pmatrix}34 & 20\\-68 & -36\end{pmatrix}$$

$$\text{ii) } \mathbf{B \circ A = BA} = \begin{pmatrix}5 & 3\\4 & 2\end{pmatrix}\begin{pmatrix}6 & 1\\-8 & -7\end{pmatrix}$$

$$=\begin{pmatrix}30-24 & 5-21\\24-16 & 4-14\end{pmatrix}$$

$$=\begin{pmatrix}6 & -16\\8 & -10\end{pmatrix}$$

iii) From (i) above

$$\mathbf{A \circ B = AB} = \begin{pmatrix}34 & 20\\-68 & -36\end{pmatrix}$$

Therefore the image of (–1, 1) under the transformation **A** o **B** is

$$\begin{pmatrix} 34 & 20 \\ -68 & -36 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -34+20 \\ 68-36 \end{pmatrix}$$

$$= \begin{pmatrix} -14 \\ 32 \end{pmatrix}$$

The image of (–1, 1) under the transformation **A** o **B** is (–14, 32).

Identity transformations and their matrices

Below is a summary of some special transformations and the matrices that produce them is given below.

1. Identity transformation

Transformation Matrix

$$\begin{cases} x^1 = x \\ y^1 = y \end{cases} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. Reflection in the x – axis

Transformation Matrix

$$\begin{cases} x^1 = x \\ y^1 = -y \end{cases} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3. Reflection in the y – axis

Transformation Matrix

$$\begin{cases} x^1 = -x \\ y^1 = y \end{cases} \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

4. Reflection in the line $y = x$

Transformation Matrix

$$\begin{cases} x^1 = y \\ y^1 = x \end{cases} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

5. Reflection in the line $y = -x$

Transformation Matrix

$$\begin{cases} x^1 = -y \\ y^1 = -x \end{cases} \qquad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Anticlockwise rotations about the origin O(0, 0)

6. Rotation through an angle 90°

Transformation Matrix

$$\begin{cases} x^1 = -y \\ y^1 = x \end{cases} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

7. Rotation through an angle of 180°

Transformation Matrix

$$\begin{cases} x^1 = -x \\ y^1 = -y \end{cases} \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

8. Rotation through an angle of 270°

Transformation Matrix

$$\begin{cases} x^1 = y \\ y^1 = -x \end{cases} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

9. Rotation through an angle θ

Transformation Matrix

$$\begin{cases} x^1 = x\cos\theta - y\sin\theta \\ y^1 = x\sin\theta + y\cos\theta \end{cases} \qquad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Note:

For clockwise rotations, replace θ by $-\theta$

Enlargement

10. Enlargement of scale factor k about the origin

Transformation Matrix

$$\begin{cases} x^1 = k \\ y^1 = k \end{cases} \qquad \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

Solved examination questions

Example 1

Write down the linear transformation **B** whose matrix **B** is $\begin{pmatrix} 6 & 1 \\ -8 & -7 \end{pmatrix}$

If $\mathbf{A} = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}$, find

a) **AB**

b) The image of a point (*x*, *y*) under the transformations

   i) **3A – 2B**

   ii) **A o B** where **A o B** means **B** followed by **A**.

Solution

The linear transformation is given by

$$\mathbf{B}: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 6 & 1 \\ -8 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\mathbf{B}: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 6x+y \\ -8x-7y \end{pmatrix}$$

Writing it in the row form we have

$\mathbf{B}: (x, y) \to (6x + y, -8x - 7y)$

a) $$\mathbf{AB} = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 6 & 1 \\ -8 & -7 \end{pmatrix} = \begin{pmatrix} 30-24 & 5-21 \\ 24-16 & 4-14 \end{pmatrix} = \begin{pmatrix} 6 & -16 \\ 8 & -10 \end{pmatrix}$$

b) i) $$3\mathbf{A} - 2\mathbf{B} = 3\begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix} - 2\begin{pmatrix} 6 & 1 \\ -8 & -7 \end{pmatrix} = \begin{pmatrix} 15 & 9 \\ 12 & 6 \end{pmatrix} - \begin{pmatrix} 12 & 2 \\ -16 & -14 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 28 & 20 \end{pmatrix}$$

Hence the image of a point (*x*, *y*) under the linear transformation **3A – 2B** is

$$\begin{pmatrix} 3 & 7 \\ 28 & 20 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x+7y \\ 28x+20y \end{pmatrix} = (3x + 7y, 28x + 20y)$$

ii) $\mathbf{A} \text{ o } \mathbf{B} = \mathbf{AB} = \begin{pmatrix} 6 & -16 \\ 8 & -10 \end{pmatrix}$ from (a) above

Hence the image of a point (*x*, *y*) under the linear transformation **A o B** is

$$\begin{pmatrix} 6 & -16 \\ 8 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6x-16y \\ 8x-10y \end{pmatrix} = (6x - 16y, 8x - 10y)$$

Example 2

$\mathbf{S} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{T} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$

respectively, are the matrices of the linear transformations **S** and **T** of the plane.

a) Find the image of the vector $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ under the combined transformation **S o T** where **S o T** means **T** followed by **S**.

b) Describe geometrically, the combined transformation **S o T**.

c) Find the point whose image under the transformation **3S – 4T** is (8, –4)

Solution

a) **S o T = ST**

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The image of the point $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ under **S o T** is

$$\mathbf{S} \text{ o } \mathbf{T}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \mathbf{ST}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ i.e. } (1, -2)$$

b) Geometrically, the combined transformation **S** o **T** is the anticlockwise rotation through 270° about the origin.
Refer to the identity transformations.

i.e. $\begin{pmatrix}2\\1\end{pmatrix} \to \begin{pmatrix}1\\-2\end{pmatrix}$

$\begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}y\\-x\end{pmatrix}$

c) $$\mathbf{3S - 4T} = 3\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} - 4\begin{pmatrix}0 & \frac{1}{2}\\-\frac{1}{2} & 0\end{pmatrix}$$
$$= \begin{pmatrix}6 & 0\\0 & 6\end{pmatrix} - \begin{pmatrix}0 & 2\\-2 & 0\end{pmatrix}$$
$$= \begin{pmatrix}6 & -2\\2 & 6\end{pmatrix}$$

Now let (a, b) be the point whose image is $(8, -4)$ under the transformation **3S – 4T** then

$$\begin{pmatrix}6 & -2\\2 & 6\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}8\\-4\end{pmatrix}$$
$$\begin{pmatrix}6a-2b\\2a+6b\end{pmatrix} = \begin{pmatrix}8\\-4\end{pmatrix}$$

$6a - 2b = 8$ ---------(1)
and $2a + 6b = -4$ ---------(2)

Solving eqns (1) and (2) simultaneously for a and b
3 × eqn(1) + eqn(2)
$20a = 20 \Rightarrow a = 1$
Substituting $x = 1$ in eqn (2)
$2(1) + 6b = -4 \Rightarrow b = -1$
Hence $(1, -1)$ is the point whose image under the linear transformation is $(8, -4)$.

Alternatively multiply the inverse of (**3S – 4T**) by $(8, -4)$ to get the object $(1, -1)$.

Example 3

a) Write down
   i) the matrix **P** of the linear transformation
   P : $(x, y) \to (x + 3y, -2x + 6y)$
   ii) the linear transformation **Q** whose matrix is $\mathbf{Q} = \begin{pmatrix}1 & -2\\-2 & 3\end{pmatrix}$

b) Find the image of the point (2, 4) under the linear transformation
   i) **4P – 3Q**
   ii) **Q** o **P** (i.e. **P** followed by **Q**)
   iii) If $\mathbf{X} = (x \quad 1)$ and $\mathbf{X}' = \begin{pmatrix}x\\1\end{pmatrix}$, where x is a real number find the truth set of $(\mathbf{XQ})\mathbf{X}' = 0$.

Solution

a) i) The transformation **P** can be written in the column form as

$$\mathbf{P}: \begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}x+3y\\-2x+6y\end{pmatrix}$$
$$\mathbf{P}: \begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}1 & 3\\-2 & 6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

Hence the matrix of the linear transformation

P is $\begin{pmatrix}1 & 3\\-2 & 6\end{pmatrix}$

ii) The linear transformation **Q** whose matrix is $\mathbf{Q} = \begin{pmatrix}1 & -2\\-2 & 3\end{pmatrix}$ is given by

$$\mathbf{Q}: \begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}1 & -2\\-2 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$
$$\mathbf{Q}: \begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}x-2y\\-2x+3y\end{pmatrix}$$

Writing it in the row form we have
Q : $(x, y) \to (x - 2y, -2x + 3y)$

b) i) $$4\mathbf{P} - 3\mathbf{Q} = 4\begin{pmatrix} 1 & 3 \\ -2 & 6 \end{pmatrix} - 3\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 12 \\ -8 & 24 \end{pmatrix} - \begin{pmatrix} 3 & -6 \\ -6 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 18 \\ -2 & 15 \end{pmatrix}$$

Hence the image of the point (2, 4) under the transformation $4\mathbf{P} - 3\mathbf{Q}$ is

$$\begin{pmatrix} 1 & 18 \\ -2 & 15 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 74 \\ 56 \end{pmatrix} \text{ i.e. } (74, 56)$$

ii) $$\mathbf{Q} \text{ o } \mathbf{P} = \mathbf{QP} = \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -2 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -9 \\ -8 & 12 \end{pmatrix}$$

Hence the image of the point (2, 4) under the linear transformation **Q** o **P** is

$$\begin{pmatrix} 5 & -9 \\ -8 & 12 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -26 \\ 32 \end{pmatrix} \text{ i.e. } (-26, 32)$$

iii) $$\mathbf{XQ} = (x, 1)\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$$

$$= (x - 2, -2x + 3)$$

Note: A (1 × 2) matrix multiplied by a (2 × 2) matrix gives a matrix of order (1 × 2).

$$\therefore \ (\mathbf{XQ})\mathbf{X}' = (x - 2, -2x + 3)\begin{pmatrix} x \\ 1 \end{pmatrix}$$

$$= [x(x - 2) + (-2x + 3)]$$
$$= (x^2 - 2x - 2x + 3)$$
$$= (x^2 - 4x + 3)$$

Note: A (1 × 2) matrix multiplied by a (2 × 1) matrix gives a matrix of order (1 × 1).

Therefore $(\mathbf{XQ})\mathbf{X}' = 0$

$$x^2 - 4x + 3 = 0$$
$$(x - 1)(x - 3) = 0$$
$$x - 1 = 0 \text{ or } x - 3 = 0$$
$$x = 1 \text{ or } 3$$

Hence the truth set of $(\mathbf{XQ})\mathbf{X}' = 0$ is $\{x : x = 1 \text{ or } 3\}$

Example 4

Two linear transformations whose matrices are **M** and **N** are defined in the *x-y* plane by

$\mathbf{M} : (x, y) \rightarrow (y, -x)$ and

$\mathbf{N} : (x, y) \rightarrow (x + 2y, 2x + 3y)$.

i) Write down the matrices **M** and **N**
ii) Find the matrix **MN**.
iii) Find the image of the point (2, −1) under the transformation **M** o **N**

Solution

i) The matrices of the linear transformations **M** and **N** are

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \mathbf{N} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

Note : $\mathbf{M} : (x, y) \rightarrow (y, -x)$
$\mathbf{M} : (x, y) \rightarrow (0x + y, -x + 0y)$

ii) $$\mathbf{MN} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$$

iii) $$\mathbf{M} \text{ o } \mathbf{N} = \mathbf{MN} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$$ from (ii) above

∴ The image of (2, −1) under the transformation **M** o **N** is

$$\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ i.e. } (1, 0)$$

Hence the image of (2, −1) under the transformation **M** o **N** is (1, 0)

Example 5

The matrix for rotation through θ ° about the origin is $\mathbf{T} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

a) Find the image P' of the point $P(\sqrt{2}, \frac{1}{\sqrt{2}})$, when it is rotated through 45° about the origin.

b) Find the point Q whose image $Q'(6, 2)$ is obtained by a transformation under the matrix $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ followed by a rotation through 90° about the origin.

Solution

a) When $\theta = 45°$,

$$\mathbf{T} = \begin{pmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

∴ The image of $P(\sqrt{2}, \frac{1}{\sqrt{2}})$ under the transformation with matrix $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \sqrt{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1-\frac{1}{2} \\ 1+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}$$

i.e. $P'(\frac{1}{2}, \frac{3}{2})$

b) Let the coordinates of the point Q be (a, b)

When $\theta = 90°$,

$$\mathbf{T} = \begin{pmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

First, transform the point Q by the matrix $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$

$$\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a+2b \\ a+4b \end{pmatrix}$$

This is followed by a rotation through 90° about the origin

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3a+2b \\ a+4b \end{pmatrix} = \begin{pmatrix} -a-4b \\ 3a+2b \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$-a-4b = 6$ ....(1)

and $3a+2b = 2$ ....(2)

Solving equation (1) and (2) simultaneously

$3 \times$ eqn(1) + eqn(2)

$-10b = 20 \Rightarrow b = -2$

Substituting $b = -2$ into eqn(1)

$-a + 8 = 6 \Rightarrow a = 2$

∴ The coordinates of the point $Q(2, -2)$

Use an alternative method to arrive this answer.

Example 6

a) The matrix **M** of a linear transformation which rotates all points (x, y) in the anticlockwise direction through an angle θ about the origin is given by $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

T is the matrix of the linear transformation which reflects the point of the plane in the line $y = x$. Find:

i) **M**, if $\theta = 120°$;

ii) the image of the point $A(1, -2)$ under the transformation **T** o **M**, correct to one decimal place.

Solution

$$\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ and } \mathbf{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

i.e. reflection in the line $y = x$

i) If $\theta = 120°$

$$\mathbf{M} = \begin{pmatrix} \cos 120° & -\sin 120° \\ \sin 120° & \cos 120° \end{pmatrix} = \begin{pmatrix} -0.5 & -0.866 \\ 0.866 & -0.5 \end{pmatrix}$$

ii) $$\mathbf{ToM} = \mathbf{TM} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -0.5 & -0.866 \\ 0.866 & -0.5 \end{pmatrix} = \begin{pmatrix} 0.866 & -0.5 \\ -0.5 & -0.866 \end{pmatrix}$$

∴ The image of $A(1,-2)$ under the transformation **T** o **M** is

$$\begin{pmatrix} 0.866 & -0.5 \\ -0.5 & -0.866 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1.866 \\ 1.232 \end{pmatrix}$$

i.e. (1.866, 1.232)

∴ The image of $A(1,-2)$ is (1.9, 1.2) correct to one decimal place.

Example 7

a) Write down the matrices **A** and **B** representing the following transformations respectively:

i) a reflection in the line $x+y=0$

ii) $(x, y) \rightarrow (\frac{1}{2}\sqrt{3}\,x - \frac{1}{2}y,\ \frac{1}{2}x + \frac{1}{2}\sqrt{3}\,y)$

b) Find the image of the point (2, 4) under the transformation **BA**.

c) Find the matrix $\mathbf{B}^3$ and identify which transformation it represents.

Solution

a) i) The matrix **A** of the linear transformation, i.e. reflection in the line $x+y=0$ or $y=-x$ is:

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

ii) The matrix **B** of the linear transformation is

$$\mathbf{B} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix}$$

b) $$\mathbf{BA} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}$$

∴ The image of (2, 4) under the transformation **BA** is

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1-2\sqrt{3} \\ -\sqrt{3}-2 \end{pmatrix}$$

i.e. $(1-2\sqrt{3}, -\sqrt{3}-2)$

c) $\mathbf{B}^3 = \mathbf{B}^2\,\mathbf{B}$ where $\mathbf{B}^2 = \mathbf{BB}$

$$\mathbf{B}^2 = \mathbf{BB} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix}\begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{B}^3 = \mathbf{B}^2\,\mathbf{B}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\mathbf{B}^3$ represents an anticlockwise rotation through 90° about the origin (Refer to identity transformations)

Example 8

A point $P(7, 3)$ is transformed to P' by the matrix $\mathbf{M} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

a) Find the coordinates of P'

b) Find the inverse of **M** and show that it returns the point P' to its original position P.

Solution

a) The coordinates of P' are given by

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix} \text{ i.e. } P'(6, 7)$$

b) The inverse of $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ is given by

$$\mathbf{M}^{-1} = \tfrac{1}{-2}\begin{pmatrix} 0 & -2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$$

Applying this inverse matrix to the position vector of P' gives

$$\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix} \text{ i.e. } (7, 3).$$

This is the coordinates of P. Therefore P' has been mapped back to P by the inverse matrix $\mathbf{M}^{-1}$.

Example 9

a) Write down the matrix **M** of the linear transformation **M**

$\mathbf{M} : (x, y) \rightarrow (kx + y, 3x + ky)$

b) If the image of $(-1, 2)$ under **M** is $(a, -7)$. Find the value of a.

c) A transformation represented by the matrix **A** takes a point $P(x, y)$ to the image $(1, -5)$. Find x and y.

Solution

a) The matrix of the linear transformation

M is $\mathbf{M} = \begin{pmatrix} k & 1 \\ 3 & k \end{pmatrix}$

b) If the image of $(-1, 2)$ under **M** is $(a, -7)$, then

$$\begin{pmatrix} k & 1 \\ 3 & k \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ -7 \end{pmatrix}$$

$$\begin{pmatrix} -k+2 \\ -3+2k \end{pmatrix} = \begin{pmatrix} a \\ -7 \end{pmatrix}$$

Comparing corresponding elements

$-3 + 2k = -7 \quad \Rightarrow k = -2$

and $-k + 2 = a \quad \Rightarrow a = -(-2) + 2 = 4$

c) If $P(x, y)$ has image $(1, -5)$ under **A**, then

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} x+2y \\ -3x+4y \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

Comparing corresponding elements

$x + 2y = 1$ ---------(1)

$-3x + 4y = -5$ --------(2)

Solving eqns (1) and (2) simultaneously gives:

$x = \frac{7}{5}$ and $y = -\frac{1}{5}$

Use an alternative method to arrive this answer.

Example 10

a) A linear transformation **M** maps $A(1, 0)$ to $A_1(-2, -4)$ and $B(0, 1)$ to $B_1(4, 1)$

i) Find the matrix **M**

ii) The image of the point $C(1, -1)$ under **M**.

b) Two linear transformations **N** and **P** are defined by

$\mathbf{N} : (x, y) \rightarrow (5x + 3y, 6x + 4y)$ and

$\mathbf{P} : (x, y) \rightarrow (4x - 3y, -6x + 5y)$

i) Write down the matrices **N** and **P**

ii) Find the matrix **NP**

iii) Deduce the inverse of **P**

c) If $\mathbf{C} = \begin{pmatrix} -2 & 5 \\ 4 & 3 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 3 & 1 \\ -2 & -3 \end{pmatrix}$, find the values of a and b such that

$$\mathbf{DC} = 2\begin{pmatrix} -3 & 7 \\ 2 & a \end{pmatrix} + \begin{pmatrix} b & 4 \\ -12 & -3 \end{pmatrix}$$

Solution

a) i) Let $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} \Rightarrow a = -2 \text{ and } c = -4$$

Also, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \Rightarrow b = 4 \text{ and } d = 1$$

Hence $\mathbf{M} = \begin{pmatrix} -2 & 4 \\ -4 & 1 \end{pmatrix}$

Use an alternative method to arrive this answer.

ii) The image of the point (1, –1) under **M** is given by:

$$\begin{pmatrix}-2 & 4\\ -4 & 1\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}=\begin{pmatrix}-6\\ -5\end{pmatrix} \text{ i.e. } (-6, -5)$$

b) i) The matrices of the linear transformations **N** and **P** are

$$\mathbf{N}=\begin{pmatrix}5 & 3\\ 6 & 4\end{pmatrix} \text{ and } \mathbf{P}=\begin{pmatrix}4 & -3\\ -6 & 5\end{pmatrix}$$

ii) $\mathbf{NP}=\begin{pmatrix}5 & 3\\ 6 & 4\end{pmatrix}\begin{pmatrix}4 & -3\\ -6 & 5\end{pmatrix}=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$

iii) $\mathbf{NP}=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}=2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=2\mathbf{I}$ where **I** is the 2 × 2 unit matrix

$\mathbf{NP}=2\mathbf{I} \Rightarrow (\tfrac{1}{2}\mathbf{N})\mathbf{P}=\mathbf{I}$

Since any matrix multiplied by its inverse gives the unit matrix, it implies that the inverse of **P** is $\tfrac{1}{2}\mathbf{N}$.

Therefore the inverse of **P** is

$$\tfrac{1}{2}\begin{pmatrix}5 & 3\\ 6 & 4\end{pmatrix}=\begin{pmatrix}\tfrac{5}{2} & \tfrac{3}{2}\\ 3 & 2\end{pmatrix}$$

c) $\mathbf{DC}=\begin{pmatrix}3 & 1\\ -2 & -3\end{pmatrix}\begin{pmatrix}-2 & 5\\ 4 & 3\end{pmatrix}$

$=\begin{pmatrix}-2 & 18\\ -8 & -19\end{pmatrix}$

$\therefore \mathbf{DC}=2\begin{pmatrix}-3 & 7\\ 2 & a\end{pmatrix}+\begin{pmatrix}b & 4\\ -12 & -3\end{pmatrix}$

$$\begin{pmatrix}-2 & 18\\ -8 & -19\end{pmatrix}=\begin{pmatrix}-6+b & 18\\ -8 & 2a-3\end{pmatrix}$$

Comparing corresponding elements

$-b+6=-2 \quad \Rightarrow \quad b=4$ and

$2a-3=-19 \quad \Rightarrow \quad a=-8$

Example 11

A reflection in a line through the origin which makes an angle θ with the positive direction of the x-axis has the matrix of transformation:

$$M(\theta)=\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}$$

An anticlockwise rotation through an angle θ about the origin has the transformation matrix $R(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

a) Write down the matrix for
   i) reflection in the line $y=x\sqrt{3}$
   ii) an anticlockwise rotation about the origin through an angle of 30°

b) Hence if $\sqrt{3}=1.732$, find correct to one decimal place the image of the point $P(3, -2)$ under a reflection in the line $y=x\sqrt{3}$ followed by an anticlockwise rotation about the origin through an angle of 30°.

c) If $L=\begin{pmatrix}2 & 2\\ -2 & 2\end{pmatrix}$ is the matrix for an anticlockwise rotation about the origin through an angle of 135° followed by an enlargement from the origin, find the scale factor of the enlargement.

Solution

a) i) $y=x\sqrt{3} \Rightarrow$ gradient of line $=\sqrt{3}$

But the gradient of a line is equal to the tangent of angle the line makes with the positive x-axis. Therefore

$\tan\theta=\sqrt{3} \Rightarrow \theta=\tan^{-1}\sqrt{3}=60°$

Hence the matrix for reflection in the line $y=x\sqrt{3}$ is

$$M(60)=\begin{pmatrix}\cos 120° & \sin 120°\\ \sin 120° & -\cos 120°\end{pmatrix}$$

$$=\begin{pmatrix}\cos 60^\circ & \sin 60^\circ\\ \sin 60^\circ & \cos 60^\circ\end{pmatrix}$$

$$M(60)=\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$$

ii) Anticlockwise rotation about the origin through an angle of 30° is

$$R(30)=\begin{pmatrix}\cos 30^\circ & -\sin 30^\circ\\ \sin 30^\circ & \cos 30^\circ\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$$

b) Reflection in the line $y=x\sqrt{3}$ followed by an anticlockwise rotation about the origin through an angle of 30°

$$R(30)\text{o}M(60)=[R(30)][M(60)]$$

$$=\begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4} & \frac{3}{4}+\frac{1}{4}\\ 6\frac{1}{4}+\frac{3}{4} & \frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4}\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{\sqrt{3}}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}=\begin{pmatrix}\frac{-1.732}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1.732}{2}\end{pmatrix}$$

$$=\begin{pmatrix}-0.866 & 0.5\\ 0.5 & 0.866\end{pmatrix}$$

Hence the image of the point $P(3,\ 2)$ under $R(30)$ o $M(60)$ is given by:

$$\begin{pmatrix}-0.866 & 0.500\\ 0.500 & 0.866\end{pmatrix}\begin{pmatrix}3\\ -2\end{pmatrix}=\begin{pmatrix}-3.6\\ -0.2\end{pmatrix}$$

c) $L=\begin{pmatrix}2 & 2\\ -2 & 2\end{pmatrix}$

The matrix for an anticlockwise rotation about the origin through an angle of 135° followed by an enlargement from the origin is given by: $k\,[R(135^\circ)]$, where k = scale factor.

$$k[R(135^\circ)]=k\begin{pmatrix}\cos 135^\circ & -\sin 135^\circ\\ \sin 135^\circ & \cos 135^\circ\end{pmatrix}$$

$$=k\begin{pmatrix}-\cos 45^\circ & -\sin 45^\circ\\ \sin 45^\circ & -\cos 45^\circ\end{pmatrix}$$

$$=k\begin{pmatrix}-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{pmatrix}$$

Hence $\begin{pmatrix}2 & 2\\ -2 & 2\end{pmatrix}=k\begin{pmatrix}-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{pmatrix}$

$$2=-k\tfrac{\sqrt{2}}{2}$$

$$k=\tfrac{-4}{\sqrt{2}}=\tfrac{-4}{\sqrt{2}}\times\tfrac{\sqrt{2}}{\sqrt{2}}=-2\sqrt{2}$$

Hence the scale factor of the enlargement is $k=-2\sqrt{2}$.

Example 12

a) Write down the matrices **A** and **B** which represent rotations about the origin, through angles α and β respectively.

b) Find the matrix which represents the transformation **AB** and describe this transformation in words.

c) Write down another matrix which represents the transformation in (b) and hence find an expression for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ in terms of $\sin\alpha$, $\cos\alpha$, $\sin\beta$ and $\cos\beta$.

Solution

a) $\mathbf{A}=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$,

$\mathbf{B}=\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}$

b) **AB**

$$=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}$$

$$=\begin{pmatrix}\cos\alpha\cos\beta-\sin\alpha\sin\beta & -\cos\alpha\sin\beta-\sin\alpha\cos\beta\\ \sin\alpha\cos\beta+\cos\alpha\sin\beta & \cos\alpha\cos\beta-\sin\alpha\sin\beta\end{pmatrix}$$

This composite transformation is a rotation through an angle β followed by a rotation through an angle α.

c) The transformation in (b) above is the same as rotation through an angle $(\alpha+\beta)$ and this can be represented by the matrix

$$\begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta & \cos(\alpha+\beta) \end{pmatrix}$$

Comparing this with the matrix **AB** above gives:

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\cos\beta$

Example 13

a) If $P'(5, 1)$ and $Q'(4, -4)$ are the images of $P(1, 2)$ and $Q(2, 1)$ respectively under the linear transformation **T**, find

i) the matrix **T** of the linear transformation;

ii) the image of the point $R(2, 3)$ under **T**.

b) Two linear transformations **M** and **N** are defined by

$M:(x,y) \rightarrow (5x+3y, 6x+4y)$ and

$N:(x,y) \rightarrow (4x-3y, -6x+5y)$.

i) Write down the matrices **M** and **N**

ii) Evaluate **MN** and hence the inverses of **M** and **N**.

Solution

a) i) Let $\mathbf{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$P(1, 2) \rightarrow P'(5, 1)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$a + 2b = 5$..............(1)

$c + 2d = 1$...............(2)

$Q(2, 1) \rightarrow (4, -4)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

$2a + b = 4$.................(3)

$2c + d = -4$...............(4)

Solving equations (1) and (3) simultaneously gives $a = 1$ and $b = 2$

Solving equations (2) and (4) simultaneously gives $c = -3$ and $d = 2$

$$\therefore \mathbf{T} = \begin{pmatrix} 1 & 2 \\ -3 & 2 \end{pmatrix}$$

ii) Image of R, $R' = \begin{pmatrix} 1 & 2 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$= \begin{pmatrix} 2+6 \\ -6+6 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

b) i) Given $\begin{matrix} \mathbf{M}:(x, y) \rightarrow (5x+3y, 6x+4y) \\ \mathbf{N}:(x, y) \rightarrow (4x-3y, -6x+5y) \end{matrix}$,

the matrix of the linear transformation **M** is

$\mathbf{M} = \begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}$ and of **N** is $\mathbf{N} = \begin{pmatrix} 4 & -3 \\ -6 & 5 \end{pmatrix}$

ii) The matrix $\mathbf{MN} = \begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ -6 & 5 \end{pmatrix}$

$$= \begin{pmatrix} 20-18 & -15+15 \\ 24-24 & -18+20 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\mathbf{MN} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\tfrac{1}{2}\mathbf{MN} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

Where **I** is the unit matrix

$\tfrac{1}{2}\mathbf{MN} = \mathbf{I}$

If **M** is the matrix then $\mathbf{M}(\tfrac{1}{2}\mathbf{N}) = \mathbf{I}$

i.e. Matrix × its inverse = identity matrix

$\frac{1}{2}$ **N** is the inverse of **M**

Inverse of **M** $=\frac{1}{2}\begin{pmatrix}5 & 3\\ 6 & 4\end{pmatrix}=\begin{pmatrix}\frac{5}{2} & \frac{3}{2}\\ 3 & 2\end{pmatrix}$

If **N** is the matrix then $(\frac{1}{2}\mathbf{M})\mathbf{N}=\mathbf{I}$

$\frac{1}{2}$ **M** is the inverse of **N**

Inverse of **N** $=\frac{1}{2}\begin{pmatrix}4 & -3\\ -6 & 5\end{pmatrix}=\begin{pmatrix}2 & -\frac{3}{2}\\ -3 & \frac{5}{2}\end{pmatrix}$

Example 14

a) Two linear transformations **P** and **Q** defined by:

$\mathbf{P}:(x, y)\rightarrow(x+y, -x)$

$\mathbf{Q}:(x, y)\rightarrow(2x+y, 3x+4y)$.

i) Write down the matrices of **P** and **Q**

ii) Find **PQ**.

iii) Find the image of the point $(-2, -1)$ under the transformation **P** o **Q**.

iv) If I is the 2×2 unit matrix, find the matrix **M** such that $(\mathbf{P}+\mathbf{Q})\mathbf{M}=\mathbf{I}$

b) Given that $\mathbf{N}=\begin{pmatrix}3 & -5\\ 4 & 2\end{pmatrix}$, find the image of the point (1, –1) under the transformation by the inverse of the matrix **N**.

Solution

a) i) Given $\mathbf{P}:(x, y)\rightarrow(x+y, -x)$

$$\text{P:}\begin{pmatrix}x\\ y\end{pmatrix}\rightarrow\begin{pmatrix}x+y\\ -x+0y\end{pmatrix}\rightarrow\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\mathbf{P}=\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}$$

Given $\mathbf{Q}:(x, y)\rightarrow(2x+y, 3x+4y)$

$$\mathbf{Q}:\begin{pmatrix}x\\ y\end{pmatrix}\rightarrow\begin{pmatrix}2x+y\\ 3x+4y\end{pmatrix}\rightarrow\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\mathbf{Q}=\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}$$

ii) $\mathbf{PQ}=\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}$

$$=\begin{pmatrix}2+3 & 1+4\\ -2+0 & -1+0\end{pmatrix}=\begin{pmatrix}5 & 5\\ -2 & -1\end{pmatrix}$$

Matrix × point = Image

Image of (–2, –1)

$$=\begin{pmatrix}5 & 5\\ -2 & -1\end{pmatrix}\begin{pmatrix}-2\\ -1\end{pmatrix}=\begin{pmatrix}-15\\ 5\end{pmatrix}$$

$= (-15, 5)$

b) If $(\mathbf{P}+\mathbf{Q})\mathbf{M}=\mathbf{I}$ then **M** is the inverse of $(\mathbf{P}+\mathbf{Q})$ since when any matrix is multiplied by its inverse the result is **I**.

$$\mathbf{P}+\mathbf{Q}=\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}+\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}=\begin{pmatrix}3 & 2\\ 2 & 4\end{pmatrix}$$

The determinant of $\mathbf{P}+\mathbf{Q}=12-4=8$

∴ The inverse of $\mathbf{P}+\mathbf{Q}$ i.e. **M** is given by:

$$\mathbf{M}=\tfrac{1}{8}\begin{pmatrix}4 & -2\\ -2 & 3\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{4}\\ -\frac{1}{4} & \frac{3}{8}\end{pmatrix}$$

Alternative method

a) Let $\mathbf{M}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ then $(\mathbf{P}+\mathbf{Q})\mathbf{M}=\mathbf{I}$

$$\begin{pmatrix}3 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$3a+2c=1$ ……….(1)

$3b+2d=0$ ……....(2)

$2a+4c=0$ ……...(3)

$2b+4d=1$ ……....(4)

Solving equations (1) and (3) and (2) and (4) simultaneously gives

$a=\frac{1}{2},\ c=-\frac{1}{4},\ b=-\frac{1}{4},\ d=\frac{3}{8}$

$$\therefore\ \mathbf{M}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{4}\\ -\frac{1}{4} & \frac{3}{8}\end{pmatrix}$$

b) Determinant of $\mathbf{N}=6-(-20)=26$

Inverse of **N**, $\mathbf{N}^{-1}=\frac{1}{26}\begin{pmatrix}2 & 5\\ -4 & 3\end{pmatrix}=\begin{pmatrix}\frac{1}{13} & \frac{5}{26}\\ -\frac{2}{13} & \frac{3}{26}\end{pmatrix}$

Image = Inverse matrix × point

$$\text{Image} = \begin{pmatrix} \frac{1}{13} & \frac{5}{26} \\ -\frac{2}{13} & \frac{3}{26} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{13}-\frac{5}{26} \\ -\frac{2}{13}-\frac{3}{26} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{26} \\ -\frac{7}{26} \end{pmatrix}$$

$\therefore$ Image of $(1, -1)$ under $\mathbf{N}^{-1}$ is $(-\frac{3}{26}, -\frac{7}{26})$

Example 15

Two linear transformations P and T are defined by:

$\mathbf{P}: (x, y) \rightarrow (3x - 2y, x + 4y)$;

$\mathbf{T}: (x, y) \rightarrow (3x + y, -2x + 3y)$

a) Write down the matrices of P and T.

b) Find the matrix **PT**.

c) Find the image of (0, 3) under the transformation **P**.

d) Find the inverse of **T**.

e) Determine the co-ordinates of the point whose image under **T** is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Solution

a) $\mathbf{P} = \begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix}$ and $\mathbf{T} = \begin{pmatrix} 3 & 1 \\ -2 & 3 \end{pmatrix}$

b) $\mathbf{PT} = \begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 13 & -3 \\ -5 & 13 \end{pmatrix}$

c) Matrix × point = image

$$\begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ 12 \end{pmatrix} = (-6, 12)$$

d) Determinant of $\mathbf{T} = 3 \times 3 + 2 \times 1 = 11$

Inverse of T, $\mathbf{T}^{-1} = \frac{1}{11}\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{11} & -\frac{1}{11} \\ \frac{2}{11} & \frac{3}{11} \end{pmatrix}$

e) Matrix × point = image

$$\begin{pmatrix} 3 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{11} & -\frac{1}{11} \\ \frac{2}{11} & \frac{3}{11} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{11} \\ \frac{8}{11} \end{pmatrix} = (\tfrac{1}{11}, \tfrac{8}{11})$$

Example 16

a) **P** and **Q** are two linear transformations in the $x - y$ plane defined by:

$\mathbf{P}: (x, y) \rightarrow (-2x + 3y, 4x + y)$,

$\mathbf{Q}: (x, y) \rightarrow (-3x - 6y, -4x - 6y)$.

i) Obtain a single transformation representing the transformation **Q** followed by **P**.

ii) Find the image of the point (2, 3) when it is transformed by **Q** followed by **P**.

iii) What is the image of the point (–1, 4) under the transformation **Q**?

b) Find the image P' of the point $P(\sqrt{2}, 2\sqrt{2})$ under an anticlockwise rotation through 225° about the origin.

Solution

a) i) The transformation **Q** followed by **P** is

$$\mathbf{PQ} = \begin{pmatrix} -2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -3 & -6 \\ -4 & -6 \end{pmatrix} = \begin{pmatrix} -6 & -6 \\ -16 & -30 \end{pmatrix}$$

$\therefore$ **P o Q**: $(x, y) \rightarrow (-6x - 6y, -16x - 30y)$

ii) $\begin{pmatrix} -6 & -6 \\ -16 & -30 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -12 & -18 \\ -32 & -90 \end{pmatrix} = \begin{pmatrix} -30 \\ -122 \end{pmatrix}$

Image $= (-30, -122)$

iii) $\begin{pmatrix} -3 & -6 \\ -4 & -6 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 & -24 \\ 4 & -24 \end{pmatrix} = \begin{pmatrix} -21 \\ -20 \end{pmatrix}$

Image $= (-21, -20)$

b) $\begin{pmatrix} \cos 225° & -\sin 225° \\ \sin 225° & \cos 225° \end{pmatrix}\begin{pmatrix} -\sqrt{2} \\ 2\sqrt{2} \end{pmatrix}$

$$=\begin{pmatrix}-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{pmatrix}\begin{pmatrix}-\sqrt{2}\\ 2\sqrt{2}\end{pmatrix}$$

$$=\begin{pmatrix}1+2\\ 1-2\end{pmatrix}=\begin{pmatrix}3\\ -1\end{pmatrix}$$

Image $= P'(3, -1)$

Example 17

The image of the point (3, 4) under a linear transformation R is (9, –6) and the image of (–1, –3) under R is (–8, 7).

a) Find
   i) the matrix **R** of the linear transformation.
   ii) $\mathbf{R}^{-1}$, the inverse of the matrix **R**.

b) If $\mathbf{S}=\begin{pmatrix}0 & 1\\ -3 & 4\end{pmatrix}$, find $\mathbf{R}^{-1}\mathbf{S}$.

Solution

a) i) Let $\mathbf{R}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}3\\ 4\end{pmatrix}=\begin{pmatrix}9\\ -6\end{pmatrix}$$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}-1\\ -3\end{pmatrix}=\begin{pmatrix}-9\\ 6\end{pmatrix}$$

$3a + 4b = 9$ ....(1)
$a + 3b = 8$ ....(2)
$3c + 4d = -6$ ....(3)
$c + 3d = -7$ ....(4)

Solving equations (1) and (2),

$$\begin{aligned}3a + 4b &= 9\\ 3a + 9b &= 24\\ \hline 5b = 15 &\Rightarrow b = 3\end{aligned}$$

Putting $b = 3$ in (2)

$a = 8 - 3(3) = -1 \Rightarrow a = -1$

Solving eqns (3) and (4) simultaneously,

$d = -3$ and $c = 2$

$$\therefore\ \mathbf{R}=\begin{pmatrix}-1 & 3\\ 2 & -3\end{pmatrix}$$

ii) Det $\mathbf{R} = (-1\times-3)-(2\times3)=3-6=-3$

$$\therefore\ \mathbf{R}^{-1}=-\frac{1}{3}\begin{pmatrix}-3 & -3\\ -2 & -1\end{pmatrix}=\begin{pmatrix}1 & 1\\ \frac{2}{3} & \frac{1}{3}\end{pmatrix}$$

b) $$\mathbf{R}^{-1}\mathbf{S}=\tfrac{1}{3}\begin{pmatrix}3 & 3\\ 2 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -3 & 4\end{pmatrix}=\tfrac{1}{3}\begin{pmatrix}-9 & 15\\ -3 & 6\end{pmatrix}$$

$$=\begin{pmatrix}-3 & 5\\ -1 & 2\end{pmatrix}$$

Linear transformation of a line by a matrix

Illustrative examples

Example 1

Find the image of i) $y = 3x$ ii) $y = 3x + 1$ under the mapping $\begin{pmatrix}2 & 3\\ 1 & 2\end{pmatrix}$

Solution

i) Let any point on $y = 3x$ be (λ, 3λ), where λ is a parameter. The image of (λ, 3λ) is given by: $\begin{pmatrix}2 & 3\\ 1 & 2\end{pmatrix}\begin{pmatrix}\lambda\\ 3\lambda\end{pmatrix}=\begin{pmatrix}11\lambda\\ 7\lambda\end{pmatrix}$

This is the position vector of any point on the image line.

When λ = 1, the point is (11, 7), and when λ = 0, the point is (0, 0)

The equation of line passing through (11, 7) and (0, 0) is $11y = 7x$.

So $11y = 7x$ is the required image of $y = 3x$.

ii) Let any point on $y = 3x + 1$ be (λ, 3λ+1), where λ is a parameter.

The image of (λ, 3λ+1) is given by:

$$\begin{pmatrix}2 & 3\\ 1 & 2\end{pmatrix}\begin{pmatrix}\lambda\\ 3\lambda+1\end{pmatrix}=\begin{pmatrix}2\lambda+9\lambda+3\\ \lambda+6\lambda+2\end{pmatrix}=\begin{pmatrix}11\lambda+3\\ 7\lambda+2\end{pmatrix}$$

This is the position vector of any point on the image line.

When $\lambda = 1$, the point is (14, 9), and when $\lambda = 0$, the point is (3, 2)
The equation of line passing through (14, 9) and (3, 2) is $11y = 7x + 1$
So $11y = 7x + 1$ is the required image of $y = 3x + 1$.

Example 2

Find the equation of the line onto which the line $x + y = 0$ is mapped onto by the transformation

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}5&12\\3&5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

Solution

$7y = 2x$ Work out the solution

Example 3

Find the line whose image is $y = 3x - 2$ under the linear transformation
T: $(x, y) \rightarrow (3x + 2y, 5x + 4y)$.

Solution

Here the image line is given and we are required to find the object line. Here we use the inverse matrix.

$\mathbf{T}^{-1}=\frac{1}{2}\begin{pmatrix}4&-2\\-5&3\end{pmatrix}$ of the transformation instead of the original matrix **T**.
Follow the procedure used in example 1 and come out with the answer $y = 1 - 2x$

Review exercise 28

1. A transformation **T** is given by $\mathbf{T}=\begin{pmatrix}1&2\\3&4\end{pmatrix}$.
   If $Q(5, 6)$ is the image of the point P under transformation **T**, find the coordinates of P.

2. a) If (5, 1) and (4, −4) are the images of $P(1, 2)$ and $Q(2, 1)$ respectively under the linear transformation **M**, find
   i) the matrix **M**
   ii) the image of the point $A(2, 3)$ under **M**.
   b) If the matrix **B** of another transformation **R** is $\mathbf{B}=\begin{pmatrix}1&-3\\-2&2\end{pmatrix}$, find **BM**.

3. a) The matrices of the linear transformations **A** and **B** are given respectively by
   $\mathbf{A}=\begin{pmatrix}-1&4\\0&3\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}2&-5\\1&-3\end{pmatrix}$
   i) Calculate the image of the point (2, −3) under the transformation **B** followed by **A**.
   b) A linear transformation is represented by 2 × 2 matrix **A**. The image of the point (1, 0) under **A** is (−13, 4) while the image of (0, 1) under **A** is (6, −19). Find **A**.

4. $Q'(3, 5)$ is the image of Q under a linear transformation, $\mathbf{T}=\begin{pmatrix}3&1\\5&2\end{pmatrix}$.
   Find the coordinates of Q.

5. A point A has image $A'(9, 8)$ under the linear transformation matrix
   $\mathbf{P}=\begin{pmatrix}1&4\\2&3\end{pmatrix}$. Find A.

6. a) If $\mathbf{M}=\begin{pmatrix}2&-1\\1&3\end{pmatrix}$, find the values of $\mathbf{M}^2$, $\mathbf{M}^3$ and $\mathbf{M}^{-1}$.
   Find x and y, given that
   $\mathbf{M}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3\\5\end{pmatrix}$

b) A transformation **T** is equivalent to an enlargement with centre at the origin, scale factor 2, followed by a reflection in the line $x + y = 0$. What matrix defines **T**. If **T** maps a point P onto (6, 2), what are the coordinates of P?

7. A transformation **M** is represented by the matrix $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.

i) Find the image of the point (–2, 5) under **M**.
ii) Find the inverse of **M**
iii) Given that the point (11, 13) is the image of the point (a, b) under **M**, find a and b.

8. A linear transformation is represented by the 2 × 2 matrix **A**. The image of the point (1, 0) under **A** is (3, 4) while the image of (0, 1) under **A** is (6, –5). Find **A.**

9. a) Find the matrix of the following linear transformations:

i) $\mathbf{N} : (x, y) \rightarrow (-2y, -x + 2y)$
ii) $\mathbf{P} : (x, y) \rightarrow (x + y, -2x)$
iii) $\mathbf{M} : (x, y) \rightarrow (3x + y, 2x - 3y)$
iv) $\mathbf{Q} : (x, y) \rightarrow (-y, 2x + y)$

b) Find i) **NM** ii) **NP** iii) **NQ** iv) **MP** v) **MQ** vi) **PQ**

10. A linear transformation P of the plane maps the points (1, 3), (–2, –3) to the points (2, 4), (–3, –11), respectively. Find the matrix of this transformation.

11. a) i) Write down the matrices of the linear transformations:

$\mathbf{T} : (x, y) \rightarrow (x + y, x + 2y)$ and
$\mathbf{S} : (x, y) \rightarrow (2x - y, 3x + y)$

ii) A point $P(1, 2)$ is transformed by **T** followed by **S**. Find the image of P
iii) A point $Q(a, b)$ is transformed by **S** followed by **T**. If its image is (1, 1), find a and b.

b) The point P is rotated through an angle of 50° anticlockwise about the origin

i) State the matrix **M** for the linear transformation.
ii) Calculate, correct to one decimal place, the image P^1 of the point P after rotation using matrix **M**

12. i) Write down the matrices of the linear transformations:

$\mathbf{Q} : (x, y) \rightarrow (4x + 2y, -3y)$ and
$\mathbf{R} : (x, y) \rightarrow (2x + 8y, 7x + 2y)$

ii) Find the matrix **Q** + **R**
iii) Find the inverses of **Q** and **R**
iv) Use the inverse of **R** to solve the following simultaneous equations

$$-2x + 8y = 20$$
$$7x + 2y = 20$$

13. a) Under a linear transformation **P** of a plane, the image of (2, 1) is (4, 2) and that of (–3, 1) is (–1, 7). Find the matrix **P** of the transformation.

b) If the matrix of another transformation **Q** is $\mathbf{Q} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$ find the image of a point (x, y) under a composite transformation **P** o **Q**, where **P** o **Q** means **Q** followed by **P**. Hence describe **P** o **Q** geometrically.

14. a) A linear transformation **M** maps
$A(1, 0) \rightarrow A^1(-2, -4)$ and
$B(0, 1) \rightarrow B^1(4, 1)$,
Find;

i) The matrix **M**;

ii) The image of the point **C** (1, -1) under **M**.

b) Two linear transformation **N** and **P** are defined by
$\mathbf{N}:(x, y)\to(2x+y,\ 3x+2y)$ and
$\mathbf{P}:(x, y)\to(2x-3y,\ -2x+y)$;

i) Write down the matrices **N** and **P**

ii) Find the matrix **NP**

iii) Find the inverse of **P**.

15. Find the matrix of the linear transformation $\mathbf{P}:(x, y)\to(2y, x+2y)$.

16. $Q'(3, 5)$ is the image of Q under a linear transformation, $\mathbf{T}=\begin{pmatrix}3 & 1\\ 5 & 2\end{pmatrix}$. Find the coordinates of Q.

17. Find the matrix for the reflection in the line $y = x$.

18. A linear transformation **T** of a vector **p** is given by $x' = 2x - y$ and $y' = 3x + 2y$. Find the image of the point $Q(2, -4)$ under **T**.

19. Find the matrix of the linear transformation $x' = 3x + y$
$y' = 2x - 3y$

20. The point $P(-1, 3)$ undergoes an anti-clockwise rotation through 60° about the origin. Find the coordinates of its image.

21. Let $\mathrm{S}:(x, y)\to(x, y)$ and $\mathbf{T}:(x, y)\to(y, x)$ be two linear transformations in the xy plane. Find a single matrix representing the composite transformation **T** o **S**.

22. a) Identify the transformation in the Cartesian plane Oxy represented by the matrix $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$.

b) Determine the matrix **A** which transforms $\begin{pmatrix}1\\1\end{pmatrix}$ onto $\begin{pmatrix}4\\1\end{pmatrix}$ and $\begin{pmatrix}1\\2\end{pmatrix}$ onto $\begin{pmatrix}7\\0\end{pmatrix}$.

23. A transformation is defined by
$\mathbf{T}:(x, y)\to(4x+y,\ 6x+3y)$.

i) Write down the matrix **T** of the transformation.

ii) Find the inverse, $\mathbf{T}^{-1}$, of the matrix **T**.

iii) If the point P is transformed by the matrix $\mathbf{T}^{-1}$ and the image is $(-2, 1)$, find the coordinates of P.

24. a) $Q(-2, 4)$ is a point in $x-y$ plane. Calculate the coordinates;

i) The image Q' of Q under an anticlockwise rotation of 90° about the origin.

ii) The image Q'' of Q' under translation by the vector $\begin{pmatrix}0\\-6\end{pmatrix}$

iii) The image Q''' of Q under a reflection in the line $x - 2 = 0$

b) Using a scale of 2cm to 2 units on both axes, draw the figure $QQ'Q''Q'''$.

c) Hence or otherwise find the equation of the line $Q''Q'''$.

25. Two linear transformation **A** and **B** are defined as
$\mathbf{A}:(x, y)\to(3x+y,\ -2x)$,
$\mathbf{B}:(x, y)\to(-x+7y,\ 6x+y)$

i) Write the matrices **A** and **B**

ii) Find the matrix **A** + **B**

iii) Find the inverse of **B**.

26. The line $y = 5x + 2$ has been transformed by the linear equations

$x' = x - 2y$ and

$y' = 3x + y$

Determine the equation of the resulting line.

27. a) Write down the matrix of the linear transformation

$\mathbf{M} : (x, y) \rightarrow (kx + y,\ \ 3x + ky)$.

b) If the image of $(-1,\ 2)$ under **M** is $(a, -7)$, find the value of a.

c) Find the inverse of **M**.

28. Find the line whose image is $y = 3x - 2$ under the linear transformation

$\mathbf{T}$: $(x, y) \rightarrow (3x + 2y, 5x + 4y)$.

29. Two transformations in the $x - y$ plane are given by

$\mathbf{A}$: $(x, y) \rightarrow (x + 2y, 3x - y)$

$\mathbf{B}$: $(x, y) \rightarrow (2y, 2x + 3y)$

i) Find a single matrix representing the transformation **AB**.

ii) Find the image of the point (0, 1) when it is transformed by **AB**.

iii) When a point (a, b) is transformed by **AB**, its image is $(-2, 3)$ Determine the value of a and b

30. The matrix **B** of the linear transformation

$\mathbf{B} : (x, y) \rightarrow (ax + by, cx + dy)$, is

given by $\begin{pmatrix} 3 & 2 \\ 7 & 4 \end{pmatrix}$

i) Determine the values of the constant coefficients a, b, c and d.

ii) Find the matrix of the inverse of the transformation **B** and use it to find the point whose image is $(-1, 2)$

31. i) Write down the matrices **A** and **B** respectively of the linear transformation

$\mathbf{A}$: $(x, y) \rightarrow (5x + y, 2x - 3y)$,

$\mathbf{B}$: $(x, y) \rightarrow (y, -x + 2y)$

ii) Use the matrices **A** and **B** to calculate:

a) the image of the point $(-1, -2)$ under the transformation $(2\mathbf{A} + 3\mathbf{B})$

b) the coordinate of the point whose image under the transformation **A** followed by **B** is (3, 2).

32. a) Write down the matrices **A** and **B** respectively of the linear transformations

$\mathbf{A}$: $(x, y) \rightarrow (-x + 5y, 3x - y)$,

$\mathbf{B}$: $(x, y) \rightarrow (4x + 7y, -2x + 6y)$

b) Use the matrices **A** and **B** to calculate the image of the point $(-1, 2)$ under the transformation $(2\mathbf{A} + 5\mathbf{B})$.

33. A matrix, **P** maps (1, 0) onto $(2, -1)$ and (1, 1) onto (5,3). Another matrix, **Q** maps (x, y) onto $(-2y, 2x - y)$. Find the matrix

a) **P**;

b) **Q**;

c) **PQ**;

d) the image of the point $(-1, 3)$ under the transformation **P** o **Q**.

CHAPTER 29

LOGIC

Compound statements

A *statement* is a verbal or written assertion and it may be a *simple statement* or a *compound statement.* For example, "Kofi is a bad boy" is a simple statement which can be either true or false. Two or more simple statements can also be combined to form a compound statement. For example, "Kofi is a bad boy" and "Kofi is a thief" are two simple statements which can be combined to form a compound statement "If Kofi is a bad boy then Kofi is a thief". In logic we use small letters to represent simple statements.

For example, p: Kofi is a student or

q: Kofi is in form five.

It must be emphasised here that, the basic property of any statement is, it is either true (T) or false (F) and cannot be true or false at the same time.

Logical connectives or operators

The basic logical operators are conjunction (AND or $\wedge$), disjunction (OR or $\vee$) and negation (NOT). These may be used to combine two simple statements to form a compound statement.

IMPLICATION ($\Rightarrow$) and EQUIVALENCE ($\Leftrightarrow$) are also logical operators.

Converse statements

The **converse** of a statement is the *reverse* form of the statement.

1. Converse of implication

Consider the following simple statements.

p: Abena has measles

q: Abena is in the hospital

The converse of the compound statement 'If Abena has measles then Abena is in the hospital' is: 'If Abena is in the hospital then Abena has measles.

It must be noted that the converse of any compound statement is *not always true.* That is if the compound statement 'If Abena has measles then Abena is in the hospital' is true it does not always follow that 'If Abena is in the hospital then Abena has measles.

i.e. if $p \Rightarrow q$ is true then the converse statement $q \Rightarrow p$ is not always true and therefore not valid.

2. Converse of conjunction (AND)

Consider the following statements:

p: Kofi is a student

q: Kofi is in form five

The converse of the compound statement: "Kofi is a student **and** in form five" is 'Kofi is in form five and a student'. This converse statement is true only when the compound statement is true.

i.e. if $p \wedge q$ is true then the converse statement $q \wedge p$ is also true.

3. Converse of disjunction (OR)

Consider the following statements:

p: Kofi is a student

q: Ama is in form five

The converse of the compound statement: "Either Kofi is a student **or** Ama is in form five" is 'Either Ama is in form five or Kofi is a student'. This converse statement is true only when the compound statement is true.

i.e. if $p \vee q$ is true then the converse statement $q \vee p$ is also true.

The truth table

The truth table is used to show whether two logical expressions are the same or not. It also shows how the truth of a compound statement depends on the truth of its component simple statements. The truth table gives all possible truth values of a compound statement in terms of the possible truth values of its components. In truth tables, we represent *truth* by ***T*** and *false* by ***F***.

And ($\wedge$) (Conjunction)

Two simple statements p and q can be combined by the word "**AND**" to form a compound statement "*p* **and** *q*". The compound statement p and q is also represented as $p \wedge q$.

For example, p: Kofi is a student and q: Kofi is in form five, can be combined as "Kofi is a student **and** in form five".

Thus we write p and q or $p \wedge q$.

The compound statement $p \wedge q$ is true only when both p and q are true. Otherwise it is false.

The table below is the truth table for $p \wedge q$.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The first two columns give the four possible values of p and q. The third column is for $p \wedge q$.

For the first row p is true (T) and q is also true (T), therefore $p \wedge q$ is true since both p and q are true. For the other three rows $p \wedge q$ is false since both p and q are not true.

Note that the compound statement $p \wedge q$ is true only when p and q are both true. If at least one of the simple statements is false then $p \wedge q$ is false. "AND" is the same as intersection ($\cap$) in sets.

'Or' (Disjunction)

When two statements are connected by the word "**OR**", the result of this combination is called the **disjunction** of the two statements. We have two types of disjunction.

1. Inclusive disjunction ($\vee$)
2. Exclusive disjunction ($\underline{\vee}$)

Inclusive Or ($\vee$) (Inclusive disjunction)

If p and q are two simple statements then the compound statement p or q written as $p \vee q$ is true when any of the statements p **or** q **or both** are true. Otherwise it is false.

For example, p: Kofi is a student and q: Ama is in form five, can be combined as: "Either Kofi is a student or Ama is in form five". This combined statement is true if any of them is true or both are true. Otherwise it is false. Below is the truth table of $p \vee q$ (p or q or both).

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The third column is $p \vee q$. Since both p and q are false in the fourth row, $p \vee q$ is also false. Note that if at least one of the two simple statements is true, then $p \vee q$ is true. Inclusive "OR" is the same as union ($\cup$) in sets.

Exclusive Or ($\veebar$) (Exclusive disjunction)

If p and q are two simple statements then the compound statement p or q written as $p \veebar q$ is true when any of the statements p **or** q is true but both **cannot** be true. Otherwise it is false. Consider the two statements: p: I am in Accra and q: I am in Kumasi.

We can write $p \veebar q$ since I cannot be in Accra and Kumasi at the same time.

Below is the truth table of $p \veebar q$ (p or q).

p	q	$p \veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

The third column is $p \veebar q$. Since both p and q are false in the fourth row, $p \veebar q$ is also false. Also since both p and q are true in the first row, $p \veebar q$ is false.

Note that if exactly one of the two simple statements is true, then $p \veebar q$ is true.

Not (~) (Negation)

Given any simple statement p, another statement called the **negation** of p can be formed by inserting in statement p the word "not". The negation of p is written as $\sim p$ (i.e. **not** p).

Consider the two simple statements;

p: Kofi is a bad boy

q: Kofi is **not** a bad boy

For these two statements, if p is true, then q must be false. q is said to be the negation of p.

i.e. $q = \sim p$. Thus if p is true then $\sim p$ is false. Also if p is false then $\sim p$ is true.

The truth table for $\sim p$ is shown below.

p	$\sim p$
T	F
F	T

"NOT" is the same as complement in sets.

Implication ($\Rightarrow$)

For any two statements p and q, $p \Rightarrow q$ means q is true only when p is true.

$p \Rightarrow q$ means "statement p **implies** the statement q" or "**if** p **then** q".

Consider the following statements:

p: Abena has measles

q: Abena is in the hospital

The compound statement '**If** Abena has measles **then** Abena is in the hospital' means $p \Rightarrow q$.

The symbol '$\Rightarrow$' is the implication sign.

$p \Rightarrow q$ is read as 'p implies q'.

For example, $3x - 2 = 10 \;\Rightarrow\; x = 4$

If p and q are two statements, then p implies q denoted as $p \Rightarrow q$ is true except when p is true and q is false, i.e. when the first statement is true and the second statement is false then $p \Rightarrow q$ is false.

Consider the truth table for $p \Rightarrow q$ below.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

For the second row, p is true but q is false. Therefore $p \Rightarrow q$ is false.
For the fourth row, although both statements are false $p \Rightarrow q$ is true.

Equivalence ($\Leftrightarrow$)

If the two statements p and q are such that $p \Rightarrow q$ is true and the converse statement $q \Rightarrow p$ is also true, then $p \Leftrightarrow q$. In particular, the statements p and q are **equivalent** if both $p \Rightarrow q$ and its converse $q \Rightarrow p$ are both true. Thus, if $p \Rightarrow q$ and $q \Rightarrow p$ then $p \Leftrightarrow q$. The symbol '$\Leftrightarrow$' is the equivalent implication sign. $p \Leftrightarrow q$ is read as 'p is equivalent to q'.

Note this carefully: If $p \Rightarrow q$, then we can write the equivalent statement $\sim q \Rightarrow \sim p$.
Thus $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$.
Note also that $\sim p \Rightarrow \sim q$ and $q \Rightarrow p$ **are not** equivalent statements of $p \Rightarrow q$.
Note that $p \Leftrightarrow q$ is true only when the statements p and q are **both** true or false. Otherwise it is false.

Consider the truth table for $p \Leftrightarrow q$ below.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Rows 1 and 4 are true since p and q are both true or both false. Rows 3 and 4 are false. Two logical statements are said to be *equivalent* if their corresponding entries on the truth table are the *same*. At this stage the student is advised to know how the logical operators combine simple statements to form compound statements and the conditions governing the logical operators as shown by the truth tables for $p \wedge q$, $p \vee q$, $p \veebar q$, $\sim p$, $p \Rightarrow q$ and $p \Leftrightarrow q$.

Illustrative example

Examples 1

Form the truth table for $(p \wedge \sim q)$ and hence $\sim(p \wedge \sim q)$

Solution

Note that for the truth tables constructed above, the first two columns are for the statements p and q and that there are enough rows in the table for the 4 possible combinations of T and F for these statements. (i.e. p is true (T) and q is true (T), p is T and q is F etc). To form the truth set of $(p \wedge \sim q)$, first form the column for $\sim q$ before the column for $(p \wedge \sim q)$ as shown below.
For $\sim(p \wedge \sim q)$, negate the column for $(p \wedge \sim q)$.

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Note that column 3 is the negation of q i.e. $\sim q$. Therefore by the rule, negate all the values under q as shown. Then combine p and $\sim q$ (column 1 and 3) using the operator $\wedge$ to get the column for $p \wedge \sim q$. Finally negate the column for $(p \wedge \sim q)$ (i.e. column 4) to get $\sim(p \wedge \sim q)$.

The student is advised to always refer to the truth tables for $\wedge, \vee, \sim, \Rightarrow$ and $\Leftrightarrow$ when in difficulty .

Example 2

Use the truth table to show that the following statements are equivalent

i) $\sim(p \vee q) \Leftrightarrow (\sim p \wedge \sim q)$

ii) $\sim(p \wedge q) \Leftrightarrow (\sim p \vee \sim q)$

iii) $(\sim p \vee q) \Leftrightarrow (p \Rightarrow q)$

Solution

i) Form the columns for $\sim(p \vee q)$ and $(\sim p \wedge \sim q)$ and compare them. If the corresponding entries are identical, then the statements are equivalent.

Consider the truth table below.

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

The $\sim(p \vee q)$ column was obtained from the $(p \vee q)$ column using the rule for negation. The $\sim p \wedge \sim q$ column was obtained by first obtaining the columns for $\sim p$ and $\sim q$ by negating p and q respectively and then combining them using $\wedge$ to get $(\sim p \wedge \sim q)$. Since the entries in columns 4 and 7 are identical it means that $\sim(p \vee q)$ and $(\sim p \wedge \sim q)$ are equivalent. **This is de Morgan's law.**

ii) Form the columns for $\sim(p \wedge q)$ and $(\sim p \vee \sim q)$. For $\sim(p \wedge q)$, first form the column $(p \wedge q)$ i.e. column 3 and negate to get $\sim(p \wedge q)$ i.e. column 4.

For $(\sim p \vee \sim q)$, first form the columns for $\sim p$ and $\sim q$ (i.e. columns 5 and 6) and then combine them using $\vee$ to get $(\sim p \vee \sim q)$ i.e. column 7 shown on the truth table below.

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Since columns 4 and 7 have identical entries it implies $\sim(p \wedge q)$ and $(\sim p \vee \sim q)$ are equivalent i.e. $\sim(p \wedge q) \Leftrightarrow (\sim p \vee \sim q)$.
This is de Morgan's law.

iii) Form the columns for $(\sim p \vee q)$ and $(p \Rightarrow q)$. For $\sim p \vee q$, first form the column for $\sim p$ by negating p and using the column for $\sim p$ and that of q, form the column $\sim p \vee q$. Also form the column $p \Rightarrow q$ using columns p and q.

Consider the truth table below.

p	q	$\sim p$	$\sim p \vee q$	$p \Rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since columns 4 and 5 have the same entries, $(\sim p \wedge q) \Leftrightarrow (p \Rightarrow q)$.

Example 3

Show that $\sim(\sim p \wedge q) \Leftrightarrow (p \vee \sim q)$.

Solution

Form the columns for $\sim(\sim p \wedge q)$ and $(p \vee \sim q)$ and compare them.

To form the table for $\sim(\sim p \wedge q)$, first form $(\sim p \wedge q)$ (i.e. bracket first) and then negate to get the column for $\sim(\sim p \wedge q)$. To form the column for $\sim p \wedge q$, first negate p to get $\sim p$ and then combine $\sim p$ and q to get $\sim p \wedge q$. Also to form the column for $(p \vee \sim q)$, first negate q to get $\sim q$ and then combine p and $\sim q$ to get $(p \vee \sim q)$.

p	q	$\sim p$	$\sim p \wedge q$	$\sim(\sim p \wedge q)$	$\sim q$	$(p \vee \sim q)$
T	T	F	F	T	F	T
T	F	F	F	T	T	T
F	T	T	T	F	F	F
F	F	T	F	T	T	T

The 5th and 7th columns have the same entries.

$\therefore \ \sim(\sim p \wedge q) \Leftrightarrow (p \vee \sim q)$

Note: The following laws are true for the logical operators $\sim, \vee, \wedge, \Rightarrow$ and $\Leftrightarrow$.

1. $p \vee p \Leftrightarrow p$ and $p \wedge p \Leftrightarrow p$
2. $[p \vee (q \vee r)] \Leftrightarrow (p \vee q) \vee r$ and
   $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
   (Associative law)
3. $(p \vee q) \Leftrightarrow (q \vee p)$ and
   $(p \wedge q) \Leftrightarrow (q \wedge p)$
   (Commutative law)
4. $\sim(p \vee q) \Leftrightarrow (\sim p \wedge \sim q)$ and
   $\sim(p \wedge q) \Leftrightarrow (\sim p \vee \sim q)$
   (de Morgan's law)
5. $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$.
6. $\sim(\sim p) = p$

Tautology

A compound statement is sometimes called a compound formula. A compound formula which has the truth value T for any truth value of any of its statement variables p and q is called a **tautology** or a valid formula. That is, a tautology always contains only T in the last column of its truth table.

For example, the compound statement:
'It is a triangle OR it is not a triangle'
is always true whether the prime statement 'It is a triangle' is true or false. We show that the compound statement is a tautology using a truth table.

Let p = It is a triangle

Therefore, $p \vee \sim p$ is the given compound statement

p	$\sim p$	$(p \vee \sim p)$
T	F	T
F	T	T

In general, $p \vee \sim p$ is always a tautology

Also the statement $p \wedge q \Rightarrow p \vee q$ is a tautology.

We verify this by constructing a truth table.

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q \Rightarrow p \vee q$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Note that last column of the truth table contains only T.

Contradiction

A contradiction is a compound statement that contains only F in the last column of its truth table for all values of the statement variables.
For example, the compound statement:
'It is a triangle AND it is not a triangle'
is a contradiction. Let p = It is a triangle
Therefore, $p \wedge \sim p$ is the given compound statement.
We verify this by constructing a truth table.

p	$\sim p$	$(p \wedge \sim p)$
T	F	F
F	T	F

Note that last column of the truth table contains only F.
Note also that. Contradiction is the negation of tautology. Thus since tautology is always true, contradiction is always false.
$\sim (p \wedge q \Rightarrow p \vee q)$ is a contradiction since $p \wedge q \Rightarrow p \vee q$ is a tautology.

Valid arguments

An argument is **valid** if and only if the conclusion follows from other statements (the premises). The premise is the given statement(s) from which other conclusions can be drawn.
Note that the premise of an argument is always assumed to be true. Note also that the truth of the conclusion is irrelevant when testing for the validity of an argument. The fact that the conclusion is true is not sufficient for an argument to be valid.
Consider the following argument:
Lagos is in Ghana
Ghana is in East Africa
Therefore Lagos is in East Africa
The first two statements are the premises and the last statement is the conclusion
Note that although the two premises are false they are assumed to be true.
Note also that the conclusion is false although the argument is valid.

The validity of an argument can be determined using the truth table

Since in a normal usage, we use an argument only in order to demonstrate that a certain conclusion follows from known premises, all we shall require of a valid argument form, therefore, is that under any assignment of truth values to the statement variables, if the premises all take value T then the conclusion also takes value T.

Illustrative examples

Example 1

Test the following simple argument for validity.

1. *If a rhombus is a quadrilateral then its interior angles sum up to 360°.*
2. *A rhombus is a quadrilateral*
3. *Therefore the interior angles are 360°.*

Solution

In this example, the first two statements form the premises of the argument while the last is the conclusion. We start by identifying the statement variables in the argument.
A rhombus is a quadrilateral = p
Interior angles sum up to 360° = q
The given argument can now be expressed in the form: $(p \Rightarrow q),\ p\ ;\ \ \therefore q$
To test the validity of this argument, we show that there are possible truth values to the statement variables p and q such that if the premises $(p \Rightarrow q)$ and p each take the value T, the conclusion q takes the value T.

To do this we construct a truth table for all the statement forms appearing in the premises or conclusion.

p	q	$(p \Rightarrow q)$
T	T	T
T	F	T
F	T	T
F	F	T

Note that it is only in the first row that the premises $(p \Rightarrow q)$ and p each take the value T, and the conclusion q also takes the value T. Hence the argument form; $(p \Rightarrow q)$, p; $\therefore q$ is valid.

Example 2

Test the following simple arguments for validity using truth tables.

i) If x is positive, y will be greater than 10.
x is not positive.
y is not greater than 10.

ii) If x is positive, y will be greater than 10.
y is not greater than 10.
x is not positive.

Solution

i) Not Valid ii) Valid

[Work out the missing steps.]

Review exercise 29

1. If p and q are any two statements show that

i) $(p \Rightarrow q)$ and $(\sim p \vee q)$ are equivalent.
ii) $(q \Rightarrow p)$ and $(\sim q \vee p)$ are equivalent.
iii) $(p \Rightarrow q)$ and $(\sim q \Rightarrow \sim p)$ are equivalent.
iv) $(p \vee q)$ and $(q \vee p)$ are equivalent.
v) $(p \wedge q)$ and $(q \wedge p)$ are equivalent
vi) $(p \veebar q)$ and $(p \vee q) \wedge \sim(p \wedge q)$ are equivalent
vii) $(p \Leftrightarrow q)$ and $(p \Rightarrow q) \wedge (q \Rightarrow p)$ are equivalent.

2. Determine whether or not the following arguments are valid using truth tables:

i) All engineers are rich
Rich people are never happy
Therefore no engineer is happy

ii) All students are lazy
Nobody who is rich is a student
Therefore lazy people are not rich

iii) No teacher is poor
Some musicians are poor
Therefore some musicians are not teachers

3. Copy and complete the following truth table.

p	q	$\sim p$	$p \vee q$	$\sim p \Rightarrow q$	$\sim p \vee q$	$p \Rightarrow q$
T	T	F				
T	F	F				
F	T	T				
F	F	T				

4. Fill in the truth table below. Write out the equivalent statements.

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$\sim p$	$\sim q$	$q \vee \sim p$	$\sim q \Rightarrow \sim p$
T	T	F						
T	F	F						
F	T	T						
F	F	T						

CHAPTER 30

CORRELATION AND REGRESSION

Introduction

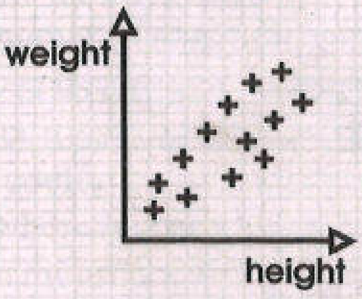
Correlation concerns the strength of the relationship between two variables. An example is the relationship between the height and weight of students in a class. **Regression** analysis determines the nature of that relationship and enables us to make predictions from it.

Scatter Diagram

In dealing with correlation, we shall be concerned with **paired values of two variables** such as the one shown below.

Height (cm)	x_1	x_2	x_3	x_4	…
Weight (kg)	y_1	y_2	y_3	y_4	…

One way to find if there is a relationship between the two variables is to draw a **scatter diagram** with the data. A scatter diagram is a dot diagram showing the relationship between two variables. Each pair of values is plotted as a point on the graph. For example, in a scatter diagram for the investigation about the relationship between heights and weights of students in a class, one axis is labelled **height** and the other is labelled **weight**.



Each point plotted on the graph shows the height and weight of one person. The plotted points are usually scattered across the diagram, hence the name **scatter diagram**.

The scatter diagram enables us to see whether there exists a relationship between the paired values (**correlation**).

Once we are satisfied that some relationship exist, we can establish the precise nature of that relationship and use it to predict values of one variable that would correspond to any given values of the other variable (**regression analysis**).

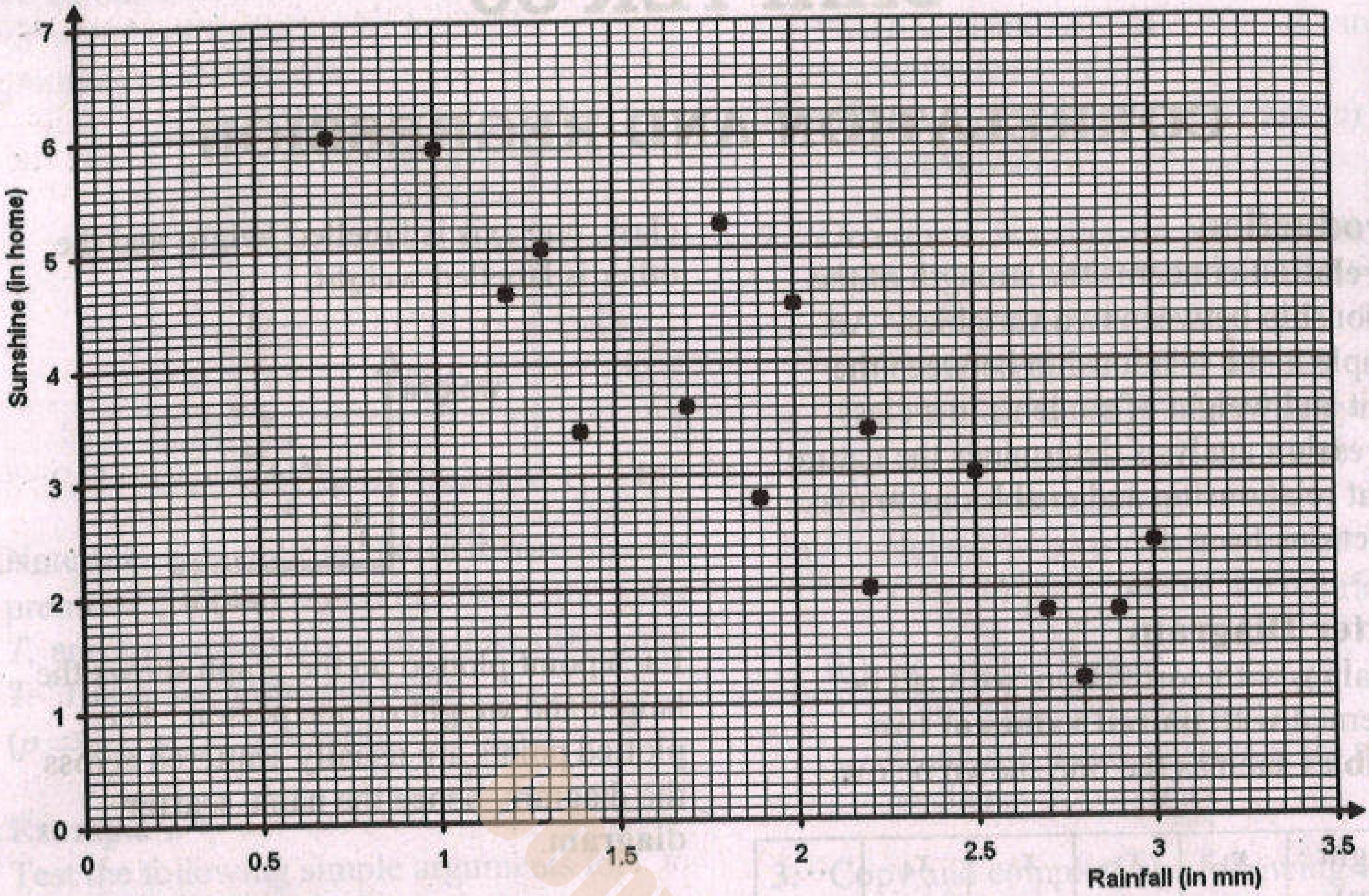
Most correlations are not so clear; hence our predictions can be nothing like so accurate.

Illustrative example

The table below gives the recorded rainfall, x, (in mm) and sunshine, y, (in hours) for 16 towns on a day in June. Draw a scatter diagram to show these data.

x	1.0	1.4	1.2	1.7	2.0	2.5	0.7	1.3	2.2	2.8	3.0	1.8	1.9	2.2	2.7	2.9
y	5.9	3.4	4.6	3.6	4.5	3.0	6.0	5.0	3.4	1.2	2.4	5.2	2.8	2.0	1.8	1.8

Solution

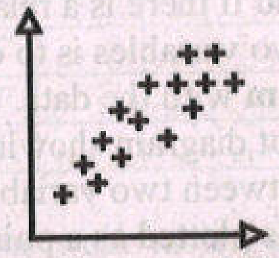


Types of correlation

When the plotted points on a scatter diagram show a definite pattern, this suggests a relationship between the variables. This relationship is called correlation between the variables. The stronger the relationship, the higher the degree of correlation and the higher the degree of correlation the more confidently we can predict values of one variable, given values of the other. There are three types of correlation; positive, negative and zero.

Positive correlation

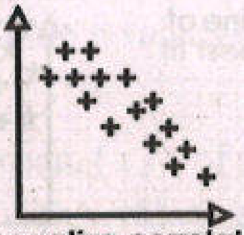
A band of points sloping *upwards* across the diagram shows a **positive correlation** between the variables. Thus, as one variable increases in size, the other tends to increase in size also. For example, height and weight are positively correlated. In general, a taller person tends to weigh more than a shorter person.



positive correlation

Negative correlation

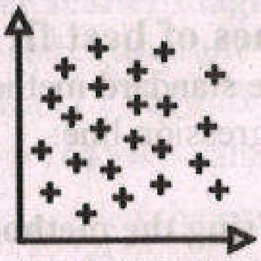
A band of points sloping *downwards* across the diagram shows a **negative correlation** between the variables. Thus, as one variable increases in size, the other tends to decrease in size. For example, the scatter diagram for rainfall and sunshine drawn above shows a negative correlation between the variables.



negative correlation

Zero or no correlation

Sometimes there is no clear tendency for values of one variable to move in any direction with changes in the other. Thus there is *no pattern* and the points are *randomly* scattered over the diagram. This shows there is **zero or no correlation** between the variables. For example, there is no correlation between a person's age and his or her house number.



no correlation

Line of best fit and strength of coefficient of correlation

Correlations vary also not only in direction (i.e. positive, negative or zero) but in strength.

Perfect correlation (unlikely to occur in statistics) implies the plotted points on the scatter diagram lie on a *straight line*. The points on a scatter diagram do not usually lie on a perfect straight line. A weaker correlation implies the points do lie away from a straight line. See diagrams below. Drawing a line that seems to fit the points *best* shows the trend in the data. This line is called the **line of best fit**. The line of best fit illustrates the link between the variables on the graph.

If you know the value of one variable you can estimate the value of the other variable from the line. The closer the correlation between the variables (i.e. if the points are close to the line of best fit), the stronger the correlation and the more accurate the estimate will be.

Coefficient of correlation

Since the amount of scatter in a scatter diagram gives us a rough measure of the strength of correlation, we need a numerical measure (a coefficient) that will be at a maximum when the correlation is strong and reduces to a minimum as the correlation weakens. Such a coefficient is called the **correlation coefficient**, usually represented by the letter "r".

The coefficient of correlation measures the closeness with which the pairs of values fit a straight line.

The correlation coefficient cannot lie outside the range between -1 and $+1$

i.e. $-1 \leq r \leq 1$.

$r = +1$ implies perfect positive correlation

$r = -1$ implies perfect negative correlation

$r = 0$ implies there is no correlation at all

The closer the correlation coefficient gets to $+1$ or -1, the stronger the correlation. The closer, it gets to 0, the weaker it is. For example, $r = +0.73$ would indicate a stronger relationship than $r = 0.45$. Also $r = -0.84$ would be stronger than either of them.

The correlation coefficient therefore describes both the **strength and direction** of the relationship between two pairs of values from two different variables. The sign gives the direction (i.e. positive or negative) and the value gives the strength.

Use the following as a guide: If *r* lies within

0.0 – 0.2 it implies very weak or negligible correlation.

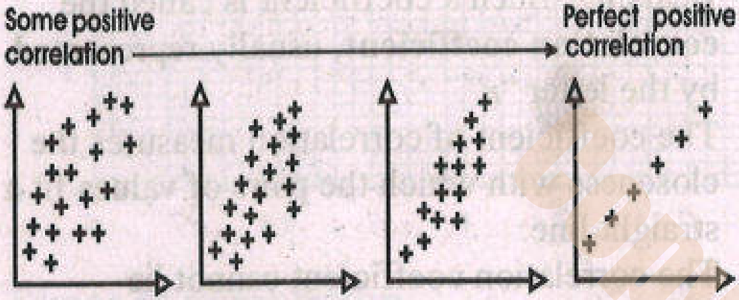
0.2 – 0.4 it implies weak or low correlation.

0.4 – 0.7 it implies moderate correlation.

0.7 – 0.9 it implies stronger or high or marked correlation.

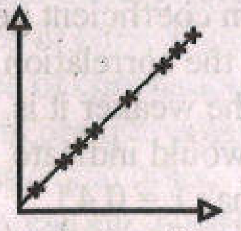
0.9 – 1.0 it implies very strong or very high correlation.

The *width* of a 'band of points' on a scatter diagram shows how closely the variables are linked. The *narrower* the band, the *closer* the correlation.

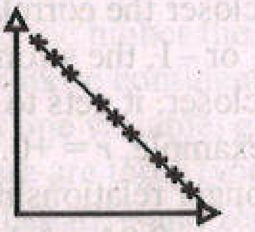


Drawing the lines of best fit – Regression line

The points on each of the scatter graphs below lie on a *straight line*.

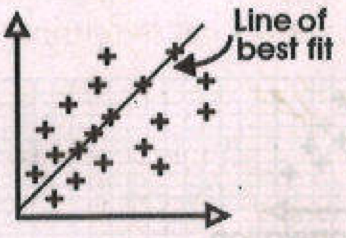


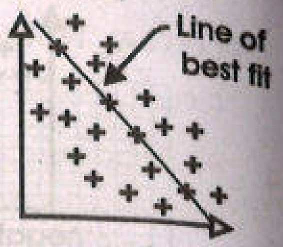
Perfect positive correlation



Perfect negative correlation

This shows there is **perfect correlation** between the two variables in each diagram. The points on a scatter diagram do not usually lie on a perfectly straight line. But, if there is a correlation between the variables, then they lie mainly in a *band* sloping across the diagram.





Drawing the line that seems to fit the points *best* shows the '*trend*' in the data. This line is called the **line of best fit**. The line of best fit drawn on the scatter diagram to show the relationship between two variables is called the **regression line**. There are two types of the regression line;

1. *The regression line of y on x* (used for predicting *y*–values given *x*–values) and
2. *The regression line of x on y* (used for predicting *x*–values given *y*–values).

Drawing lines of best fit

There are three standard methods of obtaining a regression line:

Method 1 – Using the method semi-averages

The technique consists of splitting the data into two groups, plotting the mean point for each group and joining these points with a straight line.

The steps for drawing the regression line of y on x on the scatter diagram are:

1. Draw the scatter diagram
2. Find the mean ($\bar{x}$) of all *x*-values and the mean ($\bar{y}$) of all *y*-values to get the point ($\bar{x}, \bar{y}$), called the centroid of points. i.e. $\bar{x} = \frac{\sum x}{n}$ and $\bar{y} = \frac{\sum y}{n}$, where *n* is the number of pairs.
3. Find the mean, ($\bar{x}_1$), of all the *x*-values of points with *x*-values less than $\bar{x}$ and find the mean, ($\bar{y}_1$) of their

corresponding y-values to get the point $(\bar{x}_1, \bar{y}_1)$.

4. Find the mean, $(\bar{x}_2)$, of all the x-values of points with x-values greater than $\bar{x}$ and find the mean, $(\bar{y}_2)$, of their corresponding y-values to get the point $(\bar{x}_2, \bar{y}_2)$.
5. Plot the points $(\bar{x}, \bar{y})$, $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$ on the scatter diagram.
6. Draw a straight line to pass through the centroid $(\bar{x}, \bar{y})$ and as close as possible to the points $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$.

The steps for drawing the regression line of x on y on the scatter diagram are:

1. Draw the scatter diagram
2. Find the mean $(\bar{x})$ of all x-values and the mean $(\bar{y})$ of all y-values to get the point $(\bar{x}, \bar{y})$, called the centroid of points. i.e. $\bar{x} = \dfrac{\sum x}{n}$ and $\bar{y} = \dfrac{\sum y}{n}$, where n is the number of pairs.
3. Find the mean $(\bar{y}_1)$ of all the y-values of points with y-values less than $\bar{y}$ and find the mean $(\bar{x}_1)$ of their corresponding x-values to get the point $(\bar{x}_1, \bar{y}_1)$.
4. Find the mean $(\bar{y}_2)$ of all the y-values of points with y-values greater than $\bar{y}$ and find the mean $(\bar{x}_2)$ of their corresponding x-values to get the point $(\bar{x}_2, \bar{y}_2)$.
5. Plot the points $(\bar{x}, \bar{y})$, $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$ on the scatter diagram.
6. Draw a straight line to pass through the centroid, $(\bar{x}, \bar{y})$, and as close as possible to the points $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$.

Regression equation

Like any other straight line, a regression line can be described by an equation.
This equation is called the **regression equation.** There are two types of equations corresponding to the two types of lines of regressions. For the regression line of y on x, the equation is: $y - \bar{y} = m(x - \bar{x})$
For the regression line of x on y, the equation is: $x - \bar{x} = \frac{1}{m}(y - \bar{y})$, where m is the gradient of the line (also called the coefficient of regression) and $(\bar{x}, \bar{y})$ is the centroid.

Illustrative examples

Example 1

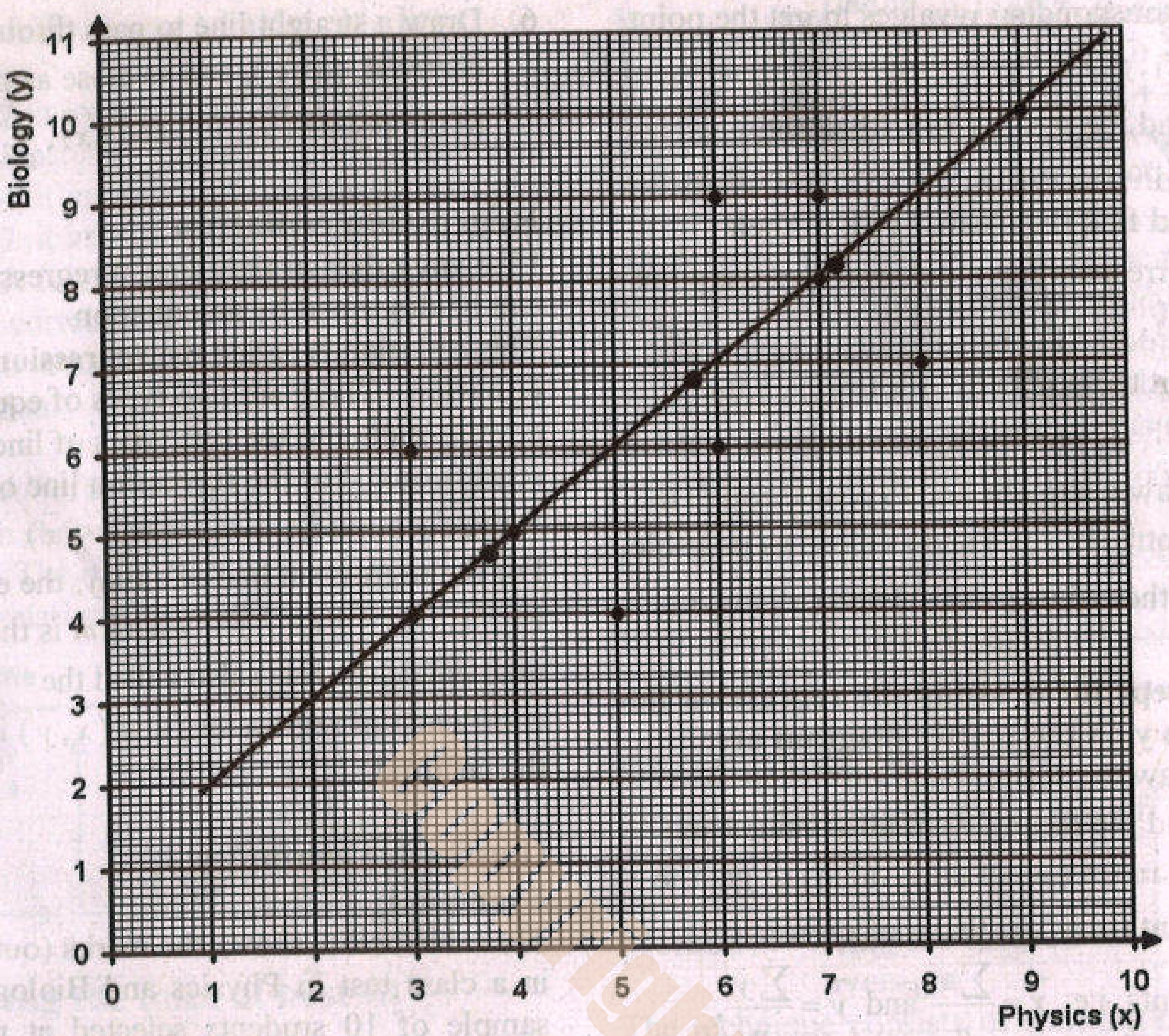
The table below shows the marks (out of 10) in a class test in Physics and Biology of a sample of 10 students selected at random from a class.

Physics(x)	3	3	4	5	6	7	7	8	9	6
Biology(y)	4	6	5	4	6	8	9	7	10	9

a) Draw a scatter diagram of the data.
b) Draw the regression line of y on x.
c) Find the equation of best fit and use it to predict the possible Biology mark of a student who scored 7 in Physics.

Solution

a) The scatter diagram is shown in the next page.



b) i) The mean of all x,

c) $\bar{x} = \frac{\sum x}{n} = \frac{58}{10} = 5.8$ and the mean of all y, $\bar{y} = \frac{\sum y}{n} = \frac{68}{10} = 6.8$

$\therefore$ The centroid, $(\bar{x}, \bar{y}) = (5.8, 6.8)$

ii) The mean of all x-values which are less than $\bar{x}$, $\bar{x}_1 = \frac{3+3+4+5}{4} = 3.75$

The mean of their corresponding y-values,

$$\bar{y}_1 = \frac{4+6+5+4}{4} = 4.75$$

$\therefore$ $(\bar{x}_1, \bar{y}_1) = (3.75, 4.75)$

iii) The mean of all x-values which are greater than $\bar{x}$,

$$\bar{x}_2 = \frac{6+7+7+8+9+6}{6} = 7.17$$

The mean of their corresponding y-values,

$$\bar{y}_2 = \frac{6+8+9+7+10+9}{6} = 8.17$$

$\therefore$ $(\bar{x}_2, \bar{y}_2) = (7.17, 8.17)$

iv) Plot the points (5.8, 6.8), (3.75, 4.75) and (7.17, 8.17) on the scatter diagram and draw a straight line through them to get the regression line of y on x.

d) The equation of the line of best fit is $y - \bar{y} = m(x - \bar{x})$ where m is the gradient of the line and $(\bar{x}, \bar{y})$ is the centroid.

From the graph, the gradient, $m = 1$

Therefore the equation is
$y - 6.8 = 1(x - 5.8)$
$y - 6.8 = x - 5.8 \quad \Rightarrow \quad y = x + 1$
From the equation, when the marks scored in Physics, $x = 7$, then the marks scored in Biology, $y = 7 + 1 = 8$

Example 2
The table below shows the height and weight of 10 soldiers selected at random to study the relationship between height and weight of soldiers.

Weight (kg)	70	63	72	60	66	70	74	65	62	67
Height (cm)	160	150	180	135	156	168	178	160	132	152

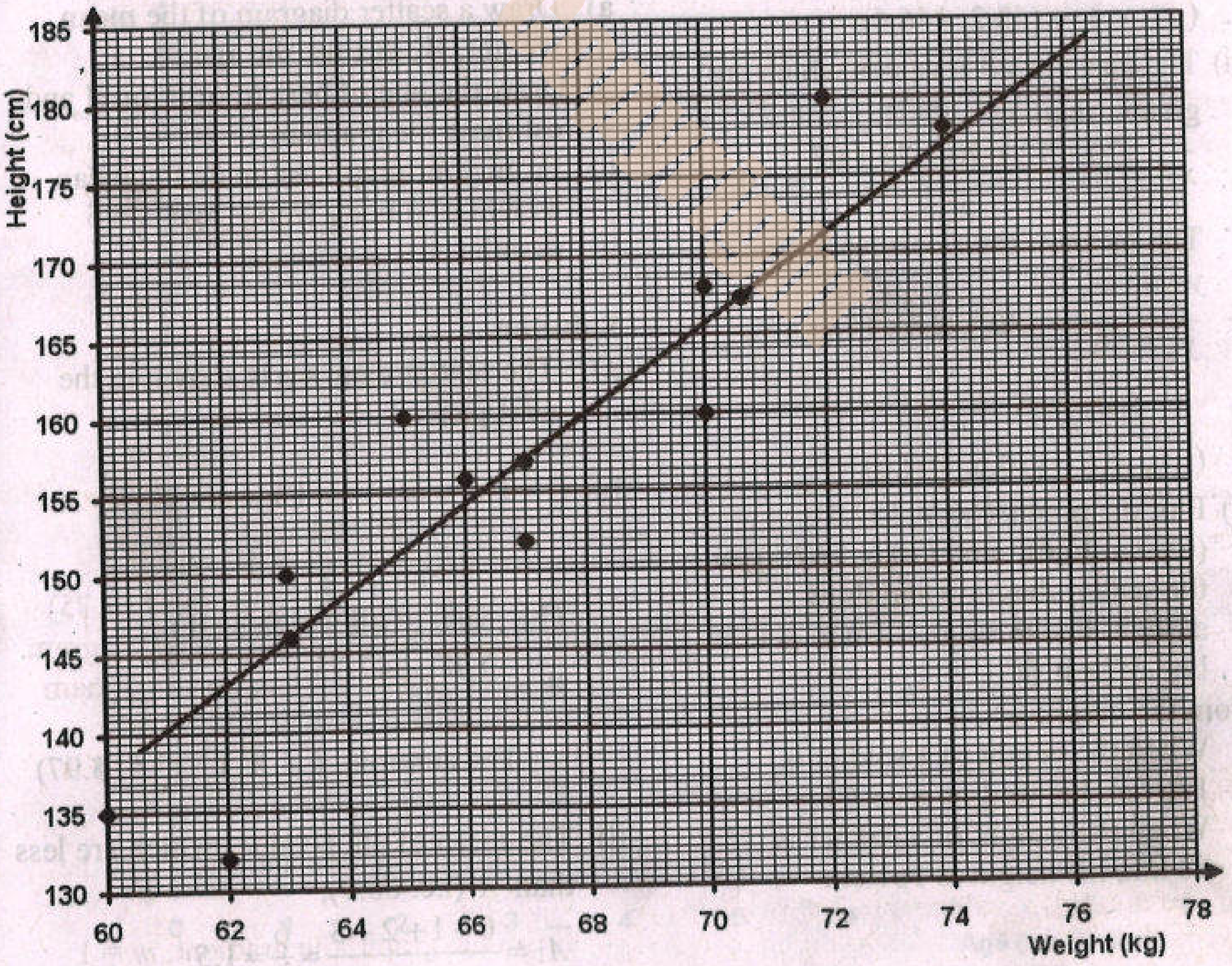
a) Using a scale of 2 cm to 5 cm on the height axis and 1 cm to 2 kg on the weight axis draw a scatter diagram of the data.

b) If $\bar{x}$ is the mean of the weights and $\bar{y}$ is the mean of the heights
  - i) find $\bar{x}$ and $\bar{y}$
  - ii) draw the line of best fit to pass through $(\bar{x}, \bar{y})$

c) Use your line to predict the value of a soldier's
  - i) weight, if the height is 176 cm
  - ii) height, if the weight is 68 kg

Solution

a) The scatter diagram is shown below.



b) i) The mean of all weights (x),

$$\bar{x} = \frac{\sum x}{n} = \frac{669}{10} = 66.9 \text{ and}$$

The mean of all heights (y),

$$\bar{y} = \frac{\sum y}{n} = \frac{1571}{10} = 157.1$$

$\therefore$ The centroid, $(\bar{x}, \bar{y}) = (66.9, 157.1)$

ii) The mean of all x-values which are less than $\bar{x}$ (i.e. 66.9),

$$\bar{x}_1 = \frac{63 + 60 + 66 + 65 + 62}{5} = 63.2$$

The mean of their corresponding y-values,

$$\bar{y}_1 = \frac{150 + 135 + 156 + 160 + 132}{5}$$

$$= 146.6$$

$\therefore$ $(\bar{x}_1, \bar{y}_1) = (63.2, 146.6)$

iii) The mean of all x-values which are greater than $\bar{x}$,

$$\bar{x}_2 = \frac{70 + 72 + 70 + 74 + 67}{5} = 70.6$$

The mean of their corresponding y-values,

$$\bar{y}_2 = \frac{160 + 180 + 168 + 178 + 152}{5}$$

$$= 167.6$$

$\therefore$ $(\bar{x}_2, \bar{y}_2) = (70.6, 167.6)$

iv) Plot the points (66.9, 157.1), (63.2, 146.6) and (70.6, 167.6) on the scatter diagram and draw a straight line through them to get the line of best fit.

c) From the graph;

i) When the height of a soldier is 176 cm, his weight is 73.6 kg

ii) When the weight of a soldier is 68 km, his height is 161 cm

Example 3

In a certain hospital, the mean weight of a group of babies born on the same day was observed over several months at intervals of one month. The results over the first seven months are shown in the table below.

Age in months (A)	Mean weight in kg (W)
0	2.50
1	3.75
2	4.20
3	5.70
4	6.40
5	7.35
6	8.25
7	9.60

a) Draw a scatter diagram of the mean weight, W, against the age, A.

b) Draw the line of best fit of W on A and estimate the gradient.

c) From your graph, estimate the mean weight of a body whose age is 5.5 months.

Solution

a) The scatter diagram is shown in the next page.

b) i) The mean of all ages (A),

$$\bar{A} = \frac{\sum A}{n} = \frac{28}{8} = 3.5 \text{ and}$$

The mean of all weights (W),

$$\bar{W} = \frac{\sum W}{n} = \frac{47.75}{8} = 5.97$$

$\therefore$ The centroid, $(\bar{A}, \bar{W}) = (3.5, 5.97)$

ii) The mean of all A-values which are less than $\bar{A}$ (i.e. 66.9),

$$\bar{A}_1 = \frac{0 + 1 + 2 + 3}{4} = \frac{6}{4} = 1.5$$

The mean of their corresponding *W*- values,

$$\overline{W}_1 = \frac{2.50+3.75+4.20+5.70}{4}$$

$$= \frac{16.15}{4} = 4.04$$

$\therefore\ (\overline{A}_1, \overline{W}_1) = (1.5, 4.04)$

iii) The mean of all *A*-values which are greater than $\overline{A}$,

$$\overline{A}_2 = \frac{4+5+6+7}{4} = \frac{22}{4} = 5.5$$

The mean of their corresponding *W*- values,

$$\overline{W}_2 = \frac{6.40+7.35+8.25+9.60}{4} = 7.9$$

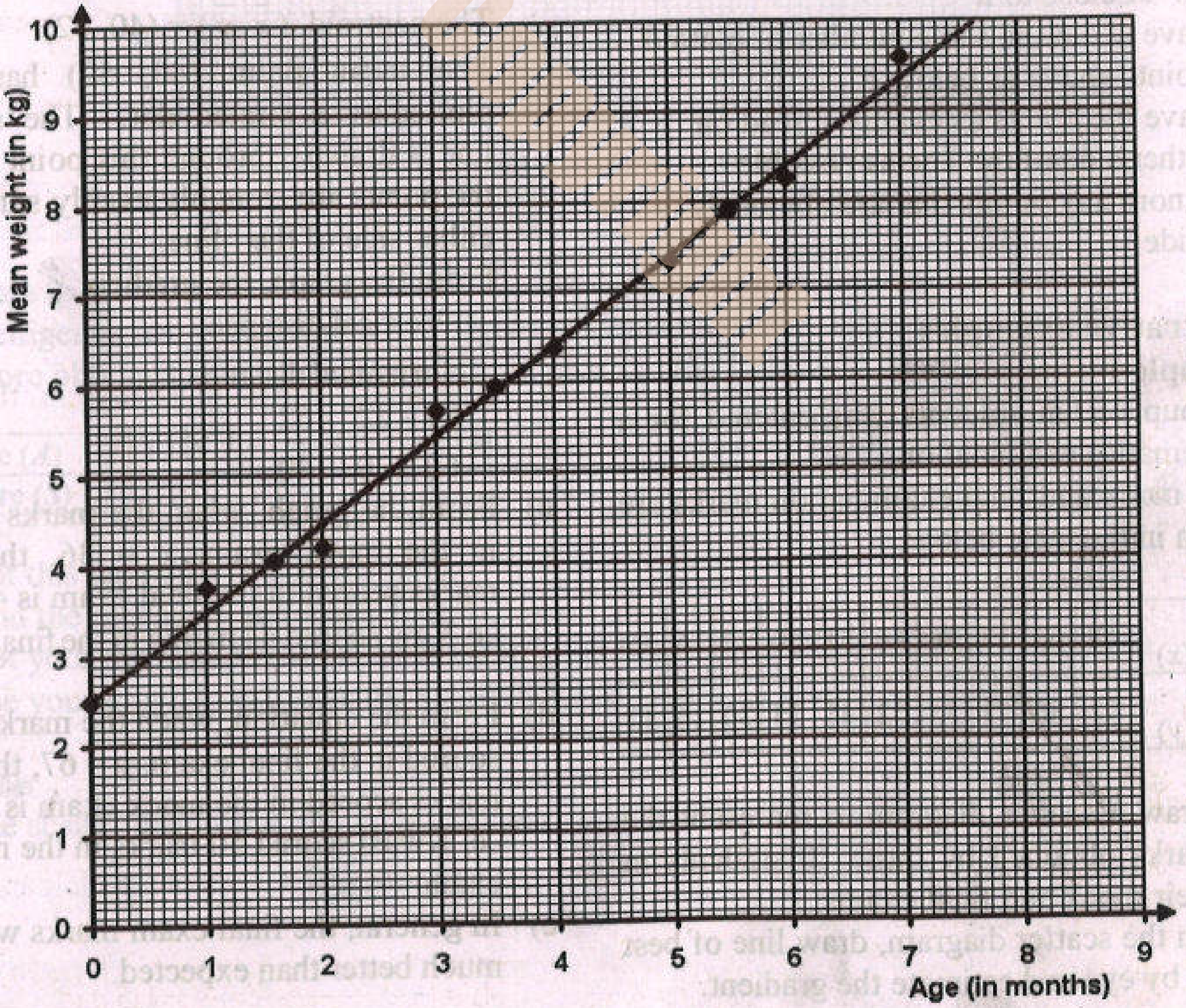
$\therefore\ (\overline{A}_2, \overline{W}_2) = (5.5, 7.9)$

iv) Plot the points (3.5, 5.97), (1.5, 4.04) and (5.5, 7.9) on the scatter diagram and draw a straight line through them to get the line of best fit.

The gradient of the line

$$= \frac{8-4}{5.5-1.5} = \frac{2}{2} = 1$$

c) From the graph, the mean weight of a body whose age is 5.5 months is 8 kg.



Method 2 – Drawing the line of best fit by eye or inspection

Another way to judge where to draw a line of best fit is simply by **eye** or **inspection**. Place a transparent ruler on the scatter diagram across the points. Move it until one of its edges fits the points best. Then you can draw the line. The main disadvantage of this method is that different people would probably draw different lines using the same data.

Use the checklist below to get the 'best fit'.

The line of best fit should:

- pass through the mean point or centroid
- match the slope shown by the plotted points
- should have as many points as possible on or close to it
- have about the same number of plotted points above as below it
- have the points as evenly spread on either side of the line as possible
- ignore any points that are obviously wide

Illustrative examples

Example 1

A group of ten students was entered for a mathematics examination.

Their mock and final examination marks are shown in the table below.

Mock exam (x)	68	63	53	36	46	31	30	22	16	35
Final exam (y)	92	80	73	55	52	45	35	30	16	42

a) Draw a scatter diagram to represent the marks obtained by these 10 students in their mock and final exams.

b) On the scatter diagram, draw line of best fit by eye and estimate the gradient.

c) Yaw was absent for the final exam but scored 36 marks in his mock exam. Estimate the mark he might obtain for his final exam.

d) Kwasi scored 67 marks in his final exam but did not take the mock exam. Estimate the mark he might have been awarded in his mock exam.

e) Comment on the mock and final exam marks for this group of ten students.

Solution

a) The scatter diagram is shown below.

b) The mean of all x, $\bar{x} = \frac{\sum x}{n} = \frac{400}{10} = 40$

and the mean of all y, $\bar{y} = \frac{\sum y}{n} = \frac{520}{10}$

$= 52$

$\therefore$ The centroid, $(\bar{x}, \bar{y}) = (40, 52)$

The mean point (40, 52) has been plotted as a circled dot. The straight line is drawn through this point so that the points are roughly evenly spread on either side of the line.

From the graph, the gradient

$$= \frac{\text{Vertical distance}}{\text{Horizontal distance}}$$

$$= \frac{84-22}{60-20} = \frac{62}{40} = 1.55$$

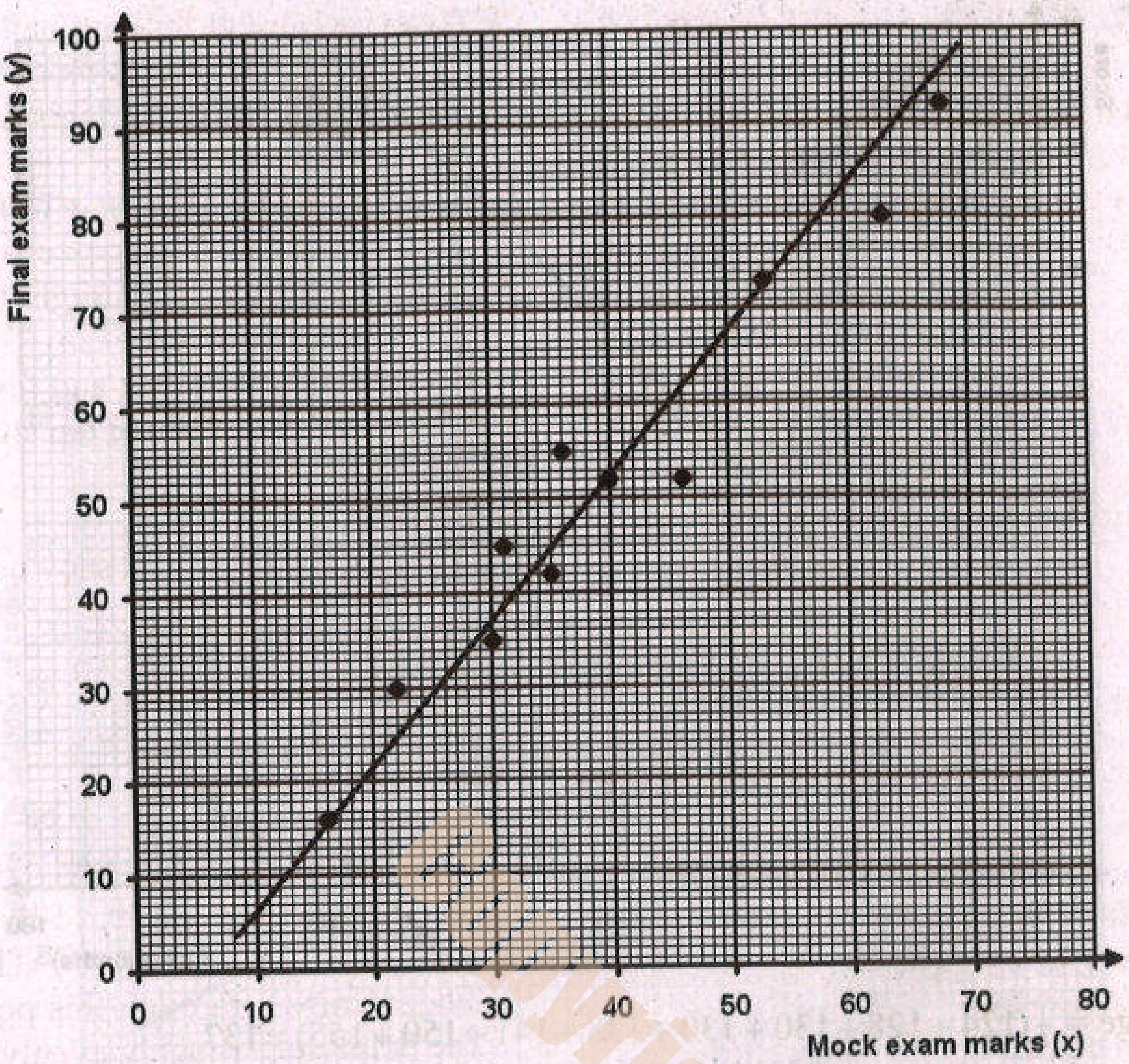
c) From the graph, when the marks scored in the mock exam, $x = 36$, then the marks scored in the final exam is 44.

i.e. Yaw scored 44 marks in the final exam.

d) From the equation, when the marks scored in the final exam, $y = 67$, then the marks scored in the mock exam is 50.

i.e. Kwasi scored 50 marks in the mock exam.

e) In general, the final exam marks were much better than expected.



Example 2

An intelligence test was given to 8 different students of various ages (in months). The score obtained (*S*) and their age (*A*) are given in the following table.

Age (*A*)	124	128	130	130	138	141	150	155
Score (*S*)	70	73	75	80	82	85	90	90

a) Plot this data on a scatter diagram.

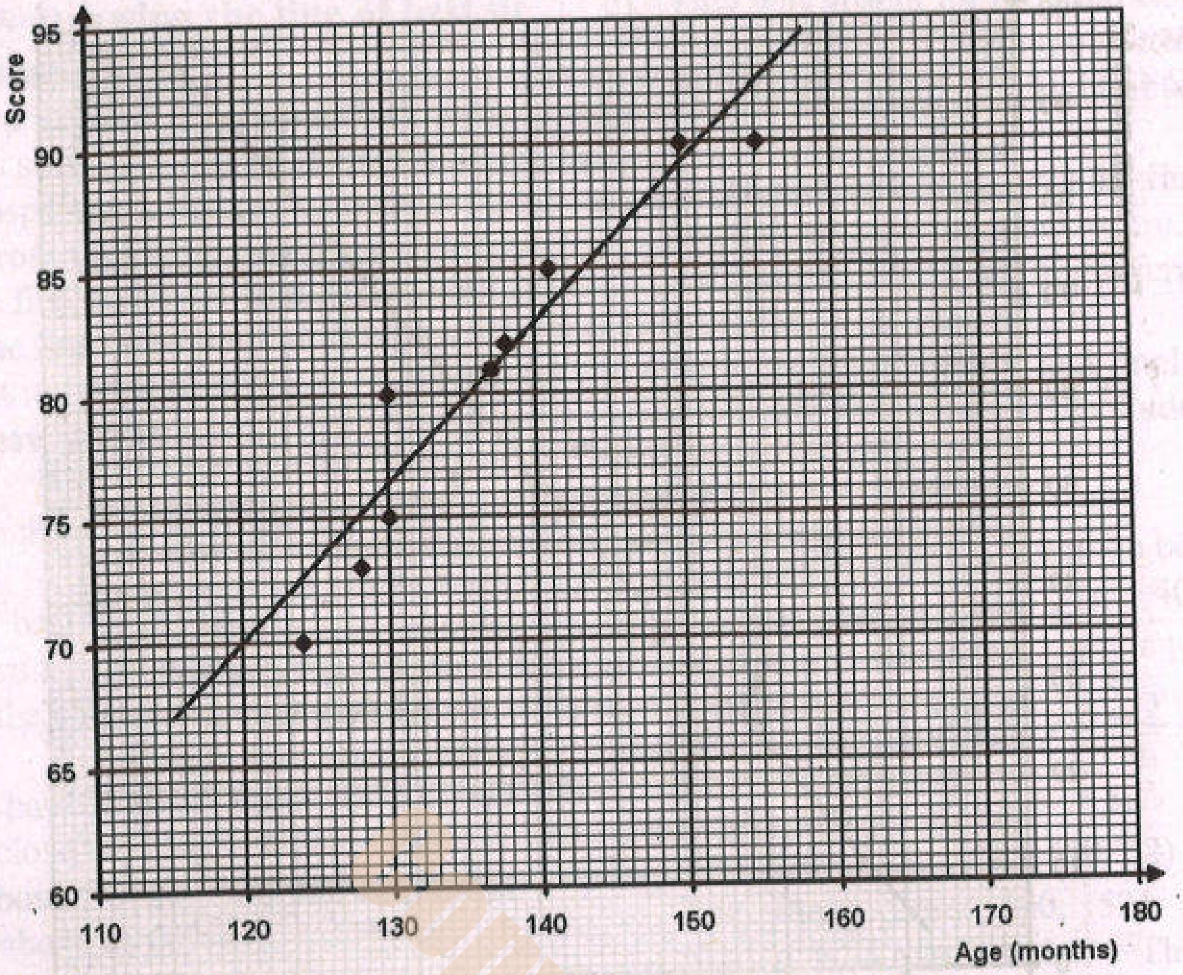
b) Find the mean age and mean score.

c) Plot your answers in (b) on the diagram, and hence draw the line of best fit by eye.

d) Use your graph to estimate the score of a pupil aged 12 years

Solution

a) The scatter diagram is shown in the next page



b) Mean age $= \frac{1}{8}(124+128+130+130+138+141+150+155) = 137$

Mean score $= \frac{1}{8}(70+73+75+80+82+85+90+90) = 80.6$

c) The mean point (137, 80.6) has been plotted as a circled dot. The straight line is drawn through this point so that the points are roughly evenly spread on either side of the line.

d) 12 years = 144 months.

From the graph, 144 months corresponds to a score of 85.

Example 3

The table below shows the annual salaries of some employees and their years of experience in a company.

Employees	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Salaries (in millions of cedis) (y)	22	23	38	33	24	22	28	26
Experience in years (x)	12	11	17	30	32	26	24	21

a) Draw a scatter diagram for the data.

b) Calculate the means $\bar{x}$ of x and $\bar{y}$ of y and plot $(\bar{x}, \bar{y})$ on the scatter diagram.

c) Draw a line of best fit passing through $(\bar{x}, \bar{y})$.

d) Use your line of best fit to estimate the annual salary of an employee with 13 years of experience

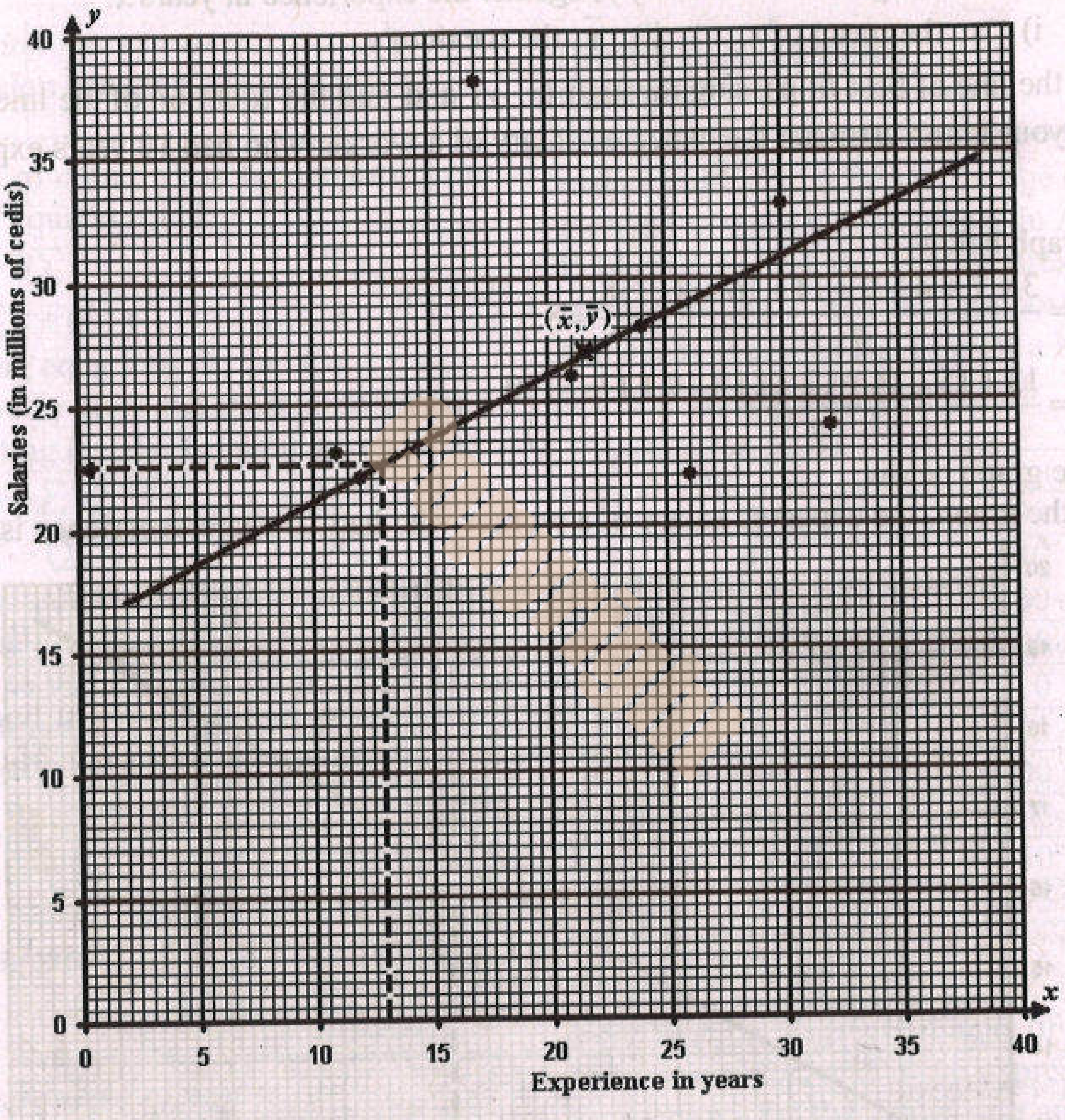
Solution

a) The scatter diagram is shown below

b) $\bar{x}=\dfrac{22+23+38+33+24+22+28+26}{8}=27,$

$\bar{y}=\dfrac{12+11+17+30+32+26+24+21}{8}=21.625$

$\therefore\ (\bar{x},\bar{y})=(21.625,\ 27)$



From the graph, the annual salary of an employee with 13 years of experience is 22.5 million

Example 4

The data in the table below shows the years of experience and the salary of seven workers selected at random in an industry.

Worker	A	B	C	D	E	F	G
Years of experience (x)	3	7	4	9	11	16	8
Salary (¢,000) (y)	13	15	14	17	15	19	17

a) Draw a scatter diagram of the salary y, against the experience in years x.

b) Find i) $\bar{x}$, the mean of x. ii) $\bar{y}$, the mean of y.

c) Draw the line of best fit passing through $(\bar{x}, \bar{y})$ and find the equation of the line.

d) From your graph estimate the expected salary of a worker who had 10 years experience.

Solution

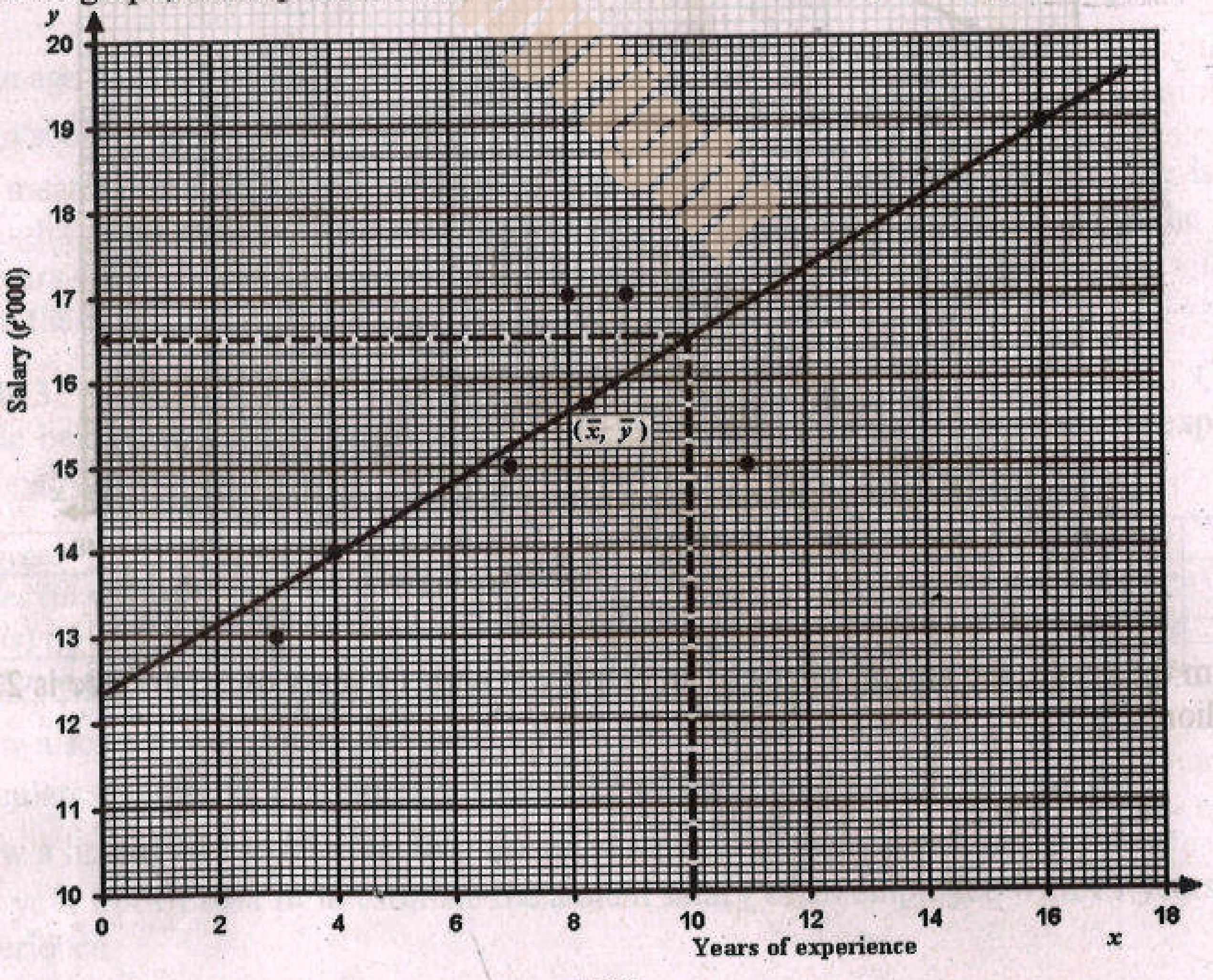
a) See graph below

b) i) $\bar{x} = \dfrac{3+7+4+9+11+16+8}{7} = \dfrac{58}{7} = 8.3$

ii) $\bar{y} = \dfrac{13+15+14+17+15+19+17}{7} = \dfrac{110}{7} = 15.7$

c) See the graph below.

d) From the graph, the expected salary of a worker who had 10 years experience is ¢16500.



Method 3 – Using least square method

The most popular method for fitting a line to a set of data is the least squares method. This method can be used to obtain the regression line of y on x or the regression line of x on y instead of using the methods described earlier. The least squares method is the most accurate.

1. *Regression line of y on x*

The regression equation of y on x is :

$y = a + bx$ ....(1)

The values of a and b can be obtained from the *normal* equations below.

$\sum y = an + b\sum x$ ....(2)

$\sum xy = a\sum x + b\sum x^2$ ....(3)

Now solving equations (2) and (3) simultaneously gives the shortcut formulae for calculating b and a from the given data:

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2} \quad \text{and} \quad a = \frac{\sum y}{n} - b\frac{\sum x}{n}$$

There are alternative formulae for determining a and b but the above formulae are simple to use. As you can see from the shortcut formulae, to estimate the least squares coefficients a and b, we need to determine the following summations:

Sum of x: $\sum x$

Sum of y : $\sum y$

Sum of x-squared : $\sum x^2$

Sum of x times y : $\sum xy$

2. *Regression line of x on y*

The regression equation of x on y is :

$x = c + dy$ ....(1)

The values of c and d can be obtained from the *normal* equations below.

$\sum x = cn + d\sum y$ ....(2)

$\sum xy = c\sum y + d\sum y^2$ ....(3)

Now solving equations (2) and (3) simultaneously gives the shortcut formulae for calculating d and c from the given data:

$$d = \frac{n\sum xy - \sum x \sum y}{n\sum y^2 - (\sum y)^2} \quad \text{and}$$

$$c = \frac{\sum x}{n} - d\frac{\sum y}{n}$$

Illustrative examples

Example 1

The following table relates the weekly maintenance cost (Gh¢) to the age (in months) of ten Kord machines in AKI-OLA Publications. Find the least squares regression line of maintenance cost on age and use it to predict the cost for a Kord machine which is 40 months old.

Age (x)	Cost (y)
5	190
10	240
15	250
20	300
30	310
30	335
30	300
50	300
50	350
60	395

Solution

Note that the regression line of y on x is being asked for; and if we express it in the form $y = a + bx$, the values of a and b can be determined using the formulae given.

Set out your calculation in the following standard layout.

x	y	xy	x^2
5	190	950	25
10	240	2400	100
15	250	3750	225
20	300	6000	400
30	310	9300	900
30	335	10050	900
30	300	9000	900
50	300	15000	2500
50	350	17500	2500
60	395	23700	3600
300	**2970**	**97650**	**12050**

Where $n = 10$

Sum of x : $\sum x = 300$

Sum of y : $\sum y = 2970$

Sum of x-squared : $\sum x^2 = 12050$

Sum of x times y : $\sum xy = 97650$

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2}$$

$$= \frac{10 \times 97650 - 300 \times 2970}{10 \times 12050 - 300^2}$$

$$= \frac{85500}{30500} = 2.8033$$

Also

$$a = \frac{\sum y}{n} - b\frac{\sum x}{n} = \frac{2970}{10} - 2.8033 \times \frac{300}{10}$$

$$= 212.90$$

Therefore the least squares regression line of cost on age is $y = 2212.90 + 2.8033x$

This can now be used to estimate the maintenance cost associated with a machine of age 40 months. Thus putting $x = 40$ gives estimated maintenance cost;

$y = 212.90 + 2.8033(40) =$ Gh¢325

Example 2

a) Draw a scatter diagram for the following pairs of values of x and y.

x	y
1	3
2	3
2	6
3	3
3	6
3	9
4	3
4	6
4	9
4	12

b) Calculate and draw the least squares regression line of

i) y on x ii) x on y

Solution

a) Plot the scatter diagram on your graph sheet.

b) Set out your calculation in the following standard layout

x	y	xy	x^2	y^2
1	3	3	1	9
2	3	6	4	9
2	6	12	4	36
3	3	9	9	9
3	6	18	9	36
3	9	27	9	81
4	3	12	16	9
4	6	24	16	36
4	9	36	16	81
4	12	48	16	144

Where $n = 10$; $\sum x = 30$; $\sum y = 60$

$\sum xy = 195$; $\sum x^2 = 100$; $\sum y^2 = 450$

i) For regression line of y on x

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2} = \frac{10 \times 195 - 30 \times 60}{10 \times 100 - 30^2}$$

$$= \frac{150}{100} = 1.5$$

Also $a = \frac{\sum y}{n} - b\frac{\sum x}{n} = \frac{60}{10} - 1.5 \times \frac{30}{10} = 1.5$

Therefore the least squares regression line of y on x is $y = 1.5 + 1.5x$

Draw this line on your scatter diagram and label it clearly.

ii) For regression line of x on y

$$d = \frac{n\sum xy - \sum x \sum y}{n\sum y^2 - \left(\sum y\right)^2} = \frac{10 \times 195 - 30 \times 60}{10 \times 450 - 60^2}$$

$$= \frac{150}{90} = 1.67$$

Also $c = \frac{\sum x}{n} - d\frac{\sum y}{n} = \frac{30}{10} - 1.67 \times \frac{60}{10} = -7$

Therefore the least squares regression line of x on y is $x = -7 + 1.67y$

Draw this line on your scatter diagram and label it clearly.

Rank correlation – Spearman's rank correlation

An alternative method of measuring correlation, based on the *ranks* of the sizes of item values is known as *rank correlation*. The measure of rank correlation most commonly used is known as *Spearman's* rank correlation coefficient. Given a set of paired data (x, y), Spearman's rank correlation coefficient, r_s, is obtained by ranking the x's among themselves usually from high to low and also ranking the y's among themselves from high to low. We next find differences between the ranks (d).

The **Spearman's rank correlation coefficient** is then given by the formula:

$$r_s = 1 - \frac{6\sum d^2}{n(n^2-1)}, \text{ where}$$

d = the rank difference for each pair of values

n = the number of pairs of data

Note: The value of r_s lies between –1 and 1.

$r_s = 1$ means positive perfect correlation.

$r_s = -1$, means negative perfect correlation.

Interpretation of correlation coefficients

If r_s lies within

0.0 – 0.2 it implies very weak or negligible correlation.

0.2 – 0.4 it implies weak or low correlation.

0.4 – 0.7 it implies moderate correlation.

0.7 – 0.9 it implies strong or high or marked correlation.

0.9 – 1.0 it implies very strong or very high correlation.

The procedure for obtaining Spearman's rank correlation coefficient is given in the examples below.

Illustrative examples

Example 1

Calculate the Spearman's rank correlation coefficient for the data below.

French (x)	12	24	30	34	47	58	68
German (y)	32	44	47	58	73	72	88

Solution

The standard layout for calculations is shown below.

Step 1. Rank the x values to give R_x values

Step 2. Rank the y values to give R_y values

Note that the rankings of the x values are performed independently of the rankings of the y values and ranking is done in descending order (i.e. rank 1 to the largest item, rank 2 to the next largest item and so on). You can also rank in ascending order (i.e. rank 1 to the smallest item, rank 2 to the next smallest item and so on).

Step 3. For each pair of ranks, calculate

$$d = R_x - R_y \text{ and } d^2$$

Step 4. Calculate $\sum d^2$

Step 5. The value of the rank correlation can then be found using the formula.

x	y	Rank of x (R_x)	Rank of y (R_y)	$d=R_x-R_y$	d^2
12	32	7	7	0	0
24	44	6	6	0	0
30	47	5	5	0	0
34	58	4	4	0	0
47	73	3	2	1	1
58	72	2	3	–1	1
68	88	1	1	0	0

$\sum d^2 = 2$ and n = number of pairs of data $= 7$

Spearman's rank correlation coefficient,

$$r_s = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6\times 2}{7(49-1)}$$

$$= 1 - \frac{12}{7\times 48} = 1 - \tfrac{1}{28} = \tfrac{27}{28} = 0.964$$

Therefore there exists a very strong correlation between French and German.

Example 2

The grade of nine students on a mid-term report (x) and final report (y) are as follows.

x	77	54	71	72	81	94	96	99	67
y	82	66	78	34	47	88	99	98	68

Calculate the Spearman's rank correlation coefficient

Solution

Ranking the data (highest as first), we have

x	y	Rank of x (R_x)	Rank of y (R_y)	$d=R_x-R_y$	d^2
77	82	5	4	1	1
54	66	9	7	2	4
71	78	7	5	2	4
72	34	6	9	–3	9
81	47	4	8	–4	16
94	88	3	3	0	0
96	99	2	1	1	1
99	98	1	2	–1	1
67	68	8	6	2	4

$\sum d^2 = 40$ and n = number of pairs of data $= 9$

Spearman's rank correlation coefficient,

$$r_s = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

$$= 1 - \frac{6\times 40}{9(81-1)}$$

$$= 1 - \frac{12}{9\times 80}$$

$$= 1 - \tfrac{1}{60} = \tfrac{59}{60} = 0.983$$

Therefore there exists a very strong correlation between mid-term report (x) and final report (y). Thus the higher the mark obtained in the mid-term report the higher the mark obtained in the final report.

Example 3

The heights and masses of 10 people are shown in the table below.

Height (cm) (x)	Mass (kg) (y)
130	30
139	48
146	38
151	53
157	44
160	37
162	52
163	68
169	61
172	59

Find out if there is any relationship between the heights and masses of the people by calculating Spearman's rank correlation coefficient.

Solution

$r_s = 0.73$

This means there is good correlation between height and weight.

Example 4

The weight x kg and height y m of 10 athletes are given in the table below.

Weight (kg) (x)	Height (m) (y)
62	1.75
58	1.80
66	1.93
52	1.65
71	1.87
60	1.71
75	1.98
53	1.68
69	1.84
54	1.79

a) Draw a scatter diagram of the data.

b) If $\bar{x}$ is the mean of the weights and $\bar{y}$ is the mean of the heights

  i) find $\bar{x}$ and $\bar{y}$

  ii) draw the line of best fit to pass through $(\bar{x}, \bar{y})$

c) Use your line to predict the weight of an athlete whose height is 2.00 m

d) Calculate the coefficient of rank correlation between x and y and comment on your result.

Solution

a) Draw the scatter diagram on a graph sheet.

b) i) The mean of all weights (x),

$$\bar{x} = \frac{\sum x}{n} = \frac{620}{10} = 62 \text{ and}$$

The mean of all heights (y),

$$\bar{y} = \frac{\sum y}{n} = \frac{18.0}{10} = 1.8$$

∴ The centroid, $(\bar{x}, \bar{y}) = (62, 1.8)$

ii) Find the mean of all x-values which are less than $\bar{x}$ (i.e. 62), $\bar{x}_1$ and the mean of the corresponding y-values, $\bar{y}_1$ ∴ $(\bar{x}_1, \bar{y}_1) = ?$

iii) Find the mean of all x-values which are greater than $\bar{x}$, $\bar{x}_2$ and the mean of the corresponding y-values, $\bar{y}_2$

∴ $(\bar{x}_2, \bar{y}_2) = ?$

iv) Plot the three points determined above on the scatter diagram and draw a straight line through them to get the line of best fit.

c) Find the weight when the height is 2.00 from the graph.

d) $r_s = 0.867$

This means there is good correlation between height and weight.

Comment: There exists a strong relationship between the height and weight of an athlete. Thus the taller the athlete, the heavier he is.

[Work out the solution]

Example 5

Two judges in a flower show had to place 8 flower arrangements A, B, C, D, E, F, G, H in order of merit. The results of each of the judges were as follows:

Order	1st Judge	2nd Judge
1st	G	G
2nd	F	D
3rd	D	F
4th	E	B
5th	A	H
6th	B	E
7th	H	A
8th	C	C

Calculate the coefficient of rank correlation for the assessment of the two judges.

Solution

Order	Rank of 1st Judge (R_x)	Rank of 2nd Judge (R_x)	$d = R_x - R_y$	d^2
A	5	7	−2	4
B	6	4	2	4
C	8	8	0	0
D	3	2	1	1
E	4	6	−2	4
F	2	3	−1	1
G	1	1	0	0
H	7	5	2	4

From the table, $\sum d^2 = 18$ and $n = 8$

Spearman's rank correlation coefficient,

$$r_s = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

$$= 1 - \frac{6\times 18}{8(64-1)}$$

$$= 1 - \frac{6\times 18}{8\times 63} = 1 - 0.214 = 0.786$$

Data with tied values

If one or more groups of data items have the same value (known as *tied values*), the ranks that would have been allocated separately to each must be averaged and this average rank given to each item with this equal value. For example, the numbers 9, 18, 18 and 25 would be allocated ranks 4, 2.5, 2.5 and 1 respectively. This is because two items have value 18 and each must be allocated the average of ranks 2 and 3 i.e. rank= $\frac{2+3}{2}$ = 2.5. Similarly, the numbers 12, 10, 12, 12, 13, 15 would be allocated ranks 4, 6, 4, 4, 2 and 1 respectively. This is because three items have value 12 and each must be allocated the average of ranks 3, 4 and 5 i.e. rank= $\frac{3+4+5}{3}$ = 4. The following example shows how to calculate rank correlation coefficient for data with tied values.

Illustrative example

The following data relate to the number of vehicles owned by and road deaths for the populations of 12 countries.

Vehicles (per 100 population), x	Road deaths (per 100,000 population), y
30	30
31	14
32	30
30	23
46	32
30	26
19	20
35	21
40	23
46	30
57	35
30	26

Calculate Spearman's rank correlation coefficient for the data, giving your answer to 2 decimal places and comment on the result.

Solution

x	y	Rank of x (R_x)	Rank of y (R_y)	d	d^2
30	30	9.5	4	5.5	30.25
31	14	7	12	−5	25
32	30	6	4	2	4
30	23	9.5	8.5	1	1
46	32	2.5	2	0.5	0.25
30	26	9.5	6.5	3	9
19	20	12	11	1	1
35	21	5	10	−5	25
40	23	4	8.5	−4.5	20.25
46	30	2.5	4	−1.5	2.25
57	35	1	1	0	0
30	26	9.5	6.5	3	9

Note 1: For the x values, two items have value 46 and each must be allocated the average of the ranks 2 and 3
i.e. rank $= \frac{2+3}{2} = 2.5$.

Similarly, four items have value 30 and each must be allocated the average of the ranks 8, 9, 10 and 11 i.e. rank $= \frac{8+9+10+11}{4} = 9.5$.

Note 2: For the y values, two items have value 26 and each must be allocated the average of the ranks 6 and 7
i.e. rank $= \frac{6+7}{2} = 6.5$.

Similarly, two items have value 23 and each must be allocated the average of the ranks 8 and 9 i.e. rank $= \frac{8+9}{2} = 8.5$.

From the table, $\sum d^2 = 127$ and $n = 12$
Spearman's rank correlation coefficient,

$$r_s = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

$$= 1 - \frac{6 \times 127}{12(144-1)}$$

$$= 1 - \frac{6 \times 127}{12 \times 143}$$

$$= 0.56$$

The results show a moderate degree of positive correlation between x and y showing that there is some correlation between vehicles owned and road deaths, but the relationship is not strong.

Review exercise 30

1. Construct a scatter diagram of the following pair of values of x and y.

x	1	2	3	4	5	6	7
y	2	1	4	7	5	6	7

a) Draw the regression line of y on x.
b) Find the equation of best fit and use it to predict the value of y when $x = 4.5$

2. Eight candidates sat for examinations in Mathematics and Physics.
Their corresponding marks were:

Mathematics (x)	Physics (y)
63	48
72	71
41	50
56	46
44	35
89	92
70	42
45	48

i) Plot these points on a scatter diagram
ii) Draw the regression line of y on x and find the equation of the regression line of y on x.
iii) Find the mark scored in Physics if the mark scored in Mathematics is 90
iv) Calculate Spearman's rank correlation coefficient for the data.

3. In an investigation into prediction using stars, a celebrated astrologist predicted the ages at which thirteen young people would first marry. The data of predicted and actual ages in years at first marriage is summarised in the table below.

Predicted age (x)	Actual age (y)
24	23
30	31
28	28
36	35
20	20
22	25
31	45
28	30
21	22
29	27
40	40
25	27
27	26

i) Draw a scatter diagram of these data.
ii) Draw the regression line of y on x and find the equation of the regression line of y on x.
iii) Find the actual age when the predicted age is 45.
iv) Calculate Spearman's rank correlation coefficient for the data.

4. An aptitude test was given to 10 sales staff and the results together with their annual sales are shown in the table below.

Aptitude score	Annual sales (¢000's)
8	20
9	25
12	30
11	29
8	22
11	28
14	32
16	40
12	25
9	26

i) Draw a scatter diagram to illustrate this information.

ii) Draw the line of best fit by eye.

iii) Estimate the annual sales of a new employee who scored 10 on the aptitude test.

iv) Calculate Spearman's rank correlation coefficient for the data.

5. Ten candidates sat for examinations in Mathematics and Physics.
Their corresponding marks were:

Mathematics(x)	Physics (y)
82	75
78	80
86	93
72	65
80	71
95	98
72	68
89	84
74	77
91	87

a) Draw a scatter diagram of the Mathematics, x, against Physics, y.

b) Draw the line of best fit of y on x and estimate the gradient.

c) From your graph, estimate the Physics mark of a student who obtains a mark of 75 in Mathematics.

d) Calculate Spearman's rank correlation coefficient for the data.

6. The heights and masses of 10 people are shown in the table below.

Height (cm) (x)	Mass (kg) (y)
130	30
139	48
146	38
151	53
157	44
160	37
162	52
163	68
169	61
172	59

a) Draw a scatter diagram of the weight, x, against the height, y.

b) Draw the line of best fit of y on x and estimate the gradient.

c) From your graph, estimate the height of a person if he weighs 50 kg.

d) Calculate Spearman's rank correlation coefficient for the data, giving your answer to 3 significant figures.

7. The length of service and the gross annual earnings of eleven employees of a company were as follows:

Length of service(months)	Gross earnings (million cedis)
14	121
18	117
36	124
24	118
13	104
83	60
108	74
41	52
33	54
17	47
79	64

Calculate Spearman's rank correlation coefficient for the data, giving your answer to 3 significant figures.

8. Calculate the Spearman's rank correlation coefficient for the data below, giving your answer to 2 decimal places.

x	20	25	50	35	40	15
y	5	15	30	25	20	10

Comment on your answer.

9. The following table shows the heights x and hip sizes of some cows.

x/m	y/m
1.5	0.27
1.4	0.30
1.5	0.24
1.6	0.22
1.7	0.28
1.3	0.29
1.5	0.22
1.5	0.22
1.4	0.24
1.7	0.25

a) Calculate Spearman's rank correlation coefficient between the heights and hip sizes

b) By using normal equations, calculate the equation of the line of regression of x on y given numerical constants correct to two decimal places.

c) Use the line of regression to estimate, correct to two decimal places, the hip size of a cow whose height is 1.5 m

10. The following table gives the heights x and the masses y of a sample of 12 male students drawn at random from the first students of a certain school.

x/m	y/m
1.40	77.5
1.26	75.5
1.44	90.0
1.20	67.0
1.32	78.0
1.40	84.0
1.48	89.0
1.30	80.0
1.24	66.0
1.34	72.5
1.30	69.5
1.36	76.0

You are given that $\sum x = 16.04$;

$\sum y = 925$

$\sum xy = 1240$ and $\sum x^2 = 21.52$

a) Calculate the equation of the line of regression of y on x given numerical constants correct to two significant figures.

b) Use the line of regression to estimate the mass of a student of height 1.38 m.

c) Calculate Spearman's rank correlation coefficient between the heights and masses. Interpret your results.

11. The table below shows the values of two variables x and y obtained from a survey.

x	11	12	14	15	16	18	19	20
y	81	73	53	53	43	29	15	9

a) i) Represent this information on a scatter diagram.

ii) Find $\bar{x}$ and $\bar{y}$, the means of x and y respectively.

iii) Draw the line of best fit to pass through $(\bar{x}, \bar{y})$.

b) Find the equation of the line in (a)(iii).

c) Use your equation in (b) to find, correct to **one** decimal place, the value of y when $x = 17$.

12. The marks of 10 students in Physics and Chemistry are in the table below:

Physics (x)	15	13	14	10	12	8	10	8	10	8
Chemistry (y)	20	12	18	10	6	12	14	7	3	1

a) Plot a scatter diagram to represent this information.

b) Find $\bar{x}$, the mean of x and $\bar{y}$, the mean of y and plot $(\bar{x}, \bar{y})$ on the diagram.

c) Draw the line of best fit to pass through $(\bar{x}, \bar{y})$.

d) From your graph, determine

i) the relationship between x and y;

ii) the mark obtained in Chemistry by a student who obtained 9 in Physics.

13. The following table shows the marks obtained by some students in the two papers of a subject.

Paper 1 (x)	51	45	23	47	18	31	27	30	21	23
Paper 2 (y)	41	38	11	28	12	14	14	17	15	17

a) Plot a scatter diagram to represent the information.

b) Find $\bar{x}$, the mean of x and $\bar{y}$, the mean of y.

c) Draw the line of best fit to pass through $(\bar{x}, \bar{y})$.

d) From your graph determine

i) the relationship between x and y;

ii) the value of y when x is 25.

CHAPTER 31

PERMUTATIONS AND COMBINATIONS

Introduction

In this topic, we are concerned with the practical problem of choosing a number of objects from a given set of objects.
Depending on the purpose for which the objects are chosen, the number of choices available to us varies. Let us understand this by the following example.
Suppose there are four students *a*, *b*, *c*, *d*. The teacher wants to select two of them. If we ask: 'In how many ways can this be done?', the question cannot be answered unless we are told the purpose behind selecting these students.
Suppose the two students are to be selected to play a game of table tennis against each other, then we have the following six possible pairs:
ab, *ac*, *ad*, *bc*, *bd*, *cd*. Hence, six different games can be played.

Now suppose these students obtain equal marks in Mathematics and two prizes (first prize Gh¢1000 and second prize, Gh¢500) are to be awarded to two of them by drawing lots. Then the number of choices is not six as in the last case. This is because *a* and *b* can play only one game with each other but if *a* and *b* are selected for the two prizes then the order in which they are selected is important. This happens because '*a*' getting first prize and '*b*' getting second prize is different from '*b*' getting first prize and '*a*' getting second prize even though the same pair of students is involved. Hence, each of the six pairs given above will give rise to two choices. Hence the prizes can be awarded in twelve different ways.

The above cases clearly illustrates that, even though we were asked to select two students out of four given students in both cases, the number of choices is depended on whether the order in which the students are selected is important or not important.

When a choice is made disregarding the order of selection, it is called **combination** and when the order is important, it is called a **permutation**.

Factorial notation

In dealing with problems of permutations and combinations, the following notation is important.

$n! = n \times (n-1) \times (n-2) \ldots 3 \times 2 \times 1$.

Thus the product of all integers from *n* to 1 inclusive.

n! is read **factorial *n*** or ***n* factorial**.

For example,

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ and

$3! = 3 \times 2 \times 1 = 6$

Also $15! = 15 \times 14 \times 13 \times 12!$

$(n+1)! = (n+1) \times n \times (n-1) \times (n-2)!$

$(n-2)! = (n-2) \times (n-3) \times (n-4)!$

$9! = 9 \times 8 \times 7!$ etc

$$\frac{12!}{9!} = \frac{12 \times 11 \times 10 \times 9!}{9!}$$

$$= 12 \times 11 \times 10$$

By definition, $0! = 1$.

Permutations

A **permutation** is an *ordered* arrangement of items or objects.

Permutation of *n* different objects

The number of permutations of *n* **different** objects taken *r* at a time is written as ^nP_r or $_nP_r$, where: $^nP_r = \dfrac{n!}{(n-r)!}$

Therefore the number of permutations of *n* different objects taken *n* (i.e. all) at a time is

$$^nP_n = \frac{n!}{(n-n)!} = \frac{n}{0!} = n! \quad \text{since } 0! = 1$$

When applying the formula for ^nP_r in questions, we must make sure that:

1. the choice to be made is a permutation and not a combination
2. once an item is chosen it cannot be repeated. For example, if we have to form a number of 3 digits using digits 1, 2, 3, 4, 5 then, if repetition of digits is allowed, we cannot use the formula of ^nP_r.

Illustrative examples

Example 1

In how many ways can the letters *A*, *B*, *C* be arranged in a row?

Solution

The letters can be arranged as *ABC*, *ACB*, *BAC*, *BCA*, *CAB*, *CBA* (taking all at a time) i.e. 6 ways.

Note that this is the same as arranging 3 different objects taking all 3 at a time

i.e. $^3P_3 = 3! = 3 \times 2 \times 1 = 6$ ways

Example 2

In how many ways can the letters *A*, *B*, *C* be arranged taken two at a time.

Solution

This is the same as arranging 3 different objects taking all 2 at a time.

i.e. $^3P_2 = \dfrac{3!}{(3-2)!} = \dfrac{3!}{1!} = 3 \times 2 \times 1 = 6$ ways

Example 3

In how many ways can the letters of *MATHS* be arranged?

Solution

Note that this is the same as arranging 5 different objects taking all 5 at a time

i.e. $^5P_5 = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways

Example 4

In how many ways can the letters of *MEASURING* be arranged?

Solution

Note that this is the same as arranging 9 different objects taking all 9 at a time

i.e. $^9P_9 = 9! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 362\,880$ ways

Example 5

In how many ways can 10 people be seated on a bench, if only 4 seats are available?

Solution

This is the same as arranging of 10 different objects taken 4 at a time i.e. $n = 10$ and $r = 4$.

The number of ways

$= {}^{10}P_4 = \dfrac{10!}{(10-4)!} = \dfrac{10!}{6!}$

$= 10 \times 9 \times 8 \times 7 = 5040$ ways

Example 6

In how many ways can 5 different objects be arranged taking 2 at a time?

Solution

The number of ways = $^5P_2 = \dfrac{5!}{(5-2)!}$

$$= \frac{5!}{3!} = 5 \times 4 = 20 \text{ ways}$$

Example 7

In the managing of a committee of a society, there are nine members. In how many ways can a chairman, a vice-chairman and a treasurer be selected from amongst them, if a person is eligible for one post only?

Solution

The number of ways = $^9P_3 = \dfrac{9!}{(9-3)!}$

$$= 9 \times 8 \times 7 = 504 \text{ ways}$$

Permutations with identical objects

1. The number of permutations of n objects taken all at a time, when p of the objects are identical and the rest are all different is given by:

   $$\frac{n!}{p!}$$

   For example, the number of ways the letters *MIME* can be arranged is

   $\dfrac{4!}{2!} = 4 \times 3 = 12$

   Note that here there are 4 letters, so $n = 4$ and 2 *M*s, so $p = 2$

2. The number of permutations of n objects taken all at a time, when p of the objects are identical and of one kind and q of them are identical and of a second kind and the rest are all different is given by:

   $$\frac{n!}{p!q!}.$$

In general, in a set of n objects, when p of them are identical, q of them are identical and r of them are identical, the number of permutations of these n objects taken all at a time is given by:

$$\frac{n!}{p!q!r!}$$

Illustrative examples

Example 1

In how many ways can the letters of the word *MAXIMUM* be arranged, when all are taken at a time.

Solution

There are 7 letters so $n = 7$. Of these, we have 3 *M*s and all the remaining letters are different.

Hence the required number of arrangements

$= \dfrac{7!}{3!} = 840.$

Example 2

In how many ways can the letters of the word *STATISTICS* be arranged, when all are taken at a time.

Solution

There are 10 letters so $n = 10$. Of these, we have 3 *S*s, 3 *T*s, 2 *I*s and all the remaining letters being different.

Hence the required number of arrangements

$= \dfrac{10!}{3!3!2!} = 50\,400.$

Example 3

Find the number of ways the letters of the word *PARALLEL* can be arranged, when all are taken at a time.

Solution

There are 8 letters so $n = 8$. Of these, we have 2 *A*s and 3 *L*s

Hence the required number of arrangements

$$= \frac{8!}{2!\,3!} = 3360.$$

Example 4

In how many ways can the letters of the word *SYNTHESIS* be arranged, when all are taken at a time?

Solution

There are 9 letters so $n = 9$. Of these, we have 3 *S*s and all the remaining letters are different.

Hence the required number of ways

$$= \frac{9!}{3!} = 60\,480.$$

Example 5

In how many ways can the letters of the word *OPTIMIZATION* be arranged?

Solution

There are 12 letters so $n = 12$. Of these, we have 2 *O*s, 2 *T*s, 3 *I*s and all the remaining letters being different. Hence the required number of ways $= \frac{12!}{2!\,2!\,3!} = 19958400$.

Example 6

How many different arrangements of the letter of the word *COMMISSION* are possible?

Solution

226800 Work out the missing steps

Conditional permutations

In certain permutations, the possible arrangements of the objects are restricted in some way or there are conditions imposed on some of the objects as in the following examples.

Illustrative examples

Example 1

In how many ways can the letters of the word *LETTER* be arranged amongst themselves? In how many arrangements will the two *T*'s be together?

Solution

First part

There are 6 letters, including 2 *T*s and 2 *E*s.

Hence the required number of arrangements

$$= \frac{6!}{2!2!} = 180.$$

Second part

If the two *T*'s must be together, they can be considered as one letter. Hence, we have 5 letters which include 2 *E*'s.

Therefore, the required number of arrangements $= \frac{5!}{2!} = 60$.

Example 2

How many different arrangements of letters can be made by using all the letters of the word *MINIMUM*? In how many of these are vowels separated?

Solution

First part

There are 7 letters, including 3 *M*s and 2 *I*s.

Hence the required number of arrangements

$$= \frac{7!}{3!\,2!} = 420.$$

Second part

Treat the three vowels *I, I, U* as one object. Now the number of arrangements of 5 letters with 3 *M*s is $\frac{5!}{3!}$.

(i.e. number of arrangements of 5 letters with 3 *M*s). Note that the 3 vowels are taken as 1 letter.

Therefore the number of arrangements with the vowels together is $3 \times \frac{5!}{3!} = 3 \times 20 = 60$.

Note that the factor 3 is introduced to allow for the three possible arrangements of the vowels among themselves (i.e. arranging the 3 vowels with 2 identical objects).

Thus the number of arrangements with the vowels separated $= 420 - 60 = 360$

Example 3

Find the number of ways in which letters of the word *SHALLOW* can be arranged if

a) the two *L*s must always be together;

b) the two *L*s must not come together.

Solution

First find the number of arrangements of the letters of the word *SHALLOW*

There are 7 letters, including 2 *L*s.

Hence the required number of arrangements $= \frac{7!}{2!} = 2520$.

a) Treat the two *L*s as one object.

Now the number of arrangements of 6 different objects is 6!

Note that the 2 *L*s are taken as 1 object.

Therefore the number of arrangements with the 2 *L*s together is $1 \times 6! = 720$

Note that the factor 1 is introduced to allow for the one possible arrangement of *LL*.

b) The number of arrangements with the 2 *L*s separated = number of arrangements of *SHALLOW* without restrictions – number of arrangements of *SHALLOW* with the 2 *L*s together $= 2520 - 720 = 1800$

Example 4

How many different arrangements of the letters of the word *BEGIN* are there which start with a vowel?

Solution

Treat the two vowels *E*, *I* as one object.

Now the number of arrangements of 4 different letters = 4!

Note that the 2 vowels are taken as 1 letter.

Therefore the number of arrangements which start with a vowel is

$$2 \times 4! = 2 \times 4 \times 3 \times 2 \times 1 = 48$$

Note that the factor 2 is introduced to allow for the two possibilities of a vowel starting.

Example 5

Find the number of ways in which letters of the word *SYLLABUS* can be arranged if

a) the two *L*'s must always be together;

b) the two *L*'s must not come together.

Solution

First find the number of arrangements of the letters of the word *SYLLABUS*.

There are 8 letters, including 2 *L*s and 2 *S*s.

Hence the required number of arrangements $= \frac{8!}{2!2!} = 10080$.

a) Treat the two *L*s as one object.

Now the number of arrangements of 7 objects with 2 *S*s is $\frac{7!}{2!}$

(i.e. arrangements of 6 different letters).

Note that the 2 *L*s are taken as 1 object.

Therefore the number of arrangements with the 2 *L*s together is $1 \times \frac{7!}{2!} = 2520$

Note that the factor 1 is introduced to allow for the one possible arrangement of *LL*.

b) The number of arrangements with the 2 *L*s separated = number of arrangements of *SYLLABUS* without any restrictions – number of arrangements of *SYLLABUS* with the 2 *L*s together $= 10080 - 2520 = 7560$

Example 6

Seven boys and two girls are to be seated in a row for a photograph so that the two girls are always together. Find the number of ways in which this can be done.

Solution

The two girls can be arranged amongst themselves in ${}^2P_2 = 2! = 2$ ways.

Now considering these two girls as a single person, we can arrange seven boys and this one person (i.e. in all 8 persons), in ${}^8P_8 = 8!$ ways. Thus, the total number of ways = 2(8!) = 80640.

Example 7

In how many ways can 5 boys and 3 girls be arranged in a row so that no two girls are adjacent to each other?

Solution

First the five boys can be arranged amongst themselves in ${}^5P_5 = 5!$ ways.

Consider the following arrangement.

$X\ B\ X\ B\ X\ B\ X\ B\ X\ B\ X$

The five boys are represented by the letter B. Around them there are six places, marked by X, such that if the three girls occupy any three of these 6 places, no two girls will be together. Therefore, the girls can be arranged in ${}^6P_3$ ways.

$$\begin{aligned}\text{Hence total number of ways} &= 5! \times {}^6P_3 \\ &= 5! \times 120 \\ &= 14400\end{aligned}$$

Example 8

Find the number of ways in which the letters of *ISOSCELES* can be arranged if the two E's are separated.

Solution

23520 [Work out the missing steps]

Cyclic permutations

1. The number of ways r objects chosen from n *different* objects can be arranged in a **circle** is: $\dfrac{n!}{(n-r)!\,r}$

2. The number of ways r objects chosen from n *different* objects can be arranged in a **ring** is: $\dfrac{n!}{2(n-r)!\,r}$

3. If $r = n$ i.e. if all n different objects are arranged in a circle then the number of ways is: $\dfrac{n!}{n} = (n-1)!$

4. If the objects are **fixed on a ring**, which can be turned over, the number of arrangements will be: $\dfrac{n!}{2n} = \dfrac{(n-1)!}{2}$

Illustrative examples

Example 1

The representatives of five countries attended a conference. In how many ways can they be seated at a round table?

Solution

The number of ways = $(5-1)! = 4! = 24$.

Example 2

In how many ways can six different coloured beads be arranged in a ring?

Solution

The number of ways = $\dfrac{(6-1)!}{2} = \dfrac{5!}{2} = 60$.

Example 3

In how many ways can five beads, chosen from eight different beads be threaded on to a ring?

Solution

Here $r = 5$ and $n = 8$

$$\text{The number of ways} = \frac{8!}{2(8-5)!\,5}$$

$$= \frac{8!}{2 \times 3! \times 5}$$

$$= \frac{8 \times 7 \times 6 \times 5 \times 4}{2 \times 5}$$

$$= 672$$

Example 4

a) In how many ways can six people be seated on a round table?

b) How many ways are there if two people must sit next to each other?

c) How many ways are there if two people must not sit next to each other?

Solution

a) The number of ways $= (6-1)!$
$= 5!$
$= 120.$

b) If two people sit next to each other. There are 2 ways of doing this, the second person being on the left or right of the first.
The third person has a choice of 4 places, the fourth person has a choice of 3 places, the fifth has 2 places and the sixth has 1 place.

The number of ways $= 2 \times 4 \times 3 \times 2 \times 1$
$= 48.$

c) The number of ways in which two people must not sit next to each other
= number of ways of arranging 6 people round a table without restriction
− number of ways in which two people must sit together
$= 120 - 48 = 72.$

Forming numbers with given digits

If the first digit of the number can be chosen in *m* different ways and after this is done, the second digit can be chosen *n* different ways, the third digit in *p* different ways etc., then the total number of ways of forming the number is $m \times n \times p \times \ldots\ldots$

Illustrative examples

Example 1

How many numbers of two different digits can be formed by using the digits 1, 2, 3, 4, 5 if repetition of digits is not allowed?

Solution

The first digit of the number can be chosen in 5 different ways. After that is done, since the second digit must be different from the first, the second digit can be chosen in 4 different ways. Hence, the number can be formed in $5 \times 4 = 20$ ways.

Alternatively, the number can be formed in

$$^5P_2 = \frac{5!}{(5-2)!} = 20 \text{ ways}$$

Example 2

How many four-digit numbers can be formed from the numbers 0, 1, 2, 3 and 4 without repetition?

Solution

Since the digit 0 cannot start a number, the first digit can be chosen from 1, 2, 3 or 4. Hence, the first digit can be chosen in 4 different ways. After that is done, since the second digit must be different from the first, the second digit can be chosen in 4 different ways (this time zero is included). The third digit can be chosen in 3 different ways and the fourth digit in 2 different ways.
Hence, the numbers can be formed in $4 \times 4 \times 3 \times 2 = 96$ ways.

Example 3

In how many different ways can we form a number of 4 digits from the digits 1, 2, 3, 4, 5, 6 if repetition is not allowed? How many of the those numbers will be greater than 3000?

Solution

The numbers can be formed in
$6 \times 5 \times 4 \times 3 = 360$ different ways
Next consider any such number. It is of the form: *abcd*. Since this number is to be greater than 3000, the first digit *a* cannot be either 1 or 2. Therefore, *a* must be chosen from 3, 4, 5 or 6. Hence *a* can be chosen in 4 ways. After this is done, the second digit can be chosen in 5 ways, the third in 4 ways and the fourth in 3 ways.
Hence, the numbers of ways in which the number formed is more than 3000 is
$4 \times 5 \times 4 \times 3 = 240$

Example 4

How many numbers of 3 digits can be formed by using the digits 1, 2, 3, 4, 5, 6, 7 if
i) repetition of digits is not allowed?
ii) repetition of digits is allowed?

Solution

i) $7 \times 6 \times 5 = 210$ numbers
ii) $7 \times 7 \times 7 = 343$ numbers
Work out the missing steps

Example 5

Three-digit numbers are to be formed from the digits 1, 2, 3, 4 and 5. If repetition of digits is allowed, in how many ways can the numbers less than 400 be formed?

Solution

$3 \times 5 \times 5 = 75$ numbers
Work out the missing steps.

Example 6

How many numbers between 3000 and 7000 can be formed using the digits 1, 3, 6, 8, without repetition?

Solution

$2 \times 3 \times 2 \times 1 = 2 \times 6 = 12$ numbers
Work out the missing steps

Combinations

A **combination** is a selection of a number of objects when the order of arrangement is not important. There is only one combination of the letters *ABC*.
The number of combinations of a set of *n* different objects, taken *r* at a time or the selection of *r* objects out of *n* objects is written as ${}^nC_r$ or $\binom{n}{r}$, where

$${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Thus the number of ways of choosing or selecting *r* objects from a total of *n* objects is ${}^nC_r$ or $\binom{n}{r}$. Note that: ${}^nC_r = \frac{{}^nP_r}{r!}$

Note that $4! = 4 \times 3 \times 2 \times 1 = 24$,
$3! = 3 \times 2 \times 1 = 6$ and by definition $0! = 1$.
$\frac{8!}{6!} = 8 \times 7$; $\frac{20!}{17!} = 20 \times 19 \times 18$; $\frac{10!}{9!} = 10$ etc.
${}^nC_r = {}^nC_{n-r}$; ${}^5C_2 = {}^5C_3$, ${}^9C_3 = {}^9C_6$ etc.

The student should at this stage know how to calculate combinations.
A calculating machine or mathematical tables can be used.

Illustrative examples

Example 1

In how many ways can a football team be selected from a class of 20 students?

Solution

A football team consists of 11 players.

11 students can be chosen from 20 in ${}^{20}C_{11}$

ways $= \dfrac{20!}{11!(20-11)!} = \dfrac{20!}{11!\,9!}$

$= 167960$ ways.

Example 2

In how many ways can four boys be chosen from six boys?

Solution

The number of ways $= {}^{6}C_{4}$

$$= \frac{6!}{4!(6-4)!} = \frac{6!}{4!\,2!}$$

$$= 15$$

Example 3

In how many ways can 9 mangoes be selected from a basket containing twelve mangoes?

Solution

The number of ways $= {}^{12}C_{9}$

$$= \frac{12!}{9!(12-9)!} = \frac{12!}{9!\,3!}$$

$$= 220$$

Example 4

In how many ways can three equal prizes be given to five boys?

Solution

The number of ways $= {}^{5}C_{3}$

$$= \frac{5!}{3!(5-3)!} = \frac{5!}{3!\,2!}$$

$$= 10$$

Example 5

In how many ways can three vacancies be filled out of 5 applicants?

Solution

The number of ways $= {}^{5}C_{3}$

$$= \frac{5!}{3!(5-3)!} = \frac{5!}{3!\,2!}$$

$$= 10$$

Example 6

In how many ways can a committee of 3 ladies and 2 men be selected from a group of 7 ladies and 5 men?

Solution

From the question,

$n(\text{ladies}) = 7$ and $n(\text{men}) = 5$

The number of ways of selecting 3 ladies out of 7 $= {}^{7}C_{3} = 35$

The number of ways of selecting 2 men out of 5 $= {}^{5}C_{2} = 10$

$\therefore$ The number of ways of selecting 3 ladies **and** 2 men $= {}^{7}C_{3} \times {}^{5}C_{2}$

$$= 35 \times 10 = 350$$

Note: 'and' means multiplication.

Example 7

A committee of 1 girl and 2 boys is to be formed out of 5 girls and 6 boys. In how many ways can this be done?

Solution

From the question,

$n(\text{girls}) = 5$ and $n(\text{boys}) = 6$

The number of ways of selecting 1 girl out of 5 $= {}^{5}C_{1} = 5$

The number of ways of selecting 2 boys out of 6 $= {}^{6}C_{2} = 15$

$\therefore$ The number of ways of selecting 1 girl **and** 2 boys $= {}^{5}C_{1} \times {}^{6}C_{2} = 5 \times 15 = 75$

Example 8

In how many ways can two red balls and three black balls be chosen from a bag containing five red and six black balls?

Solution

From the question,

$n(\text{red balls}) = 5$ and $n(\text{black balls}) = 6$

The number of ways of selecting 2 red balls out of 5 = ${}^5C_2 = 10$

The number of ways of selecting 3 black balls out of 6 = ${}^6C_3 = 20$

$\therefore$ The number of ways of selecting 2 red **and** 3 black balls = ${}^5C_1 \times {}^6C_3$

$$= 10 \times 20 = 200$$

Example 9

Five odd or four even numbers are to be selected from the numbers 1, 2, 3, 4, …, 15. How many ways can this be done?

Solution

{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}

Number of odd = 8, Number of even = 7

Ways of selecting five odd numbers $= {}^8C_5$

Ways of selecting four even numbers $= {}^7C_4$

Ways of selecting five odd **or** four even numbers $= {}^8C_5 + {}^7C_4 = 56 + 35 = 91$

Note: 'or' means addition.

Example 10

In how many ways can a committee of 6 men and 4 women be formed from a group of 8 men and 7 women?

Solution

980 [Work out the missing steps]

Example 11

Simplify $\begin{pmatrix} n+1 \\ 3 \end{pmatrix} - \begin{pmatrix} n-1 \\ 3 \end{pmatrix}$.

Solution

$$\begin{pmatrix} n+1 \\ 3 \end{pmatrix} - \begin{pmatrix} n-1 \\ 3 \end{pmatrix} = {}^{n+1}C_3 - {}^{n-1}C_3$$

$$= \frac{(n+1)!}{(n-2)!3!} - \frac{(n-1)!}{(n-4)!3!}$$

$$= \frac{(n+1)(n)(n-1)}{6} - \frac{(n-1)(n-2)(n-3)}{6}$$

$$= \tfrac{1}{6}(n-1)(n^2 + n - n^2 + 5n - 6)$$

$$= \tfrac{1}{6}(n-1)(6n-6) = (n-1)^2$$

Example 12

If ${}^{3n}C_2 = 15$, find the value of n.

Solution

$n = 2$ [Work out the missing steps]

Drawing with replacement versus drawing without replacement

Supposing in a bag containing 3 balls, you take a ball at random from the bag and then replace it, the result of the second draw does not depend on the ball you drew the first time. If you label the balls *A*, *B* and *C* then possible draws are shown below.

AA, AB, AC, BB, BC, CC

i.e. we have 6 ways of drawing 2 balls one after the other with replacement out of a total of 3.

However, if the balls are drawn at random from the box one after the other ***without replacement***, drawing of the second ball is dependent on the first draw. In such a case the possible draws will be

AB, AC, BC i.e. 3 ways = ${}^3C_2$ ways

Again, if we have 4 balls labelled *A*, *B*, *C* and *D* and we want to draw two balls one after the other with replacement, then the possible draws will be:

AA, AB, AC, AD, BB, BC, BD, CC, CD, DD

i.e. 10 ways

On the other hand, if the two balls are drawn at random from the box one after the other without replacement,

AB, AC, AD, BC, BD, CD

i.e. 6 ways = ${}^4C_2$ ways

We can therefore apply combinations to determine the number of ways of drawing objects from a collection *without replacement.*

Illustrative example

How many ways can 4 red and 2 blue balls be drawn (without replacement) from a bag of 6 red and 4 blue balls?

Solution

From the question, $n(\text{red balls}) = 6$ and $n(\text{blue balls}) = 4$

The number of ways of drawing 4 red balls out of 6 without replacement

$= {}^6C_4 = 15$

The number of ways of drawing 2 blue balls out of 4 without replacement = ${}^4C_2 = 6$

$\therefore$ The number of ways of drawing 4 red balls **and** 2 blue balls without replacement $= 15 \times 6 = 90$

Conditional combinations

Illustrative examples

Example 1

A sub-committee of 6, including a chairperson, is to be chosen from a main committee of 12. If the chairperson is to be a specified member of the main committee, in how many ways can the sub-committee be chosen?

Solution

If the chairperson is a specified member of the main committee then we are left with choosing the remaining 5 members out of 11. i.e. ${}^{11}C_5 = 462$

Hence the required number ways

$= 1 \times {}^{11}C_5 = 1 \times 462 = 462$

Note that there is only one way of choosing the chairperson.

Example 2

There are 15 boys and 10 girls, out of whom a committee of 3 boys and 2 girls is to be formed. Find the number of ways in which this can be done if

i) there is no restriction

ii) a particular boy is included

iii) a particular girl is excluded

Solution

i) 3 boys can be selected out of 15 boys in ${}^{15}C_3$ ways and 2 girls can be selected out of 10 girls in ${}^{10}C_2$ ways

Hence the required number ways

$= {}^{15}C_3 \times {}^{10}C_2 = 455 \times 45 = 20475$

ii) When a particular boy is included, we have now to select 2 boys out of 14 boys. This is done in ${}^{14}C_2$ ways. Also, 2 girls can be selected out of 10 girls in ${}^{10}C_2$ ways

Hence the required number ways

$= {}^{14}C_2 \times {}^{10}C_2 = 1 \times 147 \times 45 = 6615$

iii) When a particular girl is excluded, 2 girls are now selected out of the remaining 9 girls. This can be done in ${}^9C_2$ ways. Also, 3 boys can be selected out of 15 boys in ${}^{15}C_3$ ways.

Hence the required number ways

$= {}^9C_2 \times {}^{15}C_3 = 36 \times 455 = 16380$

Example 3

In how many ways can five boys be chosen from a class of twenty boys if the class captain has to be included?

Solution

If the class captain is to be included then we are left with choosing the remaining 4 boys out of 19 i.e. ${}^{19}C_4 = 3876$. Note that there is only one way of choosing the class captain.

The number of ways choosing five boys including the class captain $= 1 \times {}^{19}C_4$

$= 1 \times 3876 = 3876$

Examples 4

Out of 5 mathematicians and 7 physicists a committee consisting of 2 mathematicians and 3 physicists is to be formed. In how many ways can this be done if

a) Any mathematician and any physicist can be selected;

b) Two particular mathematicians cannot be on the committee;

c) One particular physicist must be selected.

Solution

From the question, n(mathematicians) = 5 and n(physicists) = 7

a) The number of ways of selecting 2 mathematicians out of 5 $= {}^5C_2 = 10$

The number of ways of selecting 3 physicists out of 7 $= {}^7C_3 = 35$

∴ The required number of ways $= {}^5C_2 \times {}^7C_3 = 10 \times 35 = 350$

b) If two particular mathematicians cannot be on the committee then we are left with 3 mathematicians from which we are to select 2. Also we shall select 3 physicists out of 7.

Now the number of ways of selecting 2 mathematicians out of 3 is ${}^3C_2 = 3$ and

The number of ways of selecting 3 physicists out of 7 is ${}^7C_3 = 35$

Hence the required number ways $= {}^3C_2 \times {}^7C_3 = 3 \times 35 = 105$

c) There will be only one way of selecting a particular physicist and the 2 additional physicists have to be selected out of the remaining 6.

The number of ways of selecting the physicists $= (1 \times {}^6C_2) = 15$

Also the number of ways of selecting 2 mathematicians out of 5 $= {}^5C_2 = 10$

Hence the required number of ways $= (1 \times {}^6C_2) \times {}^5C_2 = 15 \times 10 = 150$

Note very carefully how this question was solved.

Example 5

A committee of 5 persons is to be formed from 5 men and 4 women, find the number of ways of forming the committee if;

a) there must be only two women;

b) one particular woman must be included.

Solution

From the question,

n(women) = 4 and n(men) = 5

a) If there must be only two women, then the remaining members of the committee should be 3 men.

The number of ways of selecting 2 women out of 4 $= {}^4C_2 = 6$.

The number of ways of selecting 3 men out of 5 $= {}^5C_3 = 10$.

∴ The number of ways of selecting 2 women **and** 3 men $= {}^4C_2 \times {}^5C_3$
$= 6 \times 10 = 60$.

b) If one particular woman must be included, then we are left with selecting 4 persons from 5 men and 3 women (i.e. a total of 8 persons).
The number of ways of selecting 4 persons out of 8 persons $= {}^8C_4 = 70$.

Example 6

A committee of 3 is formed from 5 men and 3 women. Find:

a) the number of ways of forming the committee;

b) the number of ways of choosing at least one woman on the committee.

Solution

From the question,
$n(\text{men}) = 5$ and $n(\text{women}) = 3$.
Total number of persons = 8

a) The number of ways of selecting 3 persons out of 8 persons $= {}^8C_3 = 56$

b) The number of ways of choosing at least one woman on the committee
= The number of ways of choosing 3 persons out of 8 – number of ways of choosing no woman.
Number of ways of choosing no woman
= number of choosing all men
i.e. 3 out of 5 $= {}^5C_3 = 10$

∴ The number of ways of choosing at least one woman on the committee $= 56 - 10$
$= 46$

Example 7

A committee of two men and three women is to be chosen from five men and four women,

a) find the number of ways of forming the committee?

b) if one of the women refuses to serve on the same committee with a particular man, how many ways can the committee be formed?

Solution

From the question,
$n(\text{men}) = 5$ and $n(\text{women}) = 4$

a) The number of ways of selecting 2 men out of 5 $= {}^5C_2 = 10$
The number of ways of selecting 3 women out of 4 $= {}^4C_3 = 4$

∴ The number of ways of selecting 2 men **and** 3 women $= {}^5C_2 \times {}^4C_3$
$= 10 \times 4 = 40$

b) We first find the number of ways of forming the committee which include both this particular man and woman.
After choosing this particular man and woman, we are left with choosing 1 man out of 4 and choosing 2 women out of 3 i.e. ${}^4C_1 \times {}^3C_2 = 4 \times 3 = 12$
Therefore the number of ways of forming a committee not having both the particular man and woman =
The number of ways of forming a committee of 2 men and 3 women – number of ways of forming the committee which include both this particular man and woman $= 40 - 12$
$= 28$

Example 8

Find the number of ways a committee of 4 people can be chosen from a group of 5 men and 7 women when it contains:

a) only people of the same sex;

b) people of both sexes and there are at least as many women as men.

Solution
From the question,
$n(\text{men}) = 5$ and $n(\text{women}) = 7$

a) If the committee contains people of the same sex, then either they are all male or all female.
An all male committee can be chosen in $^5C_4 = 5$ ways.
An all female committee can be chosen in $^7C_4 = 35$ ways.
Number of ways of choosing a committee of same sex = Number of ways of choosing 4 men out of 5 **or** number of ways of choosing 4 women out of 7 = $^5C_4 + {^7C_4} = 5 + 35 = 40$

b) The committee could consist of 1) 1 man and 3 women or 2) 2 men and 2 women
The committee cannot consist of 4 women because people of both sexes must be included.

The table below gives the possibilities for the committee.

No. of men	No. of ways	No. of women	No. of ways
1	$^5C_1 = 5$	3	$^7C_3 = 35$
2	$^5C_2 = 10$	2	$^7C_2 = 21$

Thus the number of ways to have 1 man and 3 women = $^5C_1 \times {^7C_3} = [5 \times 35] = 175$
The number of ways to have 2 men and 2 women = $^5C_2 \times {^7C_3} = 10 \times 21 = 210$
∴ The number of ways of forming the committee = $[^5C_1 \times {^7C_3}] + [^5C_2 \times {^7C_3}]$

$$= [5 \times 35] + [10 \times 21]$$
$$= 175 + 210 = 385$$

Example 9
A committee of four is chosen from five teachers and three SS3 students. In how many ways can this be done so that the committee contains:

a) at least one teacher;
b) at least one teacher and one SS3 student.

Solution
a) The table below gives the possibilities for the committee.

No. of teachers	No. of ways	No. of SS3 students	No. of ways
1	$^5C_1 = 5$	3	$^3C_3 = 1$
2	$^5C_2 = 10$	2	$^3C_2 = 3$
3	$^5C_3 = 10$	1	$^3C_1 = 3$
4	$^5C_4 = 5$	0	$^3C_0 = 1$

∴ The number of ways of forming committee
$= [^5C_1 \times {^3C_3}] + [^5C_2 \times {^3C_2}] + [^5C_3 \times {^3C_1}] + [^5C_4 \times {^3C_0}]$
$= [5 \times 1] + [10 \times 3] + [10 \times 3] + [5 \times 1]$
$= 5 + 30 + 30 + 5 = 70$

b) Tabulating the possibilities for the committee would give the first three rows of the table above.
∴ The number of ways of forming committee = $5 + 30 + 30 = 65$.

Example 10
A company which has 10 directors has to send 3 of them to a conference.

a) In how many different ways can the 3 directors who attend be chosen?
b) It is now decided that at least one of those attending should be able to speak French. Four out of 10 directors speak French. In how many different ways can the 3 directors now be chosen?

Solution

a) The three directors may be chosen in $^{10}C_3 = 120$ ways

b) The committee could consist of : 1 French speaker and 2 non-French speakers **or** 2 French speakers and 1 non-French speaker **or** 3 French speakers.

1 French speaker and 2 non-French speakers may be chosen in $^4C_1 \times {}^6C_2 = 4 \times 15$
$= 60$ ways

2 French speakers and 1 non-French speaker may be chosen in $^4C_2 \times {}^6C_1 = 6 \times 6$
$= 36$ ways

3 French speakers may be chosen in $^4C_3$
$= 4$ ways

$\therefore$ Three directors may be chosen so that at least one of them speaks French in $60 + 36 + 4 = 100$ ways

Example 11

Find the number of ways a committee of 4 people can be chosen from 6 boys and 6 girls if it must contain:

a) 2 boys and 2 girls;

b) at least 1 boy and 1 girl.

Solution

From the question, $n(\text{boys}) = 6$ and
$n(\text{girls}) = 6$

a) The number of ways of selecting 2 boys out of $6 = {}^6C_2 = 15$.

The number of ways of selecting 2 girls out of $6 = {}^6C_2 = 15$.

$\therefore$ The number of ways of selecting 2 boys and 2 girls $= {}^6C_2 \times {}^6C_2$
$= 15 \times 15 = 225$.

b) The table below gives the possibilities for the committee.

No. of boys	No. of ways	No. of girls	No. of ways
1	$^6C_1 = 6$	3	$^6C_3 = 20$
2	$^6C_2 = 15$	2	$^6C_2 = 15$
3	$^6C_3 = 20$	1	$^6C_1 = 6$

$\therefore$ The number of ways of forming committee

$$= [{}^6C_1 \times {}^6C_3] + [{}^6C_2 \times {}^6C_2] + [{}^6C_3 \times {}^6C_1]$$
$$= [6 \times 20] + [15 \times 15] + [20 \times 6]$$
$$= 120 + 225 + 120 = 465$$

Example 12

A bag contains two white and three red balls. In how many ways can three balls be chosen if:

a) at least one ball must be white;

b) at least one ball must be red.

Solution

From the question,
$n(\text{white balls}) = 2$ and $n(\text{red balls}) = 3$

a) The table below gives the possibilities.

No. of white balls	No. of ways	No. of red balls	No. of ways
1	$^2C_1 = 2$	2	$^3C_2 = 3$
2	$^2C_2 = 1$	1	$^3C_1 = 3$

$\therefore$ The number of ways of choosing at least one ball must be white

$$= [{}^2C_1 \times {}^3C_2] + [{}^2C_2 \times {}^3C_1]$$
$$= [2 \times 3] + [1 \times 3]$$
$$= 6 + 3$$
$$= 9$$

b) The table below gives the possibilities.

No. of white balls	No. of ways	No. of red balls	No. of ways
0	${}^2C_0 = 1$	3	${}^3C_3 = 1$
1	${}^2C_1 = 2$	2	${}^3C_2 = 3$
2	${}^2C_2 = 1$	1	${}^3C_1 = 3$

∴ The number of ways of choosing at least one ball must be red
$= [{}^2C_0 \times {}^3C_3] + [{}^2C_1 \times {}^3C_2] + [{}^2C_2 \times {}^3C_1]$
$= [1 \times 1] + [2 \times 3] + [1 \times 3]$
$= 1 + 6 + 3 = 10$

Review exercise 31

1. Evaluate the following:

i) $\frac{20!}{15!\,5!}$ ii) $6!$ iii) $\frac{8!}{6!\,2!}$

iv) $\frac{9!}{2!\,3!\,4!}$ v) $\frac{15!}{12!}$ vi) $\frac{10!}{7!}$

2. Find the number of ways in which letters of the word *MATHEMATICS* can be arranged.

3. Find the number of permutations of four letters from the word *MATHEMATICS*.

4. Find the number of ways in which letters of the word *FACETIOUS* can be arranged.

5. Find the number of ways in which letters of the word *CALCULUS* can be arranged.

6. Find the number of ways in which letters of the word *PERCENTAGE* can be arranged.

7. Find the number of permutations of three letters from the word *CHEMISTRY*.

8. Find the number of ways in which letters of the word *NOTABLE* can be arranged.

9. Find the number of ways in which letters of the word *DEFLATED* can be arranged if:
a) the two *E*s must always be together;
b) the two *E*s must not come together.

10. Find the number of ways in which letters of the word *ADDING* can be arranged if
a) the two *D*s are together;
b) the two *D*s are separate.

11. Find the number of ways in which letters of the word *TAKORADI* can be arranged if the two *A*s are together.

12. In how many ways can five children be seated around a circular table?

13. In how many ways can six different coloured beads be placed on a ring?

14. Six people sit at a round table. In how many ways can this be done if two of them must sit next to each other?

15. In how many ways can four men be chosen from eight?

16. In how many ways can three men and two women be chosen from six men and four women?

17. A committee of three men and three women is to be formed from six men and four women.
- a) In how many ways can this be done?
- b) How many different committee can be formed if one particular woman will not serve on the committee if a particular man is chosen.

18. A bag contains three white and four red balls. In how many ways can three balls be chosen if:
- a) at least one ball must be white;
- b) at least one ball must be red.

19. Find the number of ways a committee of 4 people can be chosen from a group of 6 men and 8 women when it contains:
- a) only people of the same sex;
- b) people of both sexes and there are at least as many women as men.

20. A committee consisting of 1 girl and 2 boys is to be formed out of 5 girls and 6 boys. In how many ways can this be done?

21. In how many ways can 5 beads, chosen from 9 different beads be threaded on a ring?

22. In how many ways can five men chosen from 9 men be seated at a round table?

23. If the letters of the word '*CALCULUS*' are arranged, find the number of ways this can be done if the *A* and the *S* are always next to one another.

24. If the letters of the word '*PERCENTAGE*' are arranged, find the number of ways this can be done if the E^S are always next to one another.

25. If the letters of the word '*MATHEMATICS*' are arranged, find the number of ways this can be done if the T^S are always next to one another.

26. Find the number of ways a committee of 5 people can be chosen from a group of 6 men and 7 women when it contains:
- a) only people of the same sex
- b) people of both sexes and there are at least many women as men.

27. A mixed football team containing 5 men and 6 women is to be chosen from 7 men and 9 women. In how many ways can this be done?

28. In how many ways can a committee of 6 men and 4 women be formed from a group of 8 men and 7 women?

29. A committee of six is to be formed from nine women and three men. In how many ways can the members be chosen to include at least one man?

30. How many arrangements can be made of the letters *THIRTIETH* ?

31. How many numbers of 4 different digits can be formed using the digits 0, 1, 2, 3, 4, 5? How many of such numbers is less than 3000?

32. From the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 numbers of three digits are formed. Show that there are:
- i) 729 such numbers in all if repetitions are allowed.
- ii) 504 numbers having digits different from one another.

33. A number of four different digits is to be formed from the digits1, 2, 3, 4, 5, 6, 7, 8, 9. Find how many of them are:
- i) greater than 4000
- ii) less than 6000
- iii) divisible by 2
- iv) divisible by 5

34 Find the number of arrangements that can be made out of the letters of the following words, if all the letters are taken together.
- i) *PERMUTATION*
- ii) *COMBINATION*
- iii) *COMMITTEE*
- iv) *ADDITION*

35. In how many ways can the letters of the word *MISSISSIPPI* be arranged amongst themselves, so that the *P*'s are always together.

36. In how many ways can the letters of the word *STATISTICS* be arranged amongst themselves, so that the *T*'s are always together?

37. In how many ways can the letters of the word *FORMULA* be arranged amongst themselves, so that
- i) all the vowels are always together?
- ii) no two vowels are together?

38. In how many ways can the letters of the word *STRANGLE* be arranged amongst themselves, so that
- i) the vowels are not together?
- ii) the vowels are always together?

39. There are seven men and three women. Find the number of ways in which a committee of six can be formed from these if the committee is to include at least two women.

40. From 7 lawyers and 4 doctors, a committee of 6 is to be formed. In how many ways can this be done when the committee contains:
- i) exactly two doctors?
- ii) at least two doctors?

41. A committee of 5 members is to be formed from a teaching staff of 9 women and 3 men.
- a) Find the number of ways of forming the committee.
- b) What is the probability that the committee
  - i) consist of only women?
  - ii) includes exactly two men?
  - iii) includes at least one man?

42. A committee of 6 members is to be formed from a teaching staff 10 women and 4 men.
- a) Find the number of ways of forming the committee
- b) What is the probability that the committee
  - i) consists of only women?
  - ii) includes exactly three men?
  - iii) includes at least one man?

CHAPTER 32

THE BINOMIAL PROBABILITY DISTRIBUTION

Application of combinations

As we have already learnt in *chapter31*, combination is a selection of a number of objects when the order of arrangement is not important. The application of combinations in the theory of probability, therefore, involves drawing a specified number of items (say 3 balls) at random from a bag or box **with or without replacement**. The following examples show the application of combinations in solving problems.

Illustrative examples

Example 1

A bag contains 30 balls which are identical except for colour; 12 of the balls are red and 18 are blue. 4 balls are picked at random out of the bag. What is the probability that

i) They are all red?

ii) They are of the same colour?

iii) Two are red and two are blue?

Solution

From the question, n(red balls) = 12 and n(blue balls) = 18 $\Rightarrow$ $n(U) = 30$

4 balls are picked at random

i) P(all 4 balls are red)

$$= \frac{\text{Number of ways of picking 4 red balls out of 12}}{\text{Number of ways of picking 4 balls out of 30}}$$

$$= \frac{{}^{12}C_4}{{}^{30}C_4} = \frac{11}{609}$$

ii) P(all 4 balls are of the same colour)

= P(all 4 are red) or P(all 4 are blue)

$$= \frac{\text{Numberof ways of picking 4 red balls out of 12 } \textit{or} \text{ 4 balls out of 18}}{\text{Number of ways of picking 4 balls out of 30}}$$

$$= \frac{{}^{12}C_4 + {}^{18}C_4}{{}^{30}C_4} = \frac{79}{609}$$

iii) P(two red and two blue balls)

= P(all 2 are red) or P(all 2 are blue)

$$= \frac{\text{Number of ways of picking 2 red balls out of 12 } \textit{and} \text{ 2 balls out of 18}}{\text{Number of ways of picking 4 balls out of 30}}$$

$$= \frac{{}^{12}C_2 \times {}^{18}C_2}{{}^{30}C_4} = \frac{374}{1015}$$

Example 2

A box contains 7 red, 8 white and 3 blue balls. If three balls are drawn from the box, calculate the probability that:

i) two are red and one is blue;

ii) at least two are white;

iii) one of each colour.

Solution

a) From the question, n(red balls) = 7, n(white balls) = 8 and n(blue balls) = 3 $n(U) = 18$

i) P(2 red and 1 blue)

$$= \frac{\text{Number of ways of drawing 2 red balls out of 7 } \textit{and} \text{ 1 ball out of 3}}{\text{Number of ways of drawing 3 balls out of 18}}$$

$$= \frac{{}^{7}C_2 \times {}^{3}C_1}{{}^{18}C_3} = \frac{21 \times 3}{816} = \frac{21}{272}$$

ii) P(at least two white balls)

= P(2 white and 1 non-white) or P(3 white balls)

$$= \frac{\text{Number of ways of drawing 2 white balls out of 8 } \textit{and} \text{ 1 non - white ball out of 10}}{\text{Number of ways of drawing 3 balls out of 18}}$$

or $$\frac{\text{Number of ways of drawing 3 white balls out of 8}}{\text{Number of ways of drawing 3 balls out of 18}}$$

$\therefore$ P(at least two white balls)

$$= \frac{{}^{8}C_2 \times {}^{10}C_1}{{}^{18}C_3} + \frac{{}^{8}C_3}{{}^{18}C_3}$$

$$= \frac{280}{816} + \frac{56}{816} = \frac{168}{408}$$

Note 1: At least two whites means either 2 whites and one non-white or 3 whites.
Note 2: There are 10 non-white balls.

iii) P(one of each colour)
$= P$(1 red, 1 white and 1 blue)

$$= \frac{{}^{7}C_1 \times {}^{8}C_1 \times {}^{3}C_1}{{}^{18}C_3} = \frac{7 \times 8 \times 3}{816} = \frac{27}{136}$$

Example 3

A bag contains 20 small identical objects. 8 of them are black, 7 are red and the rest are white. If three of the objects are selected at random from the bag at once, find the probability that:

i) One is black, one is red and the remaining is white;
ii) Exactly two are red;
iii) None of them is white.

Solution

From the question, n(black) = 8, n(red) = 7, n(white) = 5 $\Rightarrow$ $n(U) = 20$

i) P(one is black, one is red and one is white)

$$= \frac{{}^{8}C_1 \times {}^{7}C_1 \times {}^{5}C_1}{{}^{20}C_3} = \frac{8 \times 7 \times 5}{1140} = \frac{14}{57}$$

ii) P(exactly two are red)
$= P$(two red and one non-red)

$$= \frac{{}^{7}C_2 \times {}^{13}C_1}{{}^{20}C_3} = \frac{21 \times 13}{1140} = \frac{91}{380}$$

Note that there are 13 non-red objects out of which we are selecting 1.

Alternative method

P(exactly two red)
$= P$(two red and one black) or
P(two are red and one white)

$$= \frac{{}^{7}C_2 \times {}^{8}C_1}{{}^{20}C_3} + \frac{{}^{7}C_2 \times {}^{5}C_1}{{}^{20}C_3} = \frac{91}{380}$$

iii) P(none of them is white)
$= P$(all 3 are non-whites)

$$= \frac{{}^{15}C_3}{{}^{20}C_3} = \frac{455}{1140} = \frac{91}{228}$$

Note that there are 15 non-whites in the bag.

Example 4

A bag contains 8 red, 3 white and 9 black balls. If three balls are drawn at random, determine the probability that:

a) all 3 are red;
b) all 3 are white;
c) 2 are red and 1 white;
d) at least 1 is white;
e) 1 of each colour is drawn.

Solution

From the question, n(red balls) = 8, n(white balls) = 3 and n(black balls) = 9
$n(U) = 20$

a) $P(\text{all 3 are red}) = \frac{{}^{8}C_3}{{}^{20}C_3} = \frac{56}{1140} = \frac{14}{285}$

Alternative method

$P(\text{all 3 are red}) = P(1R \cap 2R \cap 3R)$

$$= \frac{8}{20} \times \frac{7}{19} \times \frac{6}{18} = \frac{14}{285}$$

b) $P(\text{all 3 are white}) = \frac{{}^{3}C_{3}}{{}^{20}C_{3}} = \frac{1}{1140}$

c) $P(\text{2 are red and 1 is white})$

$$= \frac{{}^{8}C_{2} \times {}^{3}C_{1}}{{}^{20}C_{3}} = \frac{7}{95}$$

d) $P(\text{non-white}) = \frac{{}^{17}C_{3}}{{}^{20}C_{3}} = \frac{34}{57}$

Note that there are 17 non-white balls.

$\therefore$ $P(\text{at least one white})$

$= 1 - P(\text{non-white}) = 1 - \frac{34}{57} = \frac{23}{57}$

e) $P(\text{one of each colour})$

$$= \frac{{}^{8}C_{1} \times {}^{3}C_{1} \times {}^{9}C_{1}}{{}^{20}C_{3}} = \frac{8 \times 3 \times 9}{1140} = \frac{18}{95}$$

Example 5

A student representative council consists of 7 girls and 5 boys. If a dance committee of 4 persons is to be formed, what is the probability that the committee consist of

a) 2 boys and 2 girls

b) 3 boys and 1 girl

Solution

From the question,

$n(\text{girls}) = 7, n(\text{boys}) = 5 \Rightarrow n(U) = 12$

a) $P(\text{2 boys and 2 girls}) = \frac{{}^{5}C_{2} \times {}^{7}C_{2}}{{}^{12}C_{4}}$

$$= \frac{10 \times 21}{495} = \frac{14}{33}$$

b) $P(\text{3 boys and 1 girl}) = \frac{{}^{5}C_{3} \times {}^{7}C_{1}}{{}^{12}C_{4}}$

$$= \frac{10 \times 7}{495} = \frac{14}{99}$$

Example 6

A club has a membership of five men and four women. Three members are chosen at random from the club to form a committee. Find the probability that the committee consist of 2 men and one woman.

Solution

From the question,

$n(\text{men}) = 5, n(\text{women}) = 4 \Rightarrow n(U) = 9$

$P(\text{2 men and 1 woman}) = \frac{{}^{5}C_{2} \times {}^{4}C_{1}}{{}^{9}C_{3}}$

$$= \frac{10 \times 4}{84} = \frac{10}{21}$$

Example 7

In a class of 12 students 9 are boys. If two students are selected at random, what is the probability that they are

i) both girls

ii) a boy and a girl.

Solution

From the question,

$n(\text{boys}) = 9, n(\text{girls}) = 3$ and $n(U) = 20$

i) $P(\text{both girls}) = \frac{{}^{3}C_{2}}{{}^{12}C_{2}} = \frac{3}{66} = \frac{1}{22}$

ii) $P(\text{a boy and a girl}) = \frac{{}^{9}C_{1} \times {}^{3}C_{1}}{{}^{12}C_{2}}$

$$= \frac{9 \times 3}{66} = \frac{9}{22}$$

Example 8

A bag contains 20 balls which are identical except for colour. 8 of the balls are red and the rest are blue. 3 balls are picked at random out of

the bag. Find the probability that

a) they are all red;

b) they are of the same colour.

Solution

From the question, $n(R) = 8, n(B) = 12,$

$n(U) = 20$

a) $P(\text{they are all red}) = \frac{{}^{8}C_{3}}{{}^{20}C_{3}} = \frac{56}{1140} = \frac{14}{285}$

b) They are of the same colour means either they are all red or they are all blue.

P(they are of the same colour)

= P(they are all red) + P(they are all red)

$$= \frac{{}^{8}C_3}{{}^{20}C_3} + \frac{{}^{12}C_3}{{}^{20}C_3} = \frac{56}{1140} + \frac{220}{1140} = \frac{276}{140} = \frac{23}{95}$$

Example 9

A bag contains 6 white and 4 black balls. Three balls are drawn from the bag at random, one after the other without replacement. Find the probability that

i) there are 2 white and 1 black balls;

ii) there is **at least** 1 white ball.

Solution

i) P(2 white and 1 black)

$$= \frac{{}^{6}C_2 \times {}^{4}C_1}{{}^{10}C_3} = \frac{15 \times 4}{120} = \frac{1}{2} = 0.5$$

ii) P(at least 1 white)

= 1 − P(at least no white)

$$= 1 - \frac{{}^{4}C_3}{{}^{10}C_3} = 1 - \frac{4}{120} = \frac{29}{30} = 0.97$$

Example 10

A box contains 5 red balls and 3 blue balls. Two balls are chosen at random without replacement. Find the probability that they are of the same colour.

Solution

Let R and B denote the events of drawing red and blue balls respectively.

From the question,

$n(R) = 5$, $n(B) = 3$,

total number of balls, $n(U) = 8$

Both bulbs are of the same colour means either the 2 bulbs selected are all red or the 2 bulbs selected are all blue.

Solution

P(both bulbs are of the same colour)

= P(both are red) or P(both are green)

$$= \frac{{}^{5}C_2 + {}^{3}C_2}{{}^{8}C_2} = \frac{10+3}{28} = \frac{13}{28}$$

Alternative method

∴ P(both bulbs are of the same colour)

= P(both are red) or P(both are green)

$= P(1R \cap 2R) + P(1B \cap 2B)$

$= [P(1R) \times P(2R)] + [P(1B) \times P(2B)]$

$= \left(\frac{5}{8} \times \frac{4}{7}\right) + \left(\frac{3}{8} \times \frac{2}{7}\right) = \frac{26}{56} = \frac{13}{28}$

Binomial distribution

Any situation having only two possible, mutually exclusive outcomes, often labelled '*success*' and '*failure*', is called a **binomial situation**. The probability of success (or failure) is often known.

If p is the probability that an event will happen in any single trial (called the probability of success), then the probability that the event *will not* happen is $q = 1 - p$

The probability function for the binomial discrete variable X is given by:

$$P(X = x) = {}^{n}C_x p^x q^{n-x}$$

Where x is the number of times the event will happen or the number of successes and n is the number of independent trials

Thus the probability of getting x successes in n independent trials is given by:

$$P(X = x) = {}^{n}C_x p^x q^{n-x}$$

$${}^{n}C_x = \frac{n!}{x!(n-x)!}\text{; For example,}$$

$${}^{6}C_2 = \frac{6!}{2!(6-2)!} = \frac{6!}{2!4!} = 15$$

Note that if there are x successes, then there are $(n - x)$ failures. Therefore we raise p to the power of x and q to the power of $(n - x)$ as shown in the formula above.

Illustrative examples

Example 1

A fair coin is tossed six times. Find the probability of getting

i) exactly two heads
ii) at least two heads
iii) two tails
iv) at most two heads

Solution

Assuming getting a head is a success

Probability of success for each trial, $p = \frac{1}{2}$

Probability of failure (i.e. not getting a head), $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

The number of trials, $n = 6$

Let X denote the number of successes (i.e. number of heads)

$$\therefore \quad P(X = x) = {}^nC_x p^x q^{n-x}$$

$$= {}^6C_x\left(\tfrac{1}{2}\right)^x\left(\tfrac{1}{2}\right)^{6-x}$$

i) P(exactly 2 heads)

$$P(X = 2) = {}^6C_2\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^{6-2}$$

$$= {}^6C_2\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^4$$

$$= 15\times\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^4 = 15\times\tfrac{1}{64} = \tfrac{15}{64}$$

ii) P(at least 2 heads)

$= P$(exactly 2 or more heads)

$= P(X \geq 2)$

But this is tedious to calculate so we find an equivalent expression i.e.

P(at least 2 heads)

$= 1 - P$(less than 2 heads)

$= 1 - P(X < 2)$

$= 1 - [P(X = 0) + P(X = 1)]$

$= 1 - \left[{}^6C_0\left(\tfrac{1}{2}\right)^0\left(\tfrac{1}{2}\right)^6 + {}^6C_1\left(\tfrac{1}{2}\right)^1\left(\tfrac{1}{2}\right)^5\right]$

$= 1 - \left[\left(\tfrac{1}{2}\right)^6 + 6\left(\tfrac{1}{2}\right)^6\right] = 1 - 7\left(\tfrac{1}{2}\right)^6 = 1 - \tfrac{7}{64} = \tfrac{57}{64}$

iii) P(two tails)

Obtaining a head in a trial is called a success so we change the question to probability of success

P(two tails) $= P$(four heads)

$= P(X = 4) = {}^6C_4\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^2$

$= 15\times\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^4 = 15\times\tfrac{1}{64} = \tfrac{15}{64}$

iv) P(at most 2 heads)

$= P(X \leq 2)$

$= P(X = 0) + P(X = 1) + P(x = 2)$

$= {}^6C_0\left(\tfrac{1}{2}\right)^0\left(\tfrac{1}{2}\right)^6 + {}^6C_1\left(\tfrac{1}{2}\right)^1\left(\tfrac{1}{2}\right)^5 + {}^6C_2\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^4$

$= \left(\tfrac{1}{2}\right)^6 + 6\left(\tfrac{1}{2}\right)^6 + 15\left(\tfrac{1}{2}\right)^6 = 22\left(\tfrac{1}{2}\right)^6 = \tfrac{22}{64} = \tfrac{11}{32}$

Example 2

In a family of 6 children, if a child is likely to be a boy as a girl, find the probability of selecting

i) exactly three boys
ii) at least two girls
iii) at most two boys

Solution

Assuming selecting a boy is a success

Probability of success, $p = \frac{1}{2}$

Probability of failure (i.e. not selecting a boy), $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

The number of children, $n = 6$

Let X denote the number of successes (i.e. number of boys)

$$\therefore \quad P(X = x) = {}^nC_x p^x q^{n-x}$$

$$= {}^6C_x\left(\tfrac{1}{2}\right)^x\left(\tfrac{1}{2}\right)^{6-x}$$

i) P(exactly 3 boys)

$$P(X = 3) = {}^6C_3\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^{6-3}$$

$$= {}^6C_3\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^3$$

$$= 20\times\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^3$$

$$= 20\times\tfrac{1}{64} = \tfrac{20}{64} = \tfrac{5}{16}$$

ii) P(at least 2 girls) = P(two or more girls)

$= 1 - P(\text{less than two girls})$

$= 1 - [P(\text{no girl}) + P(\text{one girl})]$

Selecting a boy in a trial is called a success, so we change the question to probability of success

P(no girl) = P(6 boys)

$$P(X=6) = {}^6C_6\left(\tfrac{1}{2}\right)^6\left(\tfrac{1}{2}\right)^{6-6}$$

$$= {}^6C_6\left(\tfrac{1}{2}\right)^6\left(\tfrac{1}{2}\right)^0$$

$$= \left(\tfrac{1}{2}\right)^6 = \tfrac{1}{64}$$

[P(one girl) = P(5 boys)

$$P(X=5) = {}^6C_5\left(\tfrac{1}{2}\right)^5\left(\tfrac{1}{2}\right)^{6-5}$$

$$= {}^6C_5\left(\tfrac{1}{2}\right)^5\left(\tfrac{1}{2}\right)^1 = 6\left(\tfrac{1}{2}\right)^6 = \tfrac{6}{64}$$

∴ P(at least 2 girls)

$= 1 - [P(\text{no girl}) + P(\text{one girl})]$

$$= 1 - \left[\tfrac{1}{64} + \tfrac{6}{64}\right] = 1 - \tfrac{7}{64} = \tfrac{57}{64}$$

iii) P(at most 2 boys)

$= P(X \le 2)$

$= P(X=0) + P(X=1) + P(x=2)$

$$= {}^6C_0\left(\tfrac{1}{2}\right)^0\left(\tfrac{1}{2}\right)^6 + {}^6C_1\left(\tfrac{1}{2}\right)^1\left(\tfrac{1}{2}\right)^5 + {}^6C_2\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^4$$

$$= \left(\tfrac{1}{2}\right)^6 + 6\left(\tfrac{1}{2}\right)^6 + 15\left(\tfrac{1}{2}\right)^6$$

$$= 22\left(\tfrac{1}{2}\right)^6 = \tfrac{22}{64} = \tfrac{11}{32}$$

Example 3

The probability of a seed germinating is 0.9. If six seeds are sown what is the probability that

i) five or more seeds will germinate?

ii) exactly half will germinate?

Solution

Assuming a seed germinating is a success

Probability of success, $p = 0.9$

Probability of failure (i.e. not germinating), $q = 1 - 0.9 = 1 - 0.9 = 0.1$

The number of seeds, $n = 6$

Let X denote the number of success (i.e. number of germinating seeds)

$$\therefore\ P(X=x) = {}^nC_x p^x q^{n-x}$$

$$= {}^6C_x(0.9)^x(0.1)^{6-x}$$

i) P(five or more will germinate)

$$P(X \ge 5) = P(X=5) + P(X=6)$$

$$= {}^6C_5(0.9)^5(0.1)^1 + {}^6C_6(0.9)^6(0.1)^0$$

$$= 6(0.9)^5(0.1) + (0.9)^6 = 0.8857$$

ii) P(exactly half will germinate)

$P(X=3)$

[since there are 6 seeds, half of 6 is 3]

$$P(X=3) = {}^6C_3(0.9)^3(0.1)^3$$

$$= 20 \times (0.9)^3(0.1)^3$$

$$= 0.01458$$

Example 4

A farmer produces seeds in packets for sale. The probability that a seed selected will grow is 0.8. If 5 of the seeds are grown, what is the probability that:

i) less than two will grow?

ii) less than two will not grow?

Solution

Assuming a seed growing is a success

Probability of success, $p = 0.8$

Probability of failure (i.e. seed will not grow),

$q = 1 - 0.8 = 1 - 0.8 = 0.2$

The number of seeds, $n = 5$

Let X denote the number of success (i.e. number of seeds that will grow)

$$\therefore\ P(X=x) = {}^nC_x p^x q^{n-x}$$

$$= {}^5C_x(0.8)^x(0.2)^{5-x}$$

i) P(less than two will grow)

$$P(X<2) = P(X=0) + P(X=1)$$

$$= {}^5C_0(0.8)^0(0.2)^5 + {}^5C_1(0.8)^1(0.2)^4$$

$$= (0.2)^5 + 5(0.8)(0.2)^4 = 0.00672$$

ii) P(less than two will not grow)

$= P$(2 or more will grow)

$P(X \geq 2) = 1 - P(X < 2)$

But from (i) above $P(X < 2) = 0.00672$

$\therefore$ P(less than two will not grow)

$= 1 - 0.00672 = 0.99328$

Example 5

The probability of a man winning a weekly lottery is 0.02 in any one draw. Find the probability that in eight weeks he will

i) not win
ii) win exactly once
iii) win more than one

Solution

Assuming winning a weekly lottery is a success

Probability of success, $p = 0.02$

Probability of failure (i.e. not winning a lottery), $q = 1 - p = 1 - 0.02 = 0.98$

The number of weeks, $n = 8$

Let X denote the number of successes (i.e. number of wins)

$$P(X = x) = {}^nC_x p^x q^{n-x}$$
$$= {}^8C_x (0.02)^x (0.98)^{8-x}$$

i) P(not win)

$$P(X = 0) = {}^8C_0 (0.02)^0 (0.98)^{8-0}$$
$$= {}^8C_0 (0.02)^0 (0.98)^8$$
$$= (0.98)^8 = 0.8508$$

ii) P(win exactly once)

$$P(X = 1) = {}^8C_1 (0.02)^1 (0.98)^{8-1}$$
$$= {}^8C_1 (0.02)^1 (0.98)^7$$
$$= 8 \times (0.02)(0.98)^7 = 0.1389$$

iii) P(win more than one)

$$P(X > 1) = 1 - P(X \leq 1)$$
$$= 1 - [P(X = 0) + P(X = 1)]$$
$$= 1 - [0.8508 + 0.1389]$$
$$= 1 - 0.9897 = 0.0103$$

Example 6

In an examination, 90% of the candidates passed. If 8 of the candidates are selected at random, calculate the probability that

i) exactly 5 passed.
ii) at most 3 failed.

Solution

a) $P(\text{passed}) = \frac{90}{100} = 0.9$

$P(\text{failed}) = \frac{10}{100} = 0.1$

Number of candidates, $n = 8$

i) $P(\text{5 passed}) = {}^8C_5 (0.9)^5 (0.1)^3$

$= 56\,(0.59049)(0.001)$

$= 0.0331$

ii) P(at most 3 failed)

$= 1 - P$(less than two failed)

$$= 1 - \left(\begin{matrix} {}^8C_0 (0.9)^0 (0.1)^8 + {}^8C_1 (0.9)^1 (0.1)^7 + \\ {}^8C_2 (0.9)^2 (0.1)^6 \end{matrix} \right)$$

$= 1 - 0.00502 = 0.99498$

$= 0.995$ to three decimal places

Example 7

For every ten persons in a city one is left-handed. If six persons are selected at random from the city, find the probability that

i) exactly 3
ii) more than half
iii) at least 2

are left-handed.

Solution

$p = \dfrac{1}{10}$ or $0.1 \qquad q = \dfrac{9}{10}$ or 0.9

x = number of left-handed persons

i) $$P(x=3) = \binom{6}{3}(0.1)^3(0.9)^3$$
$$= {}^6C_3(0.1)^3(0.9)^3$$
$$= 0.0146$$

ii) $P(x > 3)$
$$= \binom{6}{4}(0.1)^4(0.9)^2 + \binom{6}{5}(0.1)^5(0.9)$$
$$+ \binom{6}{6}(0.1)^6$$
$$= 15(0.0001)(.081) + 6(0.000009)$$
$$+ 0.000001$$
$$= 0.0013$$

iii) $$P(x \geq 2) = 1 - [P(0) + P(1)]$$
$$= 1 - \left[\binom{6}{0}(0.1)^0(0.9)^6 + \binom{6}{1}(0.1)(0.9)^5\right]$$
$$= 1 - 0.8857$$
$$= 0.1143$$

Example 8

The probability that a marksman will hit a target is $\frac{5}{6}$. He fires nine shots. Find the probability, correct to three decimal places, that he will hit the target

i) at least seven times

ii) not more than six times

Solution

Probability of success (i.e. hitting a target),

$p = \frac{5}{6} \;\Rightarrow\; q = 1 - p = \frac{1}{6};\; n = 9$

i) 0.822 ii) 0.178

[Work out the solution]

Example 9

An investigation reveals that 0.8 patients are cured by a new method of curing a certain disease. If ten patients are treated by the new method, calculate correct to two significant figures, the probability that

i) exactly seven patients are cured

ii) at most seven patients are cured

Solution

Probability of success (i.e. being cured by the new method),

$p = 0.8 \;\Rightarrow\; q = 1 - p = 0.2;\; n = 10$

i) 0.25 ii) 0.47

[Work out the solution]

Example 10

A fair six-sided die is thrown four times. Find the probability, correct to three decimal places, of getting

i) exactly one *six* occurs

ii) at least one *six* occurs

Solution

Probability of success (i.e. obtaining a six in a single throw),

$p = \frac{1}{6} \;\Rightarrow\; q = 1 - p = \frac{5}{6};\; n = 4$

i) 0.386 ii) 0.518

[Work out the solution]

Example 11

A fair coin is tossed eight times. Find the probability of getting

i) exactly three heads

ii) an equal number of heads as tails

Solution

Probability of success (i.e. obtaining a head in a single toss),

$p = \frac{1}{2} \;\Rightarrow\; q = 1 - p = \frac{1}{2};\; n = 8$

i) 0.2188 ii) 0.2734

[Work out the solution]

Example 12

A fair die is thrown three times. Find the probability, correct to three decimal places, of getting

i) no *six*
ii) exactly two *sixes*
iii) at most two *sixes*
iv) at least one *six*

Solution

Probability of success (i.e. obtaining a six in a single throw),

$p = \frac{1}{6} \Rightarrow q = 1 - p = \frac{5}{6};\ n = 3$

i) 0.579 ii) 0.694
iii) 0.995 iv) 0.421
[Work out the solution]

Example 13

20% of articles taken from a batch of eight articles are found to be defective. Calculate the probability of getting three or more defective articles.

Solution

Probability of success (i.e. getting a defective article), $p = 0.2$

$q = 1 - 0.2 = 0.8;\ n = 8$

[Ans: 0.2032]
[Work out the solution]

Example 14

In a box of 100 items, 10 are known to be defective. If 5 items are selected at random from the box, what is the probability that 2 of them are defective?

Solution

Ans. 0.07 [Work out the solution]

Example 15

An investigation shows that out of every 8 patients treated with a new malaria vaccine, 6 patients are cured. If 5 patients are treated with the vaccine, what is the probability that:

a) 1 patient is cured,
b) exactly 3 patients are cured,
c) at least 3 patients are cured,

Solution

a) $\frac{15}{1024}$ b) $\frac{135}{512}$ c) $\frac{459}{512}$ [Work out the solution]

Example 16

25 out of every 100 bolts produced by a machine are defective. Find the probability that out of 4 bolts selected at random, 3 will **not** be defective.

Solution

Ans. $\frac{27}{64}$ [Work out the solution]

Example 17

If the probability of any particular child of a family being a boy is $\frac{1}{2}$, what is the probability that in a family of 6 children, there are equal number of boys and girls?

Solution

Ans. $\frac{5}{16}$ [Work out the solution]

Review exercise 32

1. A box contains 25 identical balls, 10 of them are black, 7 of them are red and the rest are white. If three balls are drawn at random from the box one after the other without replacement, find the probability that
   a) one is black, one is red and one is white
   b) exactly two are black
   c) none of them is red

2. A student representative council consists of 7 girls and 5 boys. If a dance committee of 4 persons is to be formed, what is the probability that the committee will consist of
   a) 2 boys and 2 girls?
   b) 3 boys and 1 girl?

3. A bag contains 12 blue and 8 red pens. 2 pens are chosen at random from the bag without replacement. What is the probability that
   i) one is red and the other is blue
   ii) both are red
   iii) both are of the same colour
   iv) both are of different colours.

4. A box contains 10 marbles; 7 of which are black and 3 are red. Two marbles are drawn one after the other without replacement. Find the probability of getting
   i) a red, then a black marble
   ii) two black marbles.

5. A bag contains 6 red and 5 green identical balls. Two balls are drawn at random one after the other without replacement. What is the probability that both balls are of the same colour?

6. A box contains 10 white and 15 green identical bulbs. Two bulbs are selected one after the other from the box without replacement. What is the probability that
   i) both bulbs are white?
   ii) the first bulb is green and the second is white?

7. A box contains identical balls of which 12 are red, 16 white and 8 blue. Three balls are drawn from the box one after the other without replacement. Find the probability that
   a) three are red;
   b) the first is blue and the other two are red;
   c) two are white and one is blue.

8. Two balls are drawn successively, without replacement from a bag containing 8 blue balls and 14 white balls. Find the probability that they are both blue.

9. A box contains 2 white and 3 blue identical marbles. Two identical marbles are picked at random, one after the other, without replacement, what is the probability of picking two marbles of different colours.

10. For every ten persons in a city one is left-handed. If six persons are selected at random from the city, find the probability that
    i) exactly 3 are left-handed.
    ii) more than half are left-handed.
    iii) at least 2 are left-handed.

11. A survey carried out among a group of secondary school students showed that 2 out of every 10 students buy textbooks. If 5 students are randomly selected from the group, find, correct to three decimal places the probability that
    i) exactly 3
    ii) less than 4
    iii) at least 2
    students buy textbooks.

12. The probability that a pupil will be caned daily in school is $\frac{2}{5}$. If a school week consists of 5 days, find, correct to three decimal places, the probability that the pupil will
    i) be caned in 2 days only;
    ii) not be caned in the week;
    iii) definitely be caned during the week.

13. 60% of candidates who took an examination passed. If 10 of the candidates are selected at random, find, correct to 3 significant figures the probability that
    a) exactly 4 passed;
    b) at least 2 passed;
    c) at most 3 passed

CHAPTER 33

SELF ASSESSMENT TEST

1. There are 40 pupils in a class. 30 of them study Biology, 10 physics and 21 study Chemistry. 15 study Physics and Biology, 10 study Chemistry and 13 study Biology and Chemistry. 2 study Physics only and 7 study Biology only.

- i) Represent this information on a Venn diagram.
- ii) How many pupils study all the three subjects?
- iii) If a pupil is selected at random, what is the probability that he studies Physics or Chemistry?

2. In a class of 100 students, 40 offer Biology and 10 offer Chemistry and Biology. 10 offer Physics and Chemistry, and 30 offer Physics and Biology. Each student offers at least one of the three subjects. The number of students who offer Biology is equal to those who offer Physics only and the sum of the number of students who offer Biology and those who offer Chemistry is equal to the number of students who offer Physics.

- a) Illustrate the information on a Venn diagram
- b) Find the number of students who offer
  - i) Chemistry only
  - ii) only two subjects
  - iii) only one subject
  - iv) Physics
  - v) Chemistry
  - vi) Chemistry and Physics only

3. In a class of 47 students, 31 study Physics, 26 study Chemistry and 16 study Biology. 3 study all the three subjects, 2 study Chemistry and Biology only and 6 study Biology only.

- a) Illustrate the information in a Venn diagram.
- b) Find the number of students who study
  - i) Chemistry only
  - ii) only one subject
  - iii) only two subjects

4. In a survey of 290 newspaper readers, 181 of them read the Daily Times, 142 read the Guardian, 117 read the Punch and each reads at least one of the three newspapers. If 75 read the Daily Times and the Guardian, 60 read the Daily Times and the Punch and 54 read the Guardian and the Punch.

- a) Illustrate the information on a Venn diagram
- b) How many readers read
  - i) all the three newspapers
  - ii) exactly two of the newspapers
  - iii) exactly one of the newspapers
  - iv) the guardian alone

5. Out of 40 boys in a class, 21 boys play Hockey, 27 play Football and 16 play Badminton. 5 play Hockey and Badminton, 18 play Hockey and Football, and 7 play Football and Badminton. 2 boys play no games at all. Find the number of boys who play:

- a) all the three games
- b) football
- c) football and Badminton but not Hockey

6. Evaluate $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

7. Simplify each of the following:

- a) $\sqrt{180}+\sqrt{20}$

b) $\frac{18}{\sqrt{2}}$.

c) $\frac{6\sqrt{2}-3\sqrt{3}}{5\sqrt{3}-2\sqrt{2}}$

d) $\left(7\sqrt{2}+4\sqrt{7}\right)\left(2\sqrt{7}-\sqrt{2}\right)$

8. Simplify $\sqrt{12}-\sqrt{75}-\sqrt{147}$

9. Simplify $\frac{\sqrt{3}}{\sqrt{3}-1}-\frac{\sqrt{3}}{\sqrt{3}+1}$

10. $\frac{5-\sqrt{3}}{2+\sqrt{3}}$ is expressed in the form $a+b\sqrt{3}$, where a and b are real numbers. Find the value of a and b.

11. Express $\frac{7+\sqrt{5}}{3-\sqrt{5}}$ in the form $p+q\sqrt{5}$, where $p, q \in R$.

12. Find A and B when $\frac{2-\sqrt{3}}{2+\sqrt{3}}$ is expressed in the form $A+B\sqrt{3}$.

13. Simplify $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}$.

14. Express $\frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}$ in the form $p+\sqrt{q}$ where, p and q are integers.

15. Simplify $\frac{3}{2+\sqrt{3}}-\frac{2}{2-\sqrt{3}}$.

16. Evaluate $\left(1+\sqrt{2}\right)^3+\left(1-\sqrt{2}\right)^3$

17. Given that $\sqrt{6}=2.449$, evaluate $\frac{\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

18. Find the value of p, if $2\sqrt{2}+\frac{p}{\sqrt{2}}=\sqrt{2}$.

19. Given that $\sqrt{2}=1.414$, evaluate $\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{8}}$.

20. Express $\frac{\sqrt{3}+2}{2\sqrt{3}-1}$ in the form $A+B\sqrt{C}$.

21. Without using tables evaluate $\sqrt{0.0675}$, leaving your answer in the form $a\sqrt{b}$.

22. Solve the equation $(3-2\sqrt{2})(3+2\sqrt{2})$

23. Simplify $\sqrt{24}-3\sqrt{6}-\sqrt{216}+\sqrt{294}$.

24. Rationalize $\frac{\sqrt{10}+2\sqrt{5}}{\sqrt{5}-\sqrt{10}}$.

25. Solve the equation $\sqrt{(x-3)^3}-64=0$

26. If f and g are real-valued functions defined by $f: x \to 3x$ and $g: x \to x^2+2$, find $f\left(g\left(\frac{1}{2}\right)\right)$.

27. Simplify $\frac{3}{2+\sqrt{3}}-\frac{2}{2-\sqrt{3}}$

28. Find the inverse, g^{-1}, of the function $g: x \to \frac{2}{x}-5, x \neq 0, x \in R$.

29. If $f(g(x)) = x^2-2x+1$ and $f(x) = x-1$, find $g(x)$.

30. The functions g and h are defined as $g: x \to \frac{x}{x-1}$ and $h: x \to x^2+2$.

Find $g \circ h$.

31. Find the range of the function $f : x \to x^2 + 2x + 3$.

32. Find the domain of the function $g(x) = \dfrac{x+1}{2-x}$, where x is real.

33. Find the inverse function $h^{-1}(x)$ of $h : x \to \dfrac{1+5x}{2-x}, x \neq 2$.

34. Given the function $g : x \to 3x - 2$, find a function h such that $gh(x) = x$.

35. If $f(x) = 3x - 2$, $g(x) = 2x + 5$ and the binary operation * is defined for all real numbers by $a * b = a + b - 2ab$, find $f(2) * g(2)$.

36. Find the domain of the function $f : x \to \dfrac{x+1}{(x+2)(x-2)}$.

37. If $f(x) = x^3 - \dfrac{1}{x^3}$ then find $f(-a) - f\left(\dfrac{1}{a}\right)$.

38. Which of the following is/are one-to-one function?

I. $f : x \to (x-2)^2$ II. $g : x \to 5 - x$

III. $h : x \to \dfrac{2x-1}{x+1}$

39. If $f(x) = x^2 + 4$ and $g(x) = (x+1)^2$, find $(fog)(x)$.

40. The function f and g have R as domain and co-domain. If $f(g(x)) = x^2 + 2x + 1$ and $f(x) = x^2$, find $g(x)$.

41. State the largest domain of the function $f : x \to \dfrac{1}{3x+2}$.

42. The functions g and f are defined by: $f : x \to x^2 - 9$ and $g : x \to \dfrac{x-1}{2-x}$, where x is a real number.

a) State the largest possible domain of each of the two functions

b) Find the range of g

c) Find $g \circ f$

d) Determine whether or not f is a one-to-one function

43. Use long division to evaluate $(3x^4 + 2x^3 - 3x^2 + 1) \div (x - 2)$.

44. Two functions, f and g are defined on the set of real numbers by:

$f : x \to \dfrac{x-3}{x+4}$, where $x \neq -4$ and

$g : x \to \dfrac{2x}{x-1}$, where $x \neq -1$. Find

i) g^{-1}, the in inverse of g

ii) the image of -1 under fog

45. Find the zeros of the function $h : x \to \dfrac{x^2 + x - 2}{x^2 - x - 12}$, $x \neq -3$ or 4?

46. Find the domain of the function $f : x \to \dfrac{x+4}{x^2 - 3x - 4}$.

47. If f and g are real-valued functions defined by $f : x \to 2x + 3$ and $g : x \to x - 2$, find $f(g(1))$.

48. Find the range of the function $g : x \to \dfrac{1}{2+x^2}$.

49. If $f : x \to 2x$ and $g : x \to 3x^2 + 1$, find $g(f(-1))$.

50. The functions g and h are defined as $f : x \rightarrow x^2 - 3$ and $g : x \rightarrow 2x + 4$. Find $fog(x)$.

51. Find the inverse function $h^{-1}(x)$ of $h : x \rightarrow \dfrac{x-1}{3}$.

52. Find the domain of the function $g(x) = \dfrac{x+1}{4-x^2}$, where x is real.

53. Find the inverse function $h^{-1}(x)$ of $h : x \rightarrow \dfrac{2}{4x+1}$, $x \neq -\frac{1}{4}$.

54. The inverse of a function f^{-1} is defined by $f^{-1} : x \rightarrow \dfrac{1}{x+4}$. Find the function f

55. If $f(x) = 5x - 2$ and $g(x) = \dfrac{1}{5}x + \dfrac{2}{5}$, show that $f \circ g = g \circ f$.

56. State the largest domain of the function $f : x \rightarrow \dfrac{2x-3}{(x^2-1)(x+2)}$.

57. Find the domain of the function $f : x \rightarrow \dfrac{x+4}{x^2+x-6}$

58. If $h(x) = \dfrac{2}{x^2-3}$, $x \neq \pm\sqrt{3}$, find $h\left(\dfrac{1}{x}\right)$.

59. Show whether or not the function $h : x \rightarrow \dfrac{x-3}{2x+1}$ is one-to-one.

60. $f(x) = 3x^2 - 12x + 6$. If $f(x) = -3$, find the values of x.

61. Two functions f and g are defined on the set R, of real numbers by $f : x \rightarrow x^2 - x - 6$ and $g : x \rightarrow x - 1$, find $f \circ g(3)$.

62. The inverse of a function f^{-1} is defined by $f^{-1} : x \rightarrow \dfrac{3x+1}{2}$. Find the function f.

63. State the largest domain of the function $f : x \rightarrow \dfrac{(x+4)(x+3)}{x^2-2x-3}$.

64. When $f(x) = px^3 - 3x^2 + 8x + 36$ is divided by $x + 2$ it leaves no remainder.

i) Find the value of p

ii) Hence express $f(x)$ in the form $f(x) = (x+2)(ax^2 + bx + c)$ where a, b and c are constants and find the zero of $f(x)$.

65. Find the remainder when:

a) $2x^3 + x - 7$ is divided by $x - 3$

b) $3x^4 + x^3 + 6$ is divided by $x + 4$

66. Factorise the following completely:

a) $2x^3 + 7x^2 - 7x - 12$

b) $3x^3 + 4x^2 - 17x - 6$

c) $2x^3 - 7x^2 - 3x + 18$

67. The remainder when $3x^3 - 6x^2 + ax - 1$ is divided by $(x + 1)$ is twice the remainder obtained when the same expression is divided by $(x - 3)$. Find the value of a.

68. If $(x - 1)$ is a factor of $f(x) = 4x^3 - 4x^2 - k$, where k is a constant,

i) Find the value of k

ii) Factorise $f(k)$ completely and state its zeros.

69. Find the values of p and q for which the expression $x = 4x^4 - 6x^2 + px + q$ is divisible by $(x^2 - 1)$ without a remainder, and hence factorise the expression completely.

70. Find the remainder when the polynomial $f(x) = x^3 - 2x^2 - 9x + 18$ is divided by $x + 1$.

71. Find zeros of the polynomial function $g : x \rightarrow (2x+1)(x+1)(x+3)$

72. When $x^3 - 2x^2 + mx + 4$ is divided by $x - 3$, the remainder is -2. Find m.

73. $x + 2$ is a factor of the polynomial $f(x) = 3x^3 - x^2 + nx + 10$. Find the value of n.

74. Given that $f(x) = 3x^3 + 4x^2 - 6x - 1$ and $g(x) = x - 1$. Find $f(x) \div g(x), x \neq 1$

75. Determine the zeros of the polynomial, $Q(x) = x^3 + x^2 - 4x - 4$

76. If $x - 3$ is a factor of $2x^3 - 5x^2 - kx + 30$, find the constant k.

77. Given that $f : x \rightarrow x^4 - 3x^2 - 7x + 11$, find $f(-1)$

78. Find the zeros of $p(x) = x^3 + 4x^2 + x - 6$

79. If $x + 1$ and $3x + 2$ are factors of $p(x) = 3x^3 + 2x^2 - 3x - 2$, find the third factor.

80. Find the remainder when the polynomial $q(x)=4x^4 - 2x^3 + x - 3x + 5$ is divided by $x-2$

81. If $x^3 + 8 = m(x)(x+2)$, find $m(x)$.

82. If $x + 3$ is factor of $x^2 - 2x + p$, find p.

83. If $x^3 + kx^2 + 3x - 5$ has remainder -3 when divided by $x - 2$, find the value of k.

84. Determine the values of p and q if $x - 1$ and $x + 2$ are factors of $2x^3 + px^2 - x + q$.

85. One of the roots of the polynomial $p(x) = x^3 + 2x^2 - 19x - 20$ is 4. Find the other roots.

86. What is the remainder when $x^4 - 3x^3 + 2x + 1$ is divided by $x - 1$.

87. If $f(x) = ax^3 + 3x^2 + bx + 15$ is divided by $x - 1$, it leaves a remainder -12 and leaves a remainder -21 when divided by $x - 2$.

a) Find the values of the constants a and b.

b) Find the zeros of $f(x)$

88. Find both solutions of the equation $x + 1 = \frac{20}{x}$

89. Find all the solutions that satisfy both the equations $x + y^2 = 9$ and $x - 2y + 3 = 0$.

90. Determine the range of values of k for which the quadratic equation $x^2 + 2kx + 4k - 3 = 0$ has two real and distinct roots.

91. Find the sum of roots of $4x^2 - 1 = 0$

92. If one of the roots of $x^2 - 3x + c = 0$ is 1, what is the other root?

93. Find the value of a for which the quadratic equation $x^2 - 5x + a = 0$ has equal roots.

94. Find the value of k for which the quadratic equation $3x^2 + kx + 12 = 0$ has equal roots.

95. The roots of the equation $3x^2 + 4x - 5 = 0$ are α, β. Find the value of $\frac{1}{\alpha} + \frac{1}{\beta}$.

96. The roots of the equation $2x^2 + 3x - 4 = 0$ are α and β. Find the value of $\frac{\beta}{\alpha} + \frac{\alpha}{\beta}$.

97. Find the value of $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$, If α and β are the roots of the equation $3x^2 - 5x + 1 = 0$

98. Find the value of $\frac{1}{\alpha^2\beta}+\frac{1}{\alpha\beta^2}$, given that the equation $4x^2 + 8x - 1 = 0$ has roots α, β.

99. Find the equation whose roots are α^2, β and $\alpha\beta^2$, given that α, and β are the roots of the equation $x^2 - 5x - 7 = 0$.

100. The roots of the equation $2x^2 \pm 4x + 1 = 0$ are α, and β. Find the equation whose roots are $\alpha - 2$, and $\beta - 2$.

101. Find the value of p if the roots of $x^2 + 6x + p = 0$ are α and $\alpha - 1$

102. Find the quadratic equation whose roots are $2+\sqrt{5}$ and $2-\sqrt{5}$.

103. Find the values of a, if the equation $ax^2 + (a + 1)x + a = 0$, has equal roots.

104. Solve the simultaneous equations $2x + 3y = 4$ and $4x^2 + 6xy + y^2 = 5$

105. If α, β are roots of the equation $2x^2 - 10x + 5 = 0$, find the value of

$$\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^2$$

106. If α and β are the roots of the equation $x^2 - 4x - 2$, find the value of $(\alpha+\beta) - (\alpha\beta)^2$.

107. Solve the equation $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 6 = 0$

108. Find the quadratic equation whose sum of roots is $\frac{1}{\alpha}$ and product of roots is $-\alpha^2$

109. One root of the equation $ax^2 - 6x + 2 = 0$, is twice the other. What is the value of a.

110. If the equation $px^2 + (7 - p)x + p = 0$, has exactly two distinct roots, what is the range of values of p.

111. a) A binary operation $*$ is defined on the set of real numbers by $a*b = a + b\sqrt{2}$. Compute each of the following, leaving your answers in surd form.

i) $-2 * 1$ ii) $3 * (2 * 3)$.

b) Find the set of real values of x and y such that $(2x - 1)(x + 2) \geq 0$

112. The operation Δ is defined on the set of non-zero numbers by $a\Delta b = \frac{a-b}{ab}$

determine whether Δ is

a) commutative b) associative

113. A binary operation $*$ is defined on the set, R of numbers by $a*b = a+b+\frac{3}{4}$, $a, b \in R$

Find

i) the identity element of the operation; ii) the inverse of a real number x;

iii) the inverse of $\frac{2}{3}$

114. A binary operation $*$ is defined on the set **R**, of real numbers by $p*q = p^2 + q^2 - pq$.

a) Determine whether or not $*$ is commutative.

b) Evaluate $-2 * 7$.

115. A binary operation is defined by $a*b = a^2 - b^2 + ab$, where a and b are real numbers. Evaluate $\sqrt{2}*\sqrt{3}$.

116. An operation Δ is defined by $a\Delta b = a - b + ab$, where $a, b \in R$.

a) Find the identity element, e.

b) Find the inverse of -2

c) Find the value of $(-2 \Delta 3) \Delta 1$

117. A binary operation Θ is defined on the set of real numbers by $p \Theta q = p + q + 2$. Find the identity element, e, of R under the operation.

118. Given that $p * q = p^2 - q^2$, where $p, q \in R$ find p when $2p * 8 = 4 * p$.

119. The operation * is defined on the set $S = (a, b, c, d, e)$ by the table below.

a) Calculate $\{(a * e) * e\} * a$.

b) Determine whether or not
   - i) S is closed under *
   - ii) * is commutative

*	*a*	*b*	*c*	*d*	*e*
a	*a*	*b*	*c*	*b*	*a*
b	*b*	*c*	*a*	*e*	*c*
c	*c*	*a*	*b*	*b*	*a*
d	*b*	*e*	*b*	*e*	*d*
e	*d*	*b*	*a*	*d*	*c*

120. A binary, operation Δ, is defined on the set of rational numbers by $a \Delta b = 2ab$.

a) Find the identity or neutral element, e.

b) Find the rational number that has no inverse under Δ.

121. An operation, ⊕, is defined on the set of real number by $a \oplus b = a + b - 2ab$, where $a, b \in R$. Which element, $a \in R$, does not have an inverse under ⊕.

122. A binary operation *, is defined on the set R of real numbers by $m * n = m + n - 2$

a) Find the identity/neutral element, e, under the operation *.

b) Find the inverse of –1 under *.

123. A binary operation Δ, defined on the set P of non-negative real numbers by $x \Delta y = \left(\sqrt{x} + \sqrt{y}\right)^2$, where $x, y \in P$. Calculate 24 Δ 18.

124. An operation, * is defined on the set R of real numbers by $a * b = a + b + 10$. Find the identity element.

125. A binary operation, Δ, is defined on the set of non-zero real numbers by $a \Delta b = ab - \frac{a}{b}$. Find the truth set of $4 \Delta y = 4$.

126. An operation, Δ, is defined on the set R of real numbers by $a \Delta b = a + b - 4ab$. Find $(2 \Delta 3) \Delta -2$.

127. The operation Θ is defined on the set R of real numbers by $p \Theta q = p + q - 4$. Find the inverse of 3 under the operation Θ.

128. A binary operation, *, is defined on the set $S = \{0, 2, 3, 4, 5, 6\}$ by $a * b = a + b$, where $a, b \in S$. Find the identity element, e, of S.

129. The following is a table for the operation ⊕ on the set $T = \{a, b, c, d\}$.

⊕	*a*	*b*	*c*	*d*
a	*b*	*d*	*a*	*c*
b	*d*	*c*	*b*	*a*
c	*a*	*a*	*c*	*d*
d	*c*	*a*	*d*	*b*

a) Which of the following is/are true?
   - I. ⊕ is commutative.
   - II. T is closed with respect to ⊕.
   - III. ⊕ is commutative and T is not closed with respect to ⊕.
   - IV. ⊕ is not commutative but T is closed with respect to ⊕.

b) What is the identity/neutral element?

130. Write each of the following in partial fractions.

a) $\dfrac{8 - x}{(2x - 1)(x + 2)}$

b) $\dfrac{5x^2+12x+10}{(x+1)(x^2+x+1)}$

c) $\dfrac{8-x}{(2x-1)(x+2)}$

d) $\dfrac{3x^3+4x^2-8x-18}{(x+1)(x-2)}$

e) $\dfrac{x^2+3x-13}{(x+2)(x-1)}$

131. Express $\dfrac{2x-1}{(x+1)(2x+1)}$ in partial fractions.

132. For what values of x is the expression $\dfrac{(x+4)(x+3)}{x^2-x-6}$ not defined?

133. Given that $\dfrac{1}{x^2-1} \equiv \dfrac{P}{x+1} + \dfrac{Q}{x-1}$, find the value of Q.

134. Express as a single fraction $\dfrac{k-1}{k} - \dfrac{x-1}{x}$, $x \neq 0, x \neq 0$.

135. Find the values of x for which $y = \dfrac{3x^2-5}{x^2+5x+6}$ is not defined.

136. Express $\dfrac{2x+1}{(x+2)(x+1)}$ in partial fractions.

137. Simplify $\dfrac{7a}{3b} - \dfrac{5-2a}{2b}$

138. If $\dfrac{3x+4}{x^2-3x+2} \equiv \dfrac{A}{x-1} + \dfrac{B}{x-2}$, find the value of $A+B$.

139. If $\dfrac{1}{x^2-1} \equiv \dfrac{R}{x+1} + \dfrac{S}{x-1}$, find the value of R.

140. If $\dfrac{1-x}{x^2+3x+2} \equiv \dfrac{A}{x+1} + \dfrac{B}{x+2}$, find the value of A and B.

141. Given that $\dfrac{2x+3}{x^2+3x+2} \equiv \dfrac{S}{x+2} + \dfrac{T}{x+1}$, $\{x \neq -1, -2\}$, where S and T are constants, find S

142. Express $\dfrac{3x}{(x-6)(x+3)}$ in partial fractions.

143. Simplify $\dfrac{5}{k^2-4} + \dfrac{2}{k+2}$

144. If $\dfrac{2x-15}{(2x-1)(x+3)} \equiv \dfrac{A}{2x-1} + \dfrac{B}{x+3}$, find $A+B$.

145. Write down the expression of $(1+x)^n$, where n is a rational number, as far as the term x^3. Use your expansion to approximate $\sqrt{1.04}$ to three decimal places.

146. Write down the expansion of $(x-2y)^4$

147. The value of the fifth term of the expansion for $(a+2)^{12}$, in descending powers of a, is 48. Find a.

148. Find the coefficient of x^2 in the binomial expansion of $\left(3+\dfrac{1}{2}x\right)^3$.

149. Find n if the coefficient of x^2 in the binomial expansion of $(1+x)^n$, where $n>0$ is 15.

150. Find the coefficient of y^3 in the binomial expansion of $(1+\frac{1}{2}y)^8$.

151. Using the fact that $(1+x)^n \approx 1+nx$, for small values of x, obtain the approximate value of $(1.01)^{100}$, without using mathematical tables.

152. The coefficient of x^3 in the expansion of $(k+3x)^4$ is 54. Find k.

153. Find the term independent of x in the binomial expansion of $\left(x+\frac{1}{x}\right)^6$.

154. The coefficient of x^3 in the expansion of $(2-x)^8$ is

155. Using the relation $(1+x)^n \approx 1+nx$, for small values of x, evaluate $\sqrt[4]{(0.9996)^5}$.

156. Determine the coefficient of x^6 in the expansion of $(x+y)^8$.

157. Find the 4th term in the expansion $(1+2x)^6$.

158. For what value of x is $(1+x)^5 = 1.05$?

159. Find the fifth term in ascending order, in the binomial expansion of $(2x-1)^6$.

160. Evaluate the third term of the binomial expansion of $(1.02)^{\frac{1}{2}}$

161. Evaluate the following

a) $\left\{\frac{1}{64}\right\}^{\frac{-5}{6}}$ b) $\frac{2^{-2}\times 2^{\frac{1}{2}}}{2^{-3}}$ c) $\left\{\frac{27}{8}\right\}^{-\frac{2}{3}}$

d) $6^{\frac{1}{2}n}\times 12^{n+1}\times 27^{-\frac{1}{2}n}\div 32^{\frac{1}{2}n}$

162. Solve the following equations:

a) $4^{x+2} = 2^{3x-1}$ b) $2^{3x-1} = \frac{1}{\sqrt[3]{128}}$

c) $8^{2x+1} = \frac{1}{512}$. d) $3^{\sqrt{x}} = 9^x$

e) $\left\{\frac{1}{4}\right\}^x = 32\sqrt{2}$ f) $2^{x^2} = 8^{2x-3}$

g) $4^{2x} = \frac{1}{2}$ h) $8^{3x} = 5^{2x+4}$

i) $3^{4x} = 81$ j) $3^{x^2+1} = 3^{x+1}$

k) $4^{2x+1} = \frac{1}{16}$ l) $27\sqrt[3]{3} = \left\{\sqrt[3]{\frac{1}{9}}\right\}^x$

m) $9^n - \frac{10}{9}(3^n) = -\frac{1}{9}$

163. Find the truth set of $\log_{10}(10+9x) - \log_{10}(11-x) = 2$.

164. Express as a single logarithm in its simplest form $\log 2 + 2\log 18 - \frac{3}{2}\log 36$.

165. Given that $\log_2 2x + \log_2 8x = 6$, find x.

166. Without using tables find the value of $\frac{\log 0.8 - \log 32 + \log 8}{\log 0.7 + \log 7 - \log 49}$, given that $\log 2 = 0.30103$.

167. Solve the following equations

i) $\log_x 3 + \log_x 27 = 2$

ii) $\log_3 x + 3\log_x 3 = 4$

iii) $\log_5 x + \log_2 8 = 0$

iv) $\log_5 x = 16\log_x 5$

168. Without using tables, evaluate $\log 6 + \log 4 + \log 20 - \log 3 - \log 16$

169. Given that $\log_2 x + 2\log_4 y = 4$, show that $xy = 16$.

Hence solve for x and y the simultaneous equations:

170. Find the value of x in
i) $\log_3 x = \frac{5}{2}$ ii) $\log_x 27 = \frac{3}{2}$

171. If log 2 = 0.30 and log 3 = 0.48, calculate
i) log 6 ii) log 432

172. Given that $\log_3 5 = 1.456$, evaluate, without using tables $\log_3 25 + \log_3 15$

173. Given that $\log_{10} a = p + q$ and $\log_{10} b = p - q$, express
a) $\log_{10} a^3 b^3$, b) $\log_{10}\left(\frac{a}{100b}\right)$, in terms of p and q.

174. Given that $\log_{10} 3 = 0.477$ and $\log_{10} 5 = 0.699$, find the value of $\log_{10}\{16\frac{2}{3}\}$.

175. Evaluate $\{\log_5 625\}^2 - \log_5 25$.

176. Find the value of x if log(2x + 3) = log43.

177. If $\log_{81} x = -\frac{1}{2}$, find the value of x.

178. Evaluate $\frac{1}{3}\log_2 64$.

179. If $\log_{10} x = -0.3010$, find x.

180. If $\log r = a$ and $\log s = b$, then $\log r\sqrt{s}$

181. Simplify 2log3 + 3log2 – 2log6

182. Find the value of x if $\log_x 128 = 7$

183. If $\log_8 2x - 1 = \log_8 y$, express x in terms of y.

184. Given that $2\log_y x = 3$, find the relation connecting x and y.

185. Solve $\log_{10} x - \frac{1}{2}\log_{10} 2 = \frac{1}{2}\log_{10}(x+4)$

186. Evaluate $3 + 2\log_3 243$.

187. Find x if $3\log_3 9 = x$.

188. What is the gradient of the line from the relation $a = Mb^c$, a and b are the variables.

189. Given that $\log_{10} 3 = 0.4771$ and $\log_{10} N = 2.4771$, find N.

190. Given that $\log_{10} 2 = 0.301$ and $\log_{10} 3 = 0.477$, find the value of x in

$$\left\{\frac{2}{3}\right\}^x = \frac{1}{16}.$$

191. a) It is believed two variables x and y are related by $y = ax^b$, where a and b are constants. For the following values, draw a suitable linear graph and use it to estimate, correct to one decimal place, the values of a and b.

x	y
2	113
3	312
4	640
5	1118

b) Use your graph in (a) to find, correct to the nearest whole number, the value of x when $y = 215$.

192. Measured values of x and y are given in the following table:

x	1	2	3	4	5	6	7
y	5.1	4.6	4.2	3.8	3.2	2.4	1.4

It is known that x and y are related by the equation $y^2 = a + bx$. Explain how a straight line graph may be drawn to represent the given equation and draw it for the values given. Use the graph to estimate the value of *a* and *b*. Estimate the greatest possible value of *x*.

193. The table shows corresponding values of variables *R* and *v* obtained in an experiment. By drawing a suitable linear graph, show that these pairs of values may be regarded as approximations to values satisfying a relation of the form $R = a + bv^2$, where *a* and *b* are constants. Use your graph to estimate the values of *a* and *b*, giving your answers to 2 significant figures.

v	5	10	15	20	25
R	149	175	219	280	359

194. The data below fits and equation of the form $y = bx^n$.

x	1	2	3	4	5
y	2	8	18	32	51

Draw a suitable graph and use it to find the values of *b* and *n*. Hence find *y* if $x = 6$.

195. a) In an AP, the 15th term is 67 and the sum of the first 16 terms is 480. Find the 12th term.

b) In a GP, the sum of the first two terms is 4 and the sum of the 4th term and the 5th term is 32. Find the 3rd term.

c) The sum of the first *n* terms of an AP is $4n^2 - 5n$. Write down, in terms of *n*, an expression for the sum of the first $(n - 1)$. Hence find the nth term, the first term and the common difference.

196. In an AP the sum of the first ten terms is 50 and the 5th term is three times the 2nd term. Find the first term and the sum of the first 20 terms.

197. The 8th term of an AP is 11 and the 15th term is 21. Find the first term, common difference and the *n*th term.

198. The 6th term of an AP is twice the 3rd term and the first term is 3. Find the common difference and the 10th term.

199. The *n*th of an AP is $12 - 4n$. Find the first term and the common difference.

200. The second and fourth terms of a geometric series with positive common ratio are 8 and 128 respectively. Find

a) the common ratio,

b) the first term

c) the sum of the first four terms

201. In a sequence, A sequence of numbers $u_1, u_2, u_3, \ldots$ satisfies the relation $U_{n+1} = 3 + \frac{1}{4}U_n$ for all positive integers $n \geq 1$. If 3, find

a) u_2, u_3 and u_4

b) an expression for u_n in terms of *n*.

c) the value of u_n as *n* increases indefinitely

d) a formula for S_n the sum of the first *n* terms.

202. Find the 11th term of the series 8, 14, 20, 26, …

203. Find the 17th term of the series 11, 10.7, 10.4, 10.1, …

204. The seventh term of a series is 29 and the eleventh term is 54. Determine the sixteenth term.

205. Find the 15th term of an arithmetic progression of which the first term is $2\frac{1}{2}$ and the tenth term is 16.

206. Find the sum of the first 11 terms of the series 4, 7, 10, 13, …

207. Three numbers are in arithmetic progression. Their sum is 9 and their product is 20¼. Determine the three numbers. Find the 10th term of the series 5, 10, 20, 40, …

208. Determine the sum of the first 7 terms of the series $\frac{1}{4}, \frac{3}{4}, 2\frac{1}{4}, 6\frac{3}{4},...$

209. Determine the sum to infinity of the series 4, 2, 1, …

210. Find the sum to infinity of the series $2\frac{1}{2}, -1\frac{1}{4}, \frac{5}{8},...$

211. The *n*th term of a geometric progression is $\left(\frac{2}{3}\right)^{n-1}$.

a) Write down the first three term.

b) Find an expression for the sum S_n of the first *n* terms.

c) Find the least value of *n* such that S_n differs from the sum to infinity by less than 0.003.

212. The *n*th term of a sequence is $n^2 - n - 2$. Find the first four terms and the term which equals 108.

213. Evaluate $\sum_{r=1}^{\infty}\left(\frac{1}{2}\right)^r$

214. The first and twentieth terms of an arithmetic sequence are 3 and 117 respectively. Find the

a) common difference

b) the 101st term

215. The sum of the first 20 terms of an arithmetic progression is –145 and the sum of the first 38 terms is –788.5. Find the seventh term of the arithmetic progression.

216. The first term of an arithmetic progression is 7, the last term is 43 and the sum of the terms of the progression is 250. Find the number of terms.

217. The first term of a geometric progression is 1 and the sum of the first 3 terms is $\frac{13}{9}$. Find the two possible values of the common ratio, state the value of the common ratio which causes the geometric progression to have a to sum infinity and find this sum to infinity.

218. The sum of the first two terms of an exponential sequence is 135 and the sum of the third and fourth terms is 60. Given that the common ratio is positive, calculate;

i) the common ratio and the first term,

ii) the limit of the sum of the first *n* terms as *n* becomes large.

219. a) The sixth and the eleventh terms of a linear sequence are respectively 23 and 48. Calculate the sum of the first twenty terms of the sequence.

b) The population of a country increases at the least rate of 3% per year. If the population at the end of 1970 was 8×10^6, calculate the population at the end of 1980.

220. The sum of *n* terms of a certain series is $(4^n - 1)$ for all values of *n*. Find the first 4 terms and the *n*th term. What type of series is it?

221. What is the common ratio of the exponential series (*G.P*): $\left(\sqrt{2}-1\right)+\left(3-2\sqrt{2}\right)+...?$

222. Given that the numbers $\frac{1}{2}, 1\frac{1}{2}, 3\frac{1}{2}, 5\frac{1}{2},...$ form a linear sequence (*A.P*), obtain its *n*th term.

223. Find the sum of the series: $3+1+\frac{1}{3}+\frac{1}{9}+...+\frac{1}{3^{n-2}}$ for large integral values of *n*.

224. By adding the same constant to each of the numbers 60, 100 and 150 respectively, a geometric sequence is obtained. What is the common ratio?

225. An exponential sequence (*G.P*) is given by $8, 2, \frac{1}{2}, \ldots$ Find the sum, S_n, for large values of n.

226. The first and fourth terms of an exponential sequence (*G.P*) are -4 and $\frac{1}{16}$ respectively. Find the common ratio.

227. Find the number of terms in the exponential sequence (*G.P*): $1, -\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{1024}$.

228. Find the sum of the exponential sequence (*G.P*): $3, 2, \frac{4}{3}, \ldots, 2^{n-1}.3^{2-n} \ldots$ for large values of n.

229. Find the sum of the first ten terms of the sequence $u_1, u_2, u_3, \ldots$ where $U_n = 2n+1, \quad n \geq 1$.

230. A sequence of numbers $u_1, u_2, u_3, \ldots$ satisfies the relation $U_{n+2} = 5U_{n+1} - 6U_n$ for all positive integers $n \geq 1$. If $u_1 = 1$ and $u_2 = 2$, find the value of u_4.

231. Find the value of x in the exponential sequence $(\sqrt{2}-1), (3-2\sqrt{2}), x \ldots$

232. The sum of the second and fifth terms of an *A.P* is 19 and the eighth term of the AP is 23. Find the twentieth term of the A.P.

233. A sequence of numbers $u_1, u_2, u_3, \ldots$ satisfies the relation $U_{n+1} = 2 - \dfrac{U_n}{3}$ for all positive integers $n \geq 1$. If $u_1 = 2\frac{1}{2}$, find

a) u_2, u_3 and u_4

b) an expression for u_n in terms of n.

c) the value of u_n as n increases indefinitely

d) a formula for S_n the sum of the first n terms.

234. Find the sum of the odd integers from 1 to 49 inclusive.

235. Find the number of terms in the *A.P* $x + 2x + \ldots + nx$.

236. A sequence of numbers $u_1, u_2, u_3, \ldots$ satisfies the relation $U_{n+1} = 1 + \dfrac{U_n}{2}$ for all positive integers $n \geq 1$. If $u_1 = 4$, find

a) u_2, u_3 and u_4

b) an expression for u_n in terms of n.

c) the value of u_n as n increases indefinitely

d) a formula for S_n the sum of the first n terms.

237. The second and fourth terms of a linear sequence are 9 and 17 respectively. Find the common difference.

238. The sum of the first and third terms of an *A.P* is 22 and that of the second and the fourth is 28.

a) Find the common difference

b) Find the third term.

239. *A*, *B* and *C* are the points (2, 3), (4, 7) and (7, 3) respectively. Find

i) the length of *AB*

ii) the coordinates of the midpoint of *AC*

iii) the coordinates of the point of trisection of *BC*

iv) the angle *BAC*

v) the area of triangle *ABC*

240. The points $A(6, 3)$ and $B(9, 0)$ are two vertices of a triangle OAB. The point P divide AB externally in the ratio 5 : 2 and the point Q divides OB externally in the ratio 5 : 2. Find the coordinates of P and Q and verify that PQ is parallel to OA.

241. The equation of the tangent to a circle at $P(4, 2)$ is $3y = x + 2$. The line $y = 3x$ passes through the centre C of the circle. Find
a) The co-ordinates of C;
b) The equation of the circle;
c) The point T_1 and T_2 where the circle cuts the y – axis.

242. Decide for each of the following pairs of lines whether:
i) they meet
ii) they are parallel
iii) they are perpendicular
a) $3y = 2x + 4$ and $3y = -2x - 7$.
b) $2y - x = 7$ and $4y = 2x - 12$
c) $x - 3y = 4$ and $3y + 9x = 4$

243. Find the equation of the line which:
a) has gradient –7 and passes through (3, 2).
b) passes through (6, 1) and (2, 13).
c) meets the x-axis at –3 and the y-axis at 5.

244. Find the acute angle between the line $y = 2x + 7$ and $2y + 3x = -3$

245. $OABC$ is a parallelogram. O is the origin and B is the point (2, 5), the equation of OA is $x - 3y = 0$ and the gradient of OC is $-\frac{3}{4}$. Calculate
i) the equation of BC,
ii) the coordinates of A,
iii) the length of OA.

246. The equation of a circle is $x^2 + y^2 - 2x + 14y + 44 = 0$. Find the radius of the circle.

247. $R(-3, 1)$ and $S(-1, 3)$ are two points on a circle. If RS is the diameter, find the equation of the circle.

248. Find the locus of a point equidistant from the points $P(-2, 1)$ and $Q(-1, 2)$.

249. Find the radius and coordinates of the centre of the circle $x^2 + y^2 - 3x = 0$.

250. The coordinates of P and Q are (5, 3) and (6, 0) respectively. Find the equation of the mediator of $\overline{PQ}$.

251. Find the equation of the circle centre (4, 5), which passes through the y-intercept of the line $5x - 2y + 6 = 0$.

252. Find the equation of the line which passes through the point (4, –3) and is perpendicular to the line $2x + 5y + 1 = 0$.

253. Find the set of all points L, in the oxy plane, which are equidistant from $P(2, -4)$ and $T(1, -1)$

254. In the oxy plane, the coordinates of A and B are respectively (–4, 6) and (–7, 9). If P is any point in the plane such that $|PA|^2 - |PB|^2 = 7$, find the equation of the locus of P.

255. Find the equation of a straight line which intercepts the x-axis at 5 and the y-axis at –2.

256. A point $P(1, 2)$ lies on a circle with centre $C(3, 5)$.
a) Find the equation of the normal at P.
b) Find the equation of the tangent at P.

257. Find the equation of the normal at P circle with centre (3, 7) and area 16π square units.

258. Find the equation of the line, which makes an angle of 135° with the positive x-axis and passes through the point (0, 4).

259. Find the equation of the line, which is parallel to the line $x + 4y - 1 = 0$ and passes through the point (1, 2).

260. A sector of circle of radius 9 cm has an arc of length 6 cm. Calculate, in cm^2, the area of the sector.

261. Calculate the perpendicular distance of the point $P(-4, 3)$ from the line $3x + 4y = 5$

262. Find the perimeter of the triangle PQR with vertices $P(-2, 3)$, $Q(2, 3)$ and $R(2, 6)$.

263. The area of the region between two concentric circles of radii 8 and 6 is…
[Take $\pi = \frac{22}{7}$]

264. The perimeter of a sector of a circle of radius 4 cm is 11.6 cm. Find, in radians, the angle of the sector.

265. If $\tan\theta = -\frac{3}{4}$, where $90° < \theta < 180°$, find the value of $\cos(60° - \theta)$.

266. Find the value of $\cos 75°$.

267. Express $\dfrac{1}{1 - \sin 60°}$ as a surd.

268. Given that $\tan 45° = 1$ and $\tan 30° = \dfrac{1}{\sqrt{3}}$, find $\tan 75°$.

269. Express $\sin(180 - x)$ in terms of x only.

270. Express $\dfrac{1}{1 + \cos 45°}$ in a form of a surd and simplify your answer.

271. Express $\dfrac{1 + \tan 60°}{1 - \tan 30°}$ in a form of a surd and rationalize your answer.

272. Express $\dfrac{1}{(1 - \sin 45°)^2}$ in the form $A + B\sqrt{C}$.

273. Evaluate $\sin\dfrac{\pi}{3}\cos\dfrac{\pi}{6} + \cos\dfrac{\pi}{3}\sin\dfrac{\pi}{6}$.

274. Find the value of $\sin^2(-30°) + \cos^2(-30°)$.

275. If $\cos x = -\dfrac{3}{5}$ for $\dfrac{\pi}{2} < x < \pi$ and $\sin y = \dfrac{5}{13}$ for $0 < y < \dfrac{\pi}{2}$, find $\cos(x + y)$.

276. If $\sin\alpha = \dfrac{3}{\sqrt{10}}$, for $0 < \alpha < \dfrac{\pi}{2}$, find $\cos 2\alpha$.

277. Find the solution set of $\cos 2\theta = 0.4823$ for $0 < \theta < 90°$.

278. Given that $\tan\theta = -\frac{3}{4}$, where $180° < \theta < 270°$, calculate, without using tables, $\sin 2\theta$.

279. If both α and β belong to the interval $(0, \frac{\pi}{2})$ and $\sin\alpha = \frac{2}{3}$, $\cos\beta = \frac{3}{5}$, find $\cos(\alpha + \beta)$.

280. Evaluate without tables $\sin 36° \cos 45° + \cos 36° \sin 54°$.

281. Simplify $\dfrac{\tan 3\theta - \tan\theta}{1 + \tan 3\theta \tan\theta}$,

282. Eliminate θ from the equations $x = 3\cos\theta$ and $y = 2\sin 2\theta$.

283. Find the smallest positive value of x for which the expression $f(x) = 4\sin(2x + 60°)$ has its greatest value.

284. Given that $f(x) = 3\cos x + 2\sin x$, where $0° \le x \le 360°$.
- a) What is the maximum value of $f(x)$?
- b) For what value of x is $f(x)$ maximum?

285. The area of a sector of a circle is 150 m^2. If the radius is $5\sqrt{2}$ m, find, in radians, the angle the sector subtends at the centre of the circle.

286. If $\tan x = \frac{1}{3}$, find $\cos x$.

287. Prove that

i) $\dfrac{1 - \cos 2A}{\sin 2A} = \tan A$

ii) $\dfrac{\sin(A + B)}{\cos(A + B)} = \tan A + \tan B$

288. Find the value of $\sin 2x$ and $\cos 2x$ given that $\cos x = \frac{3}{5}$

289. If $\sin\theta = \frac{8}{17}$ and θ is obtuse, find $\dfrac{1}{\cos\theta + 1}$.

290. Evaluate $\dfrac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$

291. Given that $f(x) = 2x^3 - 9x^2 + 12x + 3$, find the interval on which f is increasing and the interval on which f is decreasing.

292. The curve given by the equation $y = ax^4 + bx^2$ has a gradient of -2 at the point (1, 1). Find the values of a and b.

293. The tangent to a curve $y = x^2 - 5x + 3$ is parallel to the line $y = x$. Find its equation and where it cuts the x-axis

294. Differentiate $(3x + 1)^7$ with respect to x.

295. What is the derivative, with respect to x, of $\dfrac{1 - x^2}{1 + x^2}$?

296. Differentiate, with respect to x, $\dfrac{1}{x}(3 + 4x + 5x^2)$.

297. Find the derivative, with respect to x of $\dfrac{1 + x}{1 - x^2}$.

298. Find the equation of the tangent to the curve $y = 4x^2 - 12x + 3$ at the point (2, 1).

299. $f : x \to (x^2 + 3)^2$ is defined on the set R of real numbers. Find the gradient of $f(x)$ at $x = \frac{1}{2}$

300. For what values of x will the function $f : x \to x^3 - 6x + 5$ have turning points?

301. Find the gradient of the tangent to the curve $y = x^3 - 2x + 1$ at the point $(-1, 2)$.

302. The equation of a curve is $y = 9x - x^3$
- a) Find the gradient of the curve at the point where $x = 1$.
- b) Find the equation of the tangent to the curve at this point. Where does this tangent meet the line $y = x$?

303. Find the equation of the tangent to the curve $y = x^3 - 9x^2 + 20x - 8$ at (1, 4). At what points of the curve is the tangent parallel to the line $4x + y - 3 = 0$?

304. The distance, *s* metres, covered by a particle in time *t* seconds, is given by $s = t^3 - 6t^2 + 9t + 3$. Find, in ms^{-1}, the velocity of the particle when $t = 1$.

305. A particle moves along a straight line such that its distance *x* metres from a fixed point after *t* seconds is $-\frac{1}{3}t^3 + t^2 - \frac{5}{7}$.

a) Find the time *t* s when the particle is momentarily at rest.

b) Find the average acceleration of the particle between $t = 0$ and $t = 3$.

306. A particle projected from a fixed point with velocity 10 ms^{-1}, moves in a straight line such that its velocity after time *t* seconds is $v = bt^2 + 7t + c$, are where *b* and *c* constants. One second after projection, its acceleration is 1 ms^{-1}.

a) Find the values of *b* and *c*.

b) Find the time when the particle is momentarily at rest.

307. Find the coordinates of the point at which the function $f : x \rightarrow x^3 - 6x^2 + 9x + 2$ is greatest.

308. A rectangle has a perimeter of 24 cm. Find its dimensions if its area is to be a maximum.

309. A stone is projected vertically upwards from the point 0. Its distance from 0 after time *t* is $(27t - 3t^2)$. How long does it take to reach its maximum height?

310. The surface area of a sphere is $4\pi r^2$. If the radius *r* of the sphere is decreased from 5 cm to 4.9 cm, what is the approximate decrease in surface area?

311. The radius *r* of a sphere is 4 cm and is increasing at the rate of 0.5 cm/sec. Find the rate of increase of the volume.

312. Evaluate $\int_1^4 (2x^2 - 5x)dx$

313. Evaluate $\int_{-1}^1 (x^3 - x)dx$

314. Given that $\frac{dy}{dx} = 2x - \frac{1}{x^2}$, $x \neq 0$, find *y*, if $x = 1$ when $y = 1$.

315. Find $\int (x + 1)(x^2 - 2)dx$.

316. Evaluate $\int \frac{x^5 - x^3 + 1}{x^2} dx$

317. If $R = \int_{-1}^0 (x + 1)(x - 2)dx$, find *R*.

318. Given the table below, use the trapezium rule to estimate the value of

$$I = \int_1^5 (x^2 - 2x + 1)^2 dx$$

x	1	2	3	4	5
y	0	1	4	9	16

319. Find the area of the region enclosed by the curve $y + 1 = x^2$ and the *x*-axis.

320. The gradient of a curve at *P*(*x*, *y*) is $3x^2$. If the curve passes through *Q*(1, 5), find the equation of the curve.

321. Find the area enclosed by the *x*-axis, $x = -2$, $x = -3$ and the curve $y = x^3$.

322. The velocity of a particle moving in a straight line at time t seconds is given by $v = 3t^2 - 2t$. Find an expression for the distance (s metres) travelled, given that $s = 2$ when $t = 1$.

323. The acceleration after time t seconds of a particle moving in a straight line is $(2t - 7)\ \text{ms}^{-2}$. If the velocity of the particle is zero when $t = 2$, calculate the velocity of the particle when $t = 1$.

324. The finite area enclosed by the curve $y^2 = 4x$ and the line $x = 4$ is rotated through 360° about the x-axis. Find the volume of the solid generated in cubic units.

325. The acceleration of a body, in ms^{-2}, is given by $a = 2t + 3$. Find the distance s, travelled after 2 seconds, given that $v = 4$ and $s = 0$ when $t = 0$.

326. Find the area bounded by the curves $y = x^2$, $y = 2x$ and the line $x = 1$ and $x = 2$.

327. Using the trapezium rule with five ordinates at $x = 1, 2, 3, 4, 5$, estimate the value of $\int_1^5 (x-2)^2\,dx$

328. a) Find in square units, the area of the finite region bounded by the curve $y = x^3 - x$ and the x-axis, from $x = -1$ to $x = 1$

b) The gradient at any point $P(x, y)$ on a curve is $3 - 2x$. If $(0, 5)$ lies on the curve, find its equation

329. The area enclosed by the curve $y = x^2$, for which $x > 0$, $y = 1$ and $y = 3$ is rotated through four right angles about the y-axis. Find, in cubic units, the volume of the solid generated.

330. Evaluate $\int_{-3}^{3} (x - x^2)dx$.

331. The following table shows the grouped frequency distribution of the marks scored by 100 candidates in an examination.

Marks	Frequency
0 – 9	2
10 – 19	6
20 – 29	8
30 – 39	18
40 – 49	28
50 – 59	18
60 – 69	10
70 – 79	6
80 – 89	3
90 – 99	1

a) Construct a cumulative frequency table and use it to draw a cumulative curve

b) Use the curve to estimate

  i) The median of the distribution;

  ii) The lowest mark for the distribution;

  iii) The probability of selecting a candidate who scored more that 45%

332. The ages (in years) of a sample of 10 children are 2, 3, 3, 4, 5, 5, 5, 6, 7, 8.

a) Find the range of their ages.

b) Calculate, correct to 2 decimal places, the standard deviation of the ages.

333. The mean of the eight numbers 4, 6, 4, 7, x, $x + 1$, 8 and 2 is 5. Find x.

334. The table below is the marks scored students in class

Marks	0	1	2	3	4	5
Freq.	1	4	8	10	9	2

a) Find the median of the distribution.

b) Find the mode of the distribution.

c) What is the mean of the distribution?

335. The mean of n numbers is 9. If 14 is added to the sum of the n numbers the mean is 10. Find the value of n.

336. A distribution has a standard deviation 0.6. What is its variance?

337. Find the numerical difference between the median and the mean of the following numbers: 27, 41, 32, 14 and 11.

338. Estimate from the graph the number of candidate who obtained 72.5% and above.

339. Find the least pass mark if 80 candidates passed.

340. Given that the mean of the numbers 14, 20, 16, 25, 17 and 28 is 20, calculate their standard deviation.

341. Given that the standard deviation of the set of numbers $X = \{4, 5, 6, 7, 8\}$ is 1.41, find the standard deviation of the set of numbers $Y = \{12, 13, 14, 15, 16\}$.

342. In a distribution of 320 items arranged in order of magnitude, the upper quartile is the value of the..........................

343. A distribution has a lower quartile of 137 and an upper quartile of 156. Calculate the semi inter-quartile range.

344. Find the inter-quartile range of the following set of marks:
3, 3, 1, 2, 4, 6, 2, 2, 4, 3.

345. Twenty students score the following marks in a history test: 5, 2, 5, 3, 1, 6, 2, 2, 3, 4, 2, 1, 2, 2, 4, 3, 2, 2, 2, 3. What is the mode of the distribution?

346. The table below shows the distribution of goals scored by football teams in a sports festival.

No. of goal scored	1	2	3	4	5	6
Frequency	4	2	1	2	1	2

a) Calculate the mean number of goals scored.
b) Calculate the standard deviation of the distribution.

347. Some articles are measured to the nearest cm with the following results.

Length (cm)	8-10	11-13	14-18
Frequency	1	4	8

a) What is the width of the class 11–13cm?
b) The data in the table was represented on a histogram and the height for 8 – 10 cm class was 4 cm. What was the height of the 14 – 18 cm class?

348. Find the numerical difference between the median and the mode of the following numbers 3, 3, 2, 5, 5, 8, and 5.

349. A coin is tossed three times. What is the probability of getting exactly two heads?

350. An unbiased die is thrown twice. What is the probability of obtaining a sum of 9?

351. A fair coin and a fair die are tossed together. What is the probability of obtaining a head and an even number?

352. In how many ways can 9 mangoes be selected from a basket containing twelve mangoes?

353. In how many ways can 5 children be seated around a circular table?

354. Thirty cards, numbered from 31 to 60 are placed in a box. If one card is drawn at random, what is the probability that the number on the card contains the digit 5?

355. A bag contains 12 blue and 8 red pens. 2 pens are chosen at random from the bag without replacement. What is the probability that one is red and the other is blue?

356. The probabilities of three horses *P, Q, R* wining a race have been found to be 0.2, 0.5 and 0.3 respectively. What is the probability that the three horses will tie for a first position in a race?

357. Two dice are thrown. Find the probability that the two dice show different scores?

358. Find the value o f *k* for which the quadratic equation $h(x) = 4 + 12x + kx^2$ has equal roots.

359. Find the sum to infinity of the series

$$\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \ldots$$

360. If the factors of $2x^2 - 3x - 2$ are also factors of $2x^3 - x^2 - hx - 2$ find *h*.

361. Find, with respect to *x*, the first derivative of $(2 - 3x)^5$.

362. The quadratic function $f(x) = x^2 - x - 2$ has a minimum point at $\left(\frac{1}{2}, \frac{9}{4}\right)$. For what values of *x* is *f*(*x*) decreasing as *x* increases?

363. Evaluate $\int_{-3}^{-2} (2x^3 - 3x^2)dx$

364. Given the vectors $u = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, calculate $|u - v|$.

365. $\overrightarrow{AB} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$, $\overrightarrow{CB} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ are two vectors in *OXY* plane. If *U* is the midpoint of $\overrightarrow{AB}$, find vector $\overrightarrow{CU}$.

366. A sports squad consists of 8 lawn tennis player and 6 chess players. If 5 players are to be selected at random from the squad, calculate the probability that the selected group will consist of

i) lawn tennis and 2 chess players.
ii) no lawn tennis players.

367. a) In a triangle *ABC X* lies on *AB* such that *AX* : *XA* = 1 : 2 and *Y* lies on *AC* such that *AY* : *YC* = 1 : 2. Show that *XY* is parallel to *BC* and is one-third its length.

b) A particle of mass 8 kg moves with a velocity $12\mathbf{i} + 8\mathbf{j}$ ms^{-1} and collides with another particle of mass 5 kg moving with a velocity $3\mathbf{i} + 18\mathbf{j}$ ms^{-1}. If after impact they move together, find, correct to two decimal places, the magnitude of their common velocity.

368. The table below shows the masses, to the nearest kg, of 100 workers in a factory.

Mass (kg)	Frequency
50 – 59	0
60 – 69	8
70 – 79	18
80 – 89	30
90 – 99	20
100 – 109	9
110 – 119	5

i) Using an assumed mean of 84.5 kg, calculate, correct to one decimal place,
   a) the mean of the distribution;
   b) the standard deviation of the distribution.
ii) If a worker is selected at random for a blood test, what is the probability that his mass is between 70 kg and 99 kg

A note to the student

Question practice is a crucial part of the revision process and can make a significant difference to the grade you get in the final examination. By practicing questions you will quickly learn what the examiners are looking for in an answer.

The questions in **chapters 33 and 34** are designed to help you improve your potential grade, giving you the practice you need. The numerous **solved examination questions** will also help you to avoid the common mistakes students make in answering questions.

CHAPTER 34

HOW PREPARED ARE YOU?

1. Which of the following binary operations is not commutative?

A. $a \copyright b = a + b - ab$

B. $a \copyright b = 2a + 2b - ab$

C. $a \copyright b = \frac{1}{a} + \frac{1}{b}$

D. $a \copyright b = a - b + ab$

2. A function, f is defined by $f : x \rightarrow \dfrac{3x+1}{2}$

Find $f^{-1}(x)$.

A. $\frac{2x-1}{3}$ B $\frac{2x+1}{3}$ C. $\frac{2(x-1)}{3}$ D. $\frac{x-2}{3}$

3. An exponential sequence is defined by $30, 15, 7\frac{1}{2}, \ldots$ Find the sum to infinity of this sequence.

A. 15 B. 30 C. 45 D. 60

4. Find the coefficient of y^3 in the binomial expansion of $(x - 2y)^4$.

A. $-32x$ B. $-4x$ C. $4x$ D. $32x$

5. Find the derivative of the function

$g : x \rightarrow x^2 - \frac{1}{x^3}$

A. $2x - \frac{3}{x^4}$ B. $2x + \frac{3}{x^4}$

C. $2x - \frac{3}{x^2}$ D. $2x + \frac{3}{x^2}$

6. If $(3 + 4\sqrt{3})(2 - a\sqrt{3}) = -18 + 2\sqrt{3}$, find the value of a.

A. -4 B. -2 C. 2 D. 3

7. Find the equation of the line which makes an angle of 135° with the positive x-axis and passes through the point (0, 4).

A. $y + x + 4 = 0$ B. $y + x - 4 = 0$

C. $y - x + 4 = 0$ D. $y - x - 4 = 0$

8. Given that $2\log_y x = 3$, find the relation involving x and y.

A. $y^3 = x^2$ B. $y^3 = x$

C. $y^2 = x^3$ D. $y = x^3$

9. $\mathbf{M} : (x, y) \rightarrow (-y, 2x + y)$ is a linear transformation in the Oxy plane. Find **M**.

A. $\begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$ B. $\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$

C. $\begin{pmatrix} 0 & -1 \\ 2 & 2 \end{pmatrix}$ D. $\begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$

10. Given that $y = (2x^3 + 2x^2 + 1)^5$, find $\dfrac{dy}{dx}$.

A. $5(2x^2 + 4x)(2x^3 + 2x^2 + 1)^4$

B. $5(6x^2 + 4x)(2x^3 + 2x^2 + 1)^4$

C. $(6x^2 + 4x)(2x + 2x + 1)^4$

D. $5(6x^2 + 2x)(2x^3 + 2x + 1)^4$

A bag contains 8 red and 4 blue identical balls. Two of the balls are selected at random one after the other without replacement.

Use this information to answer Questions **11** *and* **12.**

11. What is the probability that they are both red?

A. $\frac{14}{33}$ B. $\frac{9}{14}$ C. $\frac{1}{3}$ D. $\frac{2}{11}$

12. If the first ball is found to be blue, what is the probability that the second ball is red?

A. $\frac{8}{33}$ B. $\frac{4}{11}$ C. $\frac{1}{3}$ D. $\frac{8}{11}$

13. If $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$, find **AB**.

A. $\begin{pmatrix} 3 & 7 \\ 1 & 2 \end{pmatrix}$ B. $\begin{pmatrix} 3 & 7 \\ 0 & 2 \end{pmatrix}$

C. $\begin{pmatrix} 3 & 7 \\ 2 & 2 \end{pmatrix}$ D. $\begin{pmatrix} 3 & 7 \\ 2 & 0 \end{pmatrix}$

14. Given that **P** = (7N, 060°), express this in the form $a\mathbf{i} + b\mathbf{j}$, where a and b are scalars.

A. $\frac{7}{2}\mathbf{i} + \frac{7}{2}\mathbf{j}$ B. $\frac{7}{2}\mathbf{i} - \frac{7}{2}\mathbf{j}$

C. $\frac{7}{2}\sqrt{3}\,\mathbf{i} + \frac{7}{2}\mathbf{j}$ D. $7\sqrt{3}\,\mathbf{i} + 7\mathbf{j}$

15. For what values of x will the function $f: x \to x^3 + 6x^2 + 5$ have turning points?

A. 0, 4 B. 0, 2 C. 0, –2 D. 0, –4

16. The mean of 12 numbers is 18. If each of the numbers is increased by 4, what is the new mean?

A. 14 B. 8 C. 2 D. 6

17. Evaluate $\int_{1}^{3} (4x^3 - 6x + 1)dx$

A. 54 B. 56 C. 57 D. 58

18. Find the value of x in the equation $8^{2x+1} = \frac{1}{512}$

A. –5 B. –2 C. –1 D. 4

19. Find the equation of the line which passes through the point (1, 3) and which is perpendicular to the line $2y - 3x + 1 = 0$.

A. $2y + 3x - 9 = 0$ B. $y - 3x - 9 = 0$

C. $3y + 2x - 11 = 0$ D. $2y + 3x - 3 = 0$

20. If $\tan\theta = -\frac{3}{4}$, where $90° \le \theta \le 180°$, find the value of $\cos(60° - \theta)$.

A. $-\frac{2}{5} + \frac{3}{10}\sqrt{3}$ B. $-\frac{2}{5} - \frac{3}{10}\sqrt{3}$

C. $\frac{2}{5} - \frac{3}{10}\sqrt{3}$ D. $\frac{2}{5} + \frac{3}{10}\sqrt{3}$

21. The second and fourth terms of a linear sequence are 9 and 17 respectively. Find the common difference.

A. 2 B. 4 C. 5 D. 8

22. V and P are related by $V = aP^n$, where a and n are constants. Find the gradient of the straight line obtained form it.

A. $\log a$ B. $\log n$ C. a D. n

23. Two smooth balls of masses 20 g and 15 g moving with speeds 9 ms^{-1} and 4 ms^{-1} respectively collide. If they were moving in the same direction and the speed of the first ball reduces to 6 ms^{-1}, find the speed of the second ball after collision.

A. 8 ms^{-1} B. 6 ms^{-1} C. 5 ms^{-1} D. 1 ms^{-1}

24. Two functions f and g are defined by $f: x \to x + 3$ and $g: x \to x^2 - 1$. Find $g \circ f$.

A. $x^2 + 2$ B. $x^2 + 3$

C. $x^2 + 3x + 8$ D. $x^2 + 6x + 8$

25. Which of the following matrices has no multiplicative inverse?

A. $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ B. $\begin{pmatrix} 3 & 4 \\ -1 & 7 \end{pmatrix}$

C. $\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$ D. $\begin{pmatrix} 5 & 2 \\ 1 & 2 \end{pmatrix}$

26. A binary operation is defined by $a * b = a^2 - b^2 + ab$, where a and b are real numbers. Evaluate $\sqrt{2} * \sqrt{3}$.

A. 17 B. $\sqrt{6} - 1$

C. $1 + \sqrt{6}$ D. $1 - \sqrt{6}$

27. If α and β are the roots of $4x^2 - 9x - 16 = 0$, which of the following statements is/are correct?

I $\alpha + \beta = \frac{9}{4}$ II $\alpha\beta = -4$

III $\alpha + \beta = -\frac{9}{4}$

A. I only B. II only

C. I and II only D. II and III only

28. Given that $f: x \to x^4 - 3x^2 - 7x + 11$, find $f(-1)$.

A. 2 B. 8 C. 14 D. 16

29. If **a** = 3**i** + 4**j** and **b** = 10**i** - 2**j**, find **b** – 2**a**.

A. 4**i** – 10**j** B. 6**i** + 6**j**
C. 6**i** – 2**j** D. 6**i** – 10**j**

30. The probability that Joe and Sarah solve a problem correctly are $\frac{2}{3}$ and $\frac{1}{5}$ respectively. If they both attempt the problem, find the probability that only one of them solves its correctly.

A. $\frac{4}{5}$ B. $\frac{3}{5}$ C. $\frac{2}{5}$ D. $\frac{2}{15}$

31. The equation of the line of best fit for two variables X and Y, where X is the independent variable, is given by $Y = 4.2X - 0.3$. Predict the value of Y when $X = 7.1$.

A. 30.12 B. 29.82 C.29.52 D.29.10

32. Given that $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}$, find 2**A** + 3**B**.

A. $\begin{pmatrix} 11 & -4 \\ 1 & 18 \end{pmatrix}$ B. $\begin{pmatrix} 5 & -1 \\ 1 & 7 \end{pmatrix}$

C. $\begin{pmatrix} 10 & -2 \\ 2 & 4 \end{pmatrix}$ D. $\begin{pmatrix} 11 & -8 \\ 1 & 18 \end{pmatrix}$

33. $\frac{5-\sqrt{3}}{2+\sqrt{3}}$ is expressed in the form $a + b\sqrt{3}$, where a and b are real numbers. Find the value of b.

A. 10 B. 7 C. –7 D. –10

34. The equation of a circle is given by $3x^2 + 6x + 3y^2 - 9y - 3 = 0$. Find the co-ordinates of the centre.

A. $(-2, \frac{3}{2})$ B. $(-1, \frac{3}{2})$
C. $(1, -\frac{3}{2})$ D. $(2, -3)$

The table below shows the distribution of marks of 20 students in a test.

Marks	0–19	20–39	40–59	60–79	80–89
Frequency	2	6	8	3	1

Use this to answer Questions **35** *to* **37**.

35. What is the lower class boundary of the median class?

A. 59.5 B. 40.5 C. 39.5 D. 19.5

36. Find the mean.

A. 47.6 B. 45.0 C. 44.8 D. 44.5

37. If a student is selected at random from the group, what is the probability that he obtained at least 59.5 marks?

A. $\frac{3}{20}$ B. $\frac{1}{5}$ C. $\frac{2}{5}$ D. $\frac{3}{5}$

A particle starts form rest and moves in a straight line in such a way that its velocity, v ms^{-1} at time t seconds id given by $v = 3t^2 - 6t$.
Use the information to answer Questions **38** *and* **39**.

38. Calculate the distance travelled in the first 4 seconds.

A. 12 m B. 16 m C. 64 m D.96 m

39. Calculate the average acceleration in the 3rd second.

A. 0 ms^{-2} B. 3 ms^{-2}
C. 6 ms^{-2} D. 9 ms^{-2}

40. Find the angle between **i** + 5**j** and 5**i** – **j**.

A.0° B. 45° C. 60° D. 90°

41. If $\overrightarrow{OA} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$ and N is the mid-point of $\overrightarrow{AB}$, find $\overrightarrow{ON}$.

A. $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$ B. $\begin{pmatrix} 2 \\ -6 \end{pmatrix}$

C. $\begin{pmatrix} 6 \\ 12 \end{pmatrix}$ D. $\begin{pmatrix} 8 \\ 27 \end{pmatrix}$

42. Given that $\begin{pmatrix} 1 & 0 \\ 2 & x \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 6 & 3 \end{pmatrix}$, find x.

A. 3 B. 1 C. 0 D. –1

43. Find the gradient of the tangent to the curve $y = x^3 - 2x + 1$ at the point $(-1, 2)$.
A. –1 B. 1 C. 5 D. 10

44. If $\dfrac{3x+4}{x^2-3x+2} \equiv \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)}$, find the value of $A + B$.
A. –3 B. 2 C. 3 D. 4

45. The gradient of a curve is given by $3x^2 + 2x$. Find the equation of the curve if the point $(-2, 2)$ lies on the curve.
A. $y = 3x^3 + x^2 + 6$ B. $y = 3x^3 + x^2 - 2$
C. $y = x^3 + x^2 + 6$ D. $y = x^3 + x^2 - 2$

46. $Q'(3, 5)$ is the image of Q under a linear transformation, $\mathbf{T} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$

Find the coordinates of Q.
A. (1, 0) B. (0, 11)
C. (6, 5) D. (14, 25)

47. A particle of mass 3 kg has an initial velocity $\mathbf{u} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ ms^{-1} and a final velocity $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ ms^{-1}. Find the change in momentum.

A. $\begin{pmatrix} -3 \\ -3 \end{pmatrix}$Ns B. $\begin{pmatrix} -3 \\ 21 \end{pmatrix}$Ns
C. $\begin{pmatrix} 15 \\ -3 \end{pmatrix}$Ns D. $\begin{pmatrix} 15 \\ 3 \end{pmatrix}$Ns

48. A force of 30 N acts on a particle of mass 6 kg which is at rest. How far will it move in 3 seconds?
A. $\frac{5}{3}$ m B. 15 m C. 16 m D. $\frac{45}{2}$ m

49. When $x^3 - 2x^2 + mx + 4$ is divided by $x - 3$, the remainder is –2. Find m.
A. –2 B. –3 C. -4 D. –5

50. An equation of the circle of radius $\sqrt{7}$ with centre at the origin is
A. $x^2 - y^2 = \sqrt{7}$ B. $x^2 - y^2 = 7$
C. $x^2 + y^2 = \sqrt{7}$ D. $x^2 + y^2 = 7$

51. If $4^{x+2} = 2^{3x+1}$, find x.
A. $\frac{3}{2}$ B. 3 C. 4 D. 5

52. The polynomial $2x^3 - 4x^2 + x - 7$ is divided by $x - 1$. Find the remainder
A. –14 B. –8 C. 0 D. 14

The equation of a circle is
$x^2 + y^2 + 2x - 6y = 15$.
Use this information to answer Questions **53** *and* **54**

53. Find the coordinates of the centre of the circle.
A. (–1, 3) B. (1, –3)
C. (–1, –3) D. (1, 3)

54. What is the radius of the circle?
A. $\sqrt{5}$ B. $\sqrt{7}$ C. 5 D. 25

55. If $y = 3x^2 - 5$, $x = 2$ and $\Delta x = 0.1$, find Δy, where Δx and Δy are the increments in x and y respectively.
A. 1.2 B. 1.3 C. 1.4 D. 1.6

56. Differentiate, with respect to x, $\dfrac{1+x}{1-x^2}$

A. $\dfrac{-1}{(1-x)^2}$ B. $\dfrac{1}{(1-x)^2}$
C. $\dfrac{1}{(1-x^2)^2}$ D. $\dfrac{1}{(1+x)^2}$

57. A rectangle has a perimeter of 24 cm. Find its dimensions if its area is to be a maximum.
A. 6 × 3 B. 6 × 6
C. 4 × 8 D. 12 × 12

58. What is $\frac{2\pi}{3}$ in degrees?
A. 60° B. 120° C. 210° D. 270°

59. Given the table below, use the trapezium rule to estimate the value of

$$I = \int_1^5 (x^2 - 2x + 1)dx$$

x	1	2	3	4	5
y	0	1	4	9	16

A. 20 B. 21 C. 22 D. 25

60. Evaluate $\int_0^1 (x^3+1)^2 dx$

A. $\frac{2}{14}$ B. $\frac{11}{14}$ C. $\frac{23}{14}$ D. $\frac{31}{14}$

61. The frequency table was constructed from the raw data

x	1	2	3	4
Frequency	1	2	3	2

A. x = 1, 2, 2, 3, 3, 3, 4, 4, 4, 4.
B. x = 1, 2, 2, 3, 3, 3, 2, 2, 2, 2.
C. x = 1, 2, 2, 3, 3, 3, 4, 4,.
D. x = 1, 2, 2, 3, 3, 4, 4.

62. Indicate which of the following statements is/are true.

I. The range is a measure of dispersion and its calculation takes into account every unit in the data.
II. The mean is always unique, that is a set of data has one and only one mean.
III. When we plot class frequencies (vertical axis) against class marks, we obtain a graph known as the ogive.

A. I only B. II only
C. III only D. I and II only

The table below shows the scores obtained by candidates in an examination.

Scores	Number of candidates
0 – 9	8
10 – 19	14
20 – 29	48
30 – 39	18
40 – 49	12

Use this information to answer Questions **63** *and* **64**.

63. What is the median class?

A. 10 – 19 B. 20 – 29
C. 48 D. 30 – 39

64. What is the probability that a candidate selected at random had at least a score of 30?

A. 0.70 B.0.30 C.0.20 D. 0.18

65. In how many ways can three vacancies be filled out of 5 applications?

A. 10 B. 15 C. 20 D.30

66. A fair die is thrown and a fair coin is tossed once. What is the probability of obtaining a 2 and a head?

A. $\frac{2}{3}$ B. $\frac{1}{2}$ C. $\frac{1}{6}$ D. $\frac{1}{12}$

67. Which of the following indicate that the events A and B are independent?

I. $P(A) = \frac{1}{4}, P(B) = \frac{1}{5}$ $P(A \cap B) = \frac{1}{20}$
II. $P(A) = \frac{1}{2}, P(B) = \frac{1}{6}$ $P(A \cap B) = \frac{2}{3}$
III. $P(A) = 0.25$ $P(B) = 0.2$ $P(A \cap B) = 0.05$

A. I and II only B. II and III only
C. I, II and III D. I and III only

68. A box contains four tickets numbered 1, 2, 3 and 4. Two tickets are drawn one after the other, without replacement and their numbers added. Find the probability that the sum is prime.

A. $\frac{7}{17}$ B. $\frac{1}{2}$ C. $\frac{9}{16}$ D. $\frac{2}{3}$

69. Find the angle between the vectors

$$\mathbf{p} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } \mathbf{q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

A. 53.13° B. 48.19°
C. 41.81 D. 36.87°

A vector **v** has initial point (4, –3) and end point (–2, 4).
Use this information to answer Questions **70** *and* **71**.

70. Find the vector representation of vector **v**.

A. $-6\mathbf{i} + 7\mathbf{j}$ B. $-7\mathbf{i} + 6\mathbf{j}$
C. $-6\mathbf{i} - 7\mathbf{j}$ D. $-7\mathbf{i} + 6\mathbf{j}$

71. Find the magnitude of the vector **v**.

A. 85 B. 42 C. $\sqrt{85}$ D. $\sqrt{42}$

72. Given that $\mathbf{p} = (5\text{N}, 030°)$, express this in the form $a\mathbf{i} + b\mathbf{j}$ where a and b are scalars.

A. $\frac{5}{2}\mathbf{i} + \frac{\sqrt{5}}{3}\mathbf{j}$ B. $\frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j}$
C. $\frac{5\sqrt{3}}{2}\mathbf{i} + \frac{5}{2}\mathbf{j}$ D. $\frac{\sqrt{3}}{2}\mathbf{i} + \frac{5}{2}\mathbf{j}$

73. A particle of mass 2 kg has an initial velocity $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ms^{-1} and final velocity $\mathbf{v} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ ms^{-1}. Find the change in momentum.

A. $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$Ns B. $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$Ns
C. $\begin{pmatrix} -6 \\ -4 \end{pmatrix}$Ns D. $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$Ns

74. Find the velocity after 2 seconds of a body of mass $\frac{1}{2}$ kg which moves from rest in a straight line under the action of a force $\mathbf{F} = (4 - \frac{1}{2}t)$ N.

A. 12 ms^{-1} B. 14 ms^{-1}
C. 16 ms^{-1} D. 18 ms^{-1}

75. Calculate the magnitude of a force required to move a particle of mass 3 kg from rest through a distance of 4m in 1 second.

A. 24 N B. 16 N C. 8 N D. 6 N

76. The distance s metres travelled by a particle in a straight line is $s = 12t^2 + 110t$. What is the velocity after 1 second?

A. 112ms^{-1} B. 134ms^{-1}
C. 144ms^{-1} D. 220ms^{-1}

77. A linear transformation **T** is defined by $\mathbf{T} : (x, y) \to (2x - y, 3x + y)$. Find the image of $Q(2, -4)$ under **T**.

A. (8, –2) B. (8, 2)
C. (0, –2) D. (0, 2)

78. In a transformation in the plane, the image of $P(5, 2)$ is $P'(7, 3)$. Find the image of $Q(-3, 0)$ under the transformation.

A. (–1, 1) B. (2, 1)
C. (1, 3) D. (0, 3)

79. Find the matrix of the linear transformation $x' = 3x + y$, $y' = 2x - 3y$.

A. $\begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}$ B. $\begin{pmatrix} 3 & 1 \\ 2 & -3 \end{pmatrix}$
C. $\begin{pmatrix} 1 & -3 \\ 3 & 2 \end{pmatrix}$ D. $\begin{pmatrix} 2 & -3 \\ 3 & 1 \end{pmatrix}$

80. If $\mathbf{A} = \begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, find $\mathbf{A}^2\mathbf{B}$.

A. $\begin{pmatrix} -3 & 12 \\ 2 & 8 \end{pmatrix}$ B. $\begin{pmatrix} -2 & -3 \\ -6 & -9 \end{pmatrix}$
C. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ D. $\begin{pmatrix} 0 & -12 \\ 2 & 9 \end{pmatrix}$

81. Solve for **X** in the matrix equation

$$\mathbf{X} + \begin{pmatrix} -3 & -2 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ -2 & 0 \end{pmatrix}$$

A. $\begin{pmatrix} -3 & -3 \\ 2 & 6 \end{pmatrix}$ B. $\begin{pmatrix} 9 & 1 \\ -2 & -6 \end{pmatrix}$
C. $\begin{pmatrix} 3 & 2 \\ 0 & -6 \end{pmatrix}$ D. $\begin{pmatrix} 9 & 3 \\ -2 & 6 \end{pmatrix}$

82. Solve $\begin{pmatrix} x - y \\ x + y \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

A. $x = 2, y = -4$ B. $x = 2, y = 4$
C. $x = -2, y = -4$ D. $x = -2, y = 4$

83. If $\mathbf{P} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$, find $\mathbf{P}^{-1}$.

A. $\begin{pmatrix} 2 & 3 \\ -3 & 5 \end{pmatrix}$ B. $\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$

C. $\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$ D. $\begin{pmatrix} -5 & -3 \\ 3 & 2 \end{pmatrix}$

84. P contains 4 elements, Q contains 3 elements, and 2 elements are common to both P and Q. State the number of elements in $P \cup Q$.

A. 5 B. 6 C. 7 D. 9

85. Which of the following binary operation is not commutative?

A. $a \Delta b = 2ab + ab$

B. $a \Delta b = \frac{1}{2}ab - ab$

C. $a \Delta b = a^2 - b^2$

D. $a \Delta b = a^2 + b^2$

86. Change 145° to radians.

A. $\frac{\pi}{4}$ B. $\frac{3\pi}{4}$ C. $\frac{29\pi}{36}$ D. $\frac{5\pi}{6}$

87. Find the remainder when $f(x) = x^3 + 2x^2 - 13x - 15$ is divided by $x + 1$.

A. –24 B. –1 C. 0 D. 25

88. $f(x) = 3x^3 + 8x^2 + 6x - k$. If $f(2) = 1$, find k.

A. –67 B. –61 C. 61 D. 67

89. If $f(x) = 2x$ and $g(x) = x^2 + 1$, find $f\,[g(2)]$.

A. 8 B. 10 C. 17. D. 20

90. The first term of an exponential sequence is 3 and the sixth term is $\frac{3}{32}$. Find the common ratio.

A. $\frac{1}{6}$ B. $\frac{1}{5}$ C. $\frac{1}{3}$ D. $\frac{1}{2}$

91. Three force $\mathbf{F}_1 = (8\text{ N}, 030°)$, $\mathbf{F}_2 = (10\text{ N}, 090°)$ and $\mathbf{F}_3 = (8\text{ N}, 150°)$ act on a particle. Find the magnitude of the resultant force.

A. 8 N B. 10 N C. 16 N D. 18 N

92. Which of the following represents an equation of the straight line passing through the point $P(4, 5)$ and perpendicular to the line $y = -\frac{2}{3}x + 7$?

A. $y = -\frac{3}{2}x + 7$ B. $y = \frac{3}{2}x + 7$

C. $y = -\frac{3}{2}x - 1$ D. $y = \frac{3}{2}x - 1$

93. Solve $\sqrt{(y-3)} - \sqrt{y} = -3$

A. 2 B. 4 C. 6 D. 8

94. Express $\dfrac{\sqrt{5} - \sqrt{2}}{\sqrt{5} + \sqrt{2}}$ in the form $a + b\sqrt{10}$ and find the value of a.

A. $^1/_7$ B. $^1/_5$ C. $^5/_3$ D. $^7/_3$

95. For what values of x is the expression $\dfrac{x^2 - 7x + 12}{x^2 - 9x + 18}$ undefined?

A. 3 and 4 B. 3 and 6

C. –3 and –4 D. –3 and –6

96. Simplify $\log_{10} \sqrt[3]{5^8}$.

A. $^3/_8 \log_{10}5$ B. $^3/_5 \log_{10}5$

C. $^5/_8 \log_{10}5$ D. $^8/_3 \log_{10}5$

97. Which of the following points does not satisfy $3x - y \geq 1$?

A. (2, 1) B. (1, 2)

C. (–1, 1) D. (1, –1)

98. Solve the equation

$$\frac{3x}{x-1} - \frac{2x}{x+2} = \frac{3x-6}{(x-1)(x+2)}$$

A. {–3, –2} B. {–3} C. {3, 2} D. {–2}

99. Solve $9^{2x+1} = 81^{3x-2}$

A. $\frac{2}{3}$ B. $\frac{4}{5}$ C. $\frac{5}{4}$ D. $\frac{3}{2}$

100. Give that $2\cos x = \sqrt{3}$, where $0° < x < 90°$, find the value of x.

A. 20° B. 30° C. 60° D. 75°

101. Solve $2\cos x - 1 = 0$, for $0 < x < 2\pi$.

A. $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$ B. $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$

C. $\left(\frac{\pi}{5}, \frac{2\pi}{5}\right)$ D. $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$

102. Find $\lim_{x \to -1} (x^4 - 3x^2 - 7x + 11)$

A. 2 B. 8 C. 14 D. 16

103. Differentiate with respect to x, $y = (x + 1)(x - 4)$.

A. $2x - 3$ B. $2x + 3$

C. $\frac{1}{3x^3}$ D. $2x$

104. Find an expression for y given that $\frac{dy}{dx} = x^2\sqrt{x}$.

A. $\frac{1}{7}x^{\frac{2}{7}} + c$ B. $\frac{2}{7}x^7 + c$

C. $\frac{2}{7}x^{\frac{7}{2}} + c$ D. $\frac{3}{7}x^{\frac{3}{2}} + c$

105. If $f(x) = -2x^3 + qx + 2$, where q is a constant and $f(-1) = 2$, find q.

A. -6 B. -2 C. 2 D. 6

106. Find the gradient of the curve $y = x^2 + 2x + 2$ at $x = -1$

A. -1 B. 0 C. 1 D. 2

107. Find the interval(s) where the curve $\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x - 3$ is decreasing.

A. $x < -2$ and $x > 1$ B. $x < -1$ and $x > 2$

C. $-2 < x < 1$ D. $-1 < x < 2$

108. A certain construction company has 75 casual workers; 30 earn ¢1,500 a day and 45 earn ¢950 a day. Calculate the average earning per day.

A. ¢1170 B. ¢1225

C. ¢1950 D. ¢2925

109. In how many ways can a committee comprising a girl and two boys be selected from a group of three girls and five boys?

A. 13 B. 30 C. 56 D. 72

The table below shows distribution of the number of goals scored by a football team during a sports festival.

No. of goals scored	1	2	3	4	5
Frequency	6	3	4	1	2

Use this information to answer Questions **110** *and* **111**.

110. Calculate the mean of the distribution.

A. 2.3 B. 2.4 C. 2.5 D. 2.6

111. Calculate the standard deviation of the distribution.

A. 0.86 B. 1.36 C. 1.73 D. 5.45

112. The sample space of a random experiment is {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}. What is the probability of obtaining a prime number?

A. $\frac{4}{12}$ B. $\frac{5}{12}$ C. $\frac{4}{11}$ D. $\frac{5}{11}$

113. A fair coin is tossed three times. Find the probability of obtaining at least two heads.

A. $\frac{1}{4}$ B. $\frac{3}{8}$ C. $\frac{1}{2}$ D. $\frac{5}{8}$

114. Two dice are thrown once. What is the probability that the same number shows up on both dice?

A. $\frac{1}{36}$ B. $\frac{2}{36}$ C. $\frac{5}{36}$ D. $\frac{1}{6}$

115. Find **a.b**, if $\mathbf{a} = 4\mathbf{i} + 7\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - 5\mathbf{j}$.

A. -23 B. -20 C. 20 D. $\frac{3}{2}$

116. Given that $\mathbf{v} = 3\mathbf{i} + k\mathbf{j}$ and $\mathbf{w} = \mathbf{i} - 2\mathbf{j}$ are orthogonal, find k.

A. $-\frac{3}{2}$ B. -1 C. 1 D. $\frac{3}{2}$

117. A force of $10\mathbf{i} + 4\mathbf{j}$ acts on a mass of 2 kg which is at rest. What is the velocity after 3 seconds.

A. $\frac{5}{3}\mathbf{i} + \frac{2}{3}\mathbf{j}$ B. $\frac{10}{3}\mathbf{i} + \frac{4}{3}\mathbf{j}$

C. $5\mathbf{i} + 2\mathbf{j}$ D. $15\mathbf{i} + 6\mathbf{j}$

118. The velocity of a particle, in ms^{-1}, after t seconds, is $v = 3t^2 + 2t - 1$. Find the acceleration after 2 seconds.
A. 10 ms^{-2} B. 13 ms^{-2}
C. 14 ms^{-2} D. 17 ms^{-2}

A particle projected form a fixed point with a velocity 10 ms^{-1} moves in a straight line such that its velocity after time t seconds is $v = bt^2 + 7t + c$ where b and c are constants. One second after projection, its acceleration is 1 ms^{-2}
Use this information above to answer Questions **119** *to* **121**.

119. Find the values of b and c.
A. $b = -4, c = 10$ B. $b = 4, c = -10$
C. $b = -3, c = 10$ D. $b = 3, c = 10$

120. Find the time when the particle is momentarily at rest.
A. $\frac{1}{3}$ s B. 1 s C. $\frac{10}{3}$ s D. 10 s

121. Find the distance it travels after 3 seconds, if its initial distance was 2m.
A. 31.5 m B. 34.5 m
C. 36.5 m D. 90.5 m

122. A force of 0.2 N acts on a mass of 10 g. Calculate its acceleration.
A.. 0.002 ms^{-2} B. 0.5 ms^{-2}
C. 2.0 ms^{-2} D. 20.0 ms^{-2}

The image of $P(5, 4)$ under a translation in the plane is P' (8, –2).
Use this information to answer Questions **123** *and* **124**.

123. Find the matrix equation of the transformation.
A. $x' = x + 3$, $y' = y - 6$ B. $x' = x - 3$, $y' = y + 6$
C. $x' = x + 13$, $y' = y - 6$ D. $x' = x + 13$, $y' = y + 6$

124. Find the image Q' of $Q(-4, 0)$ under the transformation.
A. $Q'(-1, 6)$ B. $Q'(-1, -6)$
C. $Q'(1, 6)$ D. $Q'(1, -6)$

125. Find the matrix of the linear transformation **P** : $(x, y) \rightarrow (-2y, -x + 2y)$.
A. $\begin{pmatrix} -2 & 0 \\ -1 & 2 \end{pmatrix}$ B. $\begin{pmatrix} 0 & -2 \\ -1 & 2 \end{pmatrix}$
C. $\begin{pmatrix} 0 & -1 \\ -2 & 2 \end{pmatrix}$ D. $\begin{pmatrix} 0 & -2 \\ 2 & -1 \end{pmatrix}$

126. If **A** = $\begin{pmatrix} -4 & 3 \\ 2 & -5 \end{pmatrix}$ and **B** = $\begin{pmatrix} 2 & -3 \\ 7 & -6 \end{pmatrix}$, find 3**A** + 2**B**.
A. $\begin{pmatrix} 8 & -6 \\ -1 & 10 \end{pmatrix}$ B. $\begin{pmatrix} -8 & 3 \\ 20 & -27 \end{pmatrix}$
C. $\begin{pmatrix} 8 & 3 \\ -20 & -27 \end{pmatrix}$ D. $\begin{pmatrix} -12 & 15 \\ -26 & 19 \end{pmatrix}$

127. If **A** = $\begin{pmatrix} 2 & -6 \\ -3 & 9 \end{pmatrix}$ and **B** = $\begin{pmatrix} -3 & 6 \\ -1 & 2 \end{pmatrix}$, find **AB.**
A. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ B. $\begin{pmatrix} -1 & 0 \\ -4 & 11 \end{pmatrix}$
C. $\begin{pmatrix} 0 & 0 \\ -18 & 0 \end{pmatrix}$ D. $\begin{pmatrix} -24 & 36 \\ -8 & 24 \end{pmatrix}$

128. Find $\mathbf{A}^{-1}$ if **A** = $\begin{pmatrix} 3 & 5 \\ -2 & -4 \end{pmatrix}$.
A. $\begin{pmatrix} 2 & \frac{5}{2} \\ -1 & -\frac{3}{2} \end{pmatrix}$ B. $\begin{pmatrix} -4 & -5 \\ 2 & 3 \end{pmatrix}$
C. $\begin{pmatrix} 4 & 3 \\ -2 & 5 \end{pmatrix}$ D. $\begin{pmatrix} 4 & -5 \\ -2 & -3 \end{pmatrix}$.

129. Which of the following matrices are inverses of each other?
I. $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ II. $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$ III. $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$.
A. I and II B. I and III
C. II and III D. I, II and III

130. Find the variance of the numbers 9, 3, 8, 8, 9, 8, 9 and 18

A. 16 B. 15 C. 4 D. 3.87

131. If $\log_x 121 = 2$, find the value of x.

A. 11 B. 10 C. 3 D. 2

132. A binary operation ⊗, is defined on the set R, of real numbers by $a \otimes b = a + b\sqrt{2}$. Find $(2 \otimes -3) \otimes 5$.

A. $2+5\sqrt{2}$ B. $2+2\sqrt{2}$

C. $2-3\sqrt{2}$ D. $-3+5\sqrt{2}$

133. Find the quadratic equation whose roots are $3-\sqrt{2}$ and $3+\sqrt{2}$.

A. $x^2+6\sqrt{2x}+7=0$

B. $x^2+6\sqrt{2x}-7=0$

C. $x^2+6x-7=0$

D. $x^2-6x+7=0$

134. Solve the inequality $x^2-x-6>0$

A. $x<-2$ or $x>3$ B. $-2<x<3$

C. $x>3$ D. $x>2$

135. For what values of n would you use $(1+x)^n$ to evaluate $\sqrt{0.99}$.

A. $x=0.01,\ n=\frac{1}{2}$ B. $x=0.01,\ n=-\frac{1}{2}$

C. $x=-0.01,\ n=\frac{1}{2}$ D. $x=-0.01,\ n=-\frac{1}{2}$

136. A sequence is given as -5, -2, 1, 4, … find the 23rd term.

A. 58 B. 61 C. 64 D. 71

137. Find the coefficient of x^6 in the Binomial expansion of $(2+x)^{10}$.

A. 3360 B. 210 C. 64 D. 16

138. If p and q are statements such that $p \Rightarrow q$. Which of the following is a valid conclusion from the implication?

A. $q \Rightarrow p$ B. $\sim q \Rightarrow p$

C. $\sim p \Rightarrow \sim q$ D. $\sim q \Rightarrow \sim p$

139. Two functions f and g are defined by $f : x \rightarrow \frac{x}{1+x}$ and $g : x \rightarrow x-1$, find the composite function $g \circ f$.

A. $x \rightarrow x$ B. $x \rightarrow -\frac{1}{1+x}$

C. $x \rightarrow \frac{x-1}{x}$ D. $f : x \rightarrow \frac{x}{1+x}$

The table below shows the scores of some student in an examination.

Marks	1-20	21-40	41-60	61-80	81-100
Frequency	8	14	18	48	12

Use the information to answer Question **140** *and* **141**.

140. State the median class.

A. 21 – 40 B. 41 – 60

C. 61 – 80 D. 81 – 100

141. Find the mean of the distribution.

A. 58.4 B. 58.9 C. 59.5 D. 60.0

142. The nth term of a sequence is U_n. If $U_1 = 3$ and $U_{n+1} = U_n + 5$, find U_3.

A. 3 B. 8 C. 11 D. 18

143. Express $\frac{2x+1}{(x-1)(x+1)}$ in partial fractions.

A. $\frac{2x}{(x-1)}+\frac{1}{(x+1)}$ B. $\frac{3}{(x-1)}+\frac{1}{(x+1)}$

C. $\frac{1}{2(x+1)}-\frac{3}{2(x-1)}$ D. $\frac{3}{2(x-1)}-\frac{1}{2(x+1)}$

144. A force (10N, 030°) acts on a body of mass 0.5kg. Calculate the magnitude of the acceleration of the body.

A. 5 ms^{-2} B. 10 ms^{-2}

C. 20 ms^{-2} D. 25 ms^{-2}

145. If $\left(\frac{1}{27}\right)^{2x-1} = 243\sqrt{3}$, find the value of x.

A. $\frac{12}{17}$ B. $-\frac{5}{12}$

C. $\frac{12}{17}$ D. $-\frac{5}{12}\sqrt{3}$

146. A linear transformation is given by $T:(x,y)\to(x,x-y)$. Write down the representative matrix.

A. $\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}$ B. $\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}$

C. $\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$ D. $\begin{pmatrix}-1 & 0\\ 1 & -1\end{pmatrix}$

147. The roots of the quadratic equation $2x^2-5x+m=0$ are α and β, where *m* is a real number. Find an expression for $\alpha^2+\beta^2$ in terms of *m*.

A. $25+2m$ B. $25-2m$

C. $\frac{25}{4}-m$ D. $\frac{25}{4}+m$

148. A function *f* is defined by $f:x\to\frac{x+2}{x-3}$, $x\in R$. State the largest possible domain of f^1, the inverse of *f*.

A. $\{x:x\in R, x\neq 3\}$

B. $\{x:x\in R, x\neq -3\}$

C. $\{x:x\in R, x\neq 1\}$

D. $\{x:x\in R, x\neq -1\}$

149. If $\log_2(3x+y)=1$ and $\log_2\frac{x}{y}=-2$, find the values of *x* and *y*.

A. $x=\frac{8}{3}, y=-\frac{2}{3}$ B. $x=\frac{2}{7}, y=-\frac{8}{7}$

C. $x=-2, y=8$ D. $x=1, y=4$

150. The line $y=mx-3$ is a tangent to the curve $y=1-3x+2x^3$ at the point (1, 0). Determine the value of *m*.

A. –3 B. –1 C. 3 D. 4

151. Given that $y=-\frac{\sqrt{3}}{2}$ and tan $y=\sqrt{3}$, find *y*.

A. 60 B. 120 C. 210 D. 240

152. The line $y=2x+1$ is perpendicular to the line $3y+bx-4=0$. Find the value of *b*

A. $\frac{3}{2}$ B. $\frac{1}{2}$ C. $-\frac{3}{2}$ D. $-\frac{1}{2}$

153. If $A=\begin{pmatrix}1 & -1\\ 2 & 3\end{pmatrix}$ and $B=\begin{pmatrix}4 & 1\\ 0 & 3\end{pmatrix}$, find **3A – B.**

A. $\begin{pmatrix}7 & 4\\ 6 & 12\end{pmatrix}$ B. $\begin{pmatrix}3 & -3\\ 6 & 9\end{pmatrix}$

C. $\begin{pmatrix}-1 & -4\\ 6 & 6\end{pmatrix}$ D. $\begin{pmatrix}-1 & 4\\ 6 & 6\end{pmatrix}$

154. Find the variance of the numbers 9, 3, 8, 8, 9, 8, 9 and 18

A. 16.00 B. 15.00

C. 4.00 D. 3.87

155. When a polynomial *f*(*x*) is divided by *x* + 2, the quotient is x^2+2x+1 and the remainder is 5. Find *f*(*x*).

A. $f(x)=x^3+4x^2+5x+7$

B. $f(x)=x^3+4x^2+x+5$

C. $f(x)=x^3+x^2+x+7$

D. $f(x)=14x^3+56x^2+70x$

156. A point *P* has image (0, 5) under the transformation $N:(x,y)\to x-y, 2x+3y$. Find the co-ordinates of *P*.

A. (–1, –1) B. (–1, 1)

C. (1, –1) D. (1, 1)

A circle *P* with centre *Q*(1, 2) intersects the line *L* at the point *R*(4, 6) only.

Use the information to answer Question **157** *to* **159.**

157. Calculate the radius of the circle.

A. 2 B. 3 C. 4 D. 5

158. Find the equation of the circle.

A. $x^2+y^2+4x-2y-20=0$

B. $x^2+y^2-2x-4y-20=0$

C. $x^2+y^2+2x-4y-25=0$

D. $x^2+y^2-25=0$

159. Determine the equation of the line *L*.

A. $4y-3x+24=0$

B. $4y+3x-12=0$

C. $4y+3x-36=0$

D. $3y-3x+12=0$

160. Given $\mathbf{x}=\mathbf{i}-\mathbf{j}$ and $\mathbf{y}=2\mathbf{i}+3\mathbf{j}$, find a unit vector along the vector $\mathbf{y}-\mathbf{x}$.

A. $\frac{1}{\sqrt{17}}(4i+j)$ B. $\frac{1}{\sqrt{17}}(i+j)$

C. $i+4j$ D. $4i+j$

161. Which of the following matrices has no multiplicative inverse?

A. $\begin{pmatrix}-2 & 5\\ 4 & 10\end{pmatrix}$ B. $\begin{pmatrix}2 & -5\\ 4 & 10\end{pmatrix}$

C. $\begin{pmatrix}4 & 5\\ -2 & 3\end{pmatrix}$ D. $\begin{pmatrix}4 & -2\\ -2 & 1\end{pmatrix}$

162. The 2nd and the 4th terms of an exponential sequence are $\frac{2}{9}$ and $\frac{8}{81}$ respectively.

Find the 6th term of the sequence.

A. $\frac{81}{32}$ B. $\frac{27}{24}$ C. $\frac{81}{324}$ D. $\frac{32}{729}$

163. The position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are $2\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+4\mathbf{j}$ respectively. Find $|\overrightarrow{BA}|$.

A. $\sqrt{10}$ B. $\sqrt{34}$ C. $9\sqrt{2}$ D. 6

164. A linear transformation is defined by $p:(x,y)\rightarrow(-y,\ 5y-2x)$. What is the image of the point (–2, 3) under the transformation *P*.

A. (2, 19) B. (–2. 19)

C. (–3, 11) D. (–3, 19)

165. The derivative of the function *g* is given by

$g: x\rightarrow 3x^2-\frac{4}{x^5}$.

A. $g(x)=x^3-\frac{1}{x^4}+4$

B. $g(x)=x^3-\frac{1}{x^4}+2$

C. $g(x)=x^3+\frac{1}{x^4}+2$

D. $g(x)=3x+\frac{20}{x^6}-19$

166. Three balls are drawn at random from a bag containing 3 red, 4 white and 5 black balls. Calculate the probability that the three balls drawn are made up of 2 black and 1 white balls.

A. $\frac{2}{11}$ B. $\frac{1}{11}$ C. $\frac{3}{22}$ D. $\frac{1}{22}$

167. Find the value of *x* that maximizes the function $f(x)=x^3-5x^2-x+5$.

A. $x=3.4$ B. $x=1.7$

C. $x=0.1$ D. $x=-0.1$

168. A stone is thrown vertically upwards and its height after *t* seconds is $h=45t-9t^2$. Find the greatest height reached.

A. 56.25 m B. 56.00 m

C. 45.50 m D. 45.00 m

169. Given that $\mathbf{A}=\begin{pmatrix}k & 1\\ 2 & 3\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}4 & 1\\ 2 & 1\end{pmatrix}$ and

$\mathbf{A}+\mathbf{B}=\begin{pmatrix}9 & 2\\ 4 & 4\end{pmatrix}$, find the value of *k*.

A. 9 B. 5 C. 3 D. –5

170. The equation of a curve is given by $y = 2x^2 + \frac{3}{2}x^3$. Find the gradient function at any point (x, y).

A. $\frac{2}{3}x^3 + \frac{3}{8}x^4$ B. $\frac{2}{3}x + \frac{1}{2}x^2$

C. $x^3 + \frac{9}{2}x^4$ D. $4x + \frac{9}{2}x^2$

171. A body of mass 2 kg has an initial velocity of $u = \begin{pmatrix} -5 \\ 3 \end{pmatrix}\text{ms}^{-1}$ and a final velocity $v = \begin{pmatrix} 4 \\ 2 \end{pmatrix}\text{ms}^{-1}$. Find the change in momentum.

A. $\begin{pmatrix} 18 \\ -2 \end{pmatrix}$NS B. $\begin{pmatrix} 13 \\ 1 \end{pmatrix}$NS

C. $\begin{pmatrix} 13 \\ -1 \end{pmatrix}$NS D. $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$NS

172. Evaluate $\int_1^2 \left(x^2 - \frac{1}{x^2}\right)dx$

A. $\frac{87}{24}$ B. $\frac{11}{6}$ C. $\frac{43}{24}$ D. $\frac{7}{12}$

173. A car and a bus were hired for an excursion. The probability that the car fails to turn up is $\frac{1}{3}$ and the probability that the bus fails to turn up is $\frac{1}{4}$. Find the probability that at least one of the vehicle turns up.

A. $\frac{11}{12}$ B. $\frac{1}{2}$ C. $\frac{5}{12}$ D. $\frac{1}{4}$

174. A body is released from a height 100m above the ground. How long does it take to reach the ground? [Take $g = 10\text{ms}^{-2}$].

A. $2\sqrt{5}$ s. B. 5 s. C. $5\sqrt{2}$ s. D. 10 s.

175. Four groups of students, consisting of 10, 15, 18 and 22 individuals, have mean height of 1.60, 1.57, 1.46 and 1.52 metres respectively. Calculate the mean height of all the students.

A. 23.55 m B. 2.48

C. 1.59 m D. 1.53 m

176. Differentiate $\frac{1}{x^2}\left(3 + 5x + 2x^2\right)$, with respect to x.

A. $-\frac{6}{x^3} - \frac{5}{x^2}$ B. $\frac{5}{x^2} - \frac{4}{x}$

C. $\frac{6}{x^3} + \frac{5}{x^2}$ D. $\frac{5}{x^2} - \frac{6}{x^3}$

The table below shows the probabilities of selecting boys and girls from the two classes 3A and 3B in school.

	CLASSES	
	3A	*3B*
P(Boy)	$\frac{1}{4}$	$\frac{2}{3}$
P(Girl)	$\frac{3}{4}$	$\frac{1}{3}$

A student is selected at random from each of the classes.

Use the information to answer Questions **177** *and* **178**.

177. What is the probability that both students selected are girls?

A. 1 B. $\frac{1}{4}$ C. $\frac{1}{2}$ D. $\frac{3}{16}$

178. What is the probability that two students selected will be a boy and a girl?

A. $\frac{7}{12}$ B. $\frac{59}{144}$ C. $\frac{1}{6}$ D. $\frac{1}{12}$

179. A particle is acted upon by the forces $\mathbf{F_1} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$N, $\mathbf{F_2} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$N. Find the force required to keep the particle in equilibrium.

A. $\begin{pmatrix}12\\7\end{pmatrix}$N B. $\begin{pmatrix}4\\3\end{pmatrix}$N

C. $\begin{pmatrix}-4\\-3\end{pmatrix}$N D. $\begin{pmatrix}-12\\-7\end{pmatrix}$N

180. Three vectors **r** = 3**i** – **j**, **s** = **i** + **j** and **t** = **i** – 3**j** and are such that **t** =α**r** + β**s**, find αand β.

A. $\alpha = -1,\ \beta = -2$ B. $\alpha = -\frac{1}{2}, \beta = -\frac{7}{2}$

C. $\alpha = 1,\ \beta = -2$ D. $\alpha = 1,\ \beta = 2$

181. The linear transformations *A* and *B* are defined by $A:(x, y) \rightarrow (x+y,\ 2x-y)$ and $B:(x, y) \rightarrow (x^1,\ y^1)$. When a point is transformed under *A* followed by *B*, it remain unchanged. Find the point $(x^1,\ y^1)$.

A. $(2x-y,\ x+y)$

B. $\left(\dfrac{-x-y}{3}, \dfrac{-2x+y}{3}\right)$

C. $\left(\dfrac{2x-y}{3}, \dfrac{x+y}{3}\right)$

D. $\left(\dfrac{x+y}{3}, \dfrac{2x-y}{3}\right)$

182. A binary operation $*$, is defined on the set *R* of real numbers by $a * b = \frac{1}{2}ab$. Find the identify elelment under the operation $*$.

A. 0 B. $\frac{1}{4}$ C. $\frac{1}{2}$ D. 2

183. Find the solution set of the inequality $2x^2 - x - 1 \leq 0$

A. $\{x : -\frac{1}{2} \leq x \leq 1\}$

B. $\{x : -1 \leq x \leq \frac{1}{2}\}$

C. $\{x : x \geq 1 \text{ or } x \leq -\frac{1}{2}\}$

D. $\{x : \frac{1}{2} \leq x \leq 1\}$

184. The inverse of a function *f* is given by $f^{-1} : x \rightarrow \dfrac{2x+1}{3}$. Find *f*.

A. $f : x \rightarrow 3x-1$ B. $f : x \rightarrow \dfrac{3x-1}{2}$

C. $f : x \rightarrow \dfrac{3}{2x+1}$ D. $f : x \rightarrow \dfrac{1}{2x+1}$

185. Find the equation of the line which passes through the point (1, 3) and is perpendicular to the line $2y - 3x + 1 = 0$.

A. $2y - 3x - 3 = 0$

B. $2y - 3x - 9 = 0$

C. $3y - 2x - 9 = 0$

D. $3y + 2x - 11 = 0$

186. A sequence is defined by $U_{n+1} = 3 + U_{n,} U_1 = 1$, where $n \geq 1$. Find the sum of the first ten terms of the sequence.

A. 27 B. 29 C. 145 D. 290

187. Find the set of zeros of the polynomial $P(x) = x^3 - 2x^2 - x + 2$.

A. $\{0, 1, 2\}$ B. $\{-1, 0, 1\}$

C. $\{-1, 1, 2\}$ D. $\{-2, 1, -1\}$

188. The coefficient of x^3 in the binomial expansion of $(p + 3x)^4$ is 54. Find the values of *p*.

A. $\frac{1}{2}$ B. 1 C. $\frac{3}{2}$ D. 2

189. Find the values of *k* for which the equation $x^2 - kx + 4 = 0$ has equal roots.

A. ±1 B. ±2 C. ±3 D. ±4

190. If $\sin A = \dfrac{4}{5}$ and $\cos B = \dfrac{12}{13}$, find the value of sin (*A* – *B*), when *A* is obtuse and *B* is acute.

A. $\dfrac{33}{65}$ B. $\dfrac{63}{65}$ C. $\dfrac{65}{63}$ D. $\dfrac{65}{33}$

191. Given that $f : x \to \frac{1}{3}(2x-1)$, find the function g such that $fog : x \to x$.

A. $g : x \to 3\left(\frac{1}{2}x+1\right)$

B. $g : x \to \frac{1}{2}(3x+1)$

C. $g : x \to -\frac{1}{3}(2x-1)$

D. $g : x \to x$

The probability that Kofi will receive a prize during a speech day is $\frac{2}{3}$ and the probability that Ama will receive a prize is $\frac{3}{5}$.

Use the information above to answer Questions **192** *and* **193**

192. Find the probability that only Ama will receive a prize

A. $\frac{3}{5}$ B. $\frac{8}{15}$ C. $\frac{1}{5}$ D. $\frac{2}{15}$

193. Find the probability that at **least** one of them will receive a prize.

A. $\frac{13}{15}$ B. $\frac{8}{15}$ C. $\frac{7}{15}$ D. $\frac{2}{15}$

194. Find the magnitude and direction of the vector **8i** – **15j**.

A. (17, 028°) B. (17, 062°)
C. (17, 118°) D. (17, 152°)

195. Find the maximum value of the function $f(x)=x^2-x-6$.

A. $-6\frac{1}{4}$ B. -6 C. $-5\frac{3}{4}$ D. $\frac{1}{2}$

196. When the polynomial $f(x)=2x^3-ax^2+5x+1$ is divided by $(x+2)$, the remainder is -1. Find the value of a

A. –7 B. –6 C. 6 D. 7

197. Find the limit of the function $\frac{3h+2h^2+4h^3}{h}$ as h approaches zero.

A. 0 B. 2 C. 3 D. 4

A group of students were asked how many children were in their family, and their responses are given in the table below.

No. of students	No. of children in the family
1	2
2	3
3	3
4	13
5	9
6	11
7	12
8	2
9	2
10	1

Use the information above to answer Questions **198** *to* **200**

198. How many families have a most 3 children?

A. 8 B. 13 C. 25 D. 36

199. Find the mean number of children per family, correct to the nearest whole number.

A. 4 B. 5 C. 6 D. 7

200. What is the probability of selecting a family with at least 4 children in the family?

A. 0.22 B. 0.36 C. 0.64 D. 0.86

201. A force is given by **F** = 12**i** – 5**j**. Find its magnitude and direction.

A. $(13\text{N}, 112.6^\circ)$ B. $(13\text{N}, 090^\circ)$
C. $(13\text{N}, 067.4^\circ)$ D. $(10.9\text{N}, 112.6^\circ)$

202. Find the interval where the curve $y=\frac{x^3}{3}+\frac{x^2}{2}-2x-3$ is increasing.

A. $-2<x<1$ B. $-1<x<2$
C. $x<-2$ and $x>1$ D. $x<-1$ and $x>2$

203. What is the single transformation for a reflection in the x-axis followed by a rotation through 90^0 anticlockwise about the origin O?

A. Reflection in the y-axis
B. Reflection in the line $y = -x$
C. Reflection in the line $y = x$
D. Rotation through 180^0 about the origin

204. Evaluate $\int_2^3 (x^2 - 3)dx$

A. $2\frac{1}{3}$ B. $2\frac{8}{9}$ C. $3\frac{1}{3}$ D. $3\frac{2}{3}$

205. Find the sine of the angle between the vectors $6\mathbf{i} + \mathbf{j}$ and $-\mathbf{i} + 5\mathbf{j}$

A. 0.001 B. 0.995
C. 0.999 D. 1.000

206. If $2^{\log 3^x} = \frac{1}{16}$, find the value of x.

A. 16 B. $\frac{1}{4}$ C. $\frac{1}{16}$ D. $\frac{-4}{\log 3}$

207. A force of 24 N acts on a body of mass 8 kg which is at rest. How far will it move in 5 seconds?

A. 15.0 m B. 37.5 m
C. 45.5 m D. 75.0 m

208. Given that

$$\frac{2}{(x-2)(x-3)} = \frac{P}{x-2} + \frac{Q}{x-3}, x \neq 2,3,$$

find the value of $2P + Q$.

A. -6 B. -2 C. $-\frac{3}{2}$ D. 0

209. A particle is kept at rest under the action of three forces $(k\text{N}, 120^0)$, $(4\text{N}, 000^0)$ and $(4\sqrt{3}\text{N}, 270^0)$. Calculate the value of k.

A. $2\sqrt{3}$ B. $\frac{8}{3}\sqrt{3}$ C. 8 D. $8\sqrt{3}$

210. The gradient of a curve is given by $\frac{dy}{dx} = 2x + 3$. If (1, 2) lies on the curve, find the equation of the curve.

A. $y = 2x^2 + 3x + 1$ B. $y = x^2 + 3x - 9$
C. $y = x^2 + 3x + 1$ D. $y = x^2 + 3x - 2$

211. A linear transformation is given by $(-x, y)$. What is the representative matrix?

A. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ B. $\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$

C. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ D. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

212. The linear equation $3x + 2y = 6$ is expressed in the form $\frac{x}{a} + \frac{y}{b} = 1$.

Find the values of and b.

A. $a = 3$, $b = 2$ B. $a = 2$, $b = 3$
C. $a = 2, b = \frac{1}{3}$ D. $a = \frac{1}{2}, b = \frac{1}{3}$

213. If $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$, find **AB**.

A. $\begin{pmatrix} -3 & 2 \\ -8 & -10 \end{pmatrix}$ B. $\begin{pmatrix} 2 & -3 \\ 8 & -10 \end{pmatrix}$

C. $\begin{pmatrix} 8 & -3 \\ 14 & 10 \end{pmatrix}$ D. $\begin{pmatrix} 8 & -8 \\ 14 & -14 \end{pmatrix}$

214. Two points $P(2, -3)$ and $Q(x, y)$ in the xy-plane are such that $\overrightarrow{PQ} = \mathbf{i} - \mathbf{j}$. Find the coordinates of Q.

A. (3, –4) B. (1, –2)
C. (–1, 2) D. (–3, 4)

215. Given that $P(A) = \frac{3}{4}$ and $P(A \cap B) = \frac{1}{16}$.

Find $P(B)$ if the event A and B are independent.

A. $\frac{13}{16}$ B. $\frac{3}{4}$ C. $\frac{1}{4}$ D. $\frac{1}{12}$

216. If $y = \dfrac{1}{(2-x)^2}$, find $\dfrac{dy}{dx}$.

A. $\dfrac{-2}{(2-x)^3}$ B. $\dfrac{-1}{(2-x)^3}$

C. $\dfrac{2}{(2-x)^3}$ D. $\dfrac{2}{2-x}$

217. If $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -5 & 4 \end{pmatrix}$ and $\mathbf{A} - 3\mathbf{B} = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$ find **B**.

A. $\begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ -1 & \frac{5}{3} \end{pmatrix}$ B. $\begin{pmatrix} \frac{2}{3} & \frac{4}{3} \\ \frac{-7}{3} & 1 \end{pmatrix}$

C. $\begin{pmatrix} 2 & 4 \\ -7 & 5 \end{pmatrix}$ D. $\begin{pmatrix} 2 & 2 \\ -3 & 5 \end{pmatrix}$

218. The sum of the first n terms, S_n, of an arithmetic sequence is $S_n = 5n^2 + 6n$. Find the n^th^ term.

A. $10n + 1$ B. $10n - 1$
C. $5n^2 - 4n$ D. $5n^2 + 1$

219. Calculate, correct to two decimal places the standard deviation of the set of integers 3, 2, 3, 3 and 4.

A. 0.75 B. 0.71 C. 0.63 D. 0.51

220. Given that $\mathbf{B} = \begin{pmatrix} 4 & 5 \\ 2 & 5 \end{pmatrix}$, find $\mathbf{B}^{-1}$.

A. $\dfrac{1}{10}\begin{pmatrix} -5 & 2 \\ 2 & -5 \end{pmatrix}$ B. $\dfrac{1}{10}\begin{pmatrix} -5 & 5 \\ 2 & 4 \end{pmatrix}$

C. $\dfrac{1}{10}\begin{pmatrix} -5 & 4 \\ 2 & 5 \end{pmatrix}$ D. $\dfrac{1}{10}\begin{pmatrix} 5 & -5 \\ -2 & 4 \end{pmatrix}$

221. Find the set of values of x for which

$$\frac{x^2 - 2x - 35}{(x+2)(2x+1)} = 0$$

A. $\{7, -5\}$ B. $\{7, 5\}$
C. $\{5, -7\}$ D. $\{-2, -\frac{1}{2}\}$

222. The circle with equation $x^2 + y^2 - 6x + 2y + c = 0$ passes through the point $(-1, 2)$. Find its radius.

A. 25 B. 5 C. $\sqrt{20}$ D. $\sqrt{13}$

223. The transformation T is given by $T : (x, y) \rightarrow (-x, x + y)$. Find the image of the point $(1, -1)$ under T.

A. $(-1, 0)$ B. $(-1, 1)$
C. $(0, 1)$ D. $(1, 1)$

224. Express $\dfrac{2}{x-1} - \dfrac{1}{2(x^2-1)}$ as a single rational function.

A. $\dfrac{3x-5}{x^2-1}$ B. $\dfrac{x+1}{2(x^2-1)}$

C. $\dfrac{3x-5}{2(x^2-1)}$ D. $\dfrac{4x+3}{2(x^2-1)}$

225. A body is acted upon by the forces $F_1(10\text{N}, 060^0)$ and $F_2(5\text{N}, 180^0)$. Find, correct to two decimal places, the magnitude of the resultant force.

A. 6.20N B. 8.66
C. 13.20N D. 13.23

226. The probability that Kofi becomes a wealthy person in future is $\frac{2}{3}$ and the probability that he drops out of college is $\frac{1}{4}$. Calculate the probability that he completes college but does not become a wealthy person in future.

A. $\dfrac{1}{12}$ B. $\dfrac{1}{6}$ C. $\dfrac{1}{4}$ D. $\dfrac{1}{2}$

227. If $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j}$, find the magnitude and direction of $\frac{1}{2}(\mathbf{a} - \mathbf{b})$.

A. $(10, 071.6^0)$ B. $(10, 018.4^0)$
C. $(\sqrt{10}, 071.6^0)$ D. $(\sqrt{10}, 018.4^0)$

228. The matrix representing anticlockwise rotation through an angle θ about the origin, O, is given by $m = \begin{pmatrix} \cos\theta & (-\sin\theta) \\ \sin\theta & \cos\theta \end{pmatrix}$. Find the image of the point (1, 1) when it is rotated through 60^0 anticlockwise about the origin.

A. (–1, 1) B. (1, –1)

C. $\left(\frac{1}{2}-\frac{\sqrt{3}}{2}, \frac{1}{2}+\frac{\sqrt{3}}{2}\right)$

D. $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}, -\frac{1}{2}-\frac{\sqrt{3}}{2}\right)$

229. There are 7 girls in a class of 20. Find the number of ways of selecting 3 boys and 2 girls.

A. 1638 B. 2730

C. 6006 D. 77520

230. Given that $\frac{{}^nC_4}{{}^nP_2} = \frac{1}{2}$, find the value of n.

A. 5 B. 6 C. 7 D. 8

231. In a class of 25 students, 6 study Social Studies (A), 14 study Science (B) and three study both Social Studies and Science. Find $n(A^1 \cap B^1)$.

A. 8 B. 5 C. 3 D. 2

232. A binary operation Δ is defined on set, R, of real numbers by $a\Delta b = \frac{a+b}{\sqrt{ab}}$. Evaluate $-1\Delta-2$.

A. $3\sqrt{2}$ B. $\frac{3}{2}\sqrt{2}$

C. $\sqrt{2}$ D. $-\frac{3}{2}\sqrt{2}$

233. A sequence is defined by $U_{n+1} 3U_n - 4$, $n \geq 1$. If $U_1 = 3$, find U_3.

A. 11 B. 29 C. 37 D. 43

234. Find the value of $\log_3 27 + \log_4 16 - \log_8 64$

A. 5 B. 4 C. 3 D. 2

235. Find the co-efficient of x^3 in the expansion of $\frac{1}{1+x}$

A. –6 B. –1 C. 1 D. 6

236. Given that $\tan x = 1$, find the value of $\cos 2x$.

A. $\frac{1}{2}$ B. $\frac{2}{\sqrt{2}}$ C. $\frac{1}{\sqrt{2}}$ D. 0

237. The compound statement $p \wedge q$ is true only when

A. p is true and q is false.

B. p is false and q is true.

C. p is true and q is true.

D. p is false and q is false.

238. $PQRS$ is a rhombus. P is the point (1, –2) and R is (5, 8). Find the equation of the diagonal QS.

A. $2x + 5y + 9 = 0$

B. $2x + 5y + 21 = 0$

C. $2y - 5x + 9 = 0$

D. $2y - 5x + 21 = 0$

239. Calculate, correct to two decimal places, the area enclosed by the line $3x - 5y + 4 = 0$ and the axes.

A. 0.50 B. 0.51

C. 0.53 D. 0.54

240. If the function $f(x) = x^2 - 8x + k$ has two equal roots, find the value of k.

A. 4 B. 16 C. 24 D. 64

241. A particle starts moving with a velocity of 6 ms^{-1} and accelerates at 2 ms^{-2}. Find the distance it covers in 3 s.

A. 27 m B. 24 m C. 18 m D. 9 m

242. Find the unit vector in the direction of the vector –8**i** + 6**j**.

A. 4**i** – 3**j** B. $\frac{4}{5}\mathbf{i}-\frac{3}{5}\mathbf{j}$

C. –4**i** + 3**j** D. $-\frac{4}{5}\mathbf{i}+\frac{3}{5}\mathbf{j}$

The table below shows the distribution of mangoes picked by some children.

No. of mangoes picked	0	1	2	3	4	5	6
No. of children	1	2	5	8	9	4	1

Use it to answer Questions **243** *and* **244**.

243. Find, correct to the nearest whole number, the mean number of mangoes picked.

A. 3 B. 4 C. 5 D. 6

244. If a child is selected at random, what is the probability that he picked at least 4 mangoes.

A. 0.1667 B. 0.3000

C. 0.4667 D. 0.5333

245. If $y = x^3 + \frac{1}{x^2}$, find $\frac{dy}{dx}$.

A. $\frac{x^4}{4}-\frac{2}{7}$ B. $\frac{x^4}{4}-\frac{3}{x^3}$

C. $3x^2+\frac{2}{x^3}$ D. $3x^2-\frac{2}{x^3}$

246. If the standard deviation of the set of numbers {1, 3, 5, 7} is 2.24. Find the standard deviation of set {6, 8, 10, 12}.

A. 2.24 B. 4.24 C. 5.00 D. 4.48

247. Evaluate $\frac{1-\sqrt{3}}{1+\sqrt{3}}$

A. $-2-\sqrt{3}$ B. $-2+2\sqrt{3}$

C. $-2+\sqrt{3}$ D. $1+\sqrt{3}$

248. If α + β are the roots of the quadratic equation $x^2+px+q=0$, express $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ in terms of *p* and *q*.

A. $\frac{p^2+2q}{q^2}$ B. $\frac{q^2+2p}{p^2}$

C. $\frac{p^2-2q}{q^2}$ D. $\frac{q^2-2p}{p^2}$

249. Write down the 27th term of the sequence 1951, 1954, 1957, 1960, …

A. 2032 B. 2029

C. 2005 D. 2003

250. If $\log_{10}100^x+\log_{10}\left(\frac{1}{10}\right)^{x+1}=0, x>0$, solve for *x*.

A. 1 B. 2 C. 3 D. 4

251. If $\frac{{}^nP_r}{{}^nC_r}-1=5$, find the value of *r*.

A. 2 B. 3 C. 5 D. 6

252. A body of mass 0.002 kg moving with a velocity of 90 ms^{-1} collides with a body of mass m kg moving in a the same direction with velocity 20 ms^{-1}. Find *m*, if both bodies move with a common velocity 70 ms^{-1} after collision.

A. 0.001 B. 0.002

C. 0.008 D. 0.0008

253. Find the matrix of the linear transformation $P:(x,y)\rightarrow(-3y,\ -4x+2y)$.

A. $\begin{pmatrix}-3 & 0\\ 2 & 4\end{pmatrix}$ B. $\begin{pmatrix}-3 & 0\\ -4 & 2\end{pmatrix}$

C. $\begin{pmatrix}0 & -3\\ -4 & 2\end{pmatrix}$ D. $\begin{pmatrix}0 & -3\\ 2 & -4\end{pmatrix}$

254. Find the gradient of the tangent to the curve $f(x)=3x^3-2x^2+2$ at the point (1, 3).

A. 5 B. 3 C. 2 D. 1

255. Solve the equation
$2\cos^2\theta - \cos\theta = 0$, where $0° < \theta < 90°$.
A. 20° B 30° C. 60° D. 75°

256. Evaluate $\int_1^2 \frac{x^2 + x - 2}{x - 1}\,dx$
A. 0 B. $\frac{3}{2}$ C. $\frac{5}{2}$ D. $\frac{7}{2}$

257. The length of a side of a square is 4 cm. Find the increase in the area of the square when the side expands by 0.1cm.
A. 0.81 cm^2 B. 0.41 cm^2
C. 0.08 cm^2 D. 0.01 cm^2

258. If the vectors $\overrightarrow{OA} = 3\mathbf{i} - 7\mathbf{j}$ and $\overrightarrow{OB} = 9\mathbf{i} + \mathbf{j}$, where O is the origin, find the position vector of the mid-point of the line segment $\overrightarrow{AB}$.
A. $6\mathbf{i} + 3\mathbf{j}$ B. $6\mathbf{i} - 3\mathbf{j}$
C. $3\mathbf{i} - 4\mathbf{j}$ D. $-6\mathbf{i} - 3\mathbf{j}$

259. If M and N are mutually exclusive events and $P(M) = \frac{2}{5}$ and $P(N) = \frac{3}{10}$, find $P(M \cup N)$.
A. $\frac{1}{10}$ B. $\frac{3}{25}$ C. $\frac{29}{50}$ D. $\frac{7}{10}$

260. Solve the equation $x^{2/3} - x^{1/3} - 6 = 0, x > 0$
A. 27 B. 9 C. 8 D. 3

261. If $f(x - 1) = 3x^3 - 2x - 1$, find $f(3)$
A. 19 B. 74 C. 183 D. 189

262. Find the image of the point $T(1, -3)$ under the transformation represented by the matrix $\begin{pmatrix} -2 & 1 \\ 3 & 5 \end{pmatrix}$.
A. (−12, −5) B. (−5, −12)
C. (−5, 5) D. (5, 5)

263. Determine the turning point of the curve $y = x^2 - 4x + 1$.
A. (2, −3) B. (2, −5)
C. (2, 3) D. (2, 5)

264. One integer is selected at random from the set $P\{x : 1 \le x < 9\}$, where x is an integer. What is the probability that it is a prime number?
A. $\frac{1}{4}$ B. $\frac{3}{8}$ C. $\frac{1}{2}$ D. $\frac{4}{7}$

265. Find the image of the point (3, −4) under the transformation
$S : (x, y) \rightarrow (2x - y, -x + 2y)$.
A. (10, 11) B. (10, −11)
C. (−10, 11) D. (−10, −11)

266. When $f(x) = ax^2 + 7x + 9$ is divided by $(x - 1)$, the remainder is 20. Find the value of a.
A. 4 B. 3 C. 4 D. 9

267. Given that $\int_0^1 (3x^a - 1)dx = 0$, find a
A. 4 B. 2 C. 0 D. −2

268. A stone is dropped from the top of a 50 m high building. Find in ms^{-1}, the speed with which it hits the ground. (Take $g = 10\ \text{ms}^{-2}$)
A. 10 B. $5\sqrt{10}$ C. 25 D. $10\sqrt{10}$

269. Given that $\overrightarrow{BA} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\overrightarrow{CB} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$, find $|\overrightarrow{AC}|$.
A. 5 B. 12 C. 13 D. 82

270. Given that $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & -3 \\ 4 & 1 \end{pmatrix}$, find the image P^1 of the point $P(-1, 2)$

under the transformation by the matrix $A + B$.
A. (–1, 3) B. (–1, –3)
C. (2, -1) D. (3, –1)

271. The gradient function of a curve is given by $f^1(x) = 3x^2 - 8x + 5$. If 2 is the root of the equation of the curve, find the equation of the curve.
A. $6x - 8$
B. $3x^3 - 8x^2 + 5x$
C. $x^3 - 4x^2 + 5x + 2$
D. $x^3 - 4x^2 + 5x - 2$

272. Find the sum of the first *n* terms of the sequence $\log_{10} x + \log_{10} x^2 + \log_{10} x^4 + \ldots$
A. $2^n \log_{10} x$ B. $(2^n - 1)\log_{10} x$
C. $(1 - 2^n)\log_{10} x$ D. $n \log_{10} x$

273. The forces $(12N, 090^0)$ and $(5N, 180^0)$ act on a body. Find the magnitude of a third force that must act on the body to keep the system in equilibrium.
A. 20 N B. 17 N
C. 13 N D. 7 N

274. A constant force moves a particle of mass 2kg, initially at rest, along a straight line through a distance of 6 m in 2 seconds. Calculate the magnitude of the force.
A.12 N B. 6 N C. 4 N D. 3 N

Use the following information to answer Questions **275** *and* **276**.

A class consists of eight boys and seven girls. A team of three boys and two girls is to be selected from the class.

275. Find the number of possible teams.
A. 21 B. 56 C. 77 D. 1176

276. If a particular boy and girl must be on the team, find the number of possible teams.
A. 126 B. 392 C. 420 D. 588

277. Given the matrices $A = \begin{pmatrix} 2 & -3 \\ 5 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}$, find the matrix of the transformation *A* followed by *B*.
A. $\begin{pmatrix} 31 & 18 \\ 18 & 31 \end{pmatrix}$ B. $\begin{pmatrix} 17 & -18 \\ -18 & 31 \end{pmatrix}$
C. $\begin{pmatrix} 17 & 18 \\ 18 & 31 \end{pmatrix}$ D. $\begin{pmatrix} 18 & 17 \\ 31 & 18 \end{pmatrix}$

278. If the determinant of a the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is *p*, deduce the determinant of the matrix $\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$ in terms of *p*.
A. kp^2 B. k^2p C. kp D. p

279. *ABC* is a triangle, $|AB| = 8\,\text{cm}$, $|BC| = 7\,\text{cm}$ and angle $ABC = 36^\circ$, find $|AC|$, correct to one decimal place.
A. 4.7 cm B. 14.3 cm
C. 15.0cm D. 22.4cm

280. If $\begin{vmatrix} x & -x \\ 2 & x \end{vmatrix} + \begin{vmatrix} 3 & 2 \\ -1 & -x \end{vmatrix} = 8$, find the value of *x*, where $x > 0$.
A. 2 B. 3 C. 4 D. 5

281. Let $\mu = \{1, 2, 3, \ldots, 20\}$ be the universal set, $P = \{\text{factors of } 18\}$ and $Q = \{\text{factors of } 12\}$ where $P, Q \in U$. Find $n(P \cup Q)'$.
A. 12 B. 8 C. 4 D. 0

282. A polynomial is defined by $f(x) = 2x^3 - 13x^2 + 14x + 24$. Find the remainder when *f*(*x*) is divided by $(x - 3)$.
A. –25 B. –3 C. 0 D. 3

283. Given that $\log_5(x+2)+\log_5(x-2)=1$, find x.
A. 3 B. 4 C. 5 D. 6

284. The second term of a geometric sequence is 9. If its fifth term is $\frac{1}{3}$, find the common ratio.
A. $\frac{1}{3}$ B. $\frac{1}{9}$ C. 1 D. 3

285. A straight line passing through the point (2, –1) is perpendicular to the line $2x+3y-1=0$. Find the equation of the straight line.
A. $2x-3y-8=0$ B. $2y+3x-8=0$
C. $2y-3x+8=0$ D. $2x-3y-1=0$

An operation $*$ is defined by $a*b=3ab$, where a and $b \in R$.
Use the information to answer questions **286** *and* **287**.

286. Find the identity element of $*$.
A. –2 B. $-\frac{1}{3}$ C. $\frac{1}{3}$ D. 2

287. Find the inverse of 3 under $*$.
A. 3 B. $\frac{1}{3}$ C. $\frac{1}{9}$ D. $\frac{1}{27}$

288. If $\sin\theta=\dfrac{3}{5}$, find $\tan\theta$
A. $\frac{1}{3}$ B. $\frac{3}{4}$ C. $\frac{4}{3}$ D. $\frac{3}{2}$

289. Write down the matrix representation of the linear transformation. $(x,y)\to(y,-x+y)$.

A. $\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$ B. $\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}$

C. $\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$ D. $\begin{pmatrix}-1 & 0\\ -1 & 1\end{pmatrix}$

290. Differentiate $\dfrac{5x^3+x^2}{x}$, $x\neq 0$, respect to x.
A. $10x+1$ B. $10x+2$
C. $15x^2+x$ D. $15x^2+2x$

291. The transformation M is given by $(x,y)\to(x,x+y)$, find the image of the point (1, 1) after its transformation by M.
A. (–1, 2) B. (0, 2)
C. (1, 1) D. (1, 2)

292. On the cumulative frequency curve, which of the following corresponds to the median?
A. The lower quartile
B. The 50th percentile
C. Upper quartile
D. Inter-quartile range

293. A particle is projected from the point O and it moves in a straight line in such a way that its velocity, $v$ms^{-1} after time t is given by $v=20+6t-3t^2$, calculate its acceleration when $t=3$ seconds.
A. –12 ms^{-2} B. 0 ms^{-2}
C. 6 ms^{-2} D. 12 ms^{-2}

294. Evaluate $\int_1^3(x^2-2x+3)\,dx$.
A. 4 B. $6\frac{2}{3}$ C. $11\frac{2}{3}$ D. 18

295. If $\dfrac{2x+5}{x^2-x-2}=\dfrac{A}{x-2}+\dfrac{B}{x+1}$, find the value of $A+B$.
A. 5 B. 4 C. 3 D. 2

296. Find the coordinates of the centre of a circle whose equation is $3x^2+3y^2-6x+8y-11$
A. (–3, 4) B. ((–3, $-\frac{8}{3}$)
C. (1, $-\frac{4}{3}$) D. (1, –4)

297. The second and fifth terms of a linear sequence are 8 and 23 respectively. Find the common difference of the sequence.
A. 8 B. 5 C. 4 D. 3

298. Find the coefficient of x^3 in the binomial expansion of $(2+\frac{1}{3}x)^5$.

A. $\frac{80}{3}$ B. $\frac{80}{9}$ C. $\frac{40}{27}$ D. $\frac{5}{108}$

299. Simplify $\frac{\sqrt{2}}{\sqrt{2}+1}+\frac{1}{\sqrt{2}-1}$.

A. 3 B. 4 C. $3\sqrt{2}$ D. $3+2\sqrt{2}$

300. The function f, is defined on the set R; of real numbers by $f: x \to \frac{2x+1}{x^2+5x-6}$.
Find the values of x for which f **does not** exist.

A. 6, –1 B. 5, 6
C. 3, –2 D. 1, –6

301. The roots of the equation $2x^2-5x-3=0$ are α and β. Find the values of $\alpha^2+\beta^2-\alpha^2\beta^2$.

A. $\frac{11}{2}$ B. 7 C. $\frac{37}{4}$ D. 19

Two functions f and g are defined by $f: x \to 5x-1$ and $g: x \to 2x^2+3, x \in R$.
Use the information to answer question **302** *and* **303**.

302. Find $g \circ f$

A. $10x^2+2$ B. $10x^2+14$
C. $50x^2-20x+5$ D. $50x^2-20x+4$

303. Find f^1, the inverse of f.

A. $f^1: x \to 5x+1$ B. $f^1: x \to \frac{x}{5}+1$
C. $f^1: x \to \frac{1}{5}(x-1)$ D. $f^1: x \to \frac{1}{5}(x+1)$

304. In an examination, 75% of candidate passed. What did the passed mark correspond to?

A. The 15th percentile
B. The 25th percentile
C. The 50th percentile
D. The 75th percentile

305. A distributor has the following information $\sum f=100$, $\sum fd=5$ and $\sum fd^2=900$. Calculate the standard deviation of the distribution.

A. 0.31 B. 0.71 C. 3.00 D. 8.75

306. If the vectors **4i** + **5j** and q**i** + **4j** are perpendicular where q is scalar find the value of q.

A. –5 B. $-\frac{1}{5}$ C. 4 D. 5

307. Given the matrix $\mathbf{A}=\begin{pmatrix}2 & 3\\ -1 & 4\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1 & 2\\ 3 & -2\end{pmatrix}$, find 2**A** + 3**B**.

A. $\begin{pmatrix}3 & 5\\ 2 & 2\end{pmatrix}$ B. $\begin{pmatrix}5 & 8\\ 1 & 6\end{pmatrix}$

C. $\begin{pmatrix}7 & 12\\ 7 & 2\end{pmatrix}$ D. $\begin{pmatrix}7 & 12\\ 11 & 14\end{pmatrix}$

308. In how many ways can a committee of 4 men and 3 women be formed from a group of 6 men and 5 women?

A. 25 B. 150 C. 420 D. 21,600

309. Find the magnitude and direction of the resultant of the forces $\mathbf{F_1}=6\mathbf{i}+\mathbf{j}$ and $\mathbf{F_2}=-\mathbf{i}+4\mathbf{j}$.

A. $(5\sqrt{2}, 315^0)$ B. $(5\sqrt{2}, 045^0)$
C. $(5, 045^0)$ D. $(5, 315^0)$

310. If $\mathbf{P}=\begin{pmatrix}2 & 3\\ -1 & 4\end{pmatrix}$ and $\mathbf{Q}=\begin{pmatrix}-1 & 0\\ 2 & 3\end{pmatrix}$, find PQ.

A. $\begin{pmatrix}-2 & 0\\ 2 & 6\end{pmatrix}$ B. $\begin{pmatrix}-2 & 1\\ 5 & 10\end{pmatrix}$

C. $\begin{pmatrix}4 & 9\\ 9 & 12\end{pmatrix}$ D. $\begin{pmatrix}-4 & -2\\ 6 & 2\end{pmatrix}$

311. Solve the equation $2(2^{2x+3}) = 4^{2x+1}$.

A. 2 B. 1 C. $\frac{1}{2}$ D. $\frac{1}{4}$

312. If the standard deviation of the numbers x_1, x_2, x_3 and x_4 is 1.71, find the standard deviation of $3x_1, 3x_2, 3x_3$ and $3x_4$.

A. 15.39 B. 5.13
C. 3.42 D. 1.71

313. Find the maximum value of the function $f(x) = x^3 + 3x^2 - 9x + 1$.

A. –4 B. 1 C. 12 D. 28

314. Find the inverse of the matrix $\mathbf{B} = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$

A. $\begin{pmatrix} -1 & \frac{3}{2} \\ 1 & -2 \end{pmatrix}$ B. $\begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$

C. $\begin{pmatrix} 2 & -\frac{2}{3} \\ -1 & 1 \end{pmatrix}$ D. $\begin{pmatrix} 2 & -\frac{3}{2} \\ -1 & 1 \end{pmatrix}$

315. A body of mass 10kg moving with a velocity of 5 ms^{-1} collides with another body of mass 15 kg moving in the same direction as the first but with a velocity 2 ms^{-1}. If after collision the bodies move with a common velocity v ms^{-1}, find the value of v.

A. 3.2 B. 5.3 C. 7.0 D. 8

316. If $\int_0^a x^3 dx = 4$, find a.

A. $\frac{1}{4}$ B. $\frac{1}{2}$ C. 2 D. 4

317. If $\mathbf{m} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{n} = -\mathbf{i} + 2\mathbf{j}$, find the unit vector in the direction of $\mathbf{n} + \mathbf{m}$.

A. $\mathbf{i} - \mathbf{j}$ B. $\sqrt{2}\,\mathbf{i} - \sqrt{2}\,\mathbf{j}$
C. $\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$ D. $\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$

318. Find the sum of the geometric series,

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$$

A. $\frac{1}{3}$ B. $\frac{1}{2}$ C. 1 D. 6

319. Simplify ${}^nP_r \div {}^nC_r$.

A. $r!$ B. $\frac{1}{r!}$

C. $\frac{(n-r)!}{r!}$ D. $\frac{r!}{(n-r!)}$

320. The velocity v ms-1 of a body moving in a straight line from a point O in time t seconds is given by $v = 8t - 3t^2$. Calculate the distance from O after 2 seconds.

A. 4 m B. 5 m C. 8 m D. 24 m

321. In how many ways can the letters from the word *MAINTAIN* be arranged?

A. 2520 B. 5040 C. 10080 D. 20160

322. Given that $\frac{dy}{dx} = 3x + 4$, and $y = \frac{3}{8}$ when $x = \frac{1}{2}$, find y in terms of x.

A. $y = \frac{3}{2}x^2 + 4x$

B. $y = \frac{3}{2}x^2 + 4x + 2$

C. $y = \frac{3}{2}x^2 - 4x - 2$

D. $y = \frac{3}{2}x^2 + 4x - 2$

323. Three forces $(2\text{N}, 180^0)$, $(3\sqrt{2}\text{N}, 315^0)$, and $(4\text{N}, 090^0)$ act on a body. Find the magnitude of their resultant.

A. $\sqrt{2}$ B. 2 C. $2\sqrt{2}$ D. $4\sqrt{2}$

The table below gives the distribution of marks scored by 40 students in a class test.

Marks	1-10	11-20	21-30	31-40	41-50
No. of students	12	6	8	4	10

Use the table to answer questions **324** *and* **325**.

324. Calculate the mean score.
A. 8.0 B. 19.5
C. 24.0 D. 28.5

325. What is the probability that a student selected at random scored 21 marks and above?

A. $\frac{1}{5}$ B. $\frac{3}{10}$ C. $\frac{9}{20}$ D. $\frac{11}{20}$

326. Simplify $\frac{\tan 60^0 + \tan 45^0}{\tan 60^0 - \tan 45^0}$, leaving your answer in surd form.

A. $-2-\sqrt{3}$ B. $-2+\sqrt{3}$
C. $2-\sqrt{3}$ D. $2+\sqrt{3}$

327. What is the image of the point (–3, 2) under the linear transformation

$$x^1 = -x + 3y$$
$$y^1 = 4x + 2y$$

A. (9, –8) B. (9, 16)
C. (3, 16) D. (3, 8)

328. An error of $P\%$ is made in measuring the radius of a circle. Calculate the approximate percentage error in the measurement of the circumference.

A. $\frac{1}{2}P\%$ B. $P\%$ C. $\frac{3}{2}P\%$ D. $2P\%$

329. Find the equation of the tangent to the curve $y = x^3 - 2x^2 - x + 2$ at the point (–1, 0).

A. $2y + x - 2 = 0$ B. $y - 8x - 8 = 0$
C. $y - 2x - 2 = 0$ D. $y - 6x - 6 = 0$

330. A ball is released vertically from the top of a roof, 30 m high. Assuming air resistance be negligible, calculate the velocity of the ball when it is 5 metres above the ground. [Take $g = 10\text{m}^2$].

A. 10 ms^{-1} B. $10\sqrt{5}$ ms^{-1}
C. 25 ms^{-1} D. 50 ms^{-1}

331. Select the false statement in the truth table below

	p	q	$p \Rightarrow q$	$p \Rightarrow \sim q$
A.	T	T	T	T
B.	T	F	F	T
C.	F	T	T	T
D.	F	F	T	T

332. Evaluate the third term of the binomial expansion of $(1+0.02)^{\frac{1}{2}}$.

A. 0.0005 B. 0.00005
C. –0.0005 D. –0.00005.

333. $f(x)$ and $g(x)$ are defined on the set of real numbers by

$$f(x) = 3x - 1,\ g(x) = \frac{5}{x}, x \neq 0$$

Which of the following statements is/are true about the function $g \circ f$? $g \circ f$ is

I. an odd function;
II. an even function;
III. a one-to-one function.

A. I only B. III only
C. I and III only D. II and III only

334. Find the truth set of the inequality $x^2 + 4 > 4x + 1$.

A. $\{x : x < -1 \cup x > 3\}$
B. $\{x : x < 1 \cup x > 3\}$
C. $\{x : x < -1\}$ D. { }

335. Find the value of cos 75°.

A. $\frac{\sqrt{2}(\sqrt{3}+1)}{2}$ B. $\frac{\sqrt{2}(\sqrt{3}-1)}{4}$
C. $\frac{\sqrt{2}(\sqrt{3}+1)}{4}$ D. $\frac{\sqrt{2}(\sqrt{3}-1)}{2}$

336. Make x the subject of the relation $y = 3 + 2^{x+1}$.

A. $x = \dfrac{\log y - \log 6}{\log 6}$

B. $x = \dfrac{\log y - \log 3 - \log 2}{\log 2}$

C. $x = \dfrac{\log(y-3) - \log 2}{\log 2}$

D. $x = \dfrac{\log y - \log 3 + \log 2}{\log 2}$

337. $R(-3,\ 1)$ and $S(-1,\ 3)$ are two points on a circle. If RS is a diameter, find the equation of the circle.

A. $x^2 + y^2 - 4x + 4y - 6 = 0$

B. $x^2 + y^2 - 4x + 4y + 6 = 0$

C. $x^2 + y^2 + 4x - 4y + 6 = 0$

D. $x^2 + y^2 + 6x - 2y + 8 = 0$

338. Find the locus of a point equidistant from the points $P(-2,\ 1)$ and $Q(-1,\ 2)$.

A. $y + x = 0$ B. $y - x + 3 = 0$

C. $y + x - 3 = 0$ D. $y - x = 0$

339. In triangle PQR below, angle $RPQ = 45^\circ$, angle $PQR = 60^0$ and $|RQ| = 4cm$. Find, in cm $|RP|$.

A. $\dfrac{\sqrt{6}}{2}$ B. $33\sqrt{6}$

C. $\dfrac{4\sqrt{6}}{3}$ D. $2\sqrt{6}$

340. The equation of a circle is $x^2 + y^2 - 2x + 14y + 44 = 0$. Find the radius of the circle.

A. $\sqrt{6}$ B. 7 C. $5\sqrt{2}$ D. $2\sqrt{6}$

341. Find the sum of the first six terms of the exponential series $\frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$

A. $\dfrac{42}{128}$ B. $\dfrac{21}{128}$ C. $-\dfrac{21}{128}$ D. $-\dfrac{42}{128}$

342. If $(x + 1)$ is a factor of the expression $x^3 - 2x^2 - 4x - 1$, find the other factor.

A. $x^2 - 3x - 1$ B. $x^2 + 3x - 1$

C. $x - 1$ D. $x^2 - 2x - 4$

343. Select the correct option(s) from the list below:

I. $\log_{10} 4.5 = \log_{10} 2 - \log_{10} 9$

II. $\log_{10} \dfrac{1}{\sqrt{64}} = -3\log_{10} 2$

III. $\log_4 5 = \dfrac{\log_3 4}{\log_3 5}$

A. I only B. II only

C. III only D. I and II only'

344. Express $\dfrac{1}{1 - \sin 60^0}$ as a surd.

A. $1 - \sqrt{2}$ B. $4 + 2\sqrt{3}$

C. $2 + \sqrt{3}$ D. $4 + 2\sqrt{2}$

345. If $f(x) = -2x^3 + qx + 2$, where q is a constant and $f(-1) = 2$, find q.

A. −6 B. −2 C. 2 D. 6

346. Find the general solution of $2\cos\frac{1}{2}x = 1$

A. $2n\pi \pm \frac{2}{3}\pi$ B. $n\pi \pm \frac{1}{3}\pi$

C. $4n\pi \pm \frac{2}{3}\pi$ D. $4n\pi \pm \frac{1}{3}\pi$

347. If $f[g(x)] = x^2 - 2x + 1$, and $f(x) = x - 1$, find $g(x)$.

A. $x - 1$ B. $(x-1)^2$

C. $x^2 - 2x + 2$ D. x^2

348. Find the inverse of $f : x \rightarrow \dfrac{2x-1}{x+2}, x \neq -2$

A. $\dfrac{1-2x}{2-x}, x \neq -2$ B. $\dfrac{1-2x}{2+x}, x \neq -2$

C. $\dfrac{1-2x}{x-2}, x \neq -2$ D. $\dfrac{1+2x}{x+2}, x \neq -2$

349. Given that $x = 2 + 3\sqrt{2}$ and $y = -1 + 5\sqrt{2}$, find $x^2 - y^2$.

A. $-29 + 22\sqrt{2}$ B. $29 - 22\sqrt{2}$

C. 73 D. $3 - 2\sqrt{2}$

350. Express $\dfrac{2x-1}{(x+1)(2x+1)}$ in partial fractions.

A. $\dfrac{1}{x+1} + \dfrac{1}{2x+1}$ B. $\dfrac{3}{x+1} + \dfrac{4}{2x+1}$

C. $\dfrac{2x}{x+1} + \dfrac{-1}{2x+1}$ D. $\dfrac{-3}{x+1} + \dfrac{4}{2x+1}$

351. Factorize the expression $(x + y - 1)^2 - (x - y + 1)^2$.

A. $2x(2y-1)$ B. $2y(2x-1)$

C. $4y(x-1)$ D. $4x(y-1)$

352. Given that $\dfrac{\log(3x+1)^{2x-1}}{\log(3x+1)} = 5$, find the value of x

A. –2 B. 2 C. 3 D. 4

353. If $\tan x = \dfrac{1}{3}$, find cos*x*.

A. $\dfrac{3\sqrt{10}}{10}$ B. $0.1\sqrt{10}$

C. $\dfrac{\sqrt{3}}{10}$ D. $\dfrac{1}{10}$

354. Differentiate, with respect to *x*, $\dfrac{1}{x^2}\left(3 + 5x + 2x^2\right)$.

A. $\dfrac{-6}{x^3} - \dfrac{5}{x_2}$ B. $\dfrac{5+4x}{x^2}$

C. $\dfrac{6}{x^3} - \dfrac{5}{x^2}$ D. $\dfrac{5}{x^2} - \dfrac{6}{x^2}$

355. If $y = \sqrt{x}$, find $\dfrac{dy}{dx}$ when $x = 4$

A. 4 B. 2 C. $\frac{1}{2}$ D. $\frac{1}{4}$

356. If $\int_2^a (2x+3)dx = 8$, $a = a > 0$, determine the value of *a*.

A. 2 B. 3 C. 6 D. 8

Use the information below to answer Questions **357** *and* **358**.

Given that $f(x) = 3\cos x + 2\sin x$, where $0^0 \le x \le 360^0$.

357. What is the maximum value of *f*(*x*)?

A.5 B. $\sqrt{14}$ C. $\sqrt{5}$ D. $\sqrt{13}$

358. For what value of *f*(*x*) maximum?

A. 33.7^0 B. 56.3^0 C. 213.7^0 D. 236.3^0

359. The gradient of a curve at the point *P*(1, –2) is 4. The line *y* = 4*x* + *k*, where *k*, is a constant, is tangent to the curve at *P*. Find *k*.

A. 4 B. 2 C. 4 D. -6

The table below shows the marks, out of 100, obtained by 1000 students in a test in an institution.

Use it to answer Questions **360** *to* **362**.

Marks	Frequency
1-10	20
11-20	14
21-30	18
31-40	24
41-50	48
51-60	106
61-70	211
71-80	296
81-90	216
91-100	47

360. Find the lower class boundary of the class containing the third quartile.

A. 70 B. 70.5 C. 80 D. 80.5

361. If the pass mark was 61% and a student was chosen at random, find the probability that the student passed the test.
A. 0.21 B. 0.23 C. 0.56 D. 0.77

362. Students scoring 30 marks or below were withdrawn from the institution. What percentage of the students were withdrawn?
A. 0.18% B. 0.52%
C. 1.8% D. 5.2%

363. A die has six faces. Two of these faces are numbered 1 and the remaining four faces 2, 3, 4 and 5. If such dice are tossed once, find the probability of 1 showing up on both dice.
A. $\frac{1}{9}$ B. $\frac{4}{25}$ C. $\frac{1}{36}$ D. $\frac{1}{18}$

364. Given that X and Y are independent events and that $P(x) = 0.5$, $P(Y) = c$ and $P(XUY) = 0.8$, find the value of c.
A. 0.3 B. 0.5 C. 0.6 D. 0.8

365. In how many ways can 3 books be selected from 9 different books?
A. 3 B. 12 C. 27 D. 84

366. A man had 10 coins in his pocket, 3 of which are known to be 50-cedis coin and the rest 100-cedis coins. If two coins were selected at random without replacement, what is the probability that both coins will be 50 cedis coins?
A. $\frac{2}{9}$ B. $\frac{3}{50}$ C. $\frac{1}{15}$ D. $\frac{9}{100}$

367. The equation of the line of best fit for two variables x and y, where y is the independent variable given by $x = -3.1 + 0.06y$. Predict the value of x when $y = 8.5$.
A. 8.3 B. 5.1 C. –2.6 D. –3.1

368. Which of the following statements is/are true about scatter diagrams?

I. Knowledge about the relationship between the variable enables us to predict and control events.

II. The independent variable is the variable which is not affected by changes in the other.

III. The main purpose of a scatter diagram is to determine whether there is any pattern among the points.

A. III only B. I and II only
C. II and III only D. I, II and III only.

369. Two vectors are represented by **r** = 2**a** and **s** = 3**a** where **a** = 2**i** – **j**. Find | **r** – **s** |.
A. 1 B. $\sqrt{2}$ C. $\sqrt{3}$ D. $\sqrt{5}$

370. Find the unit vector that is perpendicular to the vector 2**i** – 3**j**.
A. $\frac{2}{\sqrt{13}}(3i - 2j)$ B. $\frac{2}{\sqrt{13}}\left(i + \frac{2}{3}j\right)$
C. $\frac{2}{\sqrt{13}}\left(\frac{3}{2}i + j\right)$ D. $\frac{2}{\sqrt{13}}\left(\frac{2}{3}i + j\right)$

371. A parallelogram $PQRS$ is such that $\overrightarrow{PQ} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and $\overrightarrow{PS} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$. Find the co-ordinates of R if P is the point (0, 1)
A. (3, 3) B. (3, 4) C. (3, 5) D.(–3, 4)

372. A particle starting from rest has an acceleration (8 – 2t) ms^{-2} at time t. When will the particle come to rest again?
A. 2 sec B. 4 sec
C. 6 sec D. 8 sec

373. State the matrix M representing the linear transformation.
$M : (x, y) \rightarrow (4x - y, -3x + 5y)$.
A. $\begin{pmatrix} 4 & -3 \\ -1 & 5 \end{pmatrix}$ B. $\begin{pmatrix} 4 & -3 \\ 1 & -5 \end{pmatrix}$
C. $\begin{pmatrix} 4 & -1 \\ -3 & 5 \end{pmatrix}$ D. $\begin{pmatrix} 4 & 3 \\ 3 & -5 \end{pmatrix}$

374. Find integers p and q which would make the following matrix equation true:

$$p\begin{pmatrix}8 & 6\\ 4 & 0\end{pmatrix} - q\begin{pmatrix}3 & 3\\ 2 & 2\end{pmatrix} = \begin{pmatrix}2 & 0\\ 0 & -4\end{pmatrix}$$

A. $p = -2,\ q = -1$ B. $p = -1,\ q = 2$
C. $p = 2,\ q = 1$ D. $p = 1,\ q = 2$

375. If $A^{-1} = \frac{1}{7}\begin{pmatrix}1 & 3\\ -2 & 1\end{pmatrix}$, find **A**.

A. $\begin{pmatrix}-1 & 3\\ -2 & -1\end{pmatrix}$ B. $\begin{pmatrix}1 & -2\\ 3 & 1\end{pmatrix}$

C. $\begin{pmatrix}1 & -3\\ 2 & 1\end{pmatrix}$ D. $\begin{pmatrix}-1 & 3\\ 2 & -1\end{pmatrix}$

376. Given that $A = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$ and $B = \begin{pmatrix}4 & 3\\ 2 & 1\end{pmatrix}$, find $B^2 - A$

A. $\begin{pmatrix}20 & 14\\ 9 & 5\end{pmatrix}$ B. $\begin{pmatrix}5 & 4\\ 4 & 5\end{pmatrix}$

C. $\begin{pmatrix}1 & 1\\ 2 & 4\end{pmatrix}$ D. $\begin{pmatrix}18 & 14\\ 9 & 5\end{pmatrix}$

377. A linear transformation is represented by the matrix $\begin{pmatrix}1 & -1\\ 2 & 3\end{pmatrix}$. Find the gradient of the line $y = 2x - 1$ after it has been transformed by the matrix.

378. Find the matrix of the linear transformation $p(x, y) \to (-3y,\ -4x + 2y)$.

A. $\begin{pmatrix}-3 & 0\\ 2 & -4\end{pmatrix}$ B. $\begin{pmatrix}-3 & 0\\ 4 & 2\end{pmatrix}$

C. $\begin{pmatrix}0 & -3\\ -4 & 2\end{pmatrix}$ D. $\begin{pmatrix}0 & -3\\ 2 & -4\end{pmatrix}$

379. If $p \Rightarrow q$ then

A. $q \Rightarrow p$ B. $-p \Rightarrow -q$
C. $-q \Rightarrow -p$ D. $-q \Rightarrow p$

380. Which of the following is not a rational number?

A. 0 B. $\frac{1}{3}$ C. $\sqrt{3}$ D. 3

381. For what value of t would

$$\frac{2+\sqrt{3}}{1-\sqrt{t}} = \frac{-4}{2-\sqrt{3}}?$$

A. $\frac{5}{4}$ B. $\frac{4}{5}$ C. $\frac{-16}{25}$ D. $\frac{25}{16}$

382. A binary operation is defined on the set of real numbers by $a\Delta b = \left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)$ Find the value of x if $x \Delta 2 = 1$.

A. 3 B. $\sqrt{3}$ C. $-\sqrt{3}$ D. -1

383. If $3x^2 + 3y^2 + 2x - 6y - 8 = 0$, which of the following is true?

A. $6x + 6y - 4 = 0$ B. $6x + 6\frac{dy}{dx} + 2$

C. $6x + 6y\frac{dy}{dx} + 2 - 6\frac{dy}{dx} = 0$

D. $6x + 6y + 2 - 6\frac{dy}{dx} = 0$

384. Differentiate the function $f : x \to (x^2 - 1)^3$ with respect to x.

A. $3(x^2 - 1)^2$ B. $3x(x^2 - 1)^2$
C. $6x(x^2 - 1)^2$ D. $6(x^2 - 1)^2$

385. Find the gradient of the function $f : x \to x^2 + 2x + 3$ at the point where $x = 0$ and $y = 3$.

A. 0 B. 1 C. 2 D. 3

386. The n^th term of a sequence is defined by

$$U_n = 3^3\left(\frac{1}{3}\right)^{n-1}.\ \text{Find}\ \frac{U_5}{U_6}$$

A. 27 B. 9 C. 3 D. $\frac{1}{3}$

387. If $f(x) = 3x^2 + \frac{2}{x}$, find $\frac{df}{dx}$.

A. $x + 2x$ B. $6 + \frac{2}{x^2}$

C. $6 - \frac{2}{x^2}$ D. $6x^2 - 2x$

388. Evaluate $\int_0^2 (6x^2 - 8x + 3)dx$

A. 16 B. 12 C. 8 D. 6

389. Determine the maximum point on the curve $y = 2x^3 + \frac{5}{2}x^2 - x$.

A. $\left(-1, \frac{3}{2}\right)$ B. $\left(\frac{3}{2}, -1\right)$

C. $\left(1, -\frac{3}{2}\right)$ D. $\left(-1, -\frac{3}{2}\right)$

390. Given that $\frac{dy}{dx} = 2x + 1$, find the equation of the tangent at the point (0, 1).

A. $y = x - 1$ B. $y = -x + 1$

C. $y = 2x + 1$ D. $y = x + 1$

391. Find the derivative of $\frac{x_3}{x^2 + x} + 3x^2 + 2x$

A. 1 B. $2x + 1$ C. 3 D. $x + 2$

392. $P(-2, 1)$ and $Q(-6, 2)$ are points in a plane. Find $|PQ|$.

A. $\sqrt{15}$ B. $\sqrt{16}$ C. $\sqrt{17}$ D. $\sqrt{18}$

393. If $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$, find **BC**

A. $\begin{pmatrix} -5 & 8 \\ -18 & 29 \end{pmatrix}$ B. $\begin{pmatrix} 5 & 8 \\ 18 & 29 \end{pmatrix}$

C. $\begin{pmatrix} -5 & -18 \\ 8 & 29 \end{pmatrix}$ D. $\begin{pmatrix} -5 & 29 \\ 8 & -18 \end{pmatrix}$

394. Given that $\mathbf{A} = \begin{pmatrix} 8 & -2 \\ 5 & 3 \end{pmatrix}$, find det **A**

A. 44 B. 43 C. 34 D. 14

395. Given that $f(x) = 1 + 2x$ and $g(x) = x + 1$, find $\frac{d}{dx}(f \times g)$

A. $3 + 2x$ B. $3 + 4x$ C. $3x + 2$ D. 3

396. The radius of a circle increases at a rate of 0.2 cms^{-1}. Find the rate of change in the area of the circle with radius 4 cm.

A. 0.16 π cms^{-2} B. 0.2 π cms^{-2}

C. 0.8 π cms^{-2} D. 1.6 π cms^{-2}

397. For what value of x would the derivative of $f(x) = 2x^3 - 24x$ be equal to zero?

A. 0, 2 B. 1, –1

C. 2, –2 D. 3, –3

398. The sum of the second and fifth terms of a linear sequence is 6 and the common difference is -2. What is the first term of the sequence?

A. 4 B. 6 C. 7 D. 8

399. Find the equation of the normal to the curve with a gradient function $x^2 - 3$ at the point (1, 1).

A. $2y - x = 1$ B. $2y - x = -1$

C. $y + 2x = 3$ D. $y + 2x = -3$

400. What force is required to make a 4 kg box resting on a horizontal table, just to move when pulled horizontal? [Take $g = 10 \text{ ms}^{-2}$; co-efficient of friction = 0.3].

A. 1.2N B. 2.94N

C. 11.76N D. 39.20N

401. Determine the greatest height reached when a particle is projected vertically upwards from a given point with velocity 20 ms^{-1} [Take $g = 10 \text{ ms}^{-1}$].

A. 2m^{-2} B. 20m^{-2} C. 200m^{-2} D. 400m^{-2}

402. The distance S m which a particle covers in *t* seconds is given by $S = \frac{3}{2}t^2$. Find the acceleration of the particle.

A. 4 ms B. 3 ms C. 2 ms D. 3 ms

403. A body of mass 3 kg, moving with velocity 20 ms^{-1} collides with a body *Q* of mass 5 kg. After collision the bodies move with a velocity 35 ms^{-1}. Find the velocity of *Q* before collision.

A. 78m/s B. 62m/s C. 56m/s D. 44m/s

404. Which of the following matrices has no inverse?

I. $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ II. $\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$ III. $\begin{pmatrix} 4\sqrt{3} & 3\sqrt{2} \\ 4 & 6 \end{pmatrix}$

A. II only B. III only

C. I and II only D. II and III only.

405. Given that $f(x) = 1 + 2x$, and $g(x) = x + 1$, find $\frac{d}{dx}(fog)$.

A. $3 + 2x$ B. $3 + 4x$

C. $3x + 2$ D. 2

406. The velocity of a particle at time *t* is given by $V(t) = 3t^2 + 2t - 1$. What is the displacement of the particle at time *t*.

A. $t^3 + t^2 - t + c$ B. $\frac{1}{3}t^3 + t^2 - t + c$

C. $\frac{1}{3}t^3 + t^2 - t$ D. $t^3 + t^2 - t$

407. Find the maximum value of the function $f(x) = 10 + 3x - x^2$

A. $\frac{49}{4}$ B. $\frac{15}{4}$ C. 16 D. $\frac{65}{4}$

408. If the force **F** = (5 N, 045^0), find 3**F**.

A. (15N, 135^0) B. 15N, 045^0)

C. (5N, 135^0) D. (5N, 045^0)

409. Find the magnitude and direction of the resultant of the vectors **a** = 2**i** + 3**j** and **b** = 2**i** + **j**.

A. $(4\sqrt{2}, 030^0)$ B. $(4\sqrt{2}, 045^0)$

C. $(2\sqrt{3}, 0345^0)$ D. $(4\sqrt{3}, 030^0)$

410. Find the image of the point A(2, 5) under the linear transformation represented by the matrix $\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}$

A. $\begin{pmatrix} 16 & 17 \end{pmatrix}$ B. $\begin{pmatrix} 11 & 19 \end{pmatrix}$

C. $\begin{pmatrix} 19 & 11 \end{pmatrix}$ D. $\begin{pmatrix} 17 & 16 \end{pmatrix}$

411. A matrix $\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, where $0^0 \le \theta \le 360^0$. Find the image of (2, 2) under the linear transformation defined by **M** when $\theta = 90^0$.

A. $\begin{pmatrix} 2 & 2 \end{pmatrix}$ B. $\begin{pmatrix} 2 & -2 \end{pmatrix}$

C. $\begin{pmatrix} -2 & 2 \end{pmatrix}$ D. $\begin{pmatrix} -2 & 3 \end{pmatrix}$

412. Find the value of $\frac{x}{y}$ given that $\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

A. $-\frac{3}{2}$ B. $-\frac{2}{3}$ C. $\frac{2}{3}$ D. $\frac{3}{2}$

413. Find the angle between the vectors **a** = 4**i** – 5**j** and **b** = 5**i** + 4**j**

A. 0^0 B. 45^0 C. 60^0 D. 90^0

414. Which of the following methods could be used in a frequency distribution when the class sizes are not equal?

A. Frequency density

B. Class mid-point

C. Cumulative frequency

D. Class boundary

415. A body of mass m moving with a velocity of $v_1 = 2\mathbf{i} + 3\mathbf{j}$, brings to rest a body of mass 6 kg moving in the opposite direction with a velocity of $v_2 = 6\mathbf{i} + 9\mathbf{j}$.

A. 15 kg B. 18 kg

C. 21 kg D. 25 kg

416. Two independent event A and B are such that $P(A) = \frac{5}{12}$, $P(A \cap B) = \frac{1}{8}$, find $P(B)$.

A. $\frac{5}{12}$ B. $\frac{5}{8}$ C. $\frac{2}{3}$ D. $\frac{3}{10}$

417. The probability of Kofi and Kwame scoring from penalties are 0.6 and 0.4 respectively. If they take a penalty kick each, what is the probability that exactly one goal will be scored.

A. 1.00 B. 0.52 C. 0.24 D. 0.20

A particle is projected upwards from a point O with a speed of 10ms^{-1}.
[Take $g = 10 \text{ms}^{-2}$]

Use this information to answer Question **418** *and* **419**.

418. What is the greatest height reached by the particle?

A. 5 m B. 10 m C. 50 m D. 100 m

419. Find the time when the particle is 2.5 m from the point O.

A. $4 - 2\sqrt{2}$ sec B. $4 + 2\sqrt{2}$ sec

C. $1 - 2\sqrt{2}$ sec D. $1 + \dfrac{\sqrt{2}}{2}$ sec

420. A car of mass 10^3 kg travelling at 60 ms^{-1} is brought to rest in 2 seconds, when the brakes are applied. Find the magnitude of the breaking force.

A. 60×5^3 N B. 30×5^3 N

C. 3.0×5^4 N D. 1.2×5^5 N

421. Find the real values of x which satisfy $3 < 2x - 1$ and $2x < 5$.

A. $\{x : x > 2\}$ B. $\{x : x < \frac{5}{2}\}$

C. $\{x : 2 < x < \frac{5}{2}\}$ D. $\{x : x < 2 \text{ and } x > \frac{5}{2}\}$

422. Find the unit vector in the direction of the vector $-12\mathbf{i} + 5\mathbf{j}$.

A. $-\frac{12}{13}\mathbf{i} + \frac{5}{13}\mathbf{j}$ B. $\frac{-5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}$

C. $\frac{-1}{13}\mathbf{i} + \frac{5}{13}\mathbf{j}$ D. $\frac{-12}{13}\mathbf{i} - \frac{5}{13}\mathbf{j}$

A binary operation ∇, is defined on the set $P = \{3, 4, 5, 6, 7, 8, 9\}$ by $A \nabla B = A \cap B'$, where $A = \{5, 6, 7\}$ and $B = \{3, 6, 8\}$, $A, B \subset P$.

Use the information above to answer Questions **423** *and* **424**.

423. Find $A \nabla B$.

A. $\{5\}$ B. $\{7\}$ C. $\{5, 6\}$ D. $\{5, 7\}$

424. Find $B \nabla A$.

A. $\{3\}$ B. $\{8\}$ C. $\{3, 8\}$ D. $\{4, 9\}$

425. If the vector $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$, find the bearing of P and Q.

A. 045° B. 135° C. 180° D. 315°

426. If $\tan(A + B) = \frac{2}{5}$ and $\tan A = 3$, find the value of $\tan B$.

A. $\frac{13}{5}$ B. $\frac{3}{5}$ C. $-\frac{13}{8}$ D. $-\frac{13}{11}$

427. If the matrix $\mathbf{M} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$, find a matrix $\mathbf{N}$ such that $5\mathbf{MN} = \mathbf{I}$, where $\mathbf{I}$ is the 2×2 unit matrix.

A. $\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$ B. $\begin{pmatrix} 1 & -3 \\ -1 & 2 \end{pmatrix}$

C. $\begin{pmatrix} 1 & 3 \\ 1 & -2 \end{pmatrix}$ D. $\begin{pmatrix} -1 & -3 \\ -1 & 2 \end{pmatrix}$

428. The points $(3, -2)$ and $(-5, 6)$ are the ends of the diameter of a circle. Find the equation of the circle.

A. $x^2 + y^2 + 2x - 4y + 27 = 0$

B. $x^2 + y^2 + 2x - 4y - 27 = 0$

C. $x^2 + y^2 - 2x - 4y - 27 = 0$

D. $x^2 + y^2 - 2x + 4y - 27 = 0$

The sum of the first four terms of an exponential series is $\frac{15}{8}$ and the common ratio is $\frac{1}{2}$.
Use this information above to answer Questions **429** *and* **430**.

429. Find the first term.
A. –1 B. $\frac{15}{16}$ C. 1 D. $\frac{15}{4}$

430. Find the sum to infinity of the exponential series
A. $\frac{1}{4}$ B. $\frac{1}{2}$ C. 1 D. 2

431. If the line $ky + 2x + 7 = 0$ is parallel to the line $50y + kx - 19 = 0$, find the value of k, where $k > 0$.
A. 5 B. 10 C. 25 D. 100

432. If $F(n)$ denotes the set of factors of the natural number, n, including n but excluding 1, find a number p such that $F(12) \cap F(18) = F(p)$.
A. 2 B. 3 C. 6 D. 18

433. Factorise $f(x) = 2x^3 - 5x^2 + x + 2$ completely given that $f(-\frac{1}{2}) = 0$.
A. $(2x+1)(x-1)(x-2)$
B. $(2x+1)(x+1)(x-2)$
C. $(2x-1)(x+1)(x-2)$
D. $(2x-1)(x-1)(x-2)$

434. The expression $x^3 + 2x^2 - 3x + 4$ is divided by $(x - 1)$. Find the remainder.
A. 1 B. 2 C. 3 D. 4

435. Find the values of x such that the determinant of the matrix $\begin{pmatrix} x-2 & 1 \\ 2 & x-3 \end{pmatrix}$ is zero.
A. $x = 1, 4$ B. $x = 1, 2$
C. $x = 1, -4$ D. $x = -1, -4$

436. If $\cos\theta = -\frac{8}{17}$, and θ is obtuse, find $\tan\theta$.
A. $\frac{15}{17}$ B. $-\frac{8}{15}$ C. $-\frac{15}{17}$ D. $-\frac{15}{8}$

437. Evaluate $\int_{-2}^{3}(3x^2 - 2x - 12)dx$.
A. –30 B. –18 C. –6 D. 6

438. For what value of k will the equation $kx^2 - 8x + 4 = 0$, have repeated roots.
A. 16 B. 4 C. 2 D. –4

439. Given that $\frac{dy}{dx} = \sqrt{x}$, find y.
A. $2x^{\frac{3}{2}} + c$ B. $\frac{2}{3}x^2 + c$
C. $\frac{3}{2}x^{\frac{3}{2}} + c$ D. $\frac{2}{3}x^{\frac{3}{2}} + c$

440. A car is moving at a speed of 108 kmh^{-1}. Find its speed in ms^{-1}.
A. 3 ms^{-1} B. 18 ms^{-1}
C. 30 ms^{-1} D. 300 ms^{-1}

A particle moves along the x-axis such that its distance x m, from the origin at any time t seconds is given by $x = t^3 - 2t^2 + 5t + 3$.

Use the information to answer Questions **441** *and* **442**.

441. Find the distance covered by the particle in the 4th second.
A. 27 m B. 28 m C. 32 m D. 42 m

442. Calculate the velocity of the particle at the end of the 3rd second.
A. 12 ms^{-1} B. 15 ms^{-1}
C. 20 ms^{-1} D. 27 ms^{-1}

443. Given that $\mathbf{P} = \begin{pmatrix} 6 & -5 \\ 4 & -2 \end{pmatrix}$, find $\mathbf{P}^{-1}$, the inverse of **P**.
A. $\frac{1}{8}\begin{pmatrix} -6 & -5 \\ -4 & 2 \end{pmatrix}$ B. $\frac{1}{8}\begin{pmatrix} -2 & -5 \\ 4 & 6 \end{pmatrix}$

C. $\frac{1}{8}\begin{pmatrix}-2 & -4\\ 5 & 6\end{pmatrix}$ D. $\frac{1}{8}\begin{pmatrix}-2 & 5\\ -4 & 6\end{pmatrix}$

444. Find the equation of the line passing through the point $(-\frac{1}{2}, 2)$ and parallel to the line $4x+3y-2=0$.

A. $3x-4y+8=0$ B. $4x+3y-4=0$
C. $4x-3y-4=0$ D. $6x-8y+19=0$

445. Given that $\mathbf{A}=\begin{pmatrix}2 & -5\\ 6 & 1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}3 & -2\\ -4 & 5\end{pmatrix}$, find $2\mathbf{A}-3\mathbf{B}$.

A. $\begin{pmatrix}-5 & -4\\ 24 & -13\end{pmatrix}$ B. $\begin{pmatrix}-5 & -4\\ 0 & -13\end{pmatrix}$
C. $\begin{pmatrix}5 & -4\\ 24 & -13\end{pmatrix}$ D. $\begin{pmatrix}13 & -4\\ 24 & -13\end{pmatrix}$

446. A car moving on a straight road with constant acceleration has a velocity of 20 kmh^{-1} at one instant and 15 minutes later it had a velocity of 50 kmh^{-1}. Find the acceleration of the car.

A. 8 kmh^{-2} B. 60 kmh^{-2}
C. 120 kmh^{-2} D. 150 kmh^{-2}

447. The angle between vectors **a** and **b** is 120°. If $\mathbf{a}\cdot\mathbf{b}=-12$, find $|\mathbf{a}||\mathbf{b}|$.

A. 6 B. 12 C. 24 D. 36

448. Find the maximum value of the function $f(x)=10+3x-x^2$.

A. $\frac{49}{4}$ B. $\frac{15}{4}$ C. 16 D. $\frac{65}{4}$

449. The ages in years of five members of a club are 14, 12, 13, 11 and 15. Calculate the standard deviation of their ages.

A. 2.1 B. 2.0 C. 1.4 D. 1.3

450. The co-ordinates of the points P and Q are (–1, 3) and (x, y) respectively. If the vector $\overrightarrow{PQ}=\begin{pmatrix}y\\ -x\end{pmatrix}$, find the values of x and y.

A. $x=-2, y=1$ B. $x=1, y=2$
C. $x=1, y=-2$ D. $x=2, y=-1$

451. Find the values of a and b for which $\begin{pmatrix}a-1 & 4\\ a & b+3\end{pmatrix}=\begin{pmatrix}-3 & 4\\ a & a-2\end{pmatrix}$.

A. $a=-4, b=-7$ B. $a=-4, b=-1$
C. $a=-2, b=-7$ D. $a=-2, b=-1$

452. A particle is projected vertically upwards with a speed of 40 ms^{-1} from a point on the ground. Find the maximum height reached. [Take g = 10 ms^{-2}].

A. 70 m B. 80 m C. 90 m D. 110 m

453. Find the magnitude and direction of the vector $8\mathbf{i}-15\mathbf{j}$.

A. (17, 028°) B. (17, 062°)
C. (17, 118°) D. (17, 152°)

454. If $(x+1)$ is a factor of the polynomial $x^3+px^2+x+6=0$, find the value of p.

A. –8 B. –4 C. 4 D. 8

455. Solve the inequality $x^2-2x\geq 3$.

A. $-1\leq x\leq 3$ B. $-1\leq x<3$
C. $x\geq 3$ or $x\leq -1$ D. $x\geq 3$ or $x<-1$

The marks obtained by 10 students in a test are 3, 7, 6, 2, 8, 5, 9, 1, 4 and 10.
Use the information to answer Question **456** *and* **457**.

456. Find the mean mark.

A. 4.50 B. 5.50 C. 6.50 D. 6.75

457. Find the variance.

A. 8.25 B. 8.50 C. 9.00 D. 9.17

458. Given that $y=3x$ and $3^{x-y}=\frac{1}{81}$, find the value of x.

A. 4 B. 3 C. 2 D. 1

459. A box contains 4 red and 3 blue identical balls. If two balls are picked at random, one after the other without replacement, find the probability that one is red and the other is blue.

A. $\frac{4}{7}$ B. $\frac{2}{7}$ C. $\frac{1}{7}$ D. $\frac{1}{12}$

460. Find the cosine of the angle between the vectors **p** = 3**i** – 4**j** and **q** = 7**i** + 24**j**.

A. $\frac{44}{125}$ B. $\frac{27}{125}$ C. $\frac{4}{25}$ D. $-\frac{3}{5}$

461. Write down the matrix representation of the linear transformation?
$(x, y) \rightarrow (-3x + y, x - 5y)$.

A. $\begin{pmatrix} 3 & 1 \\ 1 & -5 \end{pmatrix}$ B. $\begin{pmatrix} -3 & 1 \\ 1 & -5 \end{pmatrix}$

C. $\begin{pmatrix} -3 & 1 \\ -1 & 5 \end{pmatrix}$ D. $\begin{pmatrix} -1 & 1 \\ 3 & 5 \end{pmatrix}$

462. Which of the following is **not** a measure of central tendency?

A. Median B. Mean
C. Variance D. Mode

463. The area of a circle is increasing at a rate of 4.0 cm^{2}s^{-1}. Find the rate of increase of its radius when the radius is 5.0 cm.
[Take $\pi = 3.142$]

A. 7.58 cms^{-1} B. 1.59cms^{-1}
C. 0.16 cms^{-1} D. 0.13 cms^{-1}

464. If two fair dice are thrown and a coin is tossed, find the probability that the coin shows a head and neither dice shows a score greater than 4.

A. $\frac{2}{9}$ B. $\frac{1}{3}$ C. $\frac{4}{9}$ D. $\frac{17}{18}$

465. Find the image of the point (–3, 1) under the linear transformation $\begin{aligned} x' &= x - 2y, \\ y' &= 2x + y. \end{aligned}$

A. (5, –5) B. (–1, –5)
C. (–5, –5) D. (–5, 1)

466. A body of mass 3 kg moving with velocity 10 ms^{-1} collides with another body of mass 6 kg moving in the opposite direction with velocity 4 ms^{-1}. If they stick together after collision, find their common velocity.

A. 6 ms^{-1} B. 3 ms^{-1}
C. 2 ms^{-1} D. $\frac{2}{3}$ ms^{-1}

467. If $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$,

find $\mathbf{P}^2 - \mathbf{Q}^2$

A. $\begin{pmatrix} 2 & 4 \\ 10 & 10 \end{pmatrix}$ B. $\begin{pmatrix} 2 & 12 \\ 22 & 24 \end{pmatrix}$

C. $\begin{pmatrix} 7 & 4 \\ 8 & 7 \end{pmatrix}$ D. $\begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix}$

468. Find the image of the point (–3, 5) when it is rotated through an angle of 45° in the anticlockwise direction.

A. $(\frac{3\sqrt{2}}{2}, 4\sqrt{2})$ B. $(\frac{\sqrt{2}}{2}, -\sqrt{2})$
C. $(-\sqrt{2}, 4\sqrt{2})$ D. $(-4\sqrt{2}, \sqrt{2})$

469. The speed, v ms^{-1} of a body moving in a straight line is given by $v = 24t - 3t^2$ at time t seconds. Find the distance at the end of 3 seconds.

A. 6 m B. 42 m
C. 81 m D. 135 m

470. The gradient function of a curve is given by $f(x) = 3x^2 - 2x - 1$. If the curve through the point (1, 1), find its equation.

A. $f(x) = x^3 - 5x^2 + 2x + 2$
B. $f(x) = x^3 - x^2 - x + 2$
C. $f(x) = x^3 - x^2 - x + 4$
D. $f(x) = x^3 - x^2 + x + 2$

471. Solve the inequality $8 - x < 7 - 2x < 4 - x$.

A. $-1 < x < 3$ B. $-3 < x < 1$
C. $x < -1$ or $x > 3$ D. $x < -3$ or $x > 1$

472. Simplify $\frac{6^{\frac{1}{2}}\times 96^{-\frac{1}{4}}}{216^{-\frac{1}{4}}}$

A. 4 B. 3 C. 2 D. 1

473. If $(x+1)$ is a factor of the polynomial $p(x)=x^3-4x^2+x+6$, find the remaining factors.

A. $(x+2)(x+3)$ B. $(x-2)(x+3)$

C. $(x-3)(x+2)$ D. $(x-3)(x-2)$

474. If $\frac{1}{2}\log(x-2)=\log 7$, find the value of x.

A. 51 B. 49 C. 47 D. 5.5

475. If $4\sin\theta=3$, $0°\le\theta\le 180°$, find θ.

A. 48.6°, 131.4° B. 41.4°, 138.6°

C. 16.2°, 163.8° D. 13.8°, 166.2°

476. Find the sum to infinity of the series $12+6+3+1\frac{1}{2}+\ldots$

A. 6 B. 12 C. $22\frac{1}{2}$ D. 24

477. Find the quadratic equation whose roots are $(3-\sqrt{3})$ and $(3+\sqrt{3})$.

A. $x^2-6x+6=0$ B. $x^2+6x+6=0$

C. $x^2-3x-9=0$ D. $x^2+3x-9=0$

478. Evaluate $\int\left(\frac{x^3-x^2+1}{x^2}\right)dx$.

A. $x^2-x+\frac{2}{x^3}+k$ B. $x^2-x-\frac{1}{x^3}+k$

C. $\frac{x^2}{2}-x-\frac{1}{x}+k$ D. $\frac{x}{2}-x+\frac{1}{x}+k$

Use the information below to answer Questions **479** *and* **480**.

The operation $*$ is defined on the set $\boldsymbol{R}$ of real numbers by $a*b=\frac{2}{5}ab$.

479. Find the identity element of the operation.

A. $-\frac{2}{5}$ B. -1 C. 1 D. $\frac{5}{2}$

480. Find the inverse of 5.

A. $\frac{125}{4}$ B. $\frac{25}{4}$ C. $\frac{5}{4}$ D. $\frac{1}{4}$

481. If α and β are the roots of the equation $2x^2-3x+1=0$, find the value of $(\alpha+1)+(\beta+1)$.

A. $\frac{5}{2}$ B. $\frac{7}{2}$ C. 5 D. 7

482. If $m=2+\sqrt{5}$, find the value of $m-\frac{1}{m}$.

A. 2 B. 4 C. $2\sqrt{5}$ D. $4\sqrt{5}$

483. If $9^{2y-6}\times 27^6=3$, find the value of y.

A. $\frac{5}{6}$ B. $-\frac{4}{5}$ C. $-\frac{5}{4}$ D. $-\frac{15}{4}$

484. Express the vector (12, 315°) in the form $x\mathbf{i}+y\mathbf{j}$.

A. $-6\sqrt{2}\mathbf{i}+6\sqrt{2}\mathbf{j}$ B. $-\frac{6\sqrt{2}}{5}\mathbf{i}+\frac{5\sqrt{2}}{6}\mathbf{j}$

C. $6\sqrt{2}\mathbf{i}-6\sqrt{2}\mathbf{j}$ D. $6\sqrt{2}\mathbf{i}+6\sqrt{2}\mathbf{j}$

485. If x^2+2x-8 is a factor of the polynomial $f(x)=ax^3-4x^2+28x-16$, find the value of a.

A. 6 B. 4 C. 3 D. –3

486. Find the equation of a circle whose centre is at the point (2, 1) and passes through the point (4, –3).

A. $x^2+y^2+4x-2y-15=0$

B. $x^2+y^2-4x-2y-15=0$

C. $x^2+y^2-4x-2y-20=0$

D. $x^2+y^2-4x+2y-20=0$

487. Find the equation of the line which passes through the point (–3, 1) and is perpendicular to the line $2x+3y+4=0$.

A. $3x-2y+11=0$ B. $2x-3y+4=0$

C. $3y-2x+7=0$ D. $2y-3x+11=0$

488. If **p** = 2**i** – 3**j**, **q** = –3**i** +4**j** and **s** = 4**i** – 11**j**, find |**p** + **q** – **s**|.

A. 4 B. 7 C. 12 D. 13

489. In calculating the mean of 8 numbers, a girl mistakenly used 17 instead of 25 as one of the numbers and obtained 20 as the mean. Find the correct mean.

A. 19 B. 21 C. 23 D. 24

490. The probability that an event will occur is $\frac{v}{u}$. What is the probability that it will **not** occur.

A. $\frac{u-1}{u}$ B $\frac{v-u}{u}$

C. $\frac{v+1}{u}$ D. $\frac{u-v}{u}$

491. A particle of mass 0.3 kg moving in a straight line with a velocity of $4\mathbf{i}-7\mathbf{j}$, is given a blow which causes it to move with a velocity of $5\mathbf{i}+2\mathbf{j}$. Find the change in momentum.

A. $0.3\mathbf{i}+2.7\mathbf{j}$ B. $0.3\mathbf{i}-1.5\mathbf{j}$

C. $1.7\mathbf{i}-1.5\mathbf{j}$ D. $2.7\mathbf{i}+0.3\mathbf{j}$

492. A number is selected at random from each of the sets $\{1, 2, 6\}$ and $\{3, 4, 5\}$. What is the probability that the product of the numbers selected is less than 12?

A. $\frac{1}{3}$ B. $\frac{2}{3}$ C. $\frac{4}{9}$ D. $\frac{5}{9}$

Use the information below to answer Questions **493** *and* **494**

The equation of a curve is given by $y=3x^2-2x+1$.

493. Find the equation of the tangent to the curve at $x=2$.

A. $10x-y-11=0$ B. $10x-y-29=0$

C. $10x+y-11=0$ D. $x+10y-11=0$

494. Find the equation of the line which is perpendicular to the tangent at the point $x=2$.

A. $10x+y-92=0$

B. $x+10y+92=0$

C. $x+10y-92=0$

D. $x+10y-88=0$

495. Find the matrix of the transformation when a point is reflected in the line $y=x$.

A. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ B. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

C. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ D. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

496. The probabilities that Kofi will pass Mathematics and Physics tests are $\frac{1}{4}$ and $\frac{1}{3}$ respectively. If the two events are independent, find the probability that he fails only one of the subjects.

A. $\frac{5}{12}$ B. $\frac{1}{2}$ C. $\frac{1}{6}$ D. $\frac{1}{12}$

497. Evaluate $\lim\limits_{x\to 2}\frac{(2x^2-3x-2)}{x^2-4}$.

A. 0 B. $\frac{1}{2}$ C. $\frac{5}{4}$ D. 2

498. Two forces of magnitude 40 N and 20 N act on a body of mass 0.5 kg, at right angles to each other. What is the magnitude of the acceleration on the body?

A. $20.0\ \text{ms}^{-2}$ B. $30.0\ \text{ms}^{-2}$

C. $69.3\ \text{ms}^{-2}$ D. $89.4\ \text{ms}^{-2}$

499. Two matrices **M** and **N** are defined by

$\mathbf{M}=\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{N}=\begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}$.

Find **M** − 2**N**.

A. $\begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}$ B. $\begin{pmatrix} 2 & 0 \\ -2 & 0 \end{pmatrix}$

C. $\begin{pmatrix} -2 & 0 \\ 3 & -2 \end{pmatrix}$ D. $\begin{pmatrix} -2 & 0 \\ -1 & -2 \end{pmatrix}$

500. Find the image of the point $P(8, -2)$ under an enlargement from the origin O, with a scale factor of 2.

A. (–4, 16) B (–2, 8)

C. (8, –2) D. (16, –4)

501. A linear transformation T is defined by $T:(x,y) \rightarrow (3x-2y, 4x+5y)$. Find the image of $(2, -1)$ under T.

A. $(4, -13)$ B. $(4, 3)$
C. $(8, 3)$ D. $(8, 13)$

502. $P = \{x : 0 \leq x \leq 12\}$ and $Q = \{x : -3 < x < 10\}$ are subsets of the universal set $\mu = \{x : -3 \leq x \leq 12\}$. If $P' \cap Q' = S$, find S.

A. $\{x : x = -3\}$
B. $\{x : x \geq -3\}$
C. $\{x : 0 \leq x < 10\}$
D. $\{x : -3 \leq x \leq 10\}$

503. Find the set of values of x for which $3 < 2x - 1$ and $\frac{2}{3}x < 3$.

A. $\{x : x < \frac{9}{2}\} \cap \{x : x > 2\}$
B. $\{x : 2 < x < \frac{9}{2}\}$
C. $\{x : x > 2\}$
D. $\{x : x < \frac{9}{2}\}$

504. Simplify $\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{27}}$.

A. $\frac{2\sqrt{3}}{3}$ B. $\frac{\sqrt{3}}{3}$
C. $\frac{2\sqrt{3}}{9}$ D. $\frac{\sqrt{3}}{9}$

505. Which of the following binary operations is **not** commutative?

A. $x * y = x + y - xy$
B. $x * y = x^2 - y^2$
C $x * y = x^2 + y^2$
D. $x * y = \frac{1}{2}xy - 3$

506. If $(x - 2)$ is a factor of the polynomial $f(x) = x^3 + x^2 - px + 8$, find the value of p.

A. 2 B. 4 C. 8 D. 10

507. Two functions f and g are defined on the set R, of real numbers by $f : x \rightarrow 1 - x$ and $g : x \rightarrow 2 - x^2$. Find $g\{f(-1)\}$.

A. -2 B. 0 C. 3 D. 4

508. If $x = \frac{1}{81}$, $p = \frac{1}{4}$ and $q = 3$, find the value of x^{pq}

A. 27 B. $\frac{2}{9}$ C. $\frac{1}{9}$ D. $\frac{1}{27}$

509. A geometric sequence is defined by $(\sqrt{2} - 1), (3 - 2\sqrt{2})$,… Find the common ratio.

A. $\sqrt{2} - 7$ B. $\sqrt{2} - 1$
C. $5\sqrt{2} - 7$ D. $3\sqrt{2} - 1$

510. A function h is defined on the set R, of real numbers by $h : x \rightarrow \frac{3}{4}x$, find h^{-1}, the inverse of h.

A. $h^{-1} : x \rightarrow \frac{3}{2}x$
B. $h^{-1} : x \rightarrow \frac{4}{3}x$.
C. $h^{-1} : x \rightarrow 3 - \frac{1}{4}x$
D. $h^{-1} : x \rightarrow \frac{3-x}{4}$

511. Find the equation of the straight line which makes intercepts -3 on the x-axis and -5 on the y-axis.

A. $5x + 3y + 1 = 0$
B. $5x + 3y - 1 = 0$
C. $5x - 3y - 15 = 0$
D. $5x + 3y + 15 = 0$

512. If α and β are the roots of the equation $3x^2 - 5x + 1 = 0$, find the value of $\alpha^2 - \alpha\beta + \beta^2$.

A. 6 B. $\frac{16}{3}$ C. $\frac{16}{9}$ D. $\frac{24}{9}$

513. Find the ratio of the term in x^5 to the term in x^3 in the binomial expansion of $(2x + 3)^6$.

A. $\frac{x^2}{15}$ B. $\frac{2x^2}{15}$
C. $\frac{x^2}{5}$ D. $\frac{2x^2}{5}$

514. Find the sum of the first n terms of the sequence 2, 4, 6, 8,……

A. n^2 B. $2n + 1$
C. $n(n + 1)$ D. $(2n + 1)^2$

515. Evaluate $\lim_{x\to 0} \frac{(x+1)^2}{x^2-2}$.

A. $-\frac{1}{4}$ B. $-\frac{1}{2}$ C. $\frac{1}{2}$ D. 2

516. Express $3\log_{10} x - \frac{1}{2}\log_{10} y + 1$ as a single logarithm.

A. $\log_{10}\frac{10x^3}{\sqrt{y}}$ B. $\log_{10}(10x^3\sqrt{y})$

C. $\log_{10}(x^3\sqrt{y})$ D. $\log_{10}\frac{x^3}{\sqrt{y}}$

517. How many different arrangements of the letter of the word *COMMISSION* are possible?

A. 5040 B. 226800

C. 604800 D. 3628800

518. A box contains 6 red, 5 white and 7 green balls. 2 balls are drawn at random, one after the other without replacement. Find the probability that a red ball and a green ball are drawn.

A. $\frac{7}{15}$ B. $\frac{6}{38}$ C. $\frac{14}{51}$ D. $\frac{21}{76}$

519. If ${}^{3n}C_2 = 15$, find the value of n.

A. 6 B. 5 C. 4 D. 2

520. Find the determinant of the matrix

$\mathbf{A} = \begin{pmatrix} -4 & 3 \\ 5 & -7 \end{pmatrix}$.

A. 43 B. 13 C. –13 D.–43

521. If $27^{x+3} \div 81^8 = \frac{1}{9}$, find the value of x.

A. 9 B. 8 C. 7 D. 6

522. The perimeter of a sector of a circle of radius 3.5 cm is 10 cm. Calculate, in radians, the angle of the sector.

A. 0.43 B. 0.86

C. 1.17 D. 2.86

523. A force of magnitude 80 N acts on a body of mass 50 kg. Find the acceleration of the body.

A. $16\ \text{m s}^{-2}$ B. $3\ \text{m s}^{-2}$

C. $1.6\ \text{m s}^{-2}$ D. $1.3\ \text{m s}^{-2}$

524. Calculate the standard deviation of the numbers 4, 1, 8, 5, 9, 6, 3 and 4.

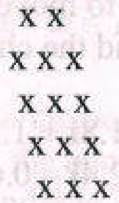
A. 2.4 B. 2.6 C. 3.2 D. 6.9

525. The probabilities that Kofi and Yola hit a target are $\frac{2}{5}$ and $\frac{3}{4}$ respectively. If they both shoot at the target, what is the probability that at least one of them hits the target?

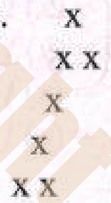
A. $\frac{3}{10}$ B. $\frac{13}{20}$ C. $\frac{7}{10}$ D. $\frac{17}{20}$

526. Which of the following scatter diagrams has the **strongest** negative correlation?

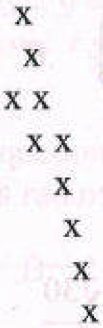
A.



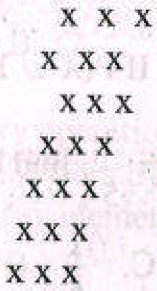
B.



C.



D.



The gradient of a curve is $3x^2 - 2x - 1$ and the curve passes through the point (1, 1).

Use the information to answer Questions 527 and 528.

527. Find the equation of the curve.
A. $y = x^3 - x^2 - x + 2$
B. $y = x^3 - x^2 - x + 4$
C. $y = 3x^3 - x^2 - x + 2$
D. $y = 3x^2 - x^2 + x - 2$

528. Find the equation of the tangent to the curve at the point (2, 4).
A. $7x - y - 18 = 0$
B. $7x + y + 18 = 0$
C. $7x + y - 10 = 0$
D. $7x - y - 10 = 0$

529. A particle of mass 0.6 kg is moving in a straight line with a velocity of $3\mathbf{i} - 4\mathbf{j}$. It is given a blow which causes it to move with a velocity of $6\mathbf{i} + 2\mathbf{j}$. Find the change in momentum.
A. $1.2\mathbf{i} + 3\mathbf{j}$ B. $1.8\mathbf{i} + 1.2\mathbf{j}$
C. $1.8\mathbf{i} + 3.6\mathbf{j}$ D. $2.4\mathbf{i} - 0.6\mathbf{j}$

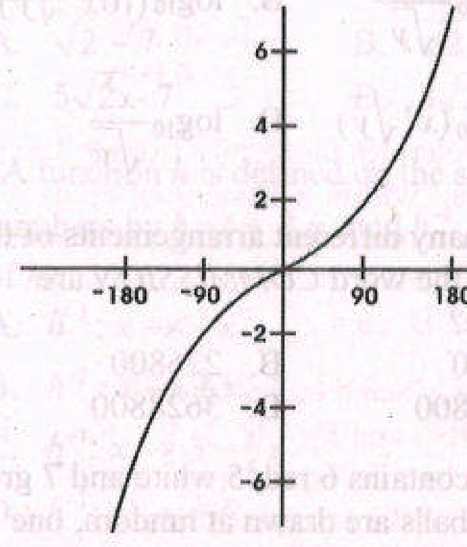
530. Given that $\mathbf{A} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 3 \\ -5 & -1 \end{pmatrix}$, find **BA**.
A. $\begin{pmatrix} 6 \\ 11 \end{pmatrix}$ B. $\begin{pmatrix} 6 \\ -11 \end{pmatrix}$
C. $\begin{pmatrix} 15 \\ -9 \end{pmatrix}$ D. $\begin{pmatrix} 15 \\ 9 \end{pmatrix}$

531. The equation of a circle is $x^2 + y^2 + 8x - 6y + 5 = 0$. Find its radius.
A. $\sqrt{95}$ B. $\sqrt{30}$
C. $2\sqrt{5}$ D. $\sqrt{19}$

532. X and Y are two events. If $P(X \cup Y) = \frac{23}{30}$, $P(X \cap Y) = \frac{1}{2}$ and $P(X) = \frac{3}{5}$, find $P(Y)$.
A. $\frac{2}{3}$ B. $\frac{4}{15}$ C. $\frac{1}{6}$ D. $\frac{1}{10}$

533. The vectors $\overrightarrow{OK}$ and $\overrightarrow{OL}$ have their initial point at (0, 0) and terminal points are (–2, 1) and (–2, –5) respectively. If M is the mid-point of $\overline{KL}$, find the vector $\overrightarrow{OM}$.
A. $2\mathbf{i} - \mathbf{j}$ B. $2\mathbf{i} - 2\mathbf{j}$
C. $4\mathbf{j}$ D. $-2\mathbf{j}$

534.



The graph above represents
A. $y = \tan\frac{x}{2}$. B. $y = 2\tan\frac{x}{2}$.
C. $y = \tan 2x$. D. $y = 2\tan x$.

535. The table below shows the distribution of ages in years of students in a class.

Ages (yrs)	20 – 29	30 – 39	40 – 49
Frequency	*y*	20	15

If the mean age of the students is 36.4 years, determine the value of *y*.
A. 7 B. 14
C. 15 D. 21

536. If $\mathbf{p} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{q} = k\mathbf{i} - 6\mathbf{j}$ are perpendicular, determine the value of the constant k.
A. 8 B. 2 C. $-\frac{1}{2}$ D. –8

537. An orange falls from a height of 18 m above the ground. Find the speed with which the orange hits the ground. [Take $g = 10 \text{ m s}^{-2}$]

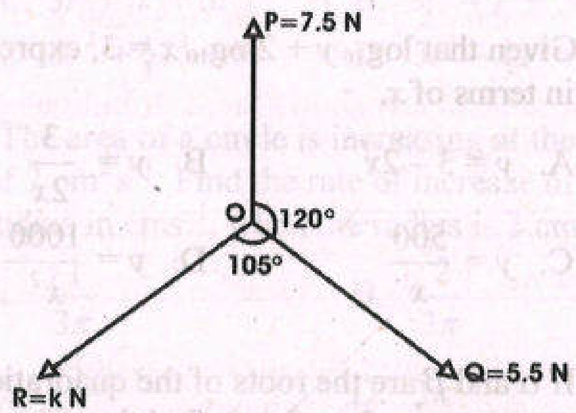
A. 9 m s^{-1} B. $6\sqrt{5} \text{ m s}^{-1}$
C. $3\sqrt{10} \text{ m s}^{-1}$ D. $6\sqrt{10} \text{ m s}^{-1}$

538. *PQR* is a triangle. $|PQ| = 2 \text{ cm}$, $|QR| = 4 \text{ cm}$ and $|PR| = 3$ cm. Calculate, correct to **three** decimal places, the cosine of the **smallest** angle.
A. 0.800 B. 0.875
C. 0.880 D. 0.900

539. A car accelerates uniformly from 8.3 m s^{-1} to 13.8 m s^{-1} within a distance of 60 m. Calculate, correct to the **nearest** second, the time it takes to cover the distance.
A. 7 s B. 6 s C. 5 s D. 4 s

540. A particle is acted on by forces $\mathbf{F}_1$(5 N, 240°) and $\mathbf{F}_2$(4 N, 315°). Find, correct to **two** decimal places, the magnitude of the resultant force.
A. 8.83 N B. 7. 17 N
C. 7. 84 N D. 7. 91 N

541.



Forces $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{OR}$ have magnitudes 7. 5 N, 5.5 N and *k* N respectively. If the forces are in equilibrium, find the value of the constant *k*.
A. 13 B. 8.4 C. 6.7 D. 4.5

Two functions *f* and *g* are defined on the set *R*, of real numbers by

$$f : x \rightarrow \frac{2x-3}{x}, x \neq 0$$

$$g : x \rightarrow 1 - \tfrac{3}{2}x$$

Use the above information to answer Questions **542** *and* **543**

542. Find f^{-1}, the inverse of *f*.

A. $f^{-1} : x \rightarrow \dfrac{3}{x-2}, \ x \neq 2$

B. $f^{-1} : x \rightarrow \dfrac{3}{2-x}, \ x \neq 2$

C. $f^{-1} : x \rightarrow \dfrac{2}{3-x}, \ x \neq 3$

D. $f^{-1} : x \rightarrow \dfrac{2}{x-3}, \ x \neq 3$

543. Find the largest possible domain of $g \circ f$.
A. $\{x : x \in R, \ x \neq \tfrac{9}{4}\}$
B. $\{x : x \in R, \ x \neq 2\}$
C. $\{x : x \in R, x \neq 0\}$
D. $\{x : x \in R, x \neq -\tfrac{9}{4}\}$

544. When the polynomial $f(x)$ is divided by $x + 2$, the quotient is $x^2 + x - 1$ and the remainder is 3. Find $f(x)$.
A. $f(x) = x^3 + 3x^2 + 3x + 1$
B. $f(x) = x^3 + 3x^2 + x - 1$
C. $f(x) = x^3 + 3x^2 + x - 5$
D. $f(x) = x^3 + 3x^2 + x + 1$

545. If the point (*p*, 0) lies on the line $2y + 3x - 9 = 0$, find the value of *p*.
A. 3 B. $\frac{9}{2}$ C. $-\frac{9}{2}$ D. –3

546. The equation of a circle is $3x^2 + 3y^2 = 4$. Find its radius
A. 4 B. 2 C $\dfrac{2\sqrt{3}}{3}$ D. $\dfrac{2}{3}$

547. A polynomial is defined by $f(x + 1) = x^3 + 3x^2 - 4x + 2$. Find $f(2)$.
A. 8 B. 2 C. –2 D. –8

548. A binary operation, *, is defined on the set of real numbers by $a * b = a + b + 4$. Find the identity element of the operation
A. 4 B. 2 C. $\frac{1}{4}$ D. –4

549. Find the range of values of x for which $7 - 6x - x^2 \le 0$.

A. $-7 \le x \le 1$ B. $-1 \le x \le 7$
C. $x \le -1,\ x \ge 7$ D. $x \le -7,\ x \ge 1$

550. A series is given by $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ Find the sum of the series.

A. 1 B. 2 C. 4 D. 8

551. Find the value of x in the equations $\log_x y = 3$ and $xy = 16$

A. 1 B. 2 C. 3 D. 4

552. Evaluate $\frac{1}{(1-\sqrt{3})^2}$, leaving your answer in the form $a + b\sqrt{3}$.

A $1+\frac{\sqrt{3}}{2}$ B $1-\frac{\sqrt{3}}{2}$
C. $\frac{1}{7}+\frac{\sqrt{3}}{14}$ D. $\frac{1}{7}-\frac{\sqrt{3}}{14}$

553. Find the value of p such that the equation $px^2 + (p+1)x + p = 0$. has equal roots.

A. $p = 1, -\frac{1}{3}$
B. $p = 1, 3$
C. $p = -1, \frac{1}{3}$
D. $p = -1, -\frac{1}{3}$

554. Evaluate $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$.

A. $\frac{4}{9}$ B. $\frac{2}{3}$ C. $\frac{3}{3}$ D. $\frac{9}{4}$

555. Find the distance of the point A (3, −2) from the mid-point of the line joining the points B (4, −2) and C(6, 6)

A. $5\sqrt{2}$ B. $2\sqrt{5}$ C. 4 D. 2

556. If $\tan\theta = x$, find $\sin\theta$, where $0 \le \theta \le 90°$

A. $\frac{1}{x}$ B. $\frac{1}{\sqrt{x^2+1}}$
C. $\frac{x}{\sqrt{x^2+1}}$ D. $\frac{x}{x^2+1}$

557. Write down the equation of a circle which has the points (2, −3) and (−1, 4) as ends of its diameter.

A. $x^2 + y^2 - x - y - 10 = 0$
B $x^2 + y^2 - x - y - 14 = 0$
C $x^2 + y^2 + x + y - 14 = 0$
D $x^2 + y^2 + x - y - 10 = 0$

558. If the binomial expansion of $(1-x)^{-\frac{1}{2}}$ is to be used to find an approximate value of $\sqrt{\frac{100}{99}}$, what will be the value of x?

A. 0.01 B. 0.001
C. −0.01 D. −0.001

559. The sum of the first n terms, S_n, of an arithmetic sequence is given by $S_n = 5n^2 + 6n$. Find the 3rd term.

A. 31 B. 32 C. 41 D. 52

560. Given that $\log_{10} y + 2\log_{10} x = 3$, express y in terms of x.

A. $y = 3 - 2x$ B $y = \frac{3}{2x}$
C. $y = \frac{500}{x}$ D. $y = \frac{1000}{x^2}$

561. If α and β are the roots of the quadratic equation $x^2 - 3x - 2 = 0$, find the value of $\frac{\alpha + \beta}{\alpha\beta}$

A. $-\frac{3}{2}$ B. −3 C. $\frac{3}{2}$ D. 3

562. Evaluate $\frac{1}{1+\cos 45°}$, leaving your answer in surd form.

A. $\sqrt{2}+1$ B. $2-\sqrt{2}$
C. $\sqrt{2}-1$ D. $\sqrt{2}-2$

563. If the gradient of the curve $y = ax^2 + 3x + 2$ at the point where $x = 2$ is 11, find the value of a

A. -2 B. 2 C. $\frac{7}{2}$ D. 4

A particle is projected from a point O, with a speed of 5 m s^{-1}. It moves in a straight line with an acceleration of $(6 - 2t)$ ms^{-2}, t seconds after leaving O.

Use the information to answer Questions **564** to **565**.

564. Calculate the speed after 2 seconds.

A. 2 m s^{-1} B. 7 m s^{-1}
C. 13 m s^{-1} D. 14 m s^{-1}

565. Calculate the maximum speed of the particle.

A. 27 m s^{-1} B. 14 m s^{-1}
C. 10 m s^{-1} D. 9 m s^{-1}

566. Find the distance of the particle after t seconds

A. $3t^2 + t^3 + 5t$ B. $3t^2 - t^3 + 5t$
C. $3t^2 + \frac{1}{3}t^3 + 5t$ D. $3t^2 - \frac{1}{3}t^3 + 5t$

567. The area of a circle is increasing at the rate of 2 $cm^2 s^{-1}$. Find the rate of increase of the radius in cms^{-1}, when the radius is 3 cm.

A. $\frac{1}{3\pi}$ B. $\frac{2}{3\pi}$

C. $\frac{3}{2\pi}$ D. 3π

568. If $\int_{-2}^{3}(kt - 1)dt = 0$, find the value of k.

A. -2 B $-\frac{2}{5}$ C. $\frac{2}{5}$ D. 2

569. Find the standard deviation of the numbers 3, 6, 2, 1, 7 and 5.

A. 2.00 B. 2.16 C. 2.50 D. 2.56

The regression line of marks obtained in Physics (y%) on marks obtained in Mathematics (x%) for students in a class is $3x - 4y + 8 = 0$.

Use the information to answer Questions **570** to **571**.

570. Find the regression coefficient of y on x.

A. 4 B $\frac{4}{3}$ C. $\frac{3}{4}$ D. $-\frac{3}{4}$

571. If a student obtained 84% in Mathematics, find the mark she obtained in Physics.

A. 81% B. 65% C. 63% D. 61%

572. Find the mean mark for the two subjects for a candidate who obtained 71% in Physics.

A. 63.1 B. 72.0 C. 78.0 D. 81.5

573. The mean of the numbers x_1, x_2, x_3 and x_4 is 6.4. Find the mean of the numbers $10 + x_1$, $10 + x_2$, $10 + x_3$ and $10 + x_4$.

A. 16. 4 B. 12.8 C. 8.2 D. 6.4

574. A bag contains 5 red and 4 green balls. If two balls are drawn at random, one after the other without replacement, what is the probability that they are of different colours?

A. $\frac{5}{8}$ B. $\frac{5}{9}$ C. $\frac{4}{9}$ D. $\frac{1}{3}$

575. Two fair dice are thrown and the score on each die recorded. Find the probability that the product of the scores on them is greater than 4.

A. $\frac{11}{12}$ B. $\frac{31}{36}$ C. $\frac{7}{9}$ D. $\frac{2}{3}$

576. In how many ways can a committee of 6 men and 4 women be formed from a group of 8 men and 7 women?

A. 3003 B. 980 C. 490 D. 245

577. Given that $\mathbf{p} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$, find the angle between **p** and **q**.

A. 0° B. 30° C. 45° D. 90°

578. Find the force which must act with the force 5 N due West to produce a force of 12 N due North

A. (13 N, 000°) B. (13 N, 023°)
C. (13 N, 067°) D. (13 N, 090°)

579. Find the constant force that will give a body of mass 5 kg at rest a velocity of 6.4 m s^{-1} in 4 s.
A. 16.0 N B. 8.0 N
C. 4. 1 N D. 3.1N

580. A ball falls from a height of 18 m above the ground. Find the speed with which it hits the ground. [Take $g = 10\ m\ s^{-2}$]
A. $9\ m\ s^{-1}$ B. $3\sqrt{10}\ m\ s^{-1}$
C. $6\sqrt{5}\ m\ s^{-1}$ D. $6\sqrt{10}\ m\ s^{-1}$

581. Two forces, 10 N and 6 N act in the direction 060° and 330° respectively. Find the component of the resultant along the x-axis.
A. $5\sqrt{3} - 3$ B. $3\sqrt{3} - 5$
C. $5 - 3\sqrt{3}$ D. $3 - 5\sqrt{3}$

In a triangle XYZ, $|XY| = 12$ cm, $|XZ| = 8$ cm and angle $YXZ = 42°$.
Use this information to answer Questions **582** *and* **583**.

582. Calculate, correct to two decimal places $|YZ|$
A. 65.31 B. 18.73 C. 8.80 D. 8.08

583. Find angle XZY.
A. 98° B. 97° C. 84° D. 91°

584. If $m\begin{pmatrix} 2 \\ -1 \end{pmatrix} + n\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$, where m and n are scalars, find the value of $(m + n)$.
A. 3 B. 2 C. 1 D –1

585. Write down the matrix of the linear transformation

$$x' = 4x + y,$$
$$y' = -5x + 2y$$

A. $\begin{pmatrix} 1 & 4 \\ 2 & -5 \end{pmatrix}$ B. $\begin{pmatrix} 1 & 2 \\ 4 & -5 \end{pmatrix}$
C. $\begin{pmatrix} 4 & -5 \\ 1 & 2 \end{pmatrix}$ D. $\begin{pmatrix} 4 & 1 \\ -5 & 2 \end{pmatrix}$

586. If $\mathbf{M} = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$, $\mathbf{N} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ and $\mathbf{I}$ is the 2×2 identity matrix, find the value of $2\mathbf{M} - \mathbf{N} + \mathbf{I}$.
A. $\begin{pmatrix} 5 & 7 \\ 3 & 3 \end{pmatrix}$ B. $\begin{pmatrix} 5 & 7 \\ 3 & 1 \end{pmatrix}$
C. $\begin{pmatrix} 4 & 8 \\ 4 & 2 \end{pmatrix}$ D. $\begin{pmatrix} 4 & 10 \\ 4 & 2 \end{pmatrix}$

587. If $\mathbf{P} = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 1 & 3 \\ 4 & 0 \end{pmatrix}$, find $\mathbf{PQ}$.
A. $\begin{pmatrix} -7 & -3 \\ 13 & 3 \end{pmatrix}$ B. $\begin{pmatrix} -7 & 3 \\ 13 & 3 \end{pmatrix}$
C. $\begin{pmatrix} 7 & 3 \\ 13 & 6 \end{pmatrix}$ D. $\begin{pmatrix} 7 & -3 \\ 13 & 3 \end{pmatrix}$

588. A linear transformation **P** is given by **P**: $(x, y) \rightarrow 2x + y, 3x + 2y$. Find the point whose image under **P** is (1, 3).
A. (–1, 3) B. (–1, –3)
C. (–3, 1) D. (–3, –1)

589. What transformation will map the point $Y(-3, 2)$ onto its image $Y^1(2, 3)$?
A. Anticlockwise rotation through 90° about the origin
B. Reflection in the line $y = x$
C. Clockwise rotation through 90° about the origin
D. Reflection in the line $y = -x$

590. If $\begin{pmatrix} 3 & 2 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, find the value of a
A. $-\frac{5}{7}$ B. $-\frac{1}{7}$ C. $\frac{1}{7}$ D. $\frac{2}{7}$

591. Evaluate the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$
A. –40 B. –12 C. 18 D. 26

592. Find the gradient of the line perpendicular to the line joining the points (–6, –1) and (2, 9)

A. $-\frac{5}{4}$ B. $-\frac{4}{5}$ C. $\frac{1}{2}$ D. $\frac{4}{5}$

593. Determine the equation of the line which passes through the point (2, –1) and is perpendicular to the line $x-4y-7=0$

A. $x+4y+2=0$ B. $3x+y-5=0$

C. $4x-y-9=0$ D. $4x+y-7=0$

594. Evaluate $\int_0^1 (x^3+x)^4(3x^2+1)dx$

A. $\frac{32}{5}$ B. 8 C. $\frac{32}{3}$ D. 32

595. Find the differential coefficient of $y=\dfrac{3x^2}{x+1}$ at the point (2, 4)

A. 24 B. 8 C. $\frac{16}{3}$ D. $\frac{8}{3}$

Use the following frequency distribution to answer Question **596 – 598.**

Interval	0-19	20-39	40-59	6079	80-99
freq	1	4	7	5	3

596. Estimate the mode of the distribution.

A. 49.5 B. 50.9 C. 51.0 D. 51.5

597. Calculate the mean of the distribution.

A. 50 B. 51.2 C. 52 D. 54.5

598. Determine the median of the distribution.

A. 51.2 B. 54.3 C. 54.5 D. 5.2

599. Find the mean deviation of the set of numbers 1, 4, 9, 16, 25.

A. 0 B. 7.6 C. 9 D. 9.5

600. Compute the standard deviation of the frequency distribution.

$\sum f=30,\ \sum fx^2=1230,\ \sum fx=150$

A. 4 B. 6 C. 16 D. 36

Theory Paper 1

1. A binary operation Δ is defined on the set R of real numbers by $a\nabla b=a^2-2ab+b^2$, where $a, b \in$ R. Find:

a) r such that $2\nabla-5=\sqrt{r}$.

b) $\dfrac{(x+1)\nabla x}{x}, x\neq 0$.

Simplify your results as far as possible.

2. a) Given that $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

Show that $\sin 2\alpha = 2\sin\alpha\cos a$.

b) Solve for x in the equation $2\sin^2 x-5\sin x-3=0$, $0^0\leq x\leq 270^0$.

3. Express $\dfrac{5-2x}{1-x^2}$ in partial fractions.

4. a) Find $\dfrac{d}{dx}\left\{\dfrac{2}{(1-x)^2}\right\}$

b) The angle of a sector of a circle is 150^0. If the radius of this circle is increasing at the rate of 0.9 cm^{-1}, find the rate of increase of the area of the sector when the radius is 5 cm, leaving your answer in terms of π.

5. The third term of a geometric series with a positive common ratio is 18 and the fifth term is 162. Find

a) the common ratio.

b) the sum of the first ten terms

6. a) Sketch the graph of the function $f: x\rightarrow x^2+2x-2$ showing clearly the co-ordinates of its turning point and the intercepts with the co-ordinates.

b) Write down the equation of the axis of symmetry.

7. The points $A(4,\ 1)$, $B(-4, 4)$ and $C(4, 6)$ are points in the $x-y$ plane. A point $\boldsymbol{P}(a,\ b)$ is the foot of the perpendicular from C to AB. By using scalar products, find:

i) $\angle ABC$

ii.) an expression in terms of a and b.

8. A box $\boldsymbol{A}$ contains 3 white and 2 blue balls. Another box $\boldsymbol{B}$ contains 4 green and 5 blue balls. A ball is picked at random from each box. Find the probability that:

i) one is green and the other is white.

ii) they are of the same colour.

9. a) A factor of the polynomial $h(x) = bx^2 - 4x + c$ where b, c are constants is $(1-2x)$. When h(x) is divided by $(1+x)$ the remainder is 3. Find

i) b and c

ii) the zeros of $h(x)$.

b) if $f(x-2) = x^3 + 3x^2 - 3x + 7$, find $f(1)$.

10. a) A variable quantity **L** is directly proportional to the square of another variable t. If **L** increases by **P**%, (**P** being small), estimate the corresponding percentage increase in it.

b) Find the area bounded by the curve $y = x^2 - 5x + 6$ and the line $y = 0$.

11. Two transformations in the xy-plane are given by $\mathbf{A} : (x, y) \to (x + 3y, 2x - 5y)$ and $\mathbf{B}(x, y) \to (4x + 3y, x - 2y)$.

i) Write down the matrix **A** and **B**.

ii) Find $(\mathbf{A} + \mathbf{B})(2\mathbf{A} - \mathbf{B})$.

iii) Find the image of the point (½ , ½)

12. a) Find the truth set of each of the following equation:

i) $2^{\sqrt{x}} = 8^x$ ii) $\log_{\sqrt{2}} 8 = x$

b) Solve the simultaneous equation $4x + 3y = 1$ and $4x^2 + 3xy + y^2 = 2$.

13. Three forces $\mathbf{F}_1 = (2\text{N},\ 180^0)$; $\mathbf{F}_2 = (2\text{N},\ 300^0)$ and $\mathbf{F}_3 = (5\text{N},\ 60^0)$ act on a body.

i) Express the forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ as column vectors.

ii) Write down a single force F_4 which will keep the body in equilibrium.

iii) If the mass of the body id 6kg, find the magnitude and direction of its acceleration.

14. a) S is the midpoint of the side QR of a triangle PQR if $\overrightarrow{PQ} = \underline{v}$ express PS in terms of $\underline{u}$ and $\underline{v}$.

b) $A(1, 2)$, $B(3, 5)$, $C(3, -6)$ and $D(x, y)$ are the vertices of the parallelogram $ABCD$. Find the value of x and y. Hence, find correct to the nearest degree, the angle between AD and DC.

Theory Paper 2

1. The function f and g are defined by

$$f : x \to \sqrt{x^2 - 1} \text{ and } g(x) = \frac{1-x}{x}, x \neq 0.$$

a) State the largest possible domain of f.

b) Determine whether or not g is a one-to-one mapping.

c) Express $\dfrac{x-9}{x^2 + 5x - 6}$ in partial fractions.

2. An error of 2% is made in measuring the radius of a sphere. Find the approximate corresponding percentage error in the measuring of its volume.

3. If $f(x) = \int\left(\frac{1}{x^2} + 2x - 3\right)dx$, find $f(x)$, given that $f(2) = 2$.

4. a) Express $\dfrac{4+\sqrt{5}}{3-2\sqrt{5}}$ in the form $p + q\sqrt{5}$, where p and q are rational numbers.

b) The function g and h are defined on the set of real numbers by $g : x \to \frac{1}{2}x$ and $h : x \to x - 5$. Find
   i) $g \circ h(2)$ ii) $h \circ g(x)$

5. Draw and indicate by shading the set of all points $P(x, y)$ the in the OXY plane that satisfy simultaneously inequalities
   $y \le x + 2$, $y \le 8 + 2x$, $x \ge 4 - 2y$

6. A committee of 5 people including at least one woman is to be selected from a group of 5 men and 4 women. Calculate, correct to three decimal places, the probability that there will be few women than men on the committee.

7. A force acting on a body of mass 3 kg changes its speed fro 10ms^{-1} to 30 ms^{-1} in 5 seconds. Find the magnitude of the force.

8. A binary operation $*$ is defined on the set, R of numbers by $a * b = a + b + \frac{3}{4}$, $a, b, \in R$. Find
   i) the identity element of the operation;
   ii) the inverse of a real number x;
   iii) the inverse of $\frac{2}{3}$

9. a) If $6 - x - x^2$ is a factor of the polynomial $g(x) = mx^3 + 5x^2kx - 18$, find the value of m and k.
   b) $M(-2, 0)$ and $N(2, 0)$ are two points in the OXY plane. A point $P(x, y)$ moves in the plane such that $|PM|^2 + |PN|^2 - 26 = 0$.
      i) Find the equation of the locus of P.
      ii) Describe the locus of P fully.

10. a) A function, f, is defined by $f(x) = 2x^3 + 3x^2$.
      i) Find the turning points and distinguish between them.
      ii) Sketch the curve $f(x) = 2x^3 + 3x^2$
   b) Calculate the finite area enclosed by the curve, the x-axis and the line $x + 2 = 0$

11. Two linear transformations defined by $T : (x, y) \to (2x - y, 3x + 2y)$ and $M : (x, y) \to (kx + y, 3x + ky)$
   a) Write down the matrices T and M.
   b) If the image of the point (2, 3) under M is $(a, -6)$, find the values of k and a.
   c) Find the image of the point (1, –2) under the transformation T followed by M.

12. a) Three forces $\mathbf{F_1} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{F_2} = 5\mathbf{i} + 3\mathbf{j}$, and $\mathbf{F_3} = 6\mathbf{i} - \mathbf{j}$ act on a particle of mass 0.0001 kg. find:
      i) the acceleration of the particle;
      ii) the acute angle between the resultant force on the particle and the vector $\mathbf{i} + \mathbf{j}$, correct to one decimal place.
   b) The position vectors of the vertices A, B and C of a triangle are $\mathbf{i} + \mathbf{j}$, $2\mathbf{i} - \mathbf{j}$ and $-\mathbf{i} + \mathbf{j}$ respectively. Find angle BAC correct to the nearest degree.

13. a) A force $\mathbf{F}$ acts on a particle of mass 0.8 kg moving an initial velocity of $5\mathbf{i} \text{ ms}^{-1}$ for 0.5 seconds. If the particle then moves with a velocity of $7\mathbf{i} + 3\mathbf{j}$, find, correct to two decimal places, the magnitude of $\mathbf{F}$.
   b) Three forces of magnitude 2 N, 5 N and 3 N act along the vectors $\mathbf{i}$, $\mathbf{i} - \mathbf{j}$ and $\mathbf{i} + \mathbf{j}$ respectively. Calculate, correct to one decimal place the resultant of the forces.

Theory Paper 3

1. A sequence of numbers $U_1, U_2, U_3 ..., U_n$ satisfies the relation $U_{n+1} = 3 + \frac{1}{4}U_n$ for all integers $n \ge 1$. If $U_1 = 5$, a) find the values of U_2, U_3 and U_4
   b) an expression for U_n in terms of n.
   c) the value of U_n for large values of n

Two function f and g are defined on the set **R** of real numbers $f: x \to \frac{1}{1+2x}, x \neq -\frac{1}{2}$ and $g: x \to \frac{2x}{x+1}, x \neq -1, x \neq -1$. Find

a) g^{-1}, the inverse of g; b) $g^{-1}of$.

3. The gradient of a curve at any point is $4x - 1$. If the point (-1, 0) lies on it, find its equation.

4. A function f is defined on the set of R, of real numbers by $f: x \to \frac{2}{3}x^3 - \frac{5}{2}x^2 - 3x + \frac{1}{6}$ Determine the values of x for x is increasing

5. If the radius of the circle $x^2 + y^2 + px + 6y - 3 = 0$ is 4, find the possible values of p.

6. If **a** = 6**i** – 5**j**, **b** = 7 **j**, and **c** = 8**i** – 2**j**. Find the values of p and q such that **c** = p**a** + q**b**.

7. Given that $I = \int_0^4 \frac{2}{x^2+1}dx$ copy and complete the table below.

x	0.0	0.5	1.0	1.5
$\frac{2}{x^2+1}$	2.00	1.60		0.62
x	2.5	3.0	3.5	4.0
$\frac{2}{x^2+1}$				0.12

Using the trapezium rule, evaluate $\int_0^4 \frac{2}{x^2+1}dx$, correct to two decimal places.

8. The table below shows the distribution of the masses of some special stones in a museum.

Mass(g)	1-20	21-40	41-60	61-80	81-100	101-120
No. of stones	7	18	22	26	17	10

a) Draw a histogram for the distribution.
b) Use the graph to estimate the mode of the distribution.

9. a) The thirteenth term of an arithmetic progression is 27 and the seventh term is three times the second term. Find
i) the first term.
ii) the common difference.
iii) the sum of the first ten terms.

b) i) Write down the first four terms of the binomial expansion of $(1 - x)^2$.
ii) use your expansion in (i) to evaluate $\sqrt{15}$, correct to three decimal places.

10. a) Using the same coordinates axes for the values of x from 0^0 to 90^0 at intervals of 15^0, draw the graphs of.
i) $y = 2 \cos 2x$, ii) $y = -\sin 2x$
[Use 2 cm to 150 m on x-axis and 2 cm to 1 unit on y-axis].

b) Use your graphs in (a) to solve
i) $8 \cos 2x = 4 \sin 2x = 0$
ii) $3 - 12 \sin 2x = 0$, correct to one decimal place.

11. a) The image of the point (2, 1) under a linear transformation T is (4, 2) and the image of (–3, 1) under T is (–1, 7). Find the matrix T of the transformation.

b) If $S = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}$, find
i) TS, where T is the matrix in (a);
ii) The image of the point (2, –3) under the transformation TS.

12. a) OAB is a triangle, where O is the origin. P and Q are mid-points of $\overline{OA}$ and $\overline{OB}$ respectively. Write down the position vectors of P and Q in terms of a and b.

b) $P(-1, 1)$, $Q(1, 3)$, $R(m, n)$ and $S(3, -3)$ are the vertices of the parallelogram $PQRS$. Calculate the values of m and n.

13. a) A force of magnitude $12N$ acts on a body of mass m kg and increases its speed from 10.8km h^{-1} to 64.8km h^{-1} in 6 seconds. Find the value of m.

b) A particle is projected vertically upwards from a point O, with a speed of 40 ms^{-1}. Find:

i) the maximum height reached;

ii) the times when the particle is 30m above the point O, correct to the nearest whole number.

14. a) The deviations of a set of numbers from 33 are 2.6, –3.2, –2.6, 1.8, 3.5, –1.6 and –0.1. Calculate, correct to two decimal places

i) the mean

ii) the standard deviation of the numbers

b) A committee of 3 doctors and 2 nurses is to be formed from 7 doctors and 5 nurses. In how many ways can this be done if

i) any doctor and any nurse can be included;

ii) one particular doctor must be on the committee?

Theory Paper 4

1. The table below shows the operation Δ on the set $\{a, b, c, d\}$.

Δ	a	b	c	d
a	b	c	a	d
b	c	d	b	a
c	a	b	c	d
d	d	a	d	c

Use it to answer the following questions:

a) State the identity element.

b) Find the inverse of the elements a and d.

c) State whether or not the operation is commutative.

2. Find from first principle the derivative of $x^2 + 2x$.

3. Express $3x^2 - 6x + 10$ in the form $a(x-b)^2 + c$, where a, b and c are integers. Hence state the minimum value of $3x^2 - 6x + 10$ and the value of x for which it occurs.

4. Find the equation of the tangent to the curve $y = \frac{1}{x}$ at the point where $x = 2$.

5. Solve the equation $2^{2x+1} - 9(2^x) + 4 = 0$.

6. Find, correct to the nearest degree, the angle between the vectors

$$\mathbf{a} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} -8 \\ -15 \end{pmatrix}.$$

7. Three men at a shooting range have probabilities $\frac{1}{6}, \frac{1}{4}$ and $\frac{1}{3}$ of hitting a bull's eye with one shot. Each man fires one shot. Find, correct to three decimal places, the probability that only one man hits the bull's eyes.

8. Evaluate $\int_1^3 \frac{x^3 + x^2 + x}{x^2} dx, x \neq 0$.

9. Two functions f and g are defined by $f : x \rightarrow 2x^2 - x + 3$ and $g : x \rightarrow 3 - x$.

Find i) $f \circ g$;

ii) the value of x for which $f \circ g(x+1) = f \circ g(x)$.

10. a) The point (r, s) is transformed by the matrix $\begin{pmatrix} 5 & -3 \\ -1 & 2 \end{pmatrix}$. If the image is $(10, -1)$, find the values of r and s.

b) Two matrices are defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 4 & -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 & -3 \\ 3 & 2 \end{pmatrix}.$$

Find the image of the point (3, 1) when it is transformed by **AB**.

c) Find the image P', of the point

$P(\sqrt{2}, 2\sqrt{2})$ under an anticlockwise rotation of 225^0 about the origin, followed by a reflection in the x-axis.

11. a) The gradient function of $y = px^2 + qx + r$ is $8x + 4$. The function has a minimum value of 1. Find the values of p, q, and r.

12. a) A particle of mass 800 g is moving in a straight line with a velocity of $(4\mathbf{i} - 3\mathbf{j})$ ms^{-1}. It is given a blow which causes it to move with a velocity of $(12\mathbf{i} + 3\mathbf{j})$ ms^{-1}. Find
   i) the change in momentum
   ii) the magnitude of the change in momentum.

b) A particle is projected vertically upwards with a speed of 25 ms^{-1} from a point on the ground. [Take g = 10 ms^{-2}]. Find
   i) the position of the particle after 4 s;
   ii) the maximum height reached;
   iii) the time taken to reach the maximum height;
   iv) the time(s) when the particle is 30m above the ground.

13. a) Two ships A and B leave a port simultaneously. A steams at 10 kmh^{-1} on a bearing 160° while B steams on a bearing of 215°. After one hour, the bearing of B from A is 260°. Find the speed of B, correct to the nearest whole number.

b) Three forces $-63\mathbf{j}$, $32.14\mathbf{i} + 38.3\mathbf{j}$ and $14\mathbf{i} - 24.25\mathbf{j}$ act on a body of mass 5 kg. Find, correct to one decimal place.
   i) the magnitude and direction of the resultant force;
   ii) the acceleration of the body.

Theory Paper 5

1. The roots of the equation $x^2 + 6x + p = 0$ are α and $\alpha - 1$. Find the value of p.

2. Find the equation of the circle whose centre is (2, 1) and passes through the point (4, –3).

3. Solve $\sqrt{x+7} - \sqrt{3x-2} = 1$

4. An error of 3 % is made in measuring the radius of a sphere. Find the percentage error in measuring its volume.

5. A chemical industry manufactures closed cylindrical metal containers to hold 50 cm^3 of a chemical. The total surface are, s cm^2, of the container is $s = (2\pi r^2 + \frac{100}{r})$ cm^2, where r is the radius of the cylinder. Find the value of r that will give the minimum surface area.

6. A bag contains 7 black and 2 white balls, and a second bag contains 1 white and 5 black balls. One of the bags is selected at random and a ball is drawn from it. Find the probability that the ball drawn is black.

7. From first principle, find the derivate of $(x+2)^2$.

8. Two forces 3 N and 4 N act on a body of mass 5 kg initially at rest on a smooth horizontal surface. If the angle between the forces is 60°, find the magnitude of the acceleration of the body.

9. a) i) Write down the binomial expansion of $(x+2y)^5$.
   ii) Use your expansion in (i) to evaluate $(1.02)^5$, correct to three decimal places.

b) When the polynomial $f(x) = 2x^3 - x^2 - mx - 2$ is divided by

$(x-1)$, the remainder is -6 .

- i) Determine the value of the constant m, hence factorize $f(x)$ completely.
- ii) Find the truth set of the equation $f(x) = 0$

10. a) A binary operation $*$ is defined on the set R, of real numbers by $x * y = x + y + 6$, where $x, y \in R$.

- i) Calculate $-3*(5*2)$.
- ii) Determine whether or not operation $*$
  - (α) is commutative
  - (β) is associative
- iii) Find the identity element of operation $*$.

b) The sum of the first twelve of an arithmetic sequence is 168. If the third term is 7, find the value of

- i) the common difference;
- ii) the first term.

11. a) Express $7 - 5x - 2x^2$ in the form $a - b(x+c)^2$. State the greatest value of the expression.

b) Find the equation of the tangents to the curve $y = x^2 - x - 2$ at the points where the curve meets the line $y = x + 1$.

12. a) Write down the linear transformation defined by the matrices

$$\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 4 & -7 \\ -5 & 2 \end{pmatrix}.$$

b) Find

- i) the inverse $\mathbf{A}^{-1}$, of the linear transformation defined by the matrix **A**;
- ii) the image of the point (3, –2) under the transformation defined by $\mathbf{A}^{-1}$.

c) Find the image of the point (4, 1) under the transformation **B** followed by **A**.

13. a) A vector $a\mathbf{i} + b\mathbf{j}$, where a and b are scalars, has its magnitude twice that of the vector $\mathbf{i} + 3\mathbf{j}$ and is parallel to the vector $3\mathbf{i} - 4\mathbf{j}$. Find the vector.

b) Two vectors are defined as $\mathbf{p} = 6\mathbf{i} + 8\mathbf{j}$ and $\mathbf{q} = -8\mathbf{i} + \frac{7}{3}\mathbf{j}$.

Find the

- i) magnitude and directions of **p** and **q**.
- ii) angle between **p** and **q**.

14. The table below shows the masses of babies on the day of delivery.

Mass (kg)	No. of Babies
1.1-1.4	1
1.5-1.8	4
1.9-2.2	6
2.3-2.6	$n+2$
2.7-3.0	13
3.1-3.4	8
3.5-3.8	5
3.9-4.2	3
4.3-4.6	1

If the mean mass of the babies is 2.8 kg, find

- a) the value of n, correct to the nearest whole number;
- b) the standard deviation of the distribution, correct to two decimal places.

15. a) A body of mass 1.5 kg moving with velocity 20 ms^{-1} collides with another body mass 3.5 kg moving with the velocity 6 ms^{-1} along the same line but in opposite direction.

- i) If they moved together after the collision, find their common velocity.
- ii) If the velocity of the 1.5 kg mass changes to 3.4 ms^{-1} but moves in the opposite direction, find the velocity of the 3.5 kg mass correct to the nearest whole number.

b) A body of mass 0.5 kg is moving in a straight line with velocity $-2\mathbf{i} + 6\mathbf{j}$. It is hit by an object which causes it to move with velocity $4\mathbf{i} - 2\mathbf{j}$. Find

- i) the change in momentum;
- ii) the magnitude of the change in momentum.

Theory Paper 6

1. Solve the simultaneous equation $3^{x+y} = 243$
   $3x + y = 9$

2. If α and β are the root of the equation $4x^2 + 7x + 3 = 0$, find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$.

3. Expand $(1-x)^{\frac{1}{3}}$ in ascending powers of *x*, up to the third term. Hence evaluate $\sqrt[3]{0.997}$, correct to three decimal places.

4. Find the angle o f slope of the tangent to the curve $y = x^2 - 3x$, at the point (2, -2).

5. Find the derivative of $3x^2 + 1$, from first principles.

6. The cost , *C*, in cedis per metre of an electric cable is given by $C = \frac{120}{x} + 600x$, where *x* is its cross-sectional area in cm^2. Find, correct to the nearest centimetre, the cross-sectional area for which the cost is least.

7. A carton contains 20 bulbs of which 3 are defective. A woman has agreed to buy the whole carton of bulbs, if she selects 3 bulbs at random, one after the other, without replacement and it contains at most 1 defective bulb. Find the probability that she does not buy the carton of bulbs.

8. Two practical *A* and *B* with masses 3 kg and 4 kg moving with velocities $u_1 = 2\mathbf{i} + 3\mathbf{j}$ and $u_2 = -\mathbf{i} + 4\mathbf{j}$ respectively collide. After collision, the particle *A* moves with velocity $2i - j$. Determine the velocity of the particle *B* after collision.

9. a) A function is define on the set *R* of real number by $f : x \rightarrow 5x - 9$. Find the inverse of *f*.

   b) A binary operation ∗ is define on the set, *R* of real numbers by $a * b = \frac{a+2}{a+b}$, $a \neq -b$.

      i. Evaluate $3 * \sqrt{2}$, leaving your answer in the form $p + r\sqrt{2}$, where *p* and *r* are rational numbers.

      ii. Find *x* such that $2 * x = \frac{3}{2}$.

   c) The sum to infinity of an exponential sequence is 9. If the second term is 2, find the value(s) of
      i) the first term;
      ii) the common ratio.

10. a) Solve $\sqrt{(3x+4)} - \sqrt{(x-3)} = 3$.

    b) When the polynomial $f(x) = x^2 + 6x + k$ is divided by $x - a$, the remainder is the same as when it is divided by $x - 2a$. Find
       i) the values of *a*;
       ii) the values of *k* if the remainder is 2.

11. a) Two transformations in the *xy*-plane are given by $A : (x, y) \rightarrow (x + 2y, 3x - y)$ and $B : (x, y) \rightarrow (2y, 2x + 3y)$.
       i) Obtain a single matrix representing the transformation **AB**.
       ii) Find the image of the point (0, 1) when it is transformed by **AB**.
       iii) When a point (*a*, *b*) is transformed by **AB**, its image is (–21, 3). Determine the values of **a** and **b**.

    b) If $\mathbf{T} = \begin{pmatrix} 4 & -2 \\ 3x & 5 \end{pmatrix}$, find the value of *x* if the determinant of **T** = 32.

12. a) If the vector *x***i** + *y***j** and 3**i** – **j** are perpendicular, find *y* when *x* = 2.

    b) Three forces 8**i** – 3**j**, 3**i** + 3**j** and **i** – 4**j** act on a body of mass 4kg. Find
       i) the resultant force;
       ii) the acceleration of the body.

c) Given that p and q are the position vectors of OP and OQ respectively, show that

$$|q-p|^2 = |p|^2 + |q|^2 - 2|p||q|\cos\theta,$$

where θ is the angle between OP and OQ.

13. a) Three forces $\mathbf{F_1} = (2\text{N}, 60^0)$, $\mathbf{F_2} = (3\text{N}, 270^0)$ and F_3 act on a body of mass 1 kg. If the body accelerates at $3\ \text{ms}^{-2}$ on a bearing of 180^0, find the force F_3.

b) Two forces are given as $\mathbf{F_1} = (3\text{N}, 090^0)$ and $\mathbf{F_2} = (4\text{N}, 150^0)$. If the force $\mathbf{F_2} - \mathbf{F_1}$ acts on a body of mass 2 kg, find the magnitude of the acceleration of the body.

Theory Paper 7

1. a) In how many ways can nine women be seated at a round table if two particular women must not sit together?

b) Out of seven women and nine men, a committee of three women and four men is to be formed. In how many ways can this be done if :

i) any woman and any woman may be included

ii) one particular man must be on the committee

2. Find, from first principle the derivatives of the function $f(x) = 2x^2$.

3. A function, f is defined by $f(x) = 4x^2 + kx + 9$. Find the value(s) of the k if the x-axis is a tangent to the curve.

4. A particle starts from rest and its acceleration, a at time t seconds is given by $a = 2 + t$.

a) Velocity of the particle after 1 second.

b) Position of the particle at any time t.

5. a) Express $2\log_{10} 5 - \frac{1}{2}\log_{10} 16 + 4\log_{10} 2$

b) Express $\dfrac{\sqrt{3} - 3\sqrt{2}}{\sqrt{3} + 3\sqrt{2}}$ in the form $a + b\sqrt{c}$, where a, b and c are rational numbers.

6. The table below shows the marks obtained by students in an examination

Marks	5	10	15	20	25
Freq	5	18	*y*	27	8

If the mean mark is 15.75, find:

a) the number of students who had 15 marks;

b) the probability of selecting 2 students who both had 15.

7. In a pentagon $ABCDE$, $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$, $\overrightarrow{DC} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ and $\overrightarrow{ED} = 2\overrightarrow{AB}$.

Find $\overrightarrow{AE}$.

8. a) Two functions f and g are defined on the set R of real numbers by $f : x \to 2x + 3$, $g : x \to \frac{1}{3}(x - 3)$.

Find i.) g^{-1} ii) fog^{-1}

b) When the polynomial $f(x) = x^3 + 5x^2 + px - 1$ is divided by $x - 2$, the remainder is 7. Find the value of p.

c) Given that $\sin A = \frac{3}{5}$ and $\cos B = \frac{2}{3}$, find the value of cos $(A + B)$, leaving your answer in surd form.

9. A body of mass 800 g lying on a smooth horizontal floor is acted upon by the forces $\mathbf{F_1} = (10\text{N}, 090^0)$, $\mathbf{F_2} = (16\text{N}, 180^0)$, $\mathbf{F_3} = (8\text{N}, 030^0)$ and $\mathbf{F_4} = (12\text{N}, 300^0)$.

a) Express all the forces acting on the body as column vectors.

b) Find, correct to two decimal places, the magnitude of
   i. the resultant force;
   ii. the acceleration with which the body starts to move.

10. The deviation of a set of given numbers from 6 are: –3, –1, 1, 0, 3, –5, 4, –2. Find, correct to two decimal places.

a) The mean

b) The variance of the numbers.

11. A welder used a rectangular sheet of metal with dimension 5 m by 3 m to make an open rectangular box by removing equal squares of sides a m from each corner of the sheet.

i) Show that the volume, V m^3, of the box is given by $V = 4a^3 - 16a^2 + 15a$.

ii) Find correct to the nearest whole number, the maximum possible volume of the box.

12. a) The volume of water in a trough is $x(2x^2 + 2x + 5)$ cm^3, when the depth is x cm. Water is flowing into the trough at a constant rate of 100 cm^3/s. Find the rate at which the level of water is rising when the depth of water in the trough is 5 cm.

b) A stone projected vertically upwards from a point 5 m above the ground covers s metres in t seconds, where

$s = 24t - 5t^2$

i) Obtain an expression for the velocity v in terms of t.

ii) Find its velocity when the stone strikes the ground.

13. a) The gradient at (0, 0) on the curve $y = Ax^3 + Bx^2 + Cx$, where A, B and C are constants, is equal to 2. Find the value of C.

b) The curve touches the x-axis at (1, 0). Find A and B.

14. If $y = 2\sin 2x - \cos 3x$, calculate the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, where $x = 0, \frac{\pi}{3}, \frac{\pi}{2}$ and π

15. A piece of copper wire of length a is cut into two parts of lengths x and $(a - x)$. The former is bent into the shape of a square, and the latter into rectangle of which the length is double the width. Show that the sum of the areas of the two figures is $\frac{1}{144}(17x^2 - 16ax + 8a^2)$. Determine the value of x for which this sum is a minimum.

16. The gradient of curve passing through the point (0, 1) is $-2x$.

a) Find the equation of the curve and sketch its graph, showing clearly the points where it cuts the x-axis.

b) Calculate the area of the region bounded by the curve and the x-axis.

c) Calculate in terms of π, the volume of the solid formed when the region in (b) is rotated through four right angles about the x-axis.

ANSWERS

Review exercise 1

1. i. $\{4, 7\}$
ii. $\{1, 2, 3, 4, 5, 7, 8\}$
iii. $\{6, 9, 10\}$ iv. $\{6, 9, 10\}$
$P^1 \cap Q^1 = (P \cup Q)^1$

2. i. $\{3, 4, 5\}$
ii. $\{2, 3, 4, 5, 10\}$
iii. $\{3, 4, 5\}$ iv. $\{3, 4, 5, 7\}$
v. $\{2, 3, 4, 5, 10\}$
vi. $\{2, 3, 4, 5, 10\}$
vii. $\{2, 3, 4, 5, 7, 10\}$
viii. $\{2, 3, 4, 5, 7, 10\}$
a. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
b. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

3. i. $\{5, 10, 15, 20, 25, 125, 625\}$
ii. $\{125, 625\}$
iii. $\{10, 15, 20, 125, 625\}$
iv. $\{\varnothing\}$

4. a. i. $\{1, 2, 3, 4, 5, 6, 7, 8, \ldots, 19\}$
ii. $\{11, 13, 17, 19\}$
iii. $\{6\}$ iv. $\{7, 11, 13, 17, 19\}$
v. $\{6, 7, 8, 9, 11, 13, 17, 19\}$
b. i. Distributive property
ii. Distributive property

5. i. $\{d, e\}$ ii. $\{a, b, c, d, e\}$
iii. $\{\varnothing\}$ v. $\{a, b\}$ vi. $\{c\}$
vii. $\{a, b, c\}$

6. i. $\{5\}$ ii. $\{2, 5, 7\}$
iii. $\{2\}$ iv. $\{7\}$
v. $\{3\}$ vi. $\{2, 3, 7\}$
vii. $\{2, 3, 7\}$ viii. $\{3\}$
a. $A^1 \cap B^1 = (A \cup B)^1$
b. $(A \cap B)^1 = A^1 \cup B^1$

7. a. i. $\{9, 18\}$
ii. $\{1, 2, 3, 4, 6, 12, 36\}$
iii. $\{36\}$
iv. $\{1, 2, 3, 4, 6, 12, 36\}$
b. (ii) and (iv)

8. a. i. $\{9\}$ ii. $\{3, 6, 12, 15\}$
iii. $\{\varnothing\}$
iv. $\{2, 4, 5, 7, 8, 9, 10, 11, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25\}$ v. $\{4, 16, 25\}$
b. Distributive property

9. b. i. 12 ii. 7 iii. 4
iv. 5 v. 7 c. $6/13$

10. i. 47
ii. 60, 35 and 13 respectively
iii. 12, 10, 5

11. b. i. 7 ii. 3 **12.** 14

13. i. 77 ii. 34 iii. 13

14. i. 7 ii .55

15. a. 4 b. 20, 16 and 12 c. 20

16. b. i. 4 ii. 7

17. ii. a. 5 b. 40 c. $33/40$

18. 3 **20.** b. i. 48 ii. 10 iii. $\frac{2}{35}$

21. b. i. 6 ii. 7 iii. 14

22. i. 19 ii. 29

23. b. $9+9-x+x+7-x+1+8-x+8=38$
c. i. 2 ii. 28

25. a. i. $E \cap C = \{\varnothing\}$
ii. $E \cap G = \{4\}$
b. i. 1 ii. 3 **26.** ii. 7

27. a. i. $\{20\}$ ii. $\{15\}$
iii. $\{3, 4, 6, 8, 9, 12, 15, 16, 18, 20\}$

28. b. 3 **29.** b. 18 c. $\frac{1}{6}$

30. 62 **31.** 82 **32.** 21

33. 35

34. a. $A = \{7, 14, 21, 28\}$
$B = \{5, 10, 15, 20, 25, 30\}$
$C = \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30\}$
b. i. $\{5, 7, 10, 14, 15, 20, 21, 25, 28, 30\}$
ii. $\{15, 30\}$
iii. $\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 18, 19, 20, 21, 22, 23, 25, 26, 28, 29\}$

35. b. i. 8 ii. 24 iii. 20

36. i. 160 ii. 170 **37.** b) 44

Review exercise 2

1. i) $3\sqrt{3}$ ii) $4\sqrt{2}$ iii) $2\sqrt{2}$
iv) $8\sqrt{2}$ v) $9\sqrt{2}$ vi) $\frac{79}{5}\sqrt{5}$
vii) $6\sqrt{2}$ viii) $9\sqrt{2}$ ix) $12\sqrt{2}$
x) $10\sqrt{5}$ xi) $15\sqrt{2}$ xii) $4\sqrt{3}$

2. $1+\frac{1}{2}\sqrt{2}$ **3.** $2\sqrt{5}$

4. i) 0 ii) $4\sqrt{7}$ iii) $-7\sqrt{6}$
iv) $6\sqrt{3}$ v) $-(1+\sqrt{2})$
vi) $3\sqrt{5}$ vii) $5\sqrt{7}$ viii) $2\sqrt{7}$
ix) $11\sqrt{2}$ x) $\frac{2\sqrt{2}}{3}$

5. 1.15 **6.** $\frac{1}{2}\sqrt{5}-\frac{1}{2}\sqrt{3}$

7. $-1+\sqrt{6}$

8. i) $-5-2\sqrt{6}$ ii) $\sqrt{3}$
iii) $2+\frac{1}{2}\sqrt{15}$

9. i) $\frac{5}{19}$ ii) $\frac{7}{5}-\frac{1}{5}\sqrt{6}$

10. $\frac{88}{19}$ **11.** $3+\sqrt{7}$

12. i) 7.243 ii) 0.153

13. i) $\sqrt{2}$ ii) $\frac{12}{17}-\frac{2}{17}\sqrt{2}$
iii) $7\sqrt{3}-16\sqrt{2}$

14. i) $1+\sqrt{2}$ ii) $5-\sqrt{3}$
iii) $4-2\sqrt{2}$ iv) $\sqrt{3}+\sqrt{7}$

15. i) $x=9$ ii) $x=20$
iii) $x=8$ iv) 5 v) $-1, 2$
vi) 9 vii) 2, 9 viii) 1, 8

16. i) $2\sqrt{3}-3$ ii) $5\sqrt{2}+8$
iii) $2\sqrt{5}+5\sqrt{15}$ iv) 12
v) $\sqrt{6}+\sqrt{2}-\sqrt{3}-1$
vi) $13+7\sqrt{3}$ vii) 4
viii) $5-3\sqrt{2}$ ix) $22-10\sqrt{5}$
x) 9 xi) $10-4\sqrt{6}$
xii) $31+12\sqrt{3}$

17. i) $\frac{3}{2}\sqrt{2}$ ii) $\frac{1}{7}\sqrt{7}$ iii) $\frac{2}{11}\sqrt{11}$
iv) $\frac{3}{5}\sqrt{10}$ v) $\frac{1}{9}\sqrt{3}$ vi) $\frac{1}{2}\sqrt{2}$
vii) $\sqrt{2}+1$ viii) $\frac{1}{23}(15\sqrt{2}-6)$
ix) $\frac{1}{3}(4\sqrt{3}+6)$ x) $-5(2+\sqrt{5})$
xi) $\frac{1}{4}(\sqrt{7}+\sqrt{3})$ xii) $4(2+\sqrt{3})$
xiii) $\sqrt{5}-2$ xiv) $\frac{1}{13}(7\sqrt{3}+2)$
xv) $3+\sqrt{5}$ xvi) $3(\sqrt{3}+\sqrt{2})$
xvii) $\frac{3}{19}(10-\sqrt{5})$ xviii) $3+2\sqrt{2})$
xix) $\frac{2}{3}(7-2\sqrt{7})$ xx) $\frac{1}{2}(1+\sqrt{5})$
xxi) $\frac{1}{4}(\sqrt{11}+\sqrt{7})$
xxii) $\frac{1}{6}(9+\sqrt{3})$
xxiii) $\frac{1}{14}(9\sqrt{2}-20)$
xxiv) $\frac{1}{6}(3\sqrt{2}+2\sqrt{3})$
xxv) $\frac{1}{2}(2+\sqrt{2})$
xxvi) $\frac{1}{42}(3\sqrt{7}-\sqrt{21})$
xxvii $\frac{1}{9}(\sqrt{30}+2\sqrt{3})$
xxviii) $\sqrt{10}-\sqrt{3}$
xxix) $4\sqrt{2}-2\sqrt{3}$
xxx) $\frac{1}{5}(7-\sqrt{6})$

18. i) $12\sqrt{2}$ ii) $7\sqrt{6}+8$

iii) $3\sqrt{2}+3\sqrt{2}+2\sqrt{2}+4$

iv) $8\sqrt{5}$ v) $10\sqrt{14}+42$

vi) $\frac{24\sqrt{6}-21}{67}$ vii) $8-2\sqrt{2}$

viii) $7-2\sqrt{10}$ ix) $7\sqrt{3}-6$

x) 8 xi) $\frac{1}{3}(\sqrt{13}+2)$ xii) $2-\sqrt{3}$

Review exercise 3

1. a) is commutative

b) 0 c) $\frac{x}{2x-1}$; $x=\frac{1}{2}$

2. a) i) –75 ii) 75

b) 0 c) $\frac{-a}{1+3a}$; $a=-\frac{1}{3}$

3. i) $\frac{290}{11}-\frac{120}{11}\sqrt{5}$

ii) $\frac{113}{4}-\frac{47}{4}\sqrt{5}$

iii) $\frac{12}{29}-\frac{5}{29}\sqrt{5}$

4. i) $11-12\sqrt{2}$

ii) Is commutative iii) 0

iv) $\frac{a}{2a-1}$; $a=\frac{1}{2}$

5. a) i) Y is closed under ⊗

ii) ⊗ is commutative

iii) t

b) $r=u$; $s=s$; $t=t$; $u=r$

c) i) $p=s$ ii) $q=s$

6. a) is commutative

b) is associative

c) $\frac{1}{2}$ d) $\frac{1}{4p}$; $p=0$

7. b) 0 c) $\frac{-p}{1+p}$; $p=-1$ d) –2

e) $p\odot(q+r)\neq(p\odot q)+(p\odot r)$

8. a) is commutative

b) (1, 0)

c) $c=\frac{a}{a^2+b^2}$; $d=\frac{-b}{a^2+b^2}$;

$(\frac{2}{5},\frac{-1}{5})$

9. a) i) S is closed under *

ii) * is commutative

b) a

c) $a=a$; $b=c$; $c=b$

10. a) ii) T is not closed under *

b) 7

11. b) i) not closed

ii) not commutative

12. i) commutative

ii) $\frac{25}{12}$ and $\frac{25}{12}$ iii) 0

iv) $(a=-1, 5)$

13. i) a) not commutative

b) not associative

ii) $y=\frac{-x}{3-x}$, $x=3$

14. a) 23 and 23 b) 0

c) $\frac{a}{2a-1}$, $a=\frac{1}{2}$ has no inverse ,

inverse of 1 is 1.

15. b) i) 4 ii) $-8-p$ iii) –10

16. a) Not closed b) 0 c) 1

17. a) Commutative b) 0

c) $\frac{a}{6a-1}$, $\frac{1}{6}$ inverse of $1=\frac{1}{5}$

18. i) a) Commutative

b) Not associative

ii) a) $\frac{-1}{6}$ b) $\frac{3}{2}$

19. i) a) commutative

b) not associative

ii) a) 19 b) 453 c) 1 459

iii) $a=-3$

20. i) Closed ii) Commutative

iii) 0 **21.** ii) b = 0

22. a) Commutative b) 0

c) $\frac{x}{5x-1}$; $x=\frac{1}{5}$

23. a) Commutative b) 0

25. i) Commutative

ii) Not associative

26. a) Associative b) 0 c) 0

Review exercise 4

1. i. and iv

2. ii, iii (one to one) and iv.

3. i. x -3 -2 -1 0 1 2 3

↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓

y -9 -5 -1 3 7 11 15

ii. x -3 -2 -1 0 1 2 3

↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓

y 8 3 0 -1 0 3 8

$x\rightarrow 4x+3$ (one-to-one)

4. i. a. $-\frac{1}{2}$ b. $-\frac{5}{3}$ c. 1 d. -11

ii. a. 1 b. 2 c. -2

iii. -4

iv. $\{-11, -4, -\frac{5}{3}, -\frac{1}{2}, \frac{1}{5}, \frac{2}{3}, 1\}$

5. i. a. 3 b. -3 c. 5 d. -5

ii. a. 0 b. -1 c. 3

iii. {-5, -3, -1, 1, 3, 5, 7}

6. i. a. -8 b. -2 c. 7 d. 1

ii. a. -3 b. -1 c. 2

iii. {-11, -8, -5, -2, 1, 4, 7}

7. i. $a=3$, $b=-1$ ii. -4 iii. 2 iv. 8

8. i. x -3 -2 -1 0 1 2 3

↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓

y -5 -3 -1 1 3 5 7

ii. x -3 -2 -1 0 1 2 3

↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓

y 7 2 -1 -2 -1 2 7

$x\rightarrow 2x+1$ (one-to-one)

9. a) $-2, 1, \frac{3}{2}$

b) i) $\frac{6}{7-x}$ ii) 28

10. a) $f^{-1}(x)=\frac{3}{2-x}$; $\{x : x\varepsilon R, x\neq 2\}$

$g^{-1}(x)=\frac{2(1-x)}{3}$; $\{x : x\varepsilon R\}$

b) i) $\frac{2x+5}{2x-4}$ ii) $\frac{3}{4}$

11. a) f is one-to-one

b) $-\frac{1}{2}, \frac{1}{4}, 4$

12. i) $\frac{10x^2-5x}{x+3}$ ii) $\frac{5(2x-1)}{2(x+1)}$

13. a) $\frac{3-2x}{x-2}$ b) $\frac{5x+8}{x+2}$

14. i) $h^{-1}=\frac{1-x}{x}$, $x=0$

ii) $\frac{-(2x^2+4x+1)}{(x+1)^2}$, 2

15. i) $g^{-1}=\frac{x}{x-2}$ ii) $\frac{3}{5}$

16. i) $\{x : x\varepsilon R, x\neq -4\}$

$f^{-1}(x)=\frac{2(1+2x)}{3-x}$

ii) $\{x : x\varepsilon R, x\neq -\frac{2}{3}\}$

$f^{-1}(x)=\frac{7-2x}{3x-5}$

iii) $\{x : x\varepsilon R, x\neq 1\}$

$g^{-1}(x)=\frac{x}{x+2}$

iv) $\{x : x\varepsilon R\}$, $g^{-1}(x)=\frac{2x-1}{3}$

v) $\{x : x\varepsilon R\}$, $h^{-1}(x)=1-x$

vi) $\{x : x\varepsilon R, x\neq -\frac{1}{3}\}$

$h^{-1}(x)=\frac{1-x}{3x}$

vii) $\{x : x\varepsilon R\}$,

$f^{-1}(x)=\frac{1}{2}\sqrt{x-5}$

viii) $\{x : x\varepsilon R, x\neq 2\}$

$f^{-1}(x)=\frac{2x-1}{x-3}$

ix) $\{x : x\varepsilon R, x\neq -\frac{1}{2}\}$

$f^{-1}(x)=\frac{3+x}{1-2x}$

x) $\{x : x\varepsilon R, x\neq -\frac{d}{c}\}$

$f^{-1}(x)=\frac{b-dx}{cx-a}$

17. a. i) $\frac{21x+11}{7x+2}$ ii) $\frac{2x-1}{x-3}$

b. i) $g(x)=3+\frac{5}{x-2}$ ii) 3

18. $k=-5$

19. i) a) 1 b) x

ii) $f^{-1}(x)=\frac{x-4}{4}$;

$g^{-1}(x)=x+1$; 1.5

20. a) $f^{-1}(x)=\frac{2x-1}{x}$

$\{x : x\varepsilon R, x\neq 0\}$

b) i) $\frac{1}{3}$ ii) $\frac{5}{3}$

21. $\{-2, -1, 2\}$

22. a) $\frac{1}{1-x}$ ii) $\frac{x-1}{x}$

23. a) $f^2=\frac{4x^2}{(x-4)^2}$; $f^{-1}=\frac{4x}{x-2}$

$x=\frac{8}{3}$ b) $a=2$; $b=-3$

24. $\{x : x\varepsilon R, x>\frac{1}{2}\}$

25. i) $Df=(x : x\in R)$,

$Dg=\{x : x\in R, x\neq 2\}$

ii) $Rf=\{y : y\geq -9\}$

iii) $Rf=\{y : y\geq -9\}$ iv) -0.9

26. i) $D_f=\{x : x\varepsilon R\}$

$D_g=\{x : x\varepsilon R, x\neq 1\}$

ii) $R_f=\{y : y\varepsilon R, y\geq 1\}$

$R_g=\{y : y\varepsilon R, y\neq -1\}$

27. $\frac{1}{5}(x+9)$

28. $h^{-1}=\frac{4+5x}{3+x}$

29. i) $\frac{4}{3}$ ii) 3 iii) $\frac{3}{4}$

30. i) $Dg=\{x : x\in R, x\neq 0\}$

$Df=\{x : x\in R, x\neq 1\}$

ii) $Rf=\{y : y\in R, y\neq 0\}$;

$Rg=\{y : y\in R, y\neq 1\}$

iv) a) $-x$ b) $\frac{1}{1-x}$

31. $f(x)=\frac{2+3x}{4-x}$ **32.** $g(x)=\frac{1-x}{x}$

33. i) $\{x : x\in R\}$ ii) $x : x\geq \frac{2}{3}$

iii) $x : x\in R$ iv) $x : x\in R, x\neq 1$

v) $x : x>\frac{3}{2}$

vi) $\{x : x\in R, -2\leq x\leq 2\}$

34. i) $y : y\in R$

ii) $\{y : y\in R, y\geq -2\}$

iii) $\{y \quad y\in R, y\neq 1\}$

iv) $\{y : y\in R, y\neq -3\}$

35. i) $x : x\in R, x\neq -5$ and $x : x\in R$

ii) a) $-4\leq y\leq 21$

b) $f^{-1}=\frac{1+5x}{1-x}$ c) $\frac{9x^2+42x-15}{1-2x+x^2}$

d) f is one-to-one

36. i) 18 ii) 18 iii) -3 iv) 11

37. -2 **38.** b) $\frac{2+x}{1-x}$

39. a) $f^{-1}=\frac{x+4}{3}$; $x\in R$

$g^{-1}=\frac{3x+1}{2x-3}$; $x\in R$, $x\neq 2$

b) i) $\frac{4x+19}{2x-1}$ ii) $\frac{29}{12}$

40. $\frac{x+11}{6}$ **41.** a) 10 b) $-\frac{15}{4}$

c) $-\frac{4}{3}$ d) 145

Review exercise 5.1

1. i) $\{2.41, -0.41\}$

ii) $\{1.215, -0.55\}$

iii) $\{-\frac{3}{2}, 6\}$ iv) $\{-1\}$

v) $\{-5.12, 3.12\}$ vi) $\{3\}$

vii) $\{-1, 3\}$ viii) $\{\frac{1}{2}, -1\}$

2. $-\frac{25}{12}$ **3.** $\{7, 11\}$

4. i) $\frac{10}{9}$ ii) $-\frac{26}{27}$ iii) 2 iv) 10

v) $-\frac{8}{9}$ vi) $\frac{4}{3}$ vii) $-\frac{10}{3}$

viii) 4 ix) 4

5. i) $x=1$, $y=2$; $x=2$, $y=1$

ii) $x=3$, $y=2$

iii) $x=5$, $y=2$

$x=-\frac{23}{5}$, $y=-\frac{14}{5}$

iv) $x=-1.86$, $y=4.72$

$x=2.17$, $y=-3.34$

v) $x=3$, $y=2$

$x=4$, $y=3$

6. $qx^2+px+1=0$

7. a) $\{k : 2<k<6\}$

b) i) -7, 11

ii) $11x^2-27x+11=0$

8. a) i) 21 ii) 4

iii) $16x^2-433x+1=0$

b) $\frac{5}{2}$; $m=-\frac{5}{2}$, $n=\frac{1}{2}$

9. i) $-\frac{11}{4}$ ii) $4x^2+11x+9=0$

10. b) $p=18$, $q=39$

11. $48x^2+152x-23=0$

12. a) $\{k : k<-8, k>0\}$ b) $\frac{2}{5}$

13. $\{n : n<-3, n>3\}$

14. a) i) $-\frac{7}{4}$ ii) $-\frac{7}{8}$ iii) $\frac{43}{8}$

b) $2x^2+9x+13$

15. a) i) -45 ii) -8

b) $x^2+45x-8=0$

16. $\frac{11}{12}$ **17.** 1

19. i) $acx^2+b(a+c)x+$
$a^2+c^2+2ac=0$

ii) $acx^2-(b^2-2ac)x+ac=0$

20. $qx^2+2px+4=0$

21. $x^2+(2p^3-5pq)x+q^2(q+p^2)=0$

22. $p=9$ or 1

23. a) i) $\frac{21}{4}$ ii) $\pm\frac{5}{4}\sqrt{17}$

b) $8x^2-95x+1=0$

24. i) $x=-1$ $y=-\frac{1}{2}$ ii) $x=1$ $y=-\frac{1}{3}$

iii) $x=-\frac{1}{3}$ $y=-\frac{1}{2}$

iv) $x=1$ $y=\frac{1}{2}$

v) $x=-3$, $y=-3$; $x=6$ $y=\frac{3}{2}$

vi) $x=1$ $y=-2$; $x=\frac{1}{4}$ $y=3$

vii) $x=1$, $y=1$; $x=3\frac{1}{2}$, $y=-4$

viii) $x=-1$, $y=2$; $x=7\frac{1}{2}$, $y=\frac{-7}{5}$

25. i) $4\sqrt{3}$ ii) $\frac{9}{4}$ iii) 2

26. $q^2=4p$ **27.** 9, -25

28. $f_{\max}=5$ $x=-1$

29. -2 ; 3 **30.** 1 or 9 **31.** 0 or 12

32. $k=9$ **33.** $p=4$ **34.** $k=0$

35. $3x^2-7x+15=0$ **36.** $k=12$

37. a) $\frac{9}{2}$ b) $\frac{5}{2}$

38. a) i) $\frac{21}{4}$ ii) -1

b) $6x^2-3x-1=0$ **39.** $\frac{25}{16}$

40. $x^2-39x+49=0$

41. $x^2+10x-125=0$

a) $\frac{40}{9}$ b) $\frac{116}{27}$

Review exercise 5.2

1. i. $p=-1$, $q=-2$

ii. $(x+1)(x+2)(x-1)$

2. $f(x)=(x+1)(2x+1)(x+2)$

3. $q=1$, $r=-5$

$f(x)=.(x-1)(2x-1)(x+2)$

4. a. $a=-12$, $b=16$ b. 9

c. $x=\{-3.65, 1.65, 2\}$

5. $p=7$, $x=\{-\frac{1}{2}, -1.30, 2.30\}$

6. a. $p=5$, $q=6$ b. -38

7. $a=1,\ b=-5$
$f(x)=(x+2)(2x-1)(x-1)$
8. $f(x)=2x^2+3x-2$
$x=-2,\ \frac{1}{2}$
9. a. $p=-3,\ q=-11$
$f(x)=(x+1)(2x-1)(x-3)$
10. $a=-1,\ b=6$
11. $a=-9,\ b=7$
$f(x)=(2x+1)(x-2)(x-3)$
12. i) -12 ii) 0 iii) 0
$(x+1)$ and $(2x+1)$; $(x-3)$
13. $a=8,\ b=5;\ (3x-5)$
14. $a=-18,\ b=8$
$f(x)=(x-2)(2x-1)(x+4)$
15. $a=-10$ **16.** $k=-34$
$f(x)=(2x-1)(2x-3)(2x+5)$
17. i. 20 ii. -4 iii. 2
18. a. $a=5,\ b=-33$
b. $f(x)=(5x+3)(x+2)(x-3)$
c. $\{-\frac{3}{5},\ -2,\ 3\}$
19. $a=2,\ b=-1,\ c=-2$
20. i) $\{3,-2,\ 1\}$
ii) $\{-4,-2,-3\}$ iii) $\{2,-3\}$
iv) $\{1.24,-3.2,1\}$ v) $\{0,1,-1\}$
vi) $\{\frac{1}{2},1,-1\}$
21. $(1-x)^2, x=1$ or $x=3$
22. $k=1,\ \frac{1}{2}\pm\sqrt{\frac{3}{4}}$ **23.** $(2x+3)$
24. $(3x-2), (x-1)$ **25.** $a=-10$
26. $x=2,-\frac{1}{2},3$
27. i) -1
ii) $(x-1)(2x+1)(2x-1)$
Zeros are $-\frac{1}{2},\frac{1}{2},1$
28. a) $a=-5\ b=-1$
b) $x=-1,\frac{3}{2},2$ **29.** $a=2$
30. $p=11,\ q=-5$; real root $=3$
31. -21 **32.** x^3+3x^2+x+1
33. $p=10$
34. i) $a=-\frac{1}{3}\ b=-\frac{4}{3}$ ii) $-\frac{2}{3}$
35. a) $a=-2\ b=-5$
b) $(x+2)(x-1)(x-3)$
c) $x=-2,1,3$
36. a) $m=6,\ n=6$ b) $x=\frac{1}{3},\frac{1}{2},1$
37. a) $a=24\ b=-35$
b) $(5x-1)(x-1)(x+6)$
c) $x=-6,\frac{1}{5},1$
38. a) $a=4\ b=70$
b) $(x+5)(x-7)$
c) $x=-5, 2, 7$
39. a) $a=7\ b=0$
b) $(3x-2)(x+2)(x+1)$
c) $x=-2,-1,\frac{2}{3}$
40. a) $a=1, b=-2, c=-1$
b) $(x+1)$ c) $x=-1, 1, 2$
41. a) $a=1\ b=-1\ c=4$
b) $(x+2)(x-2)(x-1)$
c) $x=-2, 2, 1$
42. i) $a=1, b=1, c=-2$
ii) $(x+2)(x-1)$
43. a) $a=6, b=-19$
b) $(3x+1)(2x-3)(x-2)$
c) $-\frac{1}{3},\frac{3}{2},2$
44. a) $p=2\ q=-9$ b) 4
c) $x-3,-2, 2$ **45.** $k=-8$
46. b) $(2x-1)(3x+5)(x+1)$
47. 24

Review exercise 6

1. $a=-1;\ b=6;\ c=6;\ d=1$
2. $A=2;\ B=-4;\ C=4$
3. $A=4;\ B=2;\ C=-3$
4. a) $A=2;\ B=0;\ C=-1$
b) $-1,\ \frac{3}{2}$
5. $a=2; b=\frac{50}{11}; c=\frac{16}{11}; d=-\frac{7}{11}$
6. a) $A=1;\ B=-1;\ C=2$
b) $D_f=\{x : x\varepsilon R, x\neq -2,1\}$
c) $-\frac{4}{5}$
7. $P=\frac{2}{5}$; $Q=\frac{1}{5}$
8. $\frac{-7}{x-2}+\frac{9}{x-3}$
9. $x=-4,-3$
10. $A=\frac{3}{2}$; $B=-\frac{1}{2}$
11. i) $\frac{2}{2+x}-\frac{1}{1+x^2}$ ii) $\frac{1}{1+x}+\frac{2x}{1-2x^2}$
iii) $\frac{2}{2+x}+\frac{1}{1-2x}$ iv) $\frac{-2}{1-2x}+\frac{3}{1+3x}$
v) $\frac{4}{1+2x}+\frac{2}{1-x}+\frac{3}{(1-x)^2}$
vi) $\frac{19}{5(x-3)}+\frac{6}{5(x+2)}$
vii) $\frac{16}{x-2}-\frac{14}{x-1}$
viii) $\frac{-1}{3(1+2x)}+\frac{5}{3(x+2)}-\frac{4}{(x+2)^2}$
ix) $\frac{3}{x+1}-\frac{2}{(x+1)^2}$ x) $\frac{-1}{x+1}+\frac{x}{x^2+1}$
xi) $1-\frac{3}{x-3}+\frac{3}{x+1}$
xii) $1+\frac{3}{x+4}+\frac{5}{x-2}$
xiii) $\frac{-1}{x-2}+\frac{7}{(x-2)^2}$
xiv) $\frac{-4}{9(x+1)}+\frac{4}{9(x-2)}+\frac{2}{3(x-2)^2}$
xv) $1+\frac{2}{x+3}+\frac{6}{x-2}$
xvi) $\frac{1}{x+7}-\frac{1}{x+9}$
xvii) $\frac{-3}{6(x-1)}-\frac{1}{(x-1)^2}+\frac{15x-19}{6(x^2+1)}$
xviii) $\frac{1}{x-1}-\frac{1}{x+2}+\frac{1}{(x+2)^2}$
12. 194 **13.** $-1,\ 6$
14. 0 **15.** $\frac{3}{x+1}-\frac{4}{2x+1}$
16. $\frac{1}{x-2}-\frac{1}{x+3}$
17. $P=1, Q=-1$ **18.** 6
19. i) $\frac{1}{1-x}+\frac{2}{2x+1}-\frac{3}{(2x+1)^2}$
ii) $\frac{1}{x-1}-\frac{1}{2(x-1)^2}-\frac{2x+1}{2(x^2+1)}$
iii) $\frac{1}{x-3}-\frac{1}{x+2}+\frac{2}{(x+2)^2}$
iv) $\frac{2x+3}{1+x^2}+\frac{2}{1-x}$
v) $\frac{3}{2x-1}-\frac{1}{x+2}$ vi) $\frac{2}{x+1}-\frac{1}{x}$
20. i) $\frac{4}{x+1}-\frac{3}{x-2}+\frac{1}{x+3}$
ii) $\frac{3}{x-1}+\frac{2}{(x-1)^2}+\frac{1-2x}{x^2+8}$
21. $1-\frac{2}{x-1}+\frac{5}{x-2}$
22. $(x-3)+\frac{4}{x+2}-\frac{3}{x-1}$
23. $\frac{2}{x-2}+\frac{7}{(x-2)^2}$
24. $\frac{2}{x+3}+\frac{3}{x-1}-\frac{4}{(x-1)^2}$
25. $\frac{3}{x+3}-\frac{2}{(x+3)^2}-\frac{6}{(x+3)^2}$
26. $\frac{2x+3}{x^2+2}+\frac{5}{x+1}$
27. i) $\frac{2}{x-3}-\frac{2}{x+3}$
ii) $\frac{5}{x+1}-\frac{1}{x-3}$ iii) $\frac{3}{x}+\frac{2}{x-2}-\frac{4}{x-1}$
iv) $\frac{7}{x+4}-\frac{3}{x+1}-\frac{2}{2x-1}$
v) $1+\frac{2}{x+3}+\frac{6}{x-2}$

vi) $3x-2+\frac{1}{x-2}-\frac{5}{x+2}$
vii) $\frac{4}{x+1}-\frac{7}{(x+1)^2}$
viii) $\frac{1}{x^2}+\frac{2}{x}-\frac{1}{x+3}$
ix) $\frac{5}{x-2}-\frac{10}{(x-2)^2}-\frac{4}{(x-1)^3}$
x) $\frac{2}{x-5}-\frac{3}{x+2}+\frac{4}{(x+2)^2}$
xi) $\frac{2x+3}{x^2+7}-\frac{1}{x-2}$
xii) $\frac{1}{x-4}+\frac{2-x}{x^2+3}$
28. a) $\frac{4(x-1)}{x(x-2)}$ b) $\frac{2x^2+4x+3}{(x+1)(x-1)(2x+3)}$
29. 5 **30.** 2

Review exercise 7.1

1. 9 **2.** –1
3. i) $\frac{1}{16}$ ii) $\frac{8}{27}$ iii) 27
iv) 3125 v) $\frac{4}{3}$ vi) $\frac{1}{9}$
vii) 0.09
4. $x=\frac{2y-6}{3}$ **5.** $\frac{2}{x}$
6. i) $3\sqrt{2}$ ii) $\frac{16}{9}$ iii) 2^{-n}
iv) $\frac{1}{4}$ v) $\frac{27}{64}$
7. i) $\frac{3}{2}$ ii) 125 iii) 1 iv) $6^{\frac{3}{4}}$
8. i) $-\frac{2}{5}$ ii) –2 iii) $\frac{7}{10}$
iv) 2 v) -3
9. {0.00, 0.86} **10.** {0, 2.32}
11. {0} **12.** –4 **13.** {0, 1}
14. {3.17} **15.** i) {–2} ii) {0, 2}
16. {–1} **17.** $n=1,\ m=4$
18. i) {0} ii) {2} iii) {2}
iv) {1, 2} v) {2} vi) {0}
19. i) 7 ii) $\frac{x+1}{x(x-1)}$ iii) $\frac{2x\sqrt{y}}{y-x}$
iv) $ab^{\frac{17}{6}}$ v) $\frac{81b^8}{16a^4}$
20. i) 64 ii) 0.3868 iii) $\frac{5}{4}$
21. i) $x=4, y=2$
ii) $x=2, y=3$
22. $x=1.6$
23. i) $\frac{4}{7}$ ii) 1331 iii) $\frac{27}{1000}$
iv) $\frac{9}{4}$ v) $\frac{9}{25}$ vi) 0.6
vii) $\frac{5}{4}$ viii) $\frac{27}{125}$
24. $x=-1, 2$
25. i) $x=5, y=3$
ii) $x=3, y=3$
iii) $x=2, y=1$
iv) $x=4, y=3$
v) $x=2, y=-1$
vi) $x=3, y=-2$
26. 9.64 **27.** $y=0, 1$
28. i) $x=0$ or 1 ii) $x=0$ or 2
iii) $x=0$ or -3 iv) $x=1$ or 2
v) $x=-1$ or 2 vi) $x=1$
vii) $x=0$ or -2 viii) $x=0$
ix) $x=1$ or 2 x) $x=-1$ or -2
xi) $x=-1$ xii) $x=-1$ or 4
xiv) $x=-2$ or 3
xv) $x=0$ xvi) $x=1$ or -1
29. $x=1$ or 3 **30.** $\frac{1}{27}$
31. $x=7$ **32.** i) $x=0$
ii) $x=\frac{1}{2}$ iii) $x=-3$ or 1
iv) $x=1$ or -1.667
v) $x=\frac{16}{15}$ vi) $x=-8$
vi) $x=-1$ viii) $x=\frac{1}{2}$

Review exercise 7.2

1. i) $ab=10^{x+y}$ ii) 1.55
2. 2 **3.** 5.395
4. i) {10} ii) {20}
5. $\{\frac{1}{3}, 9\}$ **6.** $y=2x$
8. i) $\frac{1}{3}$ ii) $\frac{1}{9}$ **9.** 0.2
10. 2 **11.** $\frac{3}{2}$ **12.** 1
13. i) –3 ii) –2, 4 iii) 6
14. i) $y^3=x^2$ ii) $y\le\frac{1}{x}$
15. i) 14 ii) 2 iii) 13
16. i) 0.921 ii) 0.209
iii) 1.068 iv) 1.468
v) 1.277 vi) -0.356
vii) 0.2825 viii) 1.339
x) -0.712 xi) -0.435
17. i) $\log_5 120$ ii) 0
iii) $2\log_3 6$ iv) 2
v) $\log_5 9\sqrt{6}$ vi) $\frac{1}{2}$
18. $\frac{4}{3}$ **19.** $\log\left(\frac{x^3}{\sqrt{y+1}}\right)$
20. i) $x=\frac{32}{3}$ iii) $x=5$
iv) $x=\pm 3$ v) $x=7$
vi) $x=8$ vii) $\frac{19}{47}$

viii) $x=2{,}5$ ix) $x=4$ or 1 x)
xi) $x=4$ xii) $x=\{\ \}$ or $\varnothing$ xiii)
xiv) $x=0.414$ xv) $x=1$
xvi) $x=0.75$ xvii) $x=3$ or 1
xviii) $x=1$ xix) $x=2$ or 16

Review exercise 7.3

1. i) $k=0.40,\ n=0.40,$
$t=158.52.$
2. ii) a) $c=0.4, k=0.7$
b) $q=100$
3. $n=7,\ A=0.06$
4. $a=3,\ b=0.2$
5. ii) $k=8.33,\ c=1.72$
6. $a=2.5,\ k=1.5$
a) 2.5 b) 14
7. $a=0.8,\ b=1.6$
8. $a=12.6\ \ b=1.12$
9. 504, -3
10. $n=4, a=3$
11. b) $a=4.9$ c) $b=1.2$

Review exercise 8

1. $1+14x+84x^2+280x^3+$
$560x^3+672x^5+448x^6+128x^7$,
0.8681
2. a) 2702 b) 1.0325
3. i) $128+224x+168x^2+70x^3$
ii) 127.1067
4 i) $1+2x+\frac{11}{2}x^2+\frac{13}{2}x^3$
ii) 1.04225
5. a) $p=3,\ n=5$
6. a) $1-4x+7x^2-7x^3+\frac{35}{8}x^4$
b) 0.96069
7. $32-40x+20x^2-5x^3+$
$\frac{5}{8}x^4-\frac{1}{32}x^5$; 31.2080
8. a) $1-2x-2x^2-4x^3$
b) 3.91918
9. a) $256-3072x+16128x^2-$
$48384x^3$
b) 253
10. a) $32-80x+80x^2$
b) 32.808
11. $1+\frac{1}{2}x+\frac{3}{8}x^2$, 1.01 **12.** $-8x^3$
13. i) 0.8330 ii) 85.26
iii) 1.010 iv) 1.024
v) 0.8681 vi) 535.5
14. i) 0.82 ii) 85.21 iii) 1.01
iv) 1.024 v) 0.86 vi) 535
15. i) $128+448x+672x^2+$
$560x^3+280x^4+84x^5+14x^6+x^7$
ii) $x^7-7x^6y+21x^5y^2-35x^4y^3+$
$35x^3y^4-21x^2y^5+7xy^6-y^7$
iii) $16x^4-96x^3y+216x^2y^2-$
$216xy^3+81y^4$

iv) $1-\frac{5}{2}x+\frac{5}{2}x^2-\frac{5}{4}x^3+$

$\frac{5}{16}x^4-\frac{1}{32}x^5$

v) $1-x+\frac{3}{2}x^2-\frac{10}{3}x^3+\dots$

vi) $32x^5+240x^4y+720x^3y^2+$

$1080x^2y^3+810xy^4+243y^5$

16. 15% **17.** 2% (decrease)

18. 12 % **19.** 4.5%

20. $1-2x+3x^2-4x^3+\dots$

21. i) $336x^2$ ii) $-\frac{969}{2}x^5$

iii) $-21840x^{11}$ iv) $2^{22}3^8\frac{30!}{22!8!}x^8$

v) $3360p^6q^4$ vi) $16(3a)^7b$

vii) $7920x^4$ viii) $63x^4$

22. i) $1-8x+27x^2$

ii) $1+19x+160x^2$

iii) $2-19x+85x^2$

iv) $1-68x+316x^2$

23. 0.8171 **24.** 1.062

26. i) $1-x-\frac{x^2}{2}-\frac{x^3}{2}+\dots$

ii) $\frac{1}{3}-\frac{x}{9}+\frac{x^2}{27}-\frac{x^3}{81}+\dots$

iii) $1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\dots$

27. 1.732 **28.** 3.16228

29. 1.00499 **30.** $1-3x+\frac{7}{2}x^2$

31. $1+2x+2x^2$; 7.1414

32. 3.332

33. $1-\frac{5}{2}x+\frac{19}{8}x^2-\frac{25}{16}x^3+\dots$

34. $1-x+\frac{3}{2}x^2-\frac{5}{2}x^3+\dots$; 2.238

35. 6% **36.** 3.0% **37.** 9%

38. $a=115, b=-28, c=3$

39. $a=2, b=\frac{80}{3}$

40. $16+32x+24x^2+8x^3+x^4$ and

$1-12x+54^2-108x^3+81x^4$;

504

41. i. $1+10x+45x^2+120x^3+$

$210x^4$

ii. GH¢161,000 **42.** 5

43. a) i. $1+10x+40x^2+80x^3+$

$80x^4+32x^5$

ii. 1.1041

b) $n=5, p=4$

44. $1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\frac{35}{128}x^4$

45. a) $(1+x)^4=1+4x+6x^2+4x^3$

$(2-x)^5=32-80x+80x^2-40x^3$

b) -72

46. a) $32+240x+720x^2+1080x^3+$

$810x^4+243x^5$

b) 49.94

47. a) $\frac{1}{8}-\frac{3}{16}x+\frac{3}{16}x^2-\frac{5}{32}x^3$

b) $2+\frac{1}{12}x-\frac{1}{288}x^2+\frac{5}{20736}x^3$

c) $1+\frac{3}{2}x-\frac{9}{8}x^2+\frac{27}{16}x^3$

48. $\frac{2}{12}x^2$

49. i) $1+\frac{3}{2}x+\frac{15}{16}x^2+\frac{5}{16}x^3+$

$\frac{15}{256}x^4+\frac{3}{512}x^5+\frac{1}{4096}x^6$

ii) 1.0151

50. i) $1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3$

ii) $1+x-x^2+\frac{5}{3}x^3$

iii) $1+x+x^2+x^3$

iv) $1-6x+24x^2-80x^3$

51. 0.834

52. i) $2-\frac{1}{48}x+\frac{1}{576}x^2-\frac{7}{41472}x^3$

$+\frac{35}{1990656}x^4$

ii) 1.9809

Review exercise 9.2

1. $\{x : x<\frac{3}{2} \text{ or } x>4\}$

2. i) $\{x : -5\le x\le 3\}$

ii) $\{x : x<1, x>\frac{3}{2}\}$

iii) $\{x : x<-\frac{1}{2}, x>\frac{3}{2}\}$

iv) $\{x : \frac{1}{2}<x<2\}$

v) $\{x : x<-1, x>3\}$

vi) $\{x : 1<x<2\}$

vii) $\{x : -2\le x\le 4\}$

viii) $\{x : x\le 2, x\ge 3\}$

3. $\{q : -4<q<4\}$

4. $\{k : 2<k<6\}$

5. i) $p\ge 9, p\le 1$

ii) $p\ge 5, p\le 1$

6. $-2<a<6$

7. a) $p<1, p>9$

8. i) $x>2+\sqrt{7}\ x<2-\sqrt{7}$

ii) $x=y=-\frac{1}{2}<x<\frac{1}{2}$

iii) $x>1x<-\frac{2}{5}$

iv) $x>4\ x<-2$

v) $\frac{1}{2}(-3-\sqrt{17})\le x\le\frac{1}{2}(-3+\sqrt{17})$

vi) $x>7\ x<-1$

9. i) $-1<k<1$ ii) $k<0,\ k>\frac{4}{9}$

10. $\{x : 4<x<5\}$ **11.** $-4<x<3$

Review exercise 10

1. i) $\sqrt{37}$ ii) 4 iii) 6 iv) 5

v) 13 vi) 10

2. i) $-\frac{8}{7}$; $(-\frac{9}{2}, 7)$

ii) $\frac{9}{5}$; $(\frac{13}{2}, \frac{21}{2})$

iii) $-\frac{7}{3}$; $(2, -4)$

iv) $\frac{1}{5}$; $(2, -1)$

v) $-\frac{1}{14}$; $(1, -\frac{3}{4})$

vi) $\frac{2}{5}$; $(-\frac{7}{2}, -5)$

3. i) $2x+y+5=0$

ii) $2x-y-1=0$

iii) $x-y-8=0$

4. i) $3x-y-5=0$

ii) $x-2y+9=0$

iii) $x-2y+1=0$

iv) $y-4=0$

v) $x+y-6=0$

vi) $3x-y-1=0$

5.	Gradient	x-intercept	y-intercept
i)	$-\frac{2}{3}$	$(1, 0)$	$(0, \frac{2}{3})$
ii)	-2	$(1, 0)$	$(0, 2)$
iii)	$-\frac{5}{4}$	$(0, 0)$	$(0, 0)$
iv)	$\frac{2}{3}$	$(-3, 0)$	$(0, 2)$
v)	$\frac{5}{7}$	$(\frac{7}{5}, 0)$	$(0, -\frac{5}{3})$
vi)	$\frac{1}{2}$	$(-2, 0)$	$(0, 1)$
vii)	-3	$(-2, 0)$	$(0, -6)$
viii)	$\frac{3}{2}$	$(\frac{1}{3}, 0)$	$(0, -\frac{1}{2})$
ix)	$\frac{2}{5}$	$(4, 0)$	$(0, -\frac{8}{5})$
x)	-2	$(\frac{1}{4}, 0)$	$(0, \frac{1}{2})$
xi)	$-\frac{7}{2}$	$(-\frac{6}{7}, 0)$	$(0, -3)$
xii)	$\frac{4}{3}$	$(6, 0)$	$(0, -8)$

6. i) $(5, 5)$; $(-7, 13)$

ii) $(\frac{61}{8}, \frac{13}{4})$; $(\frac{49}{2}, -8)$

7. i) 1 ii) 10 square units

8. a) $y = x$ b) $E = \left(\frac{40}{13}, \frac{40}{13}\right)$

9. i) (-1, 8) ii) (-11, 10)
iii) (-3, -14)

10. i) $x + 2y - 5 = 0$
ii) $6y - 16x - 19 = 0$
iii) $16y - 10x = 23$

11. $D = (2, -4)$

12. i) $3x - y - 6 = 0$
ii) (2, 0)

13. −2, 1 **14.** 9

15. $2x - 3y + 11 = 0$

16. $3x + 2y + 1 = 0$

17. $x - 3y - 1 = 0$

18. $y + x - 4 = 0$

19. $4y + x - 9 = 0$

20. $5x - 2y - 26 = 0$

21. $x + 2y - 9 = 0$

22. 2 **23.** $\frac{11}{5}\sqrt{5}$ **24.** 1

25. i) 45° ii) 26.6°
iii) 43.15° iv) 9.16°

26. ii) 25 square units
iii) $5\sqrt{5}$, $2\sqrt{5}$

27. $(\frac{-2}{11}, \frac{3}{11})$, (1, 4), $(4\frac{2}{11}, 1\frac{8}{11})$;
$3y - 41x = 11$, $7y + 5x = 33$,
$13y - 41x + 149 = 0$

28. $3x + y = 1$; $(\frac{7}{3}, \frac{-4}{3})$, $(\frac{-1}{3}, \frac{4}{3})$

29. $X(10, 0)$ $Y(0, 5)$ Area = 25

30. (1, 1), $y = 18 - 2x$

31. i) $5y - 3x = 9$, ii) $(7\frac{1}{5}, 0)$

32. i) $B(10, 9)$ ii) $\frac{1}{\sqrt{50}}$ iii) 6

33. i) $P(1, 7)$ $Q(5, 15)$ ii) 1:3
iii) (4, 6)

34. a) (1, 2), (5, 2) (3, 6) b) 8

35. i) $A(2, \frac{9}{4})$ $B(3, \frac{3}{2})$ ii) $\frac{5}{4}$

36. i) $9x + 13y + 19 = 0$
iii) (7, 11)

37. i) $4x - 3y + 15 = 0$
ii) $3x + 4y - 70 = 0$
iii) $A(0, 5)$ $C(10, 10)$ iv) $62\frac{1}{2}$

38. $x + y - 7 = 0$

39. $H(9, 0)$ and $G(6, -4)$

40. $2x + y = 0$

41. $4x + 5y = 0$

42. $x + 2y - 11 = 0$

43. $3x - 4y + 19 = 0$

44. a) $y = x - 3$ b) $y\sqrt{3} = x$
c) $x + y - 1 = 0$
d) $y = 2\sqrt{3} - 1 - x\sqrt{3}$
e) $y\sqrt{3} + x = 3 + \sqrt{3}$
f) $3y - x - 14 = 0$

45. a) 63.4° b) 78.7° c) 26.6°
d) 123.7° **47.** $\frac{1}{4}$ sq units

48. $2x - 3y + 8 = 0$

49. $(-\frac{4}{3}, \frac{11}{3})$, (-5, 0), (6, 0)

50. $10x - 26y - 1 = 0$

51. $x + 3y = 11$

52. $y = x - 4$

53. $(\frac{9}{10}, \frac{17}{10})$ and
$(-\frac{18}{10}, \frac{26}{10})$, $(\frac{-27}{10}, \frac{-1}{10})$ or
$(\frac{36}{10}, \frac{8}{10})$, $(\frac{27}{10}, \frac{-19}{10})$

Review exercise 11

1. a) 6 b) 1.78

2. a) 2 b) 2 c) 2.8

3. 0.36 **4.** 1.5

5. i) 61-80 ii) 20
iii) 61-80 iv) 58.9

6. i) 29.5 ii) 10 iii) $\frac{7}{20}$

7. i) 3.0 ii) 1.87

8. a) i) 6 ii) 8 iii) 4
b) i) 15.5 ii) 16.5 iii) 19
c) i) 10 ii) 7 iii) 7

9. b) i) 3.67 ii) 2 iii) 3.5

10. i) 5.29 ii) 2.87

11. i) 41 ii) 64

12. b) i) 5.27 ii) 5 iii) 6

13. 9 **14.** 4

15. i) 1180 cedis ii) 130 cedis

16. b) i) 47 ii) 44.59, 31.09

17. i) 1,100 cedis
ii) 694 cedis, 265 cedis

18. a) 41-50
c) i) 46 ii) 42.5 d) 0.14

19. b) i) 45.5 ii) 29.0

20. c) 28.5 d) 40.5

21. a) 30.75 b) i) 30.17 ii) 0.30

22. b) i) 6 ii) 0.60 c) 5.7

23. i) 3 ii) 3 iii) 2.82

24. ii) a) 128.5 b) 121.0, 136.5
iii) 15.5

25. b) i) 43.5 ii) 70 iii) 20

26. i) 41.3 ii) 6.67

27. i) 54.5 ii) 87.5 iii) 0.20

28. b) 26.2 c) 0.58

29. a) 187, 2.4 c) 94.5%

30. ii) 22.0

31. a) 142.95 b) 142.95
c) 142.90, 2.64

32. 17.58 **33.** 10.4

34. a) 39 days b) 44

Review exercise 12

1. $\frac{1}{5}$ **2.** $\frac{1}{2}$ **3.** $\frac{1}{4}$ **4.** $\frac{2}{5}$ **5.** $\frac{2}{7}$

6. a) $\frac{7}{30}$ b) $\frac{1}{15}$ c) $\frac{19}{30}$

7. i) 3.84
ii) a) $\frac{18}{31}$ b) $\frac{7}{31}$ c) $\frac{13}{31}$ d) $\frac{5}{31}$

8. i) $\frac{3}{10}$ ii) $\frac{1}{10}$ **9.** $\frac{11}{36}$

10. i) $\frac{9}{20}$ ii) $\frac{3}{20}$ iii) $\frac{17}{20}$

11. i) $\frac{51}{145}$ ii) $\frac{22}{145}$ iii) $\frac{73}{145}$ iv) $\frac{72}{145}$

12. i) $\frac{11}{26}$ ii) $\frac{1}{2}$ iii) $\frac{6}{13}$
iv) $\frac{1}{13}$ v) $\frac{12}{13}$

13. i) $\frac{1}{4}$ ii) $\frac{1}{9}$ iii) $\frac{1}{18}$ iv) $\frac{11}{36}$

14. a) i) $\frac{1}{6}$ ii) $\frac{3}{10}$ iii) 1
b) 63 **15.** i) $\frac{3}{7}$ ii) $\frac{4}{7}$

16. a) $\frac{28}{115}$ b) $\frac{27}{92}$ c) 0.355

17. $\frac{3}{8}$ **18.** 0.03 **19.** $\frac{2}{5}$

20. i) $\frac{48}{95}$ ii) $\frac{14}{95}$ iii) $\frac{47}{95}$ iv) $\frac{48}{95}$

21. a) $\frac{29}{84}$ **22.** a) $\frac{3}{10}$ b) 0

23. a) 0.4242 b) 0.1414

24. $\frac{1}{12}$ **25.** $\frac{7}{44}$

26. i. ${}^{5}/_{16}$ ii. ${}^{1}/_{32}$ iii. ${}^{43}/_{96}$

27. i. ${}^{5}/_{18}$ ii. ${}^{1}/_{6}$ iii. ${}^{1}/_{6}$

28. a. ${}^{4}/_{7}$ b. ${}^{5}/_{9}$

29. a. i. ${}^{2}/_{3}$ ii. ${}^{5}/_{9}$
b. i. ${}^{1}/_{12}$ ii. ${}^{4}/_{81}$

30. ${}^{5}/_{162}$ **31.** a. ${}^{1}/_{12}$

32. i. ${}^{7}/_{30}$ ii. ${}^{7}/_{15}$

33. a. ${}^{3}/_{7}$ b. ${}^{3}/_{14}$ c. ${}^{9}/_{14}$
d. ${}^{11}/_{14}$ e. ${}^{5}/_{14}$ f. ${}^{4}/_{7}$

34. $p(1 - q)$ **35.** $\frac{781}{1024}$

36. a) 0.56 b) 0.06 c) 0.94

37. a) $\frac{1}{729}$ b) $\frac{20}{243}$ c) $\frac{73}{729}$

38. $\frac{5}{11}$ **39.** 0.7

40. i) $\frac{3}{20}$ ii) $\frac{1}{4}$

41. a) 0.45 b) 0.1 c) 0.9

42. $\frac{1}{3}$

43. a) $\frac{11}{357}$ b) $\frac{44}{1785}$ c) $\frac{16}{357}$

44. a) $\frac{1}{12}$ b) i) $\frac{1}{6}$ ii) $\frac{5}{12}$ iii) $\frac{1}{2}$

45. a) $\frac{1}{8}$ b) $\frac{1}{72}$

46. a) $\frac{11}{36}$ b) $\frac{8}{9}$ c) $\frac{1}{6}$ **47.** $\frac{4}{33}$

48. a) $\frac{99}{1015}$ b) $\frac{44}{1015}$ c) $\frac{8}{203}$

49. $\frac{3}{5}$ **50.** $\frac{1}{8}$

51. a) i) $\frac{2}{5}$ ii) $\frac{1}{2}$ iii) $\frac{4}{5}$ b) i) $\frac{8}{25}$ ii) $\frac{1}{125}$

Review exercise 13

1. i) 5, (2, –3) ii) 3, (–2, 3)
iii) 3, (–3, 4) iv) $\frac{5}{2}$, $(-\frac{1}{2}, -2)$
v) 2, (-1, 2) vi) 3, (3, 5)
vii) 5, (-4, -1) viii) 12, (8, 4)
ix) 3, (-3, 0) x) 1, (-3, 2)
xi) 4, (-1, 4) xii) 3, (0, 3)
xiii) 2, (-1, 0) xiv) $2\sqrt{3}$, (3, 0)

2. i) $3x^2+3y^2-9x+11y+6$
ii) $x^2+y^2=25$

3. i) $x^2+y^2-5x+3=0$
ii) $x^2+y^2-5x+5y+6=0$
iii) $x^2+y^2-x-y=8$

4. a) $x-y-1=0$
b) $x^2+y^2-2x+6y-40=0$
(1,–3); $5\sqrt{2}$

5. $x^2+y^2-2x-4y=0$
$x+2y-10=0$

6. a) (-7, 10), $2\sqrt{5}$

7. a) $5x-4y-11=0$;
$4x+5y-17=0$
b) $10x+3y+20=0$;
$3x-10y+6=0$

8. $2\sqrt{5}$ **9.** 3

10. radius = $\frac{3}{2}$, $(\frac{3}{2}, 0)$

11. i) $3x-2y+1=0$
ii) $3y+2x-8=0$

12. i) $y^2=2y+12x-13$
ii) $y^2-3x^2+34x=91$
iii) $x+y=4$ iv) $x+y=0$
v) $11y=3x$ and
$99x+27y+130=0$

13. a) i) $x^2+y^2-2x-8y+15=0$
ii) Locus is a circle of radius $\sqrt{2}$ and center (1, 4)
b) i) $D(0, 5)$; $E(0, 3)$
ii) 2 iii) $x+y-3=0$

14. $x-3y-9=0$

15. $6y-6x-85=0$

16. i) $(\frac{3}{2}, -4)$, 5 ii) 6

17. a) $7x+y-5=0$
b) $3x^2+3y^2+26x-6y+26=0$
c) $8x+18y-23=0$
d) $5x^2+5y^2-46x-50y+22=0$

18. $x^2+y^2-9=0$

19. $x^2+y^2-20x-10y+100=0$ and
$x^2+y^2-40x-20y+400=0$

21. a) $y=x-1$
b) 1:3 c) (1, -3) radius $=2\sqrt{5}$

22. $x^2+y^2-6x+8y=0$,
$P(6, 0)$, $Q(-1, -7)$; $PQ=7\sqrt{2}$

23. a) $5\sqrt{2}$ b) $x-2y+1=0$

24. i) $x^2-8x+y^2-6y=0$
ii) $x^2-4x+y^2-4y+4=0$

26. $2y=x+9$; $3y=16-x$;
$x^2=2x+y^2-10y+1=0$;
$x^2-7x+y^2=1$

27. $C_1:(0,-3)$, 1; $C_2:(6,5)$, 11;
$(\frac{3}{5}, \frac{-19}{5})$

28. $x^2-4x+y^2-12y+32=0$

29. a) i) $M(2, 3)$ ii) 5
b) i) $W(3, 2)$
ii) $x^2+y^2-6x-4y-12=0$

Review exercise 14

1. i) $(\frac{15}{4}, 0)$ $x=\frac{-15}{4}$
ii) (-2, 0), $x=2$
iii) (-3, 0), $x=3$
iv) $(\frac{3}{2}, 0)$ $x=\frac{-3}{2}$

2. $4x^2+9y^2-78x+52y-$
$12xy+169=0$

3. $4x^2+9y^2+26x-78y-$
$12xy+130=0$

4. i) $y=28x$
ii) $y^2-10y+4x+21=0$

5.
i) $x-2y+15=0$, $2x+y-45=0$
ii) $x+y-12=0$, $x-y+6=0$
iii) $\frac{1}{2}x+y-6=0$, $2x-y+36=0$
iv) $2x-2y+3=0$, $2x+2y-9=0$

9. $2y+3x=0$

14. i) $y-x-3=0$ ii) $y+x-9=0$

Review exercise 15

1. i) 16 ii) 14 iii) $-3\frac{1}{2}$
iv) –37 v) 50 vi) 28

2. i) $19-7n$ ii) $14-4n$

3. $d=\frac{1}{2}$ $a=14$, $162\frac{1}{2}$

4. i) $\frac{1}{8}$ ii) $\frac{1}{6561}$ iii) –128
iv) $\frac{2187}{256}$ **5.** $154\frac{1}{3}$

6. i) $\frac{15}{4}$, $\frac{63}{16}$, $\frac{255}{64}$
ii) $U_n=4[1-(\frac{1}{4})^n]$; $n\geq1$
iii) $S_n=4[n-\frac{1}{3}+\frac{1}{3}(\frac{1}{4})^n]$; $n\geq1$

7. i) $\frac{1}{2}$ ii) 2 iii) 60 iv) 8
v) $\frac{5}{2}$ vi) $\frac{1}{2}$

8. $h, \alpha h, \alpha^2 h, \alpha^3 h$; $\alpha=\frac{r}{100}$

9. 9 **10.** $\frac{1}{2}$

11. a) 8 b) 13 **12.** 25

13. 10

14. a) 150 b) $28537\frac{1}{2}$ c) $201\frac{1}{2}$

15. 324 **16.** $a=1.4$

17. 56 **18.** 14 **19.** 512, 4096

20. GH¢10,000,
GH¢109,500

21. 11 years

22. a) GH¢599,000
b) 19 years **23.** 69.38 g

24. a) 3 b) 2 c) 59022

25. a) –4, 1 b) 1, $\frac{2}{3}$

26. $d=-\frac{1}{3}$, $a=13\frac{1}{2}$

27. b) 18, 36 **28.** a) –8 b) $\frac{1}{2}$

29. 236,600 cedis

30. $a=-\frac{5}{2}$, $d=\frac{1}{2}$

31. a) 91 b) –3 c) 28

32. i) a) $0.09+0.0009+0.000009$
$+0.00000009$

b) $0.431 + 0.0000431 +$
0.0000000431

ii) a) $\frac{1}{11}$ b) $\frac{4310}{9999}$ c) $\frac{713}{1650}$

33. $d = \frac{5}{7}$, $Un = \frac{1}{7}(72 + 5n)$

34. $a = 8, d = -4$

35. $a = 27.2, d = -2.4$

36. $1, \frac{1}{2}, 0$

37. $a = 2, d = 2, 364$

38. $a = -15, d = 15$

39. i) $32, 2^n$ ii) $\frac{1}{8}, \frac{1}{2^{n-2}}$

iii) $48, 3(-2)^{n-1}$

iv) $\frac{1}{2}, (-1)^{n-1}(\frac{1}{2})^{n-4}$

v) $\frac{1}{27}, (\frac{1}{3})^{n-2}$

40. $r = 2\ a = \frac{1}{2}$ **41.** 62 or 122

42. a) 8 b) GH¢233100

43. a) $U_2 = \frac{3}{2}, U_3 = \frac{7}{4}, U_4 = \frac{15}{8}$

b) $U_n = 2(1 - \frac{1}{2^n})$ c) 2

d) $18 + (\frac{1}{2})^9$

44. i) $d = 2$ ii) $a = 3$

45. $k = \frac{15}{2}$,

$U_n = \frac{15}{2} + (\frac{3}{2})(\frac{1}{3})^{n-1}$, $\frac{15}{2}$

46. $k = 4$, $U_n = 4 + (\frac{2}{3})^{n-1}$, 4

47. $\frac{1}{4}$; 6; $-\frac{9}{16}$

48. a) £215.89 b) 15 years

49. £10.000, £109.500

50. £3000, 11years

51. a) $\frac{403}{990}$ b) $\frac{23}{99}$ c) $\frac{1711}{4995}$ d) $\frac{7}{9}$

e) $\frac{203}{999}$ **53.** 39

54. 64 **55.** a) a=1, d=4 b) 174

56. $a = 2$ $r = \sqrt{2}$ $S_{10} = 62(\sqrt{2} + 1)$

58. a) 189 b) -255 c) $2 - (\frac{1}{2})^{19}$

d) $\frac{781}{125}$ e) $\frac{341}{1024}$

59. $\frac{1}{2}, 2$ **60.** $-\frac{1}{2}$

61. $-\frac{1}{2}$ $\frac{1}{1024}$; 13.21

62. $\frac{8}{3}\left(1 - (\frac{1}{4})^n\right)$, 4

63. $\frac{1}{2}$ **64.** 8, 4, 2, 1

65. 1, 7, 19, 37; $3n^2 - 3n + 1$

66. 16, -8 **67.** 0.8

68. $a = 2 \pm \sqrt{2}$, $r = \frac{1}{4}(2 \pm \sqrt{2})$

69. $(3n - 1) + \sqrt{3}$ **70.** $\sqrt{2} - 1$

71. $2\frac{1}{3}$ **72.** 4, 12, 36, 108, 160

73. $a = 2$ $d = \frac{3}{2}$

74. a) 160 b) $\frac{3}{2}$ c) 1300

75. 60 or 20 **76.** 27

77. a) i) 5 ii) 25 b) £460.00

78. $1\frac{1}{4}m, 2m, 2\frac{3}{4}m, 3\frac{1}{2}m, 4\frac{1}{4}m,$
$5m, 5\frac{3}{4}m, 6\frac{1}{2}m, 7\frac{1}{4}m, 8m$

79. $2n + 1, 3, 5, 7$ **80.** -2

81. a) 101 b) $\frac{9}{2}$ c) 108, 72, 48, 32

82. 23 **83.** ± 320 **84.** 10

85. a) ii) $2\left(2^n + 3n - 1\right)$ b) 11

86. -15 **87.** 19, 55 **88.** 8

Review exercise 16

1. i) $\frac{3}{5}, \frac{4}{5}$ ii) $\frac{3}{4}, \frac{3}{5}$ iii) $\frac{7}{25}, \frac{24}{7}$

2. $\frac{3}{5}, \frac{4}{5}$ **3.** $-\frac{3}{5}, \frac{3}{4}$

4. a) $\frac{\sqrt{6}-\sqrt{2}}{4}$ b) $\frac{\sqrt{6}-\sqrt{2}}{4}$ c) $\frac{\sqrt{6}+\sqrt{2}}{4}$

d) $-2-\sqrt{3}$ e) $\frac{1}{2}$

5. $\frac{3}{7}$ **6.** $\frac{12}{13}, \frac{5}{12}$

7. $A = 50°$, $C = 51°$,
$BC = 17.41$mm,
Area = 153.12 mm^2

8. $BC = 342$ km, 236.6° C from B

9. i) 11.5 m ii) 74°

10. a) 3.01 m b) 56° c) 5.0 m

11. i) $\frac{9}{5}$ ii) $-\frac{24}{7}$ **12.** 22°, 98°

13. 58.1°, 121.9°; 13.8cm, 3.2cm

14. 53.1°, 126.9°

15. i) $2\sqrt{117}$ cm ii) 16.1°

16. 7 cm, 8 cm

17. i) 2 ii) 1 iii) $\frac{3}{2}$ iv) 4 v) 2

18. i) $\sqrt{3}$ ii) $\frac{\sqrt{3}}{2}$ iii) 1 iv) 1

19. 73.3°, 59° **20.** 45°

21. 144° ; 480° ; 220° ; -972°

22. $\frac{\pi^c}{12}; \frac{\pi^c}{5}; \frac{17\pi^c}{9}; -\frac{26\pi^c}{9}$

Review exercise 17

1. $\frac{1}{10}\sqrt{30} + \frac{1}{15}\sqrt{15}$

2. i) $\theta = 60°, 300°$
ii) $\theta = 0°, 120°, 240°, 360°$

3. $-2, -\sqrt{3}$

4. a) $\frac{\sqrt{3}}{2}$ b) $-\frac{1}{2}$ **5.** $2\sqrt{3}$

6. $\frac{119}{169}$ **8.** $\frac{3\sqrt{7}}{8}$

Review exercise 18

1. b) i) 11°, 62.7° ii) 56.5°

2. b) i) 2.24 ii) 153°

3. ii) a) –2.2, 163° b) 63°

4. ii) {13°, 153°}

5. i) 48.6°, 131.4°, 210°,330°
ii) 210°, 330° iii) 235°, 305°

6. i) 30°, 150°, 210°, 330°
ii) 45°, 315° iii) 15°, 75°
iv) 109.5°, 250.5°, 360°
v) 0°, 180°, 360°
vi){57.9°, 122.1°, 237.9°}
vii){41.4°, 318.6°}
viii){20.9°, 200.9°, 339.1°}
ix){0°, 131.8°, 228.2°, 360°}
x){90°, 270°} xi){ 90°}
xii){ 60 °, 180°, 300°}
xiii){30 °, 150°, 210°, 330°}

7. i) $2; -2$ ii) $\sqrt{10}; -\sqrt{10}$
iii) $5; -5$ iv) $2; -2$
v) $25; -25$ vi) $\sqrt{85}; -\sqrt{85}$
vii) $\sqrt{5}; -\sqrt{5}$ viii) $5; -5$

Review exercise 19

1. i) –4 ii) 11 iii) 6
iv) $\frac{1}{2\sqrt{x}}$ v) –4 vi) 6
vii) $\frac{1}{4}$ viii) 3 ix) 8
x) $\frac{3}{2}$ xi) $\frac{7}{3}$ xii) $\frac{1}{2}$
xiii) 1 xiv) $\frac{1}{2}$ xv) 1
xvi) 39 xvii) $-\frac{1}{11}$ xviii) 2
xix) $\frac{1}{6}$ xx) -7 xxi) 108

2. $6x^2 + 6xh + 2h^2$, $6x^2$

3. i) $1 - \frac{1}{x^2}$ ii) $2x - 3$
iii) $\frac{-6x}{(x^2-4)^2}$ iv) $6x$
v) $2x + 5$ vi) $8x$
vii) $15x^2$ viii) $3x^2 - \frac{2}{x^3}$
ix) $2x - 3$

4. i) 5 ii) $-\frac{2}{x^3}$ iii) $8x$
iv) $-4x + 3$ v) 9 vi) $\frac{2}{3}$
vii) $18x$ viii) $21x^2$
ix) $2x + 15$

5. i) $\frac{-6}{x^4}$ ii) $\frac{-1}{2\sqrt{x^3}}$ iii) $\frac{-4}{x^5}$

iv) $\frac{3}{2\sqrt{x}}$ v) $\frac{-20}{3\sqrt[3]{x^7}}$ vi) $1-\frac{1}{x^2}$

6. i) $6x+4$ ii) $\frac{-6}{x^3}+5$

iii) $10x-4$ v) $12x^2-6x-\frac{10}{x^3}$

v) $x+\frac{1}{x^3}$ vi) $6\cos(3x-2)$

vii) $\frac{1}{2\sqrt{x}}+\frac{1}{x^2}-\frac{3}{x^4}$

viii) $\frac{3}{2}x^{\frac{1}{2}}-\frac{2}{3}x^{-\frac{1}{3}}-\frac{1}{3}x^{-\frac{4}{3}}$

ix) $\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{x^3}}$

x) $-2\sin x\cos x$

xi) $-8x\cos(x^2+1)\sin(x^2+1)$

xii) $\frac{-\cos x}{\sin^2 x}$

7. i) $4x^3-12x^2+10x-4$

ii) $4x^3+3x^2-2$

iii) $\sin x+x\cos x$

8. i) $\frac{6x^2-16x-6}{(x^2+1)^2}$ ii) $\frac{1-x^2}{(1+x^2)^2}$

iii) $\frac{3+x}{(x^2+1)^{\frac{3}{2}}}$

iv) $-\frac{6}{x^4}(x\sin 3x+\cos 3x)$

v) $15\sin 3x$ vi) $2(x+1)^{-2}$

vii) $3x^2(x^2+4)(7x^2+12)$

viii) $\frac{(x+3)(5x+3)}{2\sqrt{x}}$

9. i) $5\cos x$ ii) $-3\sin(3x+\pi/2)$

iii) $2x\sin 3x+3x^2\cos 3x$

iv) $\frac{1}{2x^{\frac{1}{2}}(x+1)^{\frac{3}{2}}}$ v) $\frac{6x(1-2x^3)}{(4x^3+1)^2}$

vi) $-\frac{x+2}{x^3(1+x^3)^{\frac{1}{2}}}$ vii) $\frac{x-2}{2(x-1)^{\frac{3}{2}}}$

viii) $-\frac{x+2}{2x^2(x+1)^{\frac{1}{2}}}$

10. i) $-\frac{8}{3}(1-8x)^{-\frac{2}{3}}$ ii) $\frac{-2}{\sqrt{(1-4x)}}$

iii) $\frac{2}{3}(2x+3)^{-\frac{2}{3}}$ iv) $10(1-2x)^{-6}$

v) $-10(4-2x)^4$ vi) $\frac{10}{(1-5x)^3}$

vii) $\frac{1}{2(3-x)^{\frac{3}{2}}}$ viii) $-3(3x+1)^{-2}$

ix) $\frac{-2}{x^3}-\frac{6}{x^4}$ x) $\frac{1}{(x-1)^{\frac{1}{2}}(x+1)^{\frac{3}{2}}}$

xi) $\frac{x(3x-x^3+2}{(x^3+1)^2}$ xii) $\frac{1}{1+\cos x}$

11. $\frac{2x+y-1}{1-x-2y}$

12. i) $-\frac{2x+y^2}{2yx+3y^2}$ ii) $-\frac{x}{y}$

iii) $-\frac{2x+y}{x+2y}$ iv) $\frac{-\cos x}{\cos y}$

v) $-\frac{2x+3y}{2y-3x}$ vi) $\frac{-y}{2(x+3)}$

13. i) $-6\sin 6x$ ii) $2\sec^2 2x$

iii) $\frac{-\cos x}{\sin^2 x}$ iv) $-\operatorname{cos}ec^2 x$

v) $-2\sin(2x-1)$

14. $1-\frac{1}{x^2}$ **15.** $\frac{16}{(1-2x)^3}$

17. $\frac{x}{y}$; $\frac{y^2-x^2}{y^3}$

18. $1+\frac{1}{y3(2x-1)}-\frac{2}{3(x+1)}$; $\frac{5}{2}$

22. $-\pi$; π^2-2

23. i) $-\frac{2}{3}x^2\sin\frac{2}{3}x+2x\cos\frac{2}{3}x$

ii) $2\cos\frac{x}{2}-x\sin\frac{x}{2}$

iii) $2x\sin 2x+2x^2\cos 2x$

iv) $-3\sec x^2(\pi/5-3x)$

v) $\cos x\cos 3x-3\sin x\sin 3x$

vi) $\frac{2-\cos x-x\sin x}{(2-\cos x)^2}$

24. $2\cos x$; $-4\sin 2x$ **27.** $-\pi$

Review exercise 20

1. i) -3 ii) $(-\frac{1}{2}, 0)$ $(2, 0)$

iii) -5, 5

2. i) -5 ii) 5 iii) 17

3. $4y+x+12=0$

4. $y=\ x-29$

5. $y+x=1$; $2y=2x-3$; $(\frac{5}{4}, -\frac{1}{4})$

6. $4y-x+1=0$; $4y+x-5=0$

7. $y-5x+1=0$ $9x+3y+19=0$

8. $y=7x-4$; $y+5x+28=0$

9. $(4,16)$; $y-8x+16=0$

10. $(\frac{1}{2}, -\frac{1}{4})$ **11.** $y+x+1=0$

12. $2y=x+1$ **13.** $k=-\frac{7}{2}$

14. $8y+121=0$ **15.** $(1, -1)$

16. $p=12$, $q=8$; $(-2, 24)$

17. $3t^2-12t+9$; a) 9 m/s

b) -3 m/s c) 3 m/s

d) $t=1$ s $t=3$ s

18. $16\frac{1}{3}$ m

19. i) a) $v=12$ $a=6$ b) 4 m

ii) a) 15 m b) -2 m/s c) 2 m/s

20. a) $v=6$ m/s; $v=0$; $v=-6$m/s

When $t=6$, P is moving
in the direction Ox, away from
O, comes to rest when $t=7$
and then reverses direction
and moves backtowards O

b) 147 m

21. $9y-27x+16=0$, $9y+2=0$
$9y+3x-4=0$, $3x-1=0$

22. a) $18x+9\sqrt{x}$ b) $x\pm\frac{1}{2}$

23. $\frac{1-y_0-2x_0}{x_0-y_0-1}$, a) $y+x=\frac{4}{3}$

b) $(0, 1)$; $(\frac{2}{3}, -\frac{1}{3})$ c) max(1),

min$(-\frac{1}{3})$ d) $(1, 0)$; $(-\frac{1}{3}, \frac{2}{3})$

24. $\frac{4}{45}$ cm/s

25. a) 4 cm b) 8π cm^2/s

26. 2800 m^2/s

27. a) 0.12π b) 0.05 cm

28. i) $p=2$; $q=1$ ii) 6

29. $a=2$, $b=4$

30. Depth$=32x^{-2}$; $W=3.42$ cm
$L=13.68$ cm; $D=2.74$ cm

31. $x=\frac{1}{3}$; max

32. i) 10 m/s ii) 50 m

33. a) 78 m/s b) 1s, $\frac{2}{5}$ s
c) 39m/s^2 d) 1.5s e) 48.5m/s

34. a) 100m/s b) 4s
c) 200m d) -100m/s

35. $(2, -11)$ minimum
$(-\frac{4}{3}, \frac{203}{27})$ maximum. Find
the intercepts on the x and y
axes and sketch the curve.

36. (0, 0) point of inflexion
$(3, -27)$ minimum

37. 10 cm, 1800 cm^3

38. 2.5cm, 262.5cm^3

39. $\frac{\sqrt{5}}{10}$ **40.** 11.42 cm^3

41. 90,000 m^2 **42.** 10 cm, 10 cm

43. 6.339 cm, 3.169 cm

44. 2513.6 cm^3/s **45.** 0.0075

46. i) a) $L\ 3(100-x)$

b) $A = 1200 - 12x^2$

ii) 3 *ha*

47. i) $v = 10 - 40t$; $a = -40$

ii) $t = \frac{1}{4}s$ $s = 25.25$ m

48. i) 5 m/s ii) 11 m/s iii) 6 m/s^2

iv) $t = \frac{1}{3}$s, 1s; $s = 1.15$ m, 10 m

49. $\frac{1}{80\pi}$ cm /s

50. (127°, 5) max; (307°, − 5) min

51. $r = 2.5$cm; $h = 5$ cm

52. $r = 2.71$cm; $h = 5.42$ cm

53. $r = 4.607$cm; $h = 9.212$ cm

54. $h = 6.67$ cm

55. $d = 24.49$cm; $h = 17.32$ cm

56. $a = 1$; $b = -2$; $c = 3$

57. $p = q = 1$; $r = 2$; (1, 0)

59. $(200\pi)^{-\frac{1}{3}}$ **60.** $\frac{1}{108}$ m^3

61. $y\sqrt{x} + 5x - 2 + \frac{5}{4}\pi$

62. $r = 4$cm; $h = 8$ cm;

$S_{\min} = 96\pi\ cm^2$

63. (−1, 25) max ; (2, −2) min

64. $\frac{11\pi}{6}$

65. i) $a = 1$; $b = -2$ ii) Minimum

66. $x = \frac{1}{3\pi}$ **67.** $f(x) = 2x^3 - 6x$

68. 6%

Review exercise 21

1. i) $\frac{2}{15}x^3 + c$ ii) $\frac{5}{24}x^4 + c$

iii) $\frac{2}{5}x^{\frac{5}{2}} + c$ iv) $-\frac{4}{3x} + c$

v) $\frac{-1}{4x^3} + c$ vi) $2\sqrt{x} + c$

vii) $2x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + c$

viii) $\frac{1}{2}x^2 - \frac{4}{3}x^3 + c$

ix) $\frac{1}{3}x^3 - x^2 + x + c$

x) $\frac{3}{2}x^2 - 5x + c$

xi) $x + \frac{1}{x} - \frac{1}{2x^2} + c$

xii) $\frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + c$

xiii) $\frac{1}{4}x^4 - \frac{3}{2}x^2 - \frac{1}{2x} + c$

xiv $-\frac{1}{x} - \frac{1}{2x^2} + c$

2. i) $\frac{16}{3}$ ii) $-\frac{24}{5}$ iii) $\frac{14}{3}$ iv) $\frac{4}{3}$

v) $\frac{124}{3}$ vi) $\frac{20}{3}$ vii) 249.6 viii) $\frac{3}{2}$

ix) $\frac{9}{8}$ x) $\frac{1}{30}$ xi) $-\frac{13}{12}$ xii) $-\frac{9}{2}$

xiii) 4 xiv) 2 xv) 1

xvi) $\frac{1}{81}$ xvii) $-\frac{1}{6}$ xviii) $\frac{5}{6}$

xix) $8\frac{2}{3}$ xx) $\frac{3}{28}$ xxi) 1

xxii $\frac{2}{9}(2\sqrt{2} - 1)$ xxiii) 0

xxiv) $\frac{31}{160}$

3. i) 3 ii) 3 iii) 2

5. $5\frac{2}{3}$

6. i) $\frac{9}{8}$ ii) $-\frac{4}{3}$ iii) $\frac{7}{9}$

7. i) $\frac{1}{6}(2x-1)^3$

ii) $\frac{1}{28}(2+7x)^4$ iii) $\frac{-8}{3}(1-x)^{\frac{3}{2}}$

iv) $-\frac{2}{5}\sqrt{(2-5x)^3}$

v) $\frac{1}{8(4x+5)^2}$ vi) $-\sqrt{(1-2x)}$

vii) $-\frac{3}{14}(3-7x)^{\frac{2}{3}}$

viii) $\frac{-3}{4(2x+1)^2} - \frac{\sqrt{(1-2x)^3}}{3}$

8. i) $\frac{1}{2}\cos 2x$

ii) $\frac{3}{4}\sin(4x - \frac{1}{2}\pi)$

iii) $-\frac{1}{2}\tan(\frac{1}{3}\pi - 2x)$

iv) $-\frac{1}{3}\cot(3x + \frac{1}{6}\pi)$

v) $\frac{1}{3}\sin 3x - 3\cos x$

vi) $\frac{1}{2}\tan 2x + \frac{1}{4}\cot 2x$

9. i) $10\frac{1}{3}$ ii) $12\frac{1}{3}$ iii) 1

iv) $\frac{\sqrt{3}}{2}$ v) $\frac{2}{3}$ vi) $\frac{31}{15}$

vii) 8 viii) $\frac{1}{3}$ ix) $\frac{16\sqrt{2}}{3}$

x) $-\frac{2}{3}$ xi) $\frac{1}{24}(2\pi - 3\sqrt{3})$

xii) 4 xiii) $57\frac{1}{6}$

xi v) $-\frac{15}{4}$ xv) $\frac{1}{3}\pi^4$ xvi) 6

10. i) $a = 2$ ii) $a = 2$ iii) $a = 2$

11. $\frac{1}{3}$

12. i) $\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$

ii) $\sin x - \frac{1}{3}\sin^3 x$

iii) $\frac{1}{3}\cos^3 x - \cos x$ iv) $\frac{5}{32}$

13. $k = 2$

Review exercise 22

1. $y = 2x^3 - 3x^2 + 11x - 7$

2. $y = 3x^2 + 2x - 5$

3. i) 26m/s ii) 53.3m

4. i) 18m/s ii) 15.75m

5. $s = \frac{3}{2}t^2 + 2t + 3$

6. 5s

7. i) (1, − 4) , (25, 20)

ii) 144 sq units

8. $\frac{1}{2}$ **9.** b) 15 c) 96.6π

10. i) 19.58 ii) 297.62π

11. a) $\frac{1}{3}$, $\frac{1}{3}$, $-\frac{2}{3}$ c) 36

12. a) $a = 3, b = 1, c = -2$

b) (25, 20) ii) $\frac{125}{54}$ sq units

13. a) 32 b) 20.84 **14.** 30

15. i) 525π ii) 75.6π iii) 55π

16. $\frac{15\pi}{2}$ cubic units

17. i) (0, 0), (1, 1)

iii) $\frac{1}{3}$ sq units

iv) $\frac{3\pi}{10}$ cubic units

18. $\frac{\pi}{30}$ cubic units

19. $\frac{412\pi}{15}$ cubic units

20. 0.921, 0.5% **21.** 14.36

22. 11.18 **23.** 9.53

24. 0.880 **25.** 1.090 **26.** 3.12

27. 1.85 **28.** 2.10 **29.** 1.30

30. 0.27 **31.** a) 0.36 b) 0.524

32. a) i) (−1, 0) ,(3, 0), (0, 3)

ii) −4, 4, 2 iii) (1, 4)

c) 10.67 sq units

33. b) 41.67 sq units; 104.17π cubic units

34. 1.3 sq units

35. a) $\frac{1}{2}$ m/s^2 b) 36 m

36. 129 m

37. $v = \frac{1}{2}t^2 - 6t$; $s = \frac{1}{6}t^3 - 3t^2$;

$v_6 = -18$ m/s; $s_6 = -72$ m

38. $6t$ m/s; 504 m

39. $v = 10(1-t)^{\frac{3}{2}} - 8;$

$s = 4(t-1)^{\frac{5}{2}} - 8t + 12$

40. $8\frac{1}{3}$ m

41. $s = 2t - \frac{1}{t+1} + 1;\ v = 2$

42. a) 10.67 b) 51; 90

43. 73.83

44. a) i) $(-2, 6), (5, 27)$

b) $57\frac{1}{6}$ sq units

c) 1326π cubic units

45. i) -28 m/s^2 ii) 0 and 12 s

iii) 18 s

46. a) 1.2 s b) $10\frac{2}{3}$ m

47. i) $(1, 0), (2, 0), (3, 0)$ ii) $\frac{1}{4}$

48. i) $20\frac{1}{4}$ ii) $5x - y - 54 = 0$

49. $\frac{32}{3}$ **50.** $4\frac{1}{2}$

51. i) $\frac{125}{6}$ ii) $k = 12$ **52.** $\frac{23}{24}$

53. i) 13 m/s ii) 14 m/s

54. i) $V = \frac{x^2}{1000} - \frac{2}{5}x + 40$ ii) 200

55. i) $a = 3\ b = -3$ ii) $y = x^3 - 3x$

56. i) $a = 2\ b = -\frac{2}{3}$

ii) $y = \frac{2}{3}x^3 - \frac{2}{3}x$

Review exercise 23

1. i) $\begin{pmatrix} -2 \\ -17 \end{pmatrix}$ ii) $\begin{pmatrix} 10 \\ 23 \end{pmatrix}$

2. $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$ **3.** $\begin{pmatrix} -5 \\ 14 \end{pmatrix}$

4. a) i) $\begin{pmatrix} 16 \\ 2 \end{pmatrix}$ ii) $2\sqrt{65}$ units

b) They are perpendicular

5. $\frac{3}{2}$

6. i) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ ii) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ iii) $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

iv) $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ v) $\begin{pmatrix} 1 \\ 11 \end{pmatrix}$ vi) $\begin{pmatrix} -3 \\ 10 \end{pmatrix}$

vii) $\begin{pmatrix} 8 \\ -9 \end{pmatrix}$

7. $C(6, 4)$

8. i) $C(5, -3)$ ii) $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

9. i) $B(-1, 1), C(7, 7)$ ii) 10units

10. 8

11. i) $Q(2, -2)$ ii) 323°

12. a) $\overrightarrow{XY} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$, $\overrightarrow{YZ} = \begin{pmatrix} -10 \\ 0 \end{pmatrix}$

$\overrightarrow{ZX} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$ b) $\overrightarrow{YL} = \begin{pmatrix} -8 \\ -4 \end{pmatrix}$

$\overrightarrow{YL}$ is perpendicular to $\overrightarrow{ZX}$

13. i) $\begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 5-p \\ 3-q \end{pmatrix}, \begin{pmatrix} p-2 \\ q-1 \end{pmatrix}$

$p = 1,\ q = 3$

14. 67.6° 16. $4\mathbf{i} - 10\mathbf{j}$

17. 30

18. $k = -1$ and $m = 2$

19. $k = \frac{7}{3}$ and $m = \frac{1}{3}$

20. $a = 1$and $b = \frac{4}{5}$

21. 11.5

22. i) $\frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ ii) $m = 5,\ n = 1;$

23. a. 320.16km b. 263.7°

24. i. $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ ii. $2\sqrt{2}$ units iii. 323°

25. i) 12.8 km; 198°

26. i) 157 km; 206°

27. i) $-14\mathbf{i} - 10\mathbf{j}$ ii) $12\mathbf{i} + 6\mathbf{j}$

iii) $-4\mathbf{i} + 10\mathbf{j}$ iv) $-30\mathbf{i} - 24\mathbf{j}$

28. i) (13, 202.6°)

ii) (13, 022.6°) iii) (5,126.9°)

iv) $(\sqrt{2}$, 315°)

29. i) $5.52\mathbf{i} + 4.31\mathbf{j}$ ii) $6.93 + 4\mathbf{j}$

iii) $3\mathbf{i} - 5.20\mathbf{j}$ iv) $-5\mathbf{i} + 8.66\mathbf{j}$

31. $R(3, 4)$ **32.** $\frac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j})$

33. $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$ **34.** $-5\mathbf{i} + 11\mathbf{j}$

35. $5\mathbf{i} - 13\mathbf{j}$ **36.** $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

37. $\begin{pmatrix} -5\sqrt{3} \\ -5 \end{pmatrix}$ **38.** 5 47. $-2\sqrt{3}$

39. a) $s = r + t$ b) $v = \frac{1}{2}(s + t)$

40. $2\mathbf{i} - 2\mathbf{j}$ **41.** 90° **42.** 1

44. 14 kmh^{-1}

Review exercise 24

1. $\frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, 34°

2. i) $\begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 5-p \\ 3-q \end{pmatrix}, \begin{pmatrix} p-2 \\ q-1 \end{pmatrix}$

$p = 1,\ q = 3$

ii) 83° iii) 8.0 units square

3. 120° **4.** 240

5. $a = 1$and $b = \frac{4}{5}$ **6.** 6

7. 17, 152°

8. $\mathbf{a}.\mathbf{b} = -4$; $\frac{-2\sqrt{5}}{25}$

9. i) $\overrightarrow{AB} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$,

$\overrightarrow{DC} = \begin{pmatrix} 6-x \\ 2-y \end{pmatrix}$ $\overrightarrow{BD} = \begin{pmatrix} x-1 \\ y-1 \end{pmatrix}$

$x = 8,\ y = 5$

ii) 18° iii) 13 sq units

10. 30° **11.** $\frac{25}{\sqrt{17}}$

12. 36.87°

13. 14 **14.** $-2\sqrt{3}$

15. i) $PQ = QR = 5,\ PR = \sqrt{2}$

ii) 81.9°, 81.9°, 16.2°

16. 8 **17.** 0.880 **18.** 90°

Review exercise 25

1. a) $P = 41.85$ N, 113°

b) $Q = 60.7$ N, 250.8°

c) $R = 47.95$ N; $S = 53.75$ N

2. a) $(9\mathbf{i} + 17\mathbf{j})$N b) 19.2 N

c) 0.469

3. a) $F_1 = \begin{pmatrix} 0 \\ -120 \end{pmatrix}$; $F_2 = \begin{pmatrix} 25 \\ 43 \end{pmatrix}$;

$F_3 = \begin{pmatrix} -0.7071K \\ 0.7071K \end{pmatrix}$

b) $K = 71.93$

4. (8.83 N, 316.58°); $P = 8.83$ N;

$Q = 0.24$ N

5. 302.4 N, at 41.4°

6. (2 N, 060°) **7.** 60° **8.** 9.83 N

9. $\begin{pmatrix} -10\sqrt{3} \\ -10 \end{pmatrix}$ **10.** $\begin{pmatrix} 10 \\ 5 \end{pmatrix}N$ **11.** 3.38N

12. $Q = 3P$ **13.** (10 N, 270°)

14. $\frac{3\sqrt{3}}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$

15. a) $\mathbf{T}_1 = \begin{pmatrix} -0.5T_1 \\ 0.866T_1 \end{pmatrix}$,

$\mathbf{T}_2 = \begin{pmatrix} 0.866T_2 \\ 0.5T_2 \end{pmatrix}$, $\mathbf{W} = \begin{pmatrix} 0 \\ -49 \end{pmatrix}$

b) $\mathbf{T} + \mathbf{T} + \mathbf{W} = 0$

c) 42.43 N, 24.50 N d)150°

16. (8 N, 141°)

17. 173.2 N; 100 N

18. 57.74 N; 115.47 N

19. $F = 86.6$ N; 100 N

20. 28.8 N **21.** $P = 86.6$ N; 50 N

22. 0.14 **23.** i) 9 N ii) 25 N

24. (13 N, 032.2°) **25.** ii) 32g

26. a) 67.4°; 22.6°

b) 180.37 N; 86.42 N

27. i)5 kg ii)2 kg **28.** 25 g

29. 50 N; 100 N **30.** $\frac{8}{27}$

31. 7.5 N; 5 m/s^2

32. $\frac{W\sqrt{3}}{6}$ **33.** 2.27 N; 187.4°

34. $\frac{\sqrt{3}}{3}$; 200 N

35. 43.3 N to wall, 156 N at angle 73.9° to floor

36. $P = 220$ N ; $Q = 260$ N

37. (13N, 067°) **38.** $5\sqrt{3}$ 3

39. 7.17 N

40. i) M= 14.4 kg

ii) Reaction at A = 124 N

41. i) 11.68 N ii) 116°

42. i) $\begin{pmatrix} -0.866T_1 \\ 0.5T_1 \end{pmatrix} + \begin{pmatrix} 0.5T_2 \\ 0.866T_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -100 \end{pmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$

ii) $T_1 = 50$ N, $T_2 = 86.6$ N

43. 17.22°N

Review exercise 26

1. i) 20 m ii) 31.25 m

iii) 1.5 s iv) $t = 2$ s, 3 s

2. 5, (17.4 m/s, 035°)

3. a) 4 m/s

b) $P = -1200$Ns, $P = 1200$Ns

4. $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$ Ns **5.** $\begin{pmatrix} 18 \\ -2 \end{pmatrix}$ Ns

6. i) 2.68 m/s; 153°

7. i) 6 m/s; 009° **8.** 3.4 m/s

9. 5.6 m/s

10. i) 6.86 m/s ii) 3.43 m/s

11. i) 6 m/s ii) 24 Ns

12. $s = 19.6t - 4.9t^2$;

$s = 19.6(t-2) - 4.9(t-2)^2$

$t = 3$ s; speed = –9.8 m/s; 9.8 m/s

13. $T = 4$ s **14.** b) $V = 2.3T$

16. $19\frac{11}{16}$ m **17.** 16.03 m/s

18. a) 4930 N b) 680 N

19. a) $\begin{pmatrix} 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 8 \\ 0 \end{pmatrix}$

b) (4.46 m/s, 051.2°)

20. i) 10 N ii) 15 m/s

21. b) i) $\frac{1}{12}$ m/s^2, $\frac{1}{72}$ m/s^2

ii) 17250 m iii) 64.7 km/h

22. $9\frac{3}{8}$ m/s^2 ii) 16.25 N iii) 95 m

23. $F_1 = 2\sqrt{5}\ N$; $F_2 = 8i + 16j$;

$F_R = 12i + 8j$; $\frac{2\sqrt{3}}{13}i + \frac{3\sqrt{3}}{13}j$;

$4i + 6j$ m/s^2

24. 8 m/s **25.** 6 m/s

26. (25 N, 180°)

27. a) 4.3 m/s; 2.9 N b) 5.3 m/s

28. a) 800 N b) 4000 N

29. 3 m/s^2; 14.3 N ; 22.9 N

30. 1.5 m/s **31.** $6\sqrt{10}$ m/s

32. 5 s **33.** 8.0 N

34. i) 8ms^{-1} ii) 2ms^{-2}

35. 8.75 m **36.** 28.3 ms^{-1}

37. 3.2ms^{-1}

Review exercise 27

1. i) a) $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix}$

2. $\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$

3. $\begin{pmatrix} 11 & -2 \\ 1 & 18 \end{pmatrix}$

4. $\begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}$

5. a) $\begin{pmatrix} 3 & -5 \\ 5 & 8 \end{pmatrix}$;

$\begin{pmatrix} 1 & -18 \\ 18 & 19 \end{pmatrix}$; $\frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$;

$x = 2, y = 1$

6. $k = \{-3, 2\}$

7. i) $\begin{pmatrix} 4 & -8 \\ -3 & -7 \end{pmatrix}$ ii) $\begin{pmatrix} 0 & 2 \\ 26 & -3 \end{pmatrix}$

Matrix multiplication is not commutative.

8. a) i) $\begin{pmatrix} & 11 \\ 11 & -1 \end{pmatrix}$

ii) $\begin{pmatrix} 5 & 1 \\ -3 & 17 \end{pmatrix}$ iii) $\begin{pmatrix} -44 & 34 \\ -48 & 40 \end{pmatrix}$

iv) $\begin{pmatrix} 0 & 16 \\ 8 & -4 \end{pmatrix}$

b) i) $\begin{pmatrix} -38 & 186 \\ 58 & -6 \end{pmatrix}$

ii) $\begin{pmatrix} -16 & 120 \\ 124 & -28 \end{pmatrix}$

9. $x = 3$; $y = 2$

10. $\begin{pmatrix} 7 & 8 \\ 3 & 3 \end{pmatrix}$ **11.** $x = -1$

12. $\begin{pmatrix} 11 & -4 \\ 1 & 18 \end{pmatrix}$ **13.** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

14. $\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$ **15.** (15, 9)

16. $\begin{pmatrix} -14 & -15 \\ 25 & 16 \end{pmatrix}$

17. $x = 5$; $y = \frac{8}{3}$

18. a) $A = \begin{pmatrix} 1 & 5 \\ -2 & 0 \end{pmatrix}$;

$B = \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix}$

b) i) $A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$;

$B = \begin{pmatrix} -1 & 7 \\ 6 & \frac{1}{y} \end{pmatrix}$

ii) $A = \begin{pmatrix} 2 & 8 \\ 4 & \frac{1}{y} \end{pmatrix}$

iii) $B = \frac{1}{\frac{1}{y}+42}\begin{pmatrix} -\frac{1}{y} & 7 \\ 6 & 1 \end{pmatrix}$

19. $x = 4$; $y = -1$ **20.** $x = 2$

21. a) –2 b) $x = -\frac{11}{2}$; $y = 5$

22. $x = 1$ or 2 or 3

23. i) -306 ii) 1 iii) 103

Review exercise 28

1. $P(-4, \frac{9}{2})$

2. a) i) $\begin{pmatrix} 1 & 2 \\ -3 & 2 \end{pmatrix}$ ii) (8, 0)

b) $\begin{pmatrix} 10 & -4 \\ -8 & 0 \end{pmatrix}$

3. a) i) (25, 33) ii) $\begin{pmatrix} 0 & -\frac{1}{3} \\ -1 & -\frac{11}{3} \end{pmatrix}$

b) $\begin{pmatrix} -13 & 6 \\ 4 & -19 \end{pmatrix}$

4. (1, 0) **5.** (1, 2)

6. a) $\begin{pmatrix} 3 & -5 \\ 5 & 8 \end{pmatrix}$; $\begin{pmatrix} 1 & -18 \\ 18 & 19 \end{pmatrix}$;

$\frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$; $x=2, y=1$

b) (−1, −3)

7. i) (−3, 11) ii) $\frac{1}{10}\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$

iii) $a=2,\ b=3$

8. $\begin{pmatrix} 3 & 6 \\ 4 & -5 \end{pmatrix}$

9. a) i) $\mathbf{N} = \begin{pmatrix} 0 & -2 \\ -1 & 2 \end{pmatrix}$

ii) $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$

iii) $\mathbf{M} = \begin{pmatrix} 3 & 1 \\ 2 & -3 \end{pmatrix}$

iv) $\mathbf{Q} = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$

b) i) $\begin{pmatrix} -4 & 6 \\ 1 & -7 \end{pmatrix}$ ii) $\begin{pmatrix} 4 & 0 \\ -5 & -1 \end{pmatrix}$

iii) $\begin{pmatrix} -4 & -2 \\ 4 & 3 \end{pmatrix}$ iv) $\begin{pmatrix} 1 & 3 \\ 8 & 2 \end{pmatrix}$

v) $\begin{pmatrix} 2 & -2 \\ -6 & -5 \end{pmatrix}$ vi) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

10. $\begin{pmatrix} 1 & \frac{1}{3} \\ 7 & -1 \end{pmatrix}$

11. a) i) $T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$;

$S = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$ ii) (1, 14)

iii) $a = \frac{1}{5};\ b = -\frac{3}{5}$

b) i) $M = \begin{pmatrix} 0.6428 & -0.7660 \\ 0.7660 & 0.6428 \end{pmatrix}$

ii) (−0.9, 2.1)

12. i) $Q = \begin{pmatrix} 4 & 2 \\ 0 & -3 \end{pmatrix}$;

$S = \begin{pmatrix} 2 & 8 \\ 7 & 2 \end{pmatrix}$ ii) $\begin{pmatrix} 2 & 10 \\ 4 & 2 \end{pmatrix}$

iii) $\begin{pmatrix} 0 & -0.33 \\ 0.5 & 0.67 \end{pmatrix}$; $\begin{pmatrix} -0.03 & 0.13 \\ 0.12 & 0.03 \end{pmatrix}$

iv) $x=2;\ y=3$

13. a) $\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ b) $(6x, 6y)$

enlargement with scale factor 6

14. a) i) $M = \begin{pmatrix} -2 & 4 \\ -4 & 1 \end{pmatrix}$

ii) (−6, −5)

b) i) $N = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$; $P = \begin{pmatrix} 2 & -3 \\ -2 & 1 \end{pmatrix}$

ii) $\begin{pmatrix} 2 & -5 \\ 2 & -7 \end{pmatrix}$ iii) $-\frac{1}{4}\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$

c) $a=-8;\ b=4$

15. $\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$ **16.** $Q(1, 0)$

17. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **18.** (8, -2)

19. $\begin{pmatrix} 3 & 1 \\ 2 & -3 \end{pmatrix}$

20. $\left(-\frac{1}{2}+\frac{3}{2}\sqrt{3},\ -\frac{\sqrt{3}}{2}+\frac{3}{2}\right)$

21. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **22.** $(x, y) \rightarrow (-y, x)$

23. i) $\begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}$ ii) $\frac{1}{6}\begin{pmatrix} 3 & -1 \\ -6 & 4 \end{pmatrix}$

iii) (−7, −9) **24.** a) i) $Q^{1}(-4,-2)$

ii) $Q^{11}(-4,-8)$ iii) $Q^{111}(6, 4)$

c) $5y-6x+16=0$

25. i) $A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$ $B = \begin{pmatrix} -1 & 7 \\ 6 & \frac{1}{y} \end{pmatrix}$

ii) $A = \begin{pmatrix} 2 & 8 \\ 4 & \frac{1}{y} \end{pmatrix}$

iii) $B = \frac{1}{\frac{1}{y}+42}\begin{pmatrix} -1 & 7 \\ 6 & 1 \end{pmatrix}$

26. $9y+8x+14=0$

27. a) $\begin{pmatrix} k & 1 \\ 3 & k \end{pmatrix}$ b) 4

c) $\begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}$

28. $y=1-2x$

29. i) $\begin{pmatrix} 4 & 8 \\ -2 & 3 \end{pmatrix}$ ii) (8, 3)

iii) $a=-\frac{15}{14}$; $a=\frac{2}{7}$

30. i) $a=3\ b=2\ c=7\ d=4$

ii) $\begin{pmatrix} -2 & 1 \\ \frac{7}{2} & -\frac{3}{2} \end{pmatrix}$; $\left(4, -\frac{13}{2}\right)$

31. i) $\mathbf{A} = \begin{pmatrix} 5 & 1 \\ 2 & -3 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$

ii) a) (−20, −1) b) $\left(\frac{15}{17}, \frac{-7}{17}\right)$

32. a) $\mathbf{A} = \begin{pmatrix} -1 & 5 \\ 3 & -1 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 4 & 7 \\ -2 & 6 \end{pmatrix}$

b) (72, 60)

33. a) $\mathbf{P} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$ b) $\mathbf{Q} = \begin{pmatrix} 0 & -2 \\ 2 & -1 \end{pmatrix}$

c) $\mathbf{PQ} = \begin{pmatrix} 6 & -7 \\ 8 & -2 \end{pmatrix}$ d) (−27, −14)

Review exercise 29

2. i) Valid ii) Not valid
iii) Not valid

Review exercise 30

1. b) $y=0.93x+0.85$, 5.05
2. ii) $y=0.86x+2.67$ iii) 79.6
3. ii) $y=1.03x+0.53$ iii) 47
4. iii) about 25,000 cedis
5. b) 0.65 c) 7.8 kg d) 0.86
6. c) 154 cm d) 0.73
7. 0.245 **8.** 0.89 **9.** a) 0.227
b) $x=1.80-1.13y$ c) 0.27
10. a) $y=17.21+44.79x$
b) 79.02 c) 0.77
11. b) $y=-0.8026x+170.03$ c)33.6
12. b) $y=1.64x-7.48$ c) 7.2
13. b) $y=0.83x-5.53$ c) 15.5

Review exercise 31

1. i) 15504 ii) 720 iii) 28
iv) 1260 v) 2730 vi) 720
2. $\frac{11!}{8}$ **3.** 2454 **4.** 362880
5. 5040 **6.** 604800 **7.** 504

8. 5040 **9.** a) 2520 b) 7560
10. a) 120 b) 240
11. 5040 **12.** 24
13. 120 **14.** 24
15. 70 **16.** 120
17. a) 80 b) 50
18. a) 31 b) 34
19. a) 85 b) 756
20. 75
21. 840 **22.** 1680 **23.** 1260
24. 40320 **25.** 907200
26. a) 27 b) 725
27. 1764 **28.** 980
29. 840 **30.** 15120
31. 300 ; 120
33. i) 2016 ii) 1680
iii) 1344 iv) 336
34. i) $\frac{11!}{2!}$ ii) $\frac{11!}{(2!)^3}$
iii) $\frac{9!}{(2!)^3}$ iv) $\frac{8!}{(2!)}$
35. $\frac{10!}{(4!)^2}$ **36.** $\frac{8!}{2!\times 3!}$
37. i) 720 ii) 1440
38. i) 30240 ii) 10080
39. 140 **40.** i) 210 ii) 371
41. a) 792 b) i) $\frac{7}{44}$ ii) $\frac{7}{22}$ iii) $\frac{37}{44}$
42. a) 3003
b) i) $\frac{10}{143}$ ii) $\frac{160}{1001}$ iii) $\frac{986}{1001}$

Review exercise 32

1. a) $\frac{14}{115}$ b) $\frac{27}{92}$ c) 0.355
2. 0.4242 b) 0.1414
3. i) $\frac{48}{95}$ ii) $\frac{14}{95}$ iii) $\frac{47}{95}$ iv) $\frac{48}{95}$
4. i. $7/30$ ii. $7/15$ **5.** $\frac{5}{11}$
6. i) $\frac{3}{20}$ ii) $\frac{1}{2}$
7. a) $\frac{11}{357}$ b) $\frac{44}{1785}$ c) $\frac{8}{119}$
8. $\frac{4}{33}$ **9.** $\frac{3}{5}$ **10.** i) 0.0146
ii) 0.0013 iii) 0.1143
11. i) 0.041 ii) 0.933 iii) 0.263
12. i) 0.346 ii) 0.078 iii) 0.922
13. a) 0.111 b) 0.998 c) 0.0548

Self Assessment Test

1. i) 5 ii) $\frac{33}{40}$
2. i) 20 ii) 70 iii) 20
iv) 70 v) 30 vi) 0
3. i) 8 ii) 24 iii) 20
4. i) 39 ii) 72 iii) 179 iv) 52

5. a) 4 b) 6 c) 3
6. $5-2\sqrt{6}$ **7.** a) $8\sqrt{5}$ b) $9\sqrt{2}$
c) $\frac{24\sqrt{6-21}}{67}$ b) $\sqrt{14}+42$
8. $4\sqrt{3}$ **9.** $\sqrt{3}$
10. $13-7\sqrt{3}$ **11.** $\frac{13}{2}+\frac{5}{2}\sqrt{5}$
12. $7-4\sqrt{3}$
13. $\sqrt{6}+\sqrt{3}-2-\sqrt{2}$
14. $3+\sqrt{6}$ **15.** $2-5\sqrt{3}$
16. 14 **17.** 4.449
18. -2 **19.** 0.3535
20. $\frac{8}{11}+\frac{5}{11}\sqrt{3}$ **21.** $\frac{3}{2}\sqrt{3}$
22. 1 **23.** 0 **24.** $-4-3\sqrt{2}$
25. 19 **26.** $6\frac{3}{4}$ **27.** $2-5\sqrt{3}$
28. $\frac{2}{5+x}$ **29.** x^2-2x+2
30. $\frac{x^2+2}{x^2+1}$ **31.** $\{y : y \in R, y \geq 2\}$
32. $\{x : x \in R, x \neq 2\}$
33. $\frac{2x-1}{x+5}$ **34.** $\frac{1}{3}(x+2)$
35. -59 **36.** $\{x : x \in R, x \neq \pm 2\}$
37. 0 **38.** II and III
39. $(x+1)^4+4$ **40.** $1+x$
41. $\{x : x \in R, x \neq -\frac{2}{3}\}$
42. a) $\{x : x \in R\}$;
$\{x : x \in R, x \neq 2\}$
b) $\{y : y \in R, x \neq 1\}$
43. $x^3+8x^2+13x+26$;
remainder is 53
44. i) $\frac{x}{x-2}, x \neq 2$ ii) $\frac{-2}{5}$
45. $x = -2$ or 1
46. $\{x : x \in R, x \neq -1, 4\}$
47. 1 **48.** $\{y : y \in R, 0 \leq y \leq \frac{1}{2}\}$
49. 13 **50.** $4x^2+16+13$
51. $3x+1$ **52.** $\{x : x \in R, x \neq \pm 2\}$
53. $\frac{2-x}{4x}, x \neq 0$ **54.** $\frac{1-4x}{x}, x \neq 0$
56. $\{x : x \in R, x \neq \pm 2, -2\}$
57. $\{x : x \in R, x \neq -3, 2\}$
58. $\frac{2x^2}{1-3x^2}$
59. *h* is not one-to-one
60. $x = 1, 3$ **61.** -4 **62.** $\frac{2x-1}{3}$
63. $\{x : x \in R, x \neq -1, 3\}$

64. i) 1 ii)
$f(x) = (x+2)(x^2-5x+18),\ -2$
65. a) 50 b) 838
66. a) $(x+1)(x+4)(2x-3)$
b) $(x-2)(x+3)(3x+1)$
c) $(x-2)(2x+3)(x-3)$
67. $-8\frac{6}{7}$ **68.** i) -1 ii) $1, \frac{1}{2}, -\frac{1}{2}$
69. $(x-1^2)(x+1)(x+5)$
70. 24 **71.** $-3, -1, -\frac{1}{2}$
72. -5 **73.** -9
74. $3x^2+7x+1$ **75.** $-1, -2, 2$
76. 13 **77.** 16 **78.** $-3, -2, 1$
79. $x-1$ **80.** 51
81. $x^2-2x+14$ **82.** -15
83. -3 **84.** $p = 5$; $q = -6$
85. $-1, -5$ **86.** 1
87. a) $a = 2, b = -32$
b) $-5, \frac{1}{2}, 3$ **88.** $-5, 4$
89. $x = -3, y = 0; x = \frac{9}{5}, y = \frac{12}{5}$
90. $k < 0, k > 3$ **91.** 0 **92.** 2
93. $\frac{25}{4}$ **94.** 12 **95.** $\frac{4}{5}$
96. $-3\frac{1}{8}$ **97.** $8\frac{8}{9}$ **98.** -32
99. $x^2+35x-343=0$
100. $2x^2+4x+17=0$
101. $\frac{35}{4}$ **102.** $x^2-4x-1=0$
103. $-\frac{1}{3}, 1$ **104.** $\frac{4}{3}$
105. 4 **106.** 12
107. 27 **108.** $ax^2-a^3x=0$
109. 4 **110.** $-7 < p < 2.3$
111. a) i) $-2+\sqrt{2}$ ii) $9+2\sqrt{2}$
b) $x \leq -2$; $x \geq \frac{1}{2}$
112. a) not commutative
b) not associative
113. i) $-\frac{3}{4}$ ii) $\frac{3}{2}-x$ iii) $\frac{5}{6}$
114. a) commutative b) 67
115. $\sqrt{6}-1$
116. $e = o$ b) $-\frac{2}{3}$ c) -23
117. -2 118) ±4
119. a) *b* b) closed
ii) commutative
120. a) $\frac{1}{2}$ b) 0

121. $\frac{1}{2}$ **122.** a) 2 b) 5

123. $42+24\sqrt{3}$ **124.** -10

125. $y=\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$

126. -19 **127.** 5 **128.** 0

129. a) I and II b) c

130. a) $\frac{3}{2x-1}-\frac{2}{x+2}$

b) $\frac{3}{x+1}+\frac{2x+7}{x^2+x+1}$

c) $\frac{3}{x}+\frac{3}{x-1}+\frac{2}{(x-1)^2}$

d) $(3x+7)+\frac{3}{x+1}+\frac{2}{x-2}$

e) $1+\frac{5}{x+2}+\frac{3}{x-1}$

131. $\frac{3}{x+1}-\frac{4}{2x+1}$

132. -2 and 3 **133.** $\frac{1}{2}$ **134.** $\frac{k-x}{kx}$

135. -3, -2 **136.** $\frac{3}{x+2}-\frac{1}{x+1}$

137. $\frac{20a-15}{6b}$ **138.** 3 **139.** $-\frac{1}{2}$

140. -3 **141.** 1

142. $\frac{2}{x-6}+\frac{4}{x+3}$ **143.** $\frac{2k+1}{k^2-4}$

144. -1

145. $1+nx+\frac{n(n-1)}{2!}x^2+$
$\frac{n(n-1)(n-2)}{3!}x^3+\ldots$; 1.020

146. $x^4-8x^3y+24x^2y^2-$
$32xy^3+16y^4$

147. $\frac{1}{\sqrt[8]{165}}$ **148.** $\frac{9}{4}$

149. 6 **150.** 7 **151.** 2.00

152. $\frac{1}{2}$ **153.** 20 **154.** -1792

155. 0.99950 **156.** $28y^2$

157. $160x^3$ **158.** 0.01

159. $60x^2$ **160.** -0.00005

161. a) 32 b) $2\sqrt{2}$ c) $\frac{4}{9}$ d) 12

162. a) 5 b) $-\frac{4}{9}$ c) -2

d) $\frac{1}{4}$ e) $-2\frac{3}{4}$ f) 3 g) $-\frac{1}{4}$

h) 2.13 i) 1 j) 0, 1 k) $-\frac{3}{2}$

l) -6 m) 0, -2

163. $\frac{1110}{109}$ **164.** log 3 **165.** 2

166. 0.69897

167. i) 9 ii) 1, 3 iii) $\frac{1}{125}$

iv) 625 or 625^{-1} **168.** 1

169. $(x, y)=(8, 2)$ or $(2, 8)$

170. i) 15.6 ii) 9

171. i) 0.78 ii) 1.64

172. 5.395

173. a) $6p$ b) $2(q-1)$

174. 1222 **175.** 14 **176.** 20

177. $\frac{1}{9}$ **178.** 2 **179.** 0.50

180. $a+\frac{1}{2}b$ **181.** log 2

182. 2 **183.** $x=4y$

184. $y^3=x^2$ **185.** $-2, 4$

186. 13 **187.** 6 **188.** c

189. 0.03 **190.** 6.841

191. a) $a=20.0$ $b=2.5$
b) 2.59

192. $a=30$ $b=-4$; $x\leq 7.5$

193. $a=1430$ $b=0.35$

194. $b=30$ $n=2$; $y=72$

195. a) 52 b) $\frac{16}{3}$

c) $4n^2-13n+9$; $8n-9$; -1; 8
b) 430

196. $\frac{9}{10}$; $200\frac{1}{19}$

197. 1; $\frac{10}{7}$; $\frac{1}{7}(10n-3)$

198. 3; 30 **199.** 8, -4

200. a) 4 b) 2 c) 170

201. a) $3\frac{1}{4}; 3\frac{13}{16}; 3\frac{61}{64}$

b) $U_n=4-\left(\frac{1}{4}\right)^{n-1}$ c) 4

d) $S_n=4n+\frac{4}{3}(1-\left(\frac{1}{4}\right)^n$

202. 68 **203.** 6.2

204. 85.25 **205.** $23\frac{1}{2}$

206. 209 **207.** $1\frac{1}{2}, 3, 4\frac{1}{2}$

208. 273.25 **209.** 8 **210.** $1\frac{2}{3}$

211. a) 1; $\frac{2}{3}$; $\frac{4}{9}$

b) $U_n=3(1-\left(\frac{2}{3}\right)^n$ c) 18

212. 2, 0, 4, 10; 11 **213.** 1

214. a) 6 b) 603

215. -2 **216.** 10

217. $r=\frac{1}{3}$ or $-\frac{4}{3}$; $S_\infty=\frac{3}{2}$

218. i) $r=\frac{2}{3}$ and $a=81$ ii) 0

219. a) 910 b) 1.043×1.03^9

220. 3, 12, 48, 192; G.P series
$3(4^{n-1})$

221. $\sqrt{2}-1$ **222.** $\frac{1}{2}(4n-5)$

223. $4\frac{1}{2}$ **224.** $\frac{5}{4}$ **225.** $\frac{32}{3}$

226. $-\frac{1}{4}$ **227.** 11 **228.** 9

229. 120 **230.** 8

231. $5\sqrt{2}-7$ **232.** 59

233. a) $1\frac{1}{6}, 1\frac{11}{18}, 1\frac{25}{54}$

b) $\frac{3}{2}+(-\frac{1}{3})^{n-1}$ c) $\frac{3}{2}$

d) $\frac{3n}{2}+\frac{3}{4}\left(1-(-\frac{1}{3})^{n-1}\right)$

234. 625 **235.** n

236. a) $3, 2\frac{1}{2}, 2\frac{1}{4}$

b) $2+(\frac{1}{2})^{n-2}$ c) 2

d) $2n+4\left(1-(\frac{1}{2})^1\right)$

237. 4 **238.** i) 3 ii) 14

239. i) $2\sqrt{5}$ units ii) $(\frac{9}{2}, 3)$

iii) $(5, 5\frac{2}{3})$ $(6, 4\frac{1}{3})$

iv) 10 sq units

240. $P(11,-2)$ $Q(15, 0)$

241. a) $(\frac{7}{3}, 7)$

b) $3x^2+3y^2-14x-42y+80=0$

c) $T_1(0, 11.73)$ and $T_2(0, 2.27)$

242. a) They meet
(gradient $\frac{2}{3}; -\frac{2}{3}$)

b) They are parallel (gradients are both $\frac{1}{2}$) c) They are perpendicular (gradients are $\frac{1}{3}, -3$)

243. a) $y=-7x+23$
b) $y=-3x+19$
c) $3y=5x+15$

244. 60.3°

245. i) $3y-x=23$ ii) $A(6, 2)$
iii) 5 units **246.** $\sqrt{6}$

247. $x^2+y^2+4x-4y+6=0$

248. $y+x=0$ **249.** $\frac{3}{2}$; $(\frac{3}{2}, 0)$

250. $3y-x+1=0$

251. $x^2+y^2-8x-10y+21=0$

252. $5x-2y-26=0$

253. $\{(x, y): x-3y=9\}$

254. $6x-6y+85=0$

255. $2x-5y-10=0$

256. i) $2y-3x-1=0$

ii) $3y+2x-8=0$

257. $x^2+y^2-6x-14y+42=0$

258. $y+x-4=0$

259. $x+4y-9=0$

260. 27 **261.** 1 **262.** 12

263. 88 **264.** 0.9

265. $-\frac{2}{5}+\frac{3}{10}\sqrt{3}$

266. $\frac{\sqrt{2}(\sqrt{3}-1)}{4}$ **267.** $4+2\sqrt{3}$

268. $2+\sqrt{3}$ **269.** sinx

270. $2-\sqrt{2}$ **271.** $3+2\sqrt{3}$

272. $6+4\sqrt{2}$ **273.** 1

274. 1 **275.** $-\frac{56}{65}$ **276.** $-\frac{4}{5}$

277. (30.6°) **278.** 0.960

279. $\frac{-8+3\sqrt{5}}{15}$ **280.** 1

281. $\tan 2\theta$ **282.** $\frac{x^2}{9}+\frac{y^2}{4}=1$

283. 15° **284.** a) $\sqrt{13}$ b) 33.7°

285. 6 **286.** $0.3\sqrt{10}$

288. $\frac{24}{25}$; $-\frac{7}{25}$

289. $\frac{17}{32}$ **290.** $2-\sqrt{3}$

291. increasing ($x<1$ or $x>2$)
decreasing ($1<x<2$)

292. $a=-2$ $b=3$

293. $x-y-6=0$

294. $21(3x+1)^6$ **295.** $\frac{-4x}{(1+x)^2}$

296. $5-\frac{3}{x^2}$ **297.** $\frac{2}{(1-x)^2}$

298. $y-4x+7=0$

299. 6.5 **300.** 0, 4 **301.** 1

302. a) 6
b) $6x-y+2=0$; $(-\frac{2}{5},\frac{2}{5})$

303. $5x-y-1=0$; (2,4); (4, −8)

304. 0 **305.** a) 2 b) -1 ms^{-2}

306. a) $b=-3$ and $c=10$ b) $3\frac{1}{3}$

307. (1, 6) **308.** 6 cm x 6 cm

309. 4.5 s **310.** 4%

311. $32\pi\text{cm}^3/\text{s}$ **312.** $4\frac{1}{2}$

313. 0 **314.** $y=x^2+\frac{1}{x}-1$

315. $\frac{x^4}{4}+\frac{x^3}{3}-x^2-2x+c$

316. $\frac{x^4}{4}-\frac{x^2}{2}+\frac{1}{x}+c$ **317.** $-\frac{7}{6}$

318. 22 **319.** $\frac{4}{3}$

320. $y=x^3+4$ **321.** 16.25

322. t^3-t^2+2 **323** 4 m/s

324. 32π **325.** $16\frac{2}{3}$

326. $\frac{4}{3}$ **327.** 10

328. a) $\frac{1}{2}$ b) $y=5+3x-x^2$

329. 40π **330.** −18

331. b) i) 50 ii) 63.5% iii) 0.49

332. a) 6 b) 1.78 **333.** 4

334. a) 3.0 b) 3 c) 2.82

335. 4 **336.** 0.36 **337.** 2

338 14 **339.** 69 **340.** 5

341. 1.41 **342.** 240th term

343. 9.5 **344.** 2 **345.** 2

346. a) 3.0 b) 1.9

347. a) 3.0 cm b) 4 cm

348. 0 **349.** $\frac{3}{8}$ **350.** $\frac{1}{9}$

351. $\frac{1}{4}$ **352.** 220 **353.** 24

354. $\frac{2}{5}$ **355.** $\frac{48}{95}$ **356.** 0.03

357. $\frac{5}{6}$ **358.** 9 **359.** $\frac{3}{4}$

360. 5 **361.** $-15(2-3x)^4$

362. $x<\frac{1}{2}$ **361.** $-51\frac{1}{2}$

362. 5 **363.** $\begin{pmatrix}4\\12\end{pmatrix}$

366. i) $\frac{60}{143}$ ii) $\frac{3}{1001}$

367. b) 14.60 m/s

368. i) a) 83.4 kg b) 15.5 kg
ii) 0.68

Theory Paper 1

1. a) $r=2401$ b) $\frac{1}{x}, x\neq 0$

2. a) $2\sin a\cos a$ b) -30° or 210°

3. $\frac{3}{2(1-x)}+\frac{7}{2(1+x)}$

4. a) $\frac{4}{(1-x)^3}$ b) $\frac{15}{4}\pi\,\text{cm}^2 s^{-1}$ or

$3.75\pi\,\text{cm}^2 s^{-1}$

5. i) $r=3$ ii) $S_{10}=59048$

6. a) (-1, -3) b) $x=-1$ or $x+1=0$

7. i) 34.6° ii) $8a-3b-14=0$

8. i) $\frac{4}{15}$ ii) $\frac{2}{9}$

9. a) i) -8,4 ii) $x=-1,\frac{1}{2}$ b) 52

10. a) $\frac{1}{2}p\%$ b) $\frac{1}{6}\text{units}^2$

11. i) $A=\begin{pmatrix}1&3\\2&-5\end{pmatrix}$ and $B=\begin{pmatrix}4&3\\1&-2\end{pmatrix}$

ii) $\begin{pmatrix}8&-33\\-29&65\end{pmatrix}$

iii) the point is (−12.5, 18)

12. a) i) the truth set is $\{0,\frac{1}{9}\}$

ii) the truth set is {6}

b) when $x=1, y=-1$

13. i) $F_1=\begin{pmatrix}0\\-2\end{pmatrix}$N,

$F_2=\begin{pmatrix}-\sqrt{3}\\1\end{pmatrix}$N,

$F_3=\begin{pmatrix}\frac{5}{2}\sqrt{3}\\2.5\end{pmatrix}$N

14. a) $\overrightarrow{PS}=\frac{1}{2}(\underline{u}+\underline{v})$

b) $(x, y)\rightarrow(1,-9)$

Theory Paper two

1. a) $\{x: x\in R, \text{except } -1<x<1\}$

b) g is a one-to-one function

c) $\frac{x-9}{x^2+5x-6}=\frac{-8}{7(x-1)}+\frac{15}{7(x+6)}$

2. 0.06 or 6%

3. $f(x)=-\frac{1}{x}+x^2-3x+\frac{9}{2}$

4. a) $-22-13\sqrt{3}$ b) i) $\frac{-3}{2}$

ii) $\frac{1}{2}x-5$

6. 0.635 **7.** 12N

8. i) $e=\frac{-3}{4}$ ii) $x^{-1}=\frac{-3}{2}-x$

iii) $\frac{-19}{10}$ or −1.9

9. $m=2$ and $k=-9$

b) i) $|PN|=x^2-4x+4+y^2$

ii) The locus of P is a circle with centre (0, 0) and radius, 3 units

10. a) i) maximum (-1, 1) and minimum (0, 0)

ii) $x=0$ or $x=-\frac{3}{2}$

b) $\frac{27}{32}$ square units

11. a) $T=\begin{pmatrix}2&-1\\3&2\end{pmatrix}$; $M=\begin{pmatrix}k&1\\3&k\end{pmatrix}$

b) $a=5$ and $k=4$ c) (15, 8)

12. a) i) $a=\begin{pmatrix}3000\\1000\end{pmatrix}\text{ms}^2$

ii) $\theta=26.6°$ b) $B\hat{A}C=117°$

13. a) $|F| = 5.77$ N

b) $R = \begin{pmatrix} 2+4\sqrt{2} \\ -\sqrt{2} \end{pmatrix}$N or

$(7.8\text{ N}, 100.5°)$

Theory Paper 3

1. a) $4\frac{1}{4}; 4\frac{1}{16}; 4\frac{1}{64}$

b) $U_n = 4 + (\frac{1}{4})^{n-1}$ c) 4

2. a) $\frac{-x}{x-2}, x \neq 2$

b) $gof = \frac{1}{1+4x}, x \neq -\frac{1}{4}$

3. $y = 2x^2 - x - 3$

4. $x < -\frac{1}{2}$ and $x < 3$

5. $p = -4$ and 4 **6.** $p = \frac{4}{3}$ and $q = \frac{2}{3}$

7. $I = 2.66$ **8.** b) mode = 66.5g

9. a) i) 3 ii) 2 iii) 120

b) i) $(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$

ii) 3.873

10. $x = 7.5°, 82.5$

11. a) (-1, 7) b) i) $TS = \begin{pmatrix} 13 & 7 \\ 11 & 5 \end{pmatrix}$

ii) (5, 7)

12. a) $\overrightarrow{OP} = \frac{1}{2}\underline{a}$; $\overrightarrow{OQ} = \frac{1}{2}\underline{d}$

b) $(5, -1)$

13. a) 4.8 kg b) i) 80 m

ii) $t = 1$ s and 7 s

14. a) i) $\bar{x} = 33.06$

ii) 2.444 (3 d.p)

b) i) 350 ii) 150

Theory Paper 4

1. a) c b) b and d c) commutative

2. $2x+2$ **3.** $3(x-1)^2 + 7$; $7x = 1$

4. $4y + x - 4 = 0$

5. $x = -1$ or 2 **6.** 115°

7. 0.431 **8.** 7.1

9. i) $18 - 11x + 2x^2$ ii) $x = \frac{9}{4}$

10. a) $r = \frac{17}{7}$, $s = \frac{5}{7}$ b) 45, 26

11. $p = 4$, $q = 4$ and $r = 3$

12. a) (6.4i +4.8j)Ns ii) 8 Ns

b) i) 60 m ii)) 31.5 m

iii) 2.55 iv) t=25 or 35

13. a) 14 km/h b) i) (67.3 N, 136.7°) ii) (13.5 m/s^2, 136.7°

Theory Paper 5

1. $\frac{35}{4}$

2. $x^2 + y^2 - 4x - 2y - 15 = 0$

3. $x = 2$ **4.** 9%

5. 2 **6.** $\frac{87}{108}$ **7.** $2(x+2)$

8. 1.1 m/s^2

9. a) i) $x^5 + 10x^4y + 40x^3y^2 +$

$80x^2y^3 + 80xy^4 + 32^5$

ii) 1.104 b) i) $m = 5$;

$f(x) - (x+1)(x-2)(2x+1)$

iii) $x = -1, 2, -\frac{1}{2}$

10. a) i) 16 ii) α) commutative

β) associative iii) –6

b) $d = 6$; $a = 1$

11. a) $\frac{81}{8} - 2(x + \frac{5}{4})^2$; $\frac{81}{8}$

b) $y - 5x + 11 = 0$;

$y + 3x + 3 = 0$

12. $A: (x, y) \to (-x + 2y, 3x - 4y)$

$B: (x, y) \to (-4x - 7y, -5x + 2y)$

b) i) $A^{-1} = \begin{pmatrix} 2 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}$

ii) $(4, 3\frac{1}{2})$ c) $(-13, 3)$

13. a) $\frac{2\sqrt{10}}{5}(3i - 4j)$

b) i) $\mathbf{p} = (10, 036.88°)$;

$\mathbf{q} = (\frac{25}{3}, 286.26°)$

ii) 110.61°

14. a) $n = 7$ b) 0.71

15. a) i) 1.8 m/s ii) 4 m/s

b) i) (3**i** – 4**j**) Ns ii) 5 Ns

Theory Paper 6

1. $x = 2$ and $y = 3$ **2.** $\frac{25}{9}$

3. 0.999 **4.** 45° **5.** $6x$

6. 0.4 cm **7.** 0.0456

8. (-**i** + 7**j**)ms^{-1}

9. a) $f^{-1} = \frac{1}{5}(x+9)$

b) i) $3 * 2 = \frac{15}{7} - \frac{5\sqrt{2}}{7}$ ii) $x = \frac{2}{3}$

c) i) 6 or 3 ii) $\frac{1}{3}$ or $\frac{2}{3}$

10. a) $x = 4$ or $x = 7$

b) i) $a = 0$ and $a = -2$

ii) $k = 2$ or k 10

11. a) i) $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ and

$B = \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}$

ii) The image is (8, 3)

iii) $a = \frac{15}{14}$ and $b = \frac{2}{7}$

b) $x = 2$

12. a) $y = -6$ b) i) $F = \begin{pmatrix} 12 \\ -4 \end{pmatrix}$

b) ii) $a = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ms^{-2}

13. a) $F_3 = (4.2N, 162°)$

b) 1.8027ms^{-2}

Theory Paper 7

1. a) 30240 b) i) 4410 ii) 1960

2. $\frac{dy}{dx} = 4x$

3. $k = 12$ or $k = -12$

4. a) $v = 2.5$ ms^{-1}

b) $s = \left(t^2 + \frac{t^3}{3}\right)m$

5. a) $2\log_{10} 10 = 2$

b) $-\frac{7}{5} + \frac{2\sqrt{6}}{5}$

6. a) $y = 42$ b) $p = \frac{287}{1600}$ or 0.1739

7. $\overrightarrow{AE} = \begin{pmatrix} 3 \\ -10 \end{pmatrix}$

8. a) i) $g^{-1}(x) = 3x + 3$

ii) $fog^{-1} = 6x + 9$

b) i) $p = -10$

c) $Cos(A + B) = \frac{1}{5}\left[\frac{3}{8} - \sqrt{5}\right]$

9. a) $F_1 = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$N, $F_2 = \begin{pmatrix} 0 \\ -16 \end{pmatrix}$N,

$F_3 = \begin{pmatrix} 4 \\ 6.9282 \end{pmatrix}$N,

$F_4 = \begin{pmatrix} -10.3923 \\ 6 \end{pmatrix}$N

b) i) The resultant force,

$F = \begin{pmatrix} 3.61 \\ -3.07 \end{pmatrix}$N

ii) The acceleration with which the body starts to move, $a = 5.93$ ms^{-2}

10. a) 6.63 b) 7.98

11. ii) 4 m^3

12. a) 0.59 ms^{-1}

b) i) $v = 24 - 10t$

ii) 26 m/s

13. $c = 2$

14.

	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	
$\frac{dy}{dx}$	4	−2	7	4
$\frac{d^2y}{dx^2}$	9	−15.9	0	−9

15. $x = \frac{8}{17}a$

16. a) $y = x^2 + 1$

b) $\frac{4}{3}$ sq.units

c) $\frac{16}{15}\pi$ cubic units

Chapter 34 – Answers to objective test

1 D	61 C	121 C	181 D	241 A	301 B	361 D	421 C	481 B	541 D
2 A	62 D	122 D	182 D	242 D	302 C	362 D	422 A	482 B	542 B
3 D	63 B	123 A	183 A	243 A	303 D	363 D	423 D	483 C	543 C
4 D	64 B	124 B	184 B	244 C	304 B	364 A	424 C	484 A	544 B
5 B	65 A	125 B	185 D	245 D	305 C	365 D	425 B	485 D	545 A
6 C	66 D	126 B	186 C	246 A	306 A	366 C	426 D	486 B	546 C
7 B	67 D	127 A	187 C	247 C	307 C	367 C	427 C	487 A	547 B
8 A	68 D	128 A	188 A	248 C	308 B	368 A	428 B	488 D	548 D
9 D	69 D	129 B	189 D	249 B	309 B	369 D	429 C	489 B	549 D
10 B	70 A	130 B	190 B	250 A	310 C	370 C	430 D	490 D	550 B
11 A	71 C	131 A	191 B	251 B	311 B	371 B	431 B	491 A	551 B
12 A	72 B	132 B	192 C	252 D	312 C	372 D	432 C	492 B	552 A
13 B	73 A	133 D	193 A	253 C	313 D	373 C	433 A	493 A	553 A
14 C	74 A	134 A	194 D	254 A	314 D	374 D	434 D	494 C	554 D
15 D	75 A	135 C	195 A	255 C	315 A	375 C	435 A	495 D	555 B
16 C	76 B	136 B	196 B	256 D	316 C	376 A	436 A	496 A	556 C
17 D	77 B	137 A	197 C	257 A	317 D	377 D	437 A	497 C	557 B
18 B	78 D	138 C	198 A	258 B	318 B	378 C	438 B	498 D	558 A
19 C	79 B	139 B	199 B	259 D	319 A	379 C	439 D	499 C	559 A
20 A	80 D	140 C	200 D	260 A	320 C	380 C	440 C	500 D	560 D
21 B	81 B	141 B	201 A	261 C	321 B	381 D	441 B	501 C	561 A
22 D	82 A	142 C	202 C	262 B	322 D	382 A	442 C	502 A	562 B
23 A	83 B	143 D	203 C	263 A	323 A	383 C	443 D	503 B	563 B
24 D	84 A	144 C	204 C	264 C	324 C	384 C	444 B	504 C	564 C
25 C	85 C	145 B	205 C	265 B	325 D	385 C	445 B	505 B	565 B
26 B	86 C	146 C	206 D	266 C	326 D	386 C	446 C	506 D	566 D
27 C	87 B	147 C	207 B	267 B	327 A	387 C	447 C	507 A	567 A
28 D	88 D	148 A	208 B	268 D	328 B	388 D	448 A	508 D	568 D
29 A	89 B	149 B	209 C	269 C	329 D	389 A	449 C	509 B	569 B
30 B	90 D	150 C	210 D	270 A	330 B	390 D	450 B	510 B	570 C
31 C	91 D	151 D	211 C	271 D	331 A	391 D	451 C	511 D	571 B
32 A	92 D	152 A	212 B	272 B	332 D	392 C	452 B	512 C	572 D
33 C	93 B	153 C	213 D	273 C	333 B	393 A	453 D	513 B	573 A
34 B	94 D	154 B	214 A	274 B	334 B	394 C	454 B	514 C	574 B
35 C	95 B	155 A	215 D	275 D	335 B	395 B	455 C	515 B	575 C
36 D	96 D	156 D	216 C	276 A	336 C	396 D	456 B	516 A	576 B
37 B	97 C	157 D	217 B	277 C	337 C	397 C	457 A	517 B	577 D
38 B	98 C	158 B	218 A	278 B	338 A	398 D	458 C	518 C	578 C
39 C	99 C	159 C	219 C	279 A	339 D	399 A	459 A	519 D	579 B
40 D	100 B	160 B	220 D	280 B	340 A	400 C	460 D	520 B	580 D
41 A	101 D	161 D	221 A	281 A	341 B	401 B	461 B	521 C	581 A
42 D	102 D	162 D	222 B	282 D	342 A	402 B	462 C	522 B	582 D
43 B	103 A	163 A	223 A	283 A	343 B	403 D	463 D	523 C	583 C
44 C	104 C	164 D	224 D	284 A	344 C	404 A	464 A	524 A	584 C
45 C	105 C	165 C	225 B	285 C	345 C	405 D	465 C	525 D	585 D
46 A	106 B	166 A	226 C	286 C	346 A	406 A	466 D	526 C	586 A
47 C	107 C	167 D	227 D	287 D	347 C	407 A	467 A	527 A	587 D
48 D	108 A	168 A	228 C	288 B	348 A	408 B	468 D	528 D	588 A
49 D	109 B	169 B	229 C	289 B	349 A	409 B	469 C	529 C	589 C
50 D	110 B	170 D	230 B	290 A	350 B	410 A	470 B	530 A	590 B
51 B	111 B	171 A	231 A	291 D	351 D	411 C	471 C	531 C	591 C
52 B	112 D	172 B	232 D	292 B	352 C	412 B	472 B	532 A	592 B
53 A	113 C	173 C	233 A	293 A	353 A	413 D	473 D	533 B	593 D
54 C	114 D	174 A	234 C	294 B	354 A	414 A	474 A	534 C	594 A
55 A	115 A	175 D	235 B	295 D	355 D	415 B	475 A	535 A	595 D
56 B	116 D	176 D	236 B	296 C	356 B	416 D	476 D	536 A	596 D
57 B	117 D	177 B	237 C	297 B	357 D	417 B	477 A	537 D	597 C
58 B	118 C	178 B	238 C	298 C	358 A	418 A	478 C	538 B	598 D
59 C	119 C	179 C	239 C	299 A	359 D	419 D	479 D	539 C	599 B
60 C	120 C	180 C	240 B	300 D	360 D	420 C	480 C	540 B	600 A